

# Mathematics

## Python

```
from sympy import *
from dtumathtools import *
init_printing()
```

## 1st semester

### ▼ $\sum$ rules

$$\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$$

$$\sum_{i=1}^n c = n \times c, \text{ where } c \text{ is a constant}$$

$$\sum_{i=1}^n i = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i^3 = 1^3 + 2^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

### ▼ Determinant

$$A \in \mathbb{C}^{m \times m}$$

In the case that  $m = 2$ :

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\det(A) = ad - bc$$

In any other case:

$$\det(A) = \sum_{i=1}^m (-1)^{i+j} a_{ij} \det(A_{ij})$$

Where  $i$  is the row number and  $j$  the column number,  $a_{ij}$  is the element of the Matrix with row  $i$  and column  $j$ , and  $A_{ij}$  is the submatrix of  $A$  with the row  $i$  and column  $j$  removed.

```
A = Matrix([[1,3,-3],[2,1,4],[2,-1,8]])
#if det=0 linearly dependent
det(A)
```

### ▼ Eigenvalues & Eigenvectors

$$Av = \lambda v$$

where  $A \in \mathbb{C}^{m \times m}$  (Square Matrix)  
eigenvectors  $\rightarrow v \in \mathbb{C}^{m \times 1}$  (Column Vector)  
eigenvalues  $\rightarrow \lambda \in \mathbb{C}^{m \times m}$  (Diagonal Matrix)

The eigenvalues are found by solving the following system:

$$\det(A - \lambda I) = 0$$

where  $I$  is an identity matrix with the same size as  $A$

```
A = Matrix([[-1,4,4],[0,7,8],[0,-4,-5]])
#the vectors can be multiplied by k and still be correct
A.eigenvecs()
```

## Matrixes and Vector

### ▼ Basic stuff

#### ▼ Rank, Kernel, and nullity of a matrix

For an arbitrary matrix  $A \in \mathbb{F}^{m \times n}$

$\text{rank}(A)$  or  $\rho(A)$  denotes the rank of a matrix  $A$ , the maximum number of rows or columns it contains. We can compute the rank by reducing the matrix to its reduced row echelon form, since the rank of a matrix is equal to the number of pivots.

$\ker(A)$  denotes the kernel of a matrix  $A$ , which is the set of vectors that satisfy,

$$A \cdot v = 0$$

$\text{null}(A)$  denotes the nullity of a matrix, which is the dimension of the kernel of that matrix.

$$\text{null}(A) = \dim \ker(A)$$

Lastly we can assume the following to be true,

$$\rho(A) + \text{null}(A) = n$$

### ▼ Span

The span of a vector space is the set of all possible linear combinations of vectors.

To compute the span of the kernel of a matrix  $A$  we would solve the following,

$$A\mathbf{x} = 0$$

Where  $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Then, using a system of equations and changing  $x_1, \dots, x_{n-1}$  to variables  $a, b, \dots, z$  we can reach,

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = a \begin{bmatrix} c_{1,a} \\ \vdots \\ c_{n,a} \end{bmatrix} + b \begin{bmatrix} c_{1,b} \\ \vdots \\ c_{n,b} \end{bmatrix} + \dots + z \begin{bmatrix} c_{1,z} \\ \vdots \\ c_{n,z} \end{bmatrix}$$

### ▼ Magnitude/Norm:

$$\mathbf{x} = [x_1, x_2, \dots, x_n]$$

#### ▼ Real

$$\mathbf{x} \in R^n$$

$$\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

#### ▼ Complex

$$\mathbf{x} \in C^n$$

$$\sqrt{\langle \mathbf{x}^*, \mathbf{x} \rangle}$$

### ▼ Orthogonal Vectors

Vectors are only orthogonal when they are perpendicular, we can show that they are perpendicular if the inner product between them is 0:

$$x_1, x_2 \in \mathbb{R}^n$$

the vectors are only orthogonal if:  
 $\langle x_1, x_2 \rangle = 0$

### ▼ Orthonormal basis

Vectors only constitute an orthonormal basis when they are orthogonal and normalized (have norm of 1), they can be normalized by doing the following:

$$u_1 = \frac{x_1}{|x_1|}$$

Keep in mind the code keeps the  $i$  in the normalized vectors

```
v1 = Matrix([I, 1, 1, 0])
u1 = v1.normalized()
```

### ▼ Orthogonal Projection on Vectors

The projection of a vector  $x$  on another vector  $y$  is given by:

$$\text{Proj}_y(x) = \frac{\langle x, y \rangle}{\langle y, y \rangle} y$$

where  $u = \frac{y}{|y|}$

### ▼ Orthogonal Projection on planes

The projection of a vector  $x$  on a plane  $Y$ , let  $u_1$  and  $u_2$  be an orthonormal basis for  $Y$ :

$$Y = \text{span}(u_1, u_2)$$

$$U = [u_1, u_2]$$

$$P = UU^*$$

$$\text{proj}_Y(x) = Px$$

For any vector  $x \notin Y$ ,  $x - \text{proj}(x)$  belongs to  $Y^\perp$  (complement of  $Y$ )

### ▼ Gram Schmidt

This procedure takes in a number of linearly independent vectors and orthonormalizes them (makes them perpendicular to each other).

$$v_1, v_2, v_3 \in \mathbb{R}$$

$$A = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$$

$$\det(A) \neq 0$$

$$u_1 = v_1$$

$$u_2 = v_2 - \text{proj}_{u_1}(v_2)$$

$$u_3 = v_3 - \text{proj}_{u_1}(v_3) - \text{proj}_{u_2}(v_3)$$

Keep in mind Gram-Schmidt process does not normalize them.

```
v1 = Matrix([1,1,1,1])
v2 = Matrix([3*I,I,I,3*I])
v3 = Matrix([2,0,-2,4])
GramSchmidt([v1, v2, v3], orthonormal = True)
```

### ▼ Diagonalizing Matrixes

$$A \in \mathbb{C}^{m \times m}$$

$$A = Q\Lambda Q^{-1}$$

$$Q = \begin{bmatrix} x_1 & x_2 & \dots & x_m \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix}$$

where  $x_1$  is an eigenvector of  $A$   
with a corresponding eigenvalue of  $\lambda_1$

```
A = Matrix([[4,-1,-1],[6,-3,-1],[-6,5,3]])
#will return True or False
A.is_diagonalizable()
```

```
A = Matrix([[4,-1,-1],[6,-3,-1],[-6,5,3]])
V, Lambda = A.diagonalize()
#be careful columns might be switched
V, Lambda
```

## ▼ Types of Matrixes

### ▼ Unitary Matrix

A unitary matrix is a complex square matrix whose columns (and rows) are orthonormal. It has the remarkable property that its inverse is equal to its conjugate transpose.

$$A \in \mathbb{C}^{n \times n}$$

$A$  is considered a unitary matrix if:

$$A^{-1} = A^*$$

### ▼ Normal Matrix

A matrix is said to be a Normal matrix if the pre and post-matrix multiplication of conjugate transpose of it with itself is commutative

$$A \in \mathbb{C}^{n \times n}$$

$A$  is considered a normal matrix if:

$$AA^* = A^*A$$

### ▼ Invertible Matrix

An invertible matrix is a square matrix such that the product of the matrix and its inverse generates the identity matrix.

$$A \in \mathbb{C}^{n \times n}$$

$A$  is considered a invertible if:

$$\det(A) \neq 0$$

### ▼ Orthogonal Matrix

An orthogonal matrix, or orthonormal matrix, is a real square matrix whose columns and rows are orthonormal vectors.

$$A \in \mathbb{R}^{n \times n}$$

$A$  is considered orthogonal if:

$$A^T = A^{-1} \text{ or } AA^T = I$$

### ▼ Symmetric Matrix

A symmetric matrix is a square matrix that is equal to its transpose.

$$A \in \mathbb{C}^{n \times n}$$

$A$  is considered symmetric if:

$$A = A^T$$

### ▼ Hermitian Matrix

A Hermitian Matrix is a complex square matrix that is equal to its conjugate transpose (adjoint)

$$A \in \mathbb{C}^{n \times n}$$

$A$  is considered Hermitian if:

$$A = A^*$$

### ▼ Description of Sets

Let  $A$  be a subset of  $\mathbb{R}$ , defined,

$$A \subseteq \mathbb{R}^n$$

#### ▼ Sets that are open, closed or neither

- Open set:

$$]a, b[$$

- Closed set:

$$[a, b]$$

- Neither set:

$$[a, b[ \text{ or } ]a, b]$$

#### ▼ Sets that are bounded or unbounded

- Bounded set:

A set that has finite bounds, like  $a$  and  $b$ , for

$$a \neq \infty \wedge b \neq \infty$$

- Unbounded set:

A set that has at least one bound, like  $a$  and  $b$ , for

$$]-\infty, b] \vee ]a, \infty] \vee ]-\infty, \infty[$$

### ▼ Connected sets

A set  $B \subset \mathbb{R}^n$  is said to be connected if, for each choice of points  $\underline{x}_1, \underline{x}_2 \in B$ , there exists a continuous function  $r : [0, 1] \rightarrow B$  such that  $r(0) = \underline{x}_1$  and  $r(1) = \underline{x}_2$ .

#### ▼ Simply connected sets

Let  $U \subseteq \mathbb{R}^n$  be a connected set. Let  $\underline{x}$  and  $\underline{y}$  be any 2 points in  $U$ , and  $r_i : [0, 1] \rightarrow U, i = 1, 2$ , be 2 continuous curves from  $\underline{x}$  to  $\underline{y}$  in  $U$ , that is,  $r_i([0, 1]) \subset U, i = 1, 2, \underline{x} = r_1(0) = r_2(0)$  and  $\underline{y} = r_1(1) = r_2(1)$ . If for any 2 such curves  $r_1$  can be continuously deformed into  $r_2$  without leaving  $U$ , the set  $U$  is simply connected.

A 2D set with a “hole” wouldn’t be a simply connected set

#### ▼ Star-shaped sets

A subset  $S \in \mathbb{R}^n$  is called a star-shaped set if there exists a point (a “center” of  $S$ )  $\underline{x} \in S$  such that the line segment between  $\underline{x}$  and any other point in  $S$  is contained in  $S$ .

### ▼ Standard Equations for the Three Typical Conic Sections

#### ▼ Ellipse

An ellipse in the  $(x, y)$  plane with center  $(c_1, c_2)$ , semi-axes  $a$  and  $b$ , and symmetry axes  $x = c_1$  and  $y = c_2$  has the standard equation:

$$\frac{(x - c_1)^2}{a^2} + \frac{(y - c_2)^2}{b^2} = 1$$

#### ▼ Hyperbola

A hyperbola in the  $(x, y)$  plane with center  $(c_1, c_2)$ , semi-axes  $a$  and  $b$ , and symmetry axes  $x = c_1$  and  $y = c_2$  has the standard equation

$$\frac{(x - c_1)^2}{a^2} - \frac{(y - c_2)^2}{b^2} = 1$$

Alternatively (if it isn't horizontally but vertically oriented):

$$\frac{(y - c_2)^2}{a^2} - \frac{(x - c_1)^2}{b^2} = 1$$

#### ▼ Hyperbola

A parabola in the  $(x, y)$  plane with its vertex (stationary point)  $(c_1, c_2)$ , and symmetry axis  $x = c_1$  has the standard equation:

$$y - c_2 = a(x - c_1)^2$$

Alternatively, if the parabola is not vertically but horizontally oriented, in which case the symmetry axis becomes  $y = c_2$ :

$$x - c_1 = a(y - c_2)^2$$

## Functions

### ▼ Types of functions

#### ▼ Surjective

- When the image set is equal to the co-domain

#### ▼ Injective

- When there are no two inputs that give the same output

#### ▼ Bijective

- When it is both surjective and injective

### ▼ Dot and inner product in $\mathbb{R}^n$

The dot product of two vectors  $\mathbf{x}$  and  $\mathbf{y}$  in  $\mathbb{R}^n$  is defined as,

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=0}^n x_i y_i$$

The norm of a vector  $\mathbf{x}$  in  $\mathbb{R}^n$  is defined as,

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\sum_{i=0}^n x_i^2}$$

Although this notation is often used, the notation known as the standard inner product is better

$$\langle \mathbf{x}, \mathbf{y} \rangle$$

The known formula for the angle  $\theta$  between two vectors is,

$$\cos(\theta) = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|}$$

### ▼ Inner product in $\mathbb{C}^n$

The standard inner product of two vectors  $\mathbf{x}$  and  $\mathbf{y}$  in  $\mathbb{C}^n$  is defined as,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=0}^n x_i \bar{y}_i$$

The norm of a vector  $\mathbf{x}$  in  $\mathbb{C}^n$  is defined as,

$$\|\mathbf{x}\| = \sqrt{\sum_{i=0}^n (a_i^2 + b_i^2)}$$

▼ Inner products and norms in abstract places

Let  $V$  be a vector space over  $\mathbb{F}$ . If  $\mathbb{F} = \mathbb{R}$ , we say that  $V$  is a real vector space, and if  $\mathbb{F} = \mathbb{C}$ , we say that  $V$  is a complex vector space.

A norm on a vector space  $V$  is a function that assigns each vector  $\mathbf{x} \in V$  a non-negative real number  $\|\mathbf{x}\| \in [0, \infty[$  that we call the norm of  $\mathbf{x}$  and consider the length of  $\mathbf{x}$ . The norm function has to satisfy the following properties:

- non-negativity:  $\|\mathbf{x}\|$  is a (real and) non-negative,
- non-degeneracy:  $\|\mathbf{x}\| = 0$  if and only if  $\mathbf{x} = 0$ ,
- scaling:  $\|c\mathbf{x}\| = |c|\|\mathbf{x}\|$ ,
- triangle inequality:  $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

A vector space  $V$  with a norm is called a normed vector space or simply a normed space.

An inner product is function that assigns to each pair of vectors  $(\mathbf{x}, \mathbf{y})$  a scalar in  $\mathbb{F}$ , denoted by  $\langle \mathbf{x}, \mathbf{y} \rangle$ . The inner product function  $(\mathbf{x}, \mathbf{y}) \rightarrow \langle \mathbf{x}, \mathbf{y} \rangle$  has to satisfy the following properties:

- non-negativity:  $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ ,
- non-degeneracy:  $\langle \mathbf{x}, \mathbf{x} \rangle = 0$  if and only if  $\mathbf{x} = 0$ ,
- (conjugate) symmetry:  $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ ; note, that for a real vector space, this property is just symmetry,  $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ ,
- Linearity:  $\langle c\mathbf{x} + d\mathbf{y}, \mathbf{z} \rangle = c\langle \mathbf{x}, \mathbf{z} \rangle + d\langle \mathbf{y}, \mathbf{z} \rangle$

A vector space  $V$  with an inner product is called an inner space

▼ Vector space  $P_n([a, b])$  equipped with the  $L^2$ -inner product

To calculate the inner product of two polynomials  $p$  and  $q$  we use the following formula,

$$\langle p, q \rangle = \int_a^b p(x)q(x)dx$$

▼ Quadratic form

The quadratic form is a way of writing second-degree equations with multiple variables with of matrixes.

$$x^T = [x_1, x_2, \dots, x_n]$$

$$A \in \mathbb{R}^{n \times n}$$

Where the Matrix A is Symmetric

$$b \in \mathbb{R}^n$$

$$q(x) = x^T Ax + b^T x + c$$

▼ Changing the Quadratic forms' basis so as not to have mixed terms

▼ Non-symmetric  $\mathbf{A}$  matrix

If we are given a quadratic form with a  $\mathbf{A}$  matrix that is not symmetric we can convert it into a symmetric as,

$$\mathbf{A}' = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

▼ With linear terms

We know the quadratic form to be,

$$q(x) = \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x} + c$$

We also know that for every real quadratic form there is an orthogonal diagonalization that removes all mixed terms, the symmetric  $\mathbf{A}$  matrix must be diagonalized as,

$$\mathbf{\Lambda} = \mathbf{Q}^{-1} \mathbf{A} \mathbf{Q}$$

Since  $\mathbf{A}$  is symmetric, its eigen vectors are all orthogonal, and therefore the  $\mathbf{Q} = \mathbf{Q}^T$  matrix can be rewritten as,

$$\mathbf{\Lambda} = \mathbf{Q}^T \mathbf{A} \mathbf{Q}$$

After normalizing the eigenvectors in  $\mathbf{Q}$  we can then change the coordinate system to,

$$\tilde{\mathbf{x}} = \mathbf{Q}^T \mathbf{x}$$

Leaving us then with the final formula,

$$q(\tilde{\mathbf{x}}) = \tilde{\mathbf{x}}^T \mathbf{\Lambda} \tilde{\mathbf{x}} + \mathbf{b}^T \tilde{\mathbf{x}} + c$$

▼ With no linear terms

We know the quadratic form to be,

$$q(x) = \mathbf{x}^T \mathbf{A} \mathbf{x} + c$$

We also know that for every real quadratic form, there is an orthogonal diagonalization that removes all mixed terms, the symmetric  $n \times n$   $\mathbf{A}$  matrix, all we need to do is find its eigenvalues as,

$$\det(\mathbf{A} - \lambda \mathbf{I}_n) = 0$$

Finally we can then simplify it to,

$$q(\tilde{\mathbf{x}}) = \tilde{x}_1^2 \lambda_1 + \tilde{x}_2^2 \lambda_2 + \cdots + \tilde{x}_n^2 \lambda_n + c$$

▼ Derivative Table

---

$$y = f(x)$$

$$\frac{dy}{dx} = f'(x)$$

$$k, \text{ any constant}$$

$$0$$

$$x$$

$$1$$

$$x^2$$

$$2x$$

$$x^n, \text{ any constant n}$$

$$nx^{n-1}$$

$$e^x$$

$$e^x$$

$$e^{kx}$$

$$ke^{kx}$$

$$e^{f(x)}$$

$$f'(x)e^{f(x)}$$

$$a^x$$

$$\ln(a)a^x$$

$$\ln(x)$$

$$\frac{1}{x}$$

$$\ln(f(x))$$

$$\frac{f'(x)}{f(x)}$$

$$\log_a(x)$$

$$\frac{1}{x \ln(a)}$$

$$\sin(x)$$

$$\cos(x)$$

$$\sin(kx)$$

$$k \cos(kx)$$

$$\cos(x)$$

$$-\sin(x)$$

$$\cos(kx)$$

$$-k \sin(kx)$$

$$g(x) + h(x)$$

$$g'(x) + h'(x)$$

$$g(x) - h(x)$$

$$g'(x) - h'(x)$$

$$g(x)h(x)$$

$$g'(x)h(x) + g(x)h'(x)$$

$$\frac{g(x)}{h(x)}$$

$$\frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2}$$

$$\frac{1}{f(x)}$$

$$-\frac{f'(x)}{f(x)^2}$$

$$(f(x))^n, \text{ any constant } n$$

$$n(f'(x))(f(x))^{n-1}$$

$$\frac{g(h(x))}{g(x) \cdot h(x)}$$

$$g'(h(x))h'(x)$$

## ▼ Plotting in Python

```
A1 = Matrix([[11,-12],[-12,4]])
b1 = Matrix([[-20],[40]])
q1 = xt*A1*x + b1.transpose()*x + c
dtuplot.plot3d(q1, (x1,-10,10),(x2,-10,10), wireframe = True, rendering_kw = {"color": "red", "alpha": 0.5})
```

## ▼ Level curves

To find the level curve of a function  $f(x, y)$  at  $c$  we simply do:

$$f(x, y) = c, c \in \mathbb{R}$$

Example of plotting level curves on python:

```

A1 = Matrix([[11,-12],[-12,4]])
b1 = Matrix([[-20],[40]])
q1 = xt*A1*x + b1.transpose()*x + c
zvals = [-100,-75,-50,-25,0,25,50,75,100]
dtuplot.plot_contour(q1, (x1,-2,6.5),(x2,-4.5,6.5), rendering_kw={"levels":zvals, "alpha":0.5}, is_filled=False)

```

## ▼ Gradient Vector

the gradient vector is given by a vector whose elements constitute the partial derivatives of each variable

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

## ▼ Gradient method

```

def gradient_method(x0,f,a,n):
    c=0
    xn=x0
    grad = Matrix.vstack(dtutools.gradient(f[0]))
    while c<n:
        xn1=xn-a*grad.subs({x1:xn[0],x2:xn[1],x3:xn[2]}) 
        xn=xn1
        c+=1
    return xn

```

```

x1,x2,x3 = symbols("x1,x2,x3")
f1 = 5*x1**2 + 5*x2**2 + 8*x3**2 +8*x1*x2-4*x1*x3+4*x2*x3-22*x1-32*x2-20*x3+53
f = Matrix([[f1]])
x0 = Matrix([[2],[2],[2]])
a=0.02
gradient_method(x0,f,a,10)

```

## ▼ Directional derivative of a function at a point in a specific direction:

The point is given by  $c$

$$c = (c_1, c_2)$$

The direction is given by a vector  $e$ , in case it is not normalized we need to do it

$$e = (e_1, e_2)$$

The directional derivative is the inner product of the gradient and the given direction vector.

$$\nabla_e f(c) = \langle e, \nabla f(x) \rangle$$

## ▼ Jacobian matrix

The Jacobian Matrix of a function  $f$  with  $k$  outputs and  $n$  variables

$$f = (f_1, f_2, \dots, f_k)$$

$$J_f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \cdots & \frac{\partial f_k}{\partial x_n} \end{bmatrix}$$

```
x1,x2,x3 = symbols("x1,x2,x3")
f1 = x1**2 + x2**2 + x3**2
f2 = exp(x1 + x2)*cos(x3)
f = Matrix([[f1],[f2]])
J_f = Matrix.vstack(dtutools.gradient(f[0]).T, dtutools.gradient(f[1]).T)
J_f
```

## ▼ Hessian matrix

The Hessian Matrix of a function  $f$  with  $k$  outputs and  $n$  variables

$$f = (f_1, f_2, \dots, f_k)$$

$$H_f = \begin{bmatrix} \frac{\partial^2 f_1}{\partial x_1^2} & \frac{\partial^2 f_1}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f_1}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f_2}{\partial x_2 \partial x_1} & \frac{\partial^2 f_2}{\partial x_2^2} & \cdots & \frac{\partial^2 f_2}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f_k}{\partial x_n \partial x_1} & \frac{\partial^2 f_k}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f_k}{\partial x_n^2} \end{bmatrix}$$

```
x1,x2,x3 = symbols("x1,x2,x3")
f1 = x1**2 + x2**2 + x3**2
f2 = exp(x1 + x2)*cos(x3)
f = Matrix([[f1],[f2]])
H2 = dtutools.hessian(f)
H2
```

## ▼ Taylor Approximation

### Function of one variable -

#### ▼ The tangent for a function of one variable

Taylor polynomial of first degree — at the point  $x_0$  is defined by:

$$P_{1,f,x_0}(x) = P_1(x) = f(x_0) + f'(x_0)(x - x_0), x \in \mathbb{R}.$$

The graph of the function  $P_1$ , i.e., the straight line given by

$$y = f(x_0) + f'(x_0)(x - x_0), x \in \mathbb{R}.$$

It's called the tangent of the function  $f$  at the point  $x_0$ .

The form of the polynomial  $P_1 = P_{1,f,x_0}$  in (4.2) implies that two very particular properties hold, namely,

$$P_1(x) = f(x_0) \text{ and } P'_1(x) = f'(x_0)$$

#### ▼ Taylor polynomials for functions of one variable

The Taylor polynomial of  $K$ th degree,

$P_{K,f,x_0} = P_K$  at the point  $x_0$  is defined by:

$$P_K(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(K)}(x_0)}{K!}(x - x_0)^K$$

we can rewrite as a sum

$$P_K(x) = \sum_{k=0}^K \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

▼ We can do it in python as (and it will plot it),

```
f = "whatever function you want"
x0 = "whatever x0 is"
k_list = "whatever k values for PK"
pl = plot(f,xlim = (-5,5), ylim = (-3,3), show=False, legend = True)
for K in k_list:
    newseries = series(f,x,x0,K+1).remove0()
    display(Eq(Function(f'P_{K}')(x), newseries))
    newplot = plot(newseries,label = f"n = {K}", show=False)
    pl.extend(newplot)
pl.show()
```

▼ Taylor's formula for functions of one variable

In order to estimate the difference between the function value  $f(x)$  and the corresponding approximations  $P_K(x)$  for  $x$  belonging to a suitable interval containing the point  $x_0$ . In order to do so, we define the so-called remainder term by,

$$R_K(x) := f(x) - P_K(x) = f(x) - \sum_{k=0}^K \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

There exists a scalar  $\xi$  between  $x$  and  $x_0$  such that,

$$R_K(x) = f(x) - P_K(x) = \frac{f^{(K+1)}(\xi)}{(K+1)!}(x - x_0)^{K+1}$$

### Function of several variables -

▼ The tangent plane for functions of several variables

Taylor polynomial of first degree — at the point  $x_0$  is defined by:

$$P_{1,f,x_0}(x) = P_1(x) = f(x_0) + \langle (x - x_0), \nabla f(x_0) \rangle, \quad x \in \mathbb{R}^n$$

The graph of the function  $P_1$ , i.e, the set of points  $(x, P_1(x))$ ,  
 $x \in \mathbb{R}^n$ , is called the tangent plane of the function  $f$  at the point  $x_0$ .

The polynomial  $P_1 = P_{1,f,x_0}$  can also be written with the help of the Jacobian matrix as such,

$$P_1 = P_{1,f,x_0} = f(x_0) + J(x_0) \times (x - x_0)$$

▼ Taylor polynomials for functions of multiple variables

The Taylor polynomial of first and second degree, where  $J(x_0)$  and  $H(x_0)$  represent the Jacobian and Hessian Matrix of  $f(x)$  at the point  $x_0$ .

$$P_1 = P_{1,f,x_0} = f(x_0) + J(x_0) \cdot (x - x_0)$$

$$P_2 = P_{1,f,x_0} + \frac{1}{2}(x - x_0)^T \cdot \left( H(x_0) \cdot (x - x_0) \right)$$

▼ We can do the Taylor polynomial of second degree in python as such,

```

f = #function
x0 = []#values of the expansion point
n = #order of the Taylor polynomial you want
Pn = dtutools.taylor(f,[x,x0[0],y,x0[1]],degree=n+1)

```

▼ To substitute x into actual values we do the following,

```

y11,y12,y13 = 3.1,0.2,-pi/2
taylorN(g,x,x0).subs({x1:y11,x2:y12,x3:y13})

```

▼ Taylor's formula for functions of multiple variables

In order to estimate the difference between the function value  $f(x)$  and the corresponding approximations  $P_K(x)$  for  $x$  belonging to a suitable interval containing the point  $x_0$ . In order to do so, we define the so-called remainder term by,

$$R_1(x) := f(x) - P_1(x) = \varepsilon(x - x_0) \|x - x_0\|$$

$$R_2(x) := f(x) - P_2(x) = \varepsilon(x - x_0) \|x - x_0\|^2$$

for some function  $\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}$  for which  
 $\varepsilon(x - x_0) \rightarrow 0$  whenever  $x \rightarrow x_0$ .

▼ Solving Limits with Taylor's Formulas

When we find ourselves with a limit of a fraction where both the numerator and denominator tend to 0 when  $x$  tends to a certain value  $x_0$  like the following,

$$\lim_{x \rightarrow x_0} \left( \frac{f(x)}{a(x - x_0)^n} \right)$$

We can calculate its finite solution using Taylor's formula as,

$$\lim_{x \rightarrow x_0} \left( \frac{P_{n,f,x_0} + (x - x_0)^n \varepsilon(x)}{a(x - x_0)^n} \right)$$

Where  $\varepsilon(x)$  is an epsilon function that tends to 0 as  $x$  tends to  $x_0$  and  $a$  can be any constant.

▼ Arguing for the continuity of a function

A function is continuous if it's composed of standard operations (multiplication, subtraction and/or addition, raised to a positive power) between known continuous functions (Polynomial, logarithmic, exponential, sin and cosine functions). It's also true for division, but we have to be careful when the denominator (bottom of the fraction).

▼ Arguing for the differentiability of a function

If a function is differentiable, its derivatives exist and are continuous everywhere. The derivative exists if it's composed of elementary operations (not necessarily division) between elementary functions.

For piecewise functions, the derivative must be continuous everywhere, including weird points.

▼ L'Hôpital's rule for solving limits

For functions  $f$  and  $g$  which are defined in an open interval  $I$  and differentiable on  $I \setminus \{c\}$  for a (possibly infinite) accumulation point on  $c$  of  $I$ , if  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0 \vee \pm\infty$ , and  $g'(x) \neq 0$  for all  $x \in I \setminus \{c\}$ , and  $\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$  exists, then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

▼ Stationary points and Extremum values

Let  $A \subset \mathbb{R}^n$  be a set, and consider a function  $f : A \rightarrow \mathbb{R}$ . Then, if  $f$  attains its minimum or maximum at the point  $x_0 \in A$ , one of the following possibilities occur:

- i.  $x_0 \in A \cap \partial A$  ( $x_0$  is a boundary point in the domain)

▼ To compute we:

Parametrize the boundary as  $r(t)$ , find the values of  $t_0$  for  $(f(r(t)))' = 0$ ,  $r(t_0)$  are the coordinates of the possible extremum, and  $f(r(t_0))$  is the value of the possible extremum.

▼ We can calculate in Python as such (note this will only give you possible extremum i. not ii. nor iii.):

```
x,y,t=symbols('x,y,t', real=True)
def f(x,y):
    return x*y*(2-x-y)+1
#change f(x,y) for whatever f you have
def r(t):
    return cos(t),sin(t)
#change r(t) for whatever parametric fits the boundary
def g(t):
    return f(*r(t))
#the * is there to separate the tuple from r(t)
eq1 = Eq(diff(g(t),t),0)
a = solve(eq1,t)
for i in a:
    print(simplify(Matrix(r5(i))),simplify(g(i)))
```

ii.  $x_0 \in A \cap A^\circ$  is a point where  $f$  is not differentiable.

▼ To compute we:

Find  $f(x)$  for the  $x$  where  $f$  is not differentiable, and that will give you the values of a possible extremum.

iii.  $x_0 \in A \cap A^\circ$  is a point where  $f$  is differentiable, and  $\nabla f(x_0) = 0$  (this is a stationary point)

▼ To compute we:

Solve  $\nabla f(x_0) = 0$ , where  $x_0$  will be the coordinates of the possible extremum.

▼ We can calculate in Python as such(note this will only give you possible extremum iii. not i. nor ii.):

```
x,y=symbols('x,y', real=True)
def f(x,y):
    return x*y*(2-x-y)+1
#change f(x,y) for whatever f you have
eq1 = Eq(diff(f(x,y),x),0)
eq2 = Eq(diff(f(x,y)),y),0
solve((eq1,eq2),(x,y))
```

- Functions that are differentiable everywhere:

A function that is differentiable everywhere can only have extrema at stationary points.

- Functions that are not differentiable everywhere:

A function that is not differentiable everywhere can have extrema at stationary points and/or where the function is not differentiable

- Functions that are bounded and closed:

A function that is not differentiable everywhere can have extrema at stationary points, where the function is not differentiable and/or at the boundary,

▼ Second partial derivative test (for finding out if the stationary point is a minimum maximum or saddle point)

Let  $U \subseteq \mathbb{R}^n$  be an open set, and assume that  $x_0 \in U$  is a stationary point for a function  $f : U \rightarrow \mathbb{R}$ . Assume that  $f$  is twice continuously differentiable, that is assume that  $\nabla f$  is a  $C^1$  vector function. Denote the Hessian matrix at  $x_0$  by  $H_f(x_0)$ . Then the following hold:

- If  $H_f(x_0)$  is a positive definite (i.e., all eigenvalues are positive), then  $x_0$  is a strict local minimum.
- If  $H_f(x_0)$  is a negative definite (i.e., all eigenvalues are negative), then  $x_0$  is a strict local maximum.
- If  $H_f(x_0)$  has both positive and negative eigenvalues, then  $x_0$  is a saddle point, that is,  $f(x_0)$  is neither a minimum nor a maximum value (not even in a small neighborhood of  $x_0$ )
- If  $H_f(x_0)$  is a singular (i.e., has  $\lambda = 0$  as an eigenvalue) and all non-zero eigenvalues have the same sign, then a detailed examination is necessary to decide whether  $f(x_0)$  is a minimum value, a maximum value, or a saddle point.

▼ Convergent vs Divergent

- An arbitrary sequence  $S_n$  is convergent when, for  $n \rightarrow \infty$ , then  $S_n \rightarrow N \in \mathbb{R}$
- An arbitrary series  $S_k$  is convergent when for  $n \rightarrow \infty$ , then  $S_k \rightarrow N \in \mathbb{R}$
- If they are not convergent, they must be divergent

# Integration

▼ Integral table

$$\int k \, dx = kx + c$$

any constant k

$$\int x \, dx = \frac{1}{2}x^2 + c$$

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + c$$

for any constant n

$$\int \frac{1}{x} \, dx = \ln x + c$$

$$\int f^n(x) \, dx = \frac{1}{n+1} \frac{1}{f'(x)} f^{n+1}(x) + c$$

$$\int f(g(x))g'(x) \, dx = \int f(t) + c \, dt, \text{ where } t = g(x)$$

$$\int f(x)g(x) \, dx = F(x)g(x) - \int F(x)g'(x) \, dx + c$$

$$\int e^{ax} \, dx = \frac{1}{a}e^{ax} + c$$

$$\int xe^{ax} \, dx = \left( \frac{x}{a} - \frac{1}{a^2} \right) e^{ax} + c$$

$$\int \ln x \, dx = x \ln x - x + c$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int \sin x \cos x \, dx = -\frac{1}{2} \cos^2 x + c$$

$$\int \tan x \, dx = -\ln(\cos x) + c$$

▼ Riemann sums

▼ The area under a graph  $f(x)$ , in the period of  $[0, 1]$

Is designated as  $A$  and it can be calculated,

$$A \approx \sum_{k=1}^n A_k = A_1 + A_2 + \dots + A_n$$

Any area  $A_k$  can be calculated as the height times the width  $h_k \times w_k$ , so we can rewrite it as:

$$A \approx \sum_{k=1}^n h_k \times w_k = h_1 \times w_1 + h_2 \times w_2 + \dots + h_n \times w_n$$

The 'height',  $h_k$  can be rewritten as  $f(x_k)$ , and the 'width'  $w_k$  can be rewritten as  $\frac{1}{n}$ , so we can rewrite as:

$$f(x_1)\frac{1}{n} + f(x_2)\frac{1}{n} + \dots + f(x_n)\frac{1}{n} = \sum_{k=1}^n f(x_k)\frac{1}{n}$$

$\frac{1}{n}$  can be rewritten as  $\Delta x$ , so the form is:

$$A \approx \sum_{k=1}^n f(x_k)\Delta x$$

▼ The area under a graph  $f(x)$ , in the period of  $[0, x_f]$

Is designated as  $A$  and it can be calculated,

$$A \approx \sum_{k=1}^n f(x_k)\frac{1}{n}x_f$$

▼ The area under a graph  $f(x)$ , in the period of  $[x_0, x_f]$

Is designated as  $A$  and it can be calculated,

$$A \approx \sum_{k=1}^n f(x_k)\frac{x_f - x_0}{n}$$

## ▼ Riemann integrable functions

A function is considered Riemann integrable if,

- $f(x)$  is continuous everywhere over the interval  $[x_0, x_f]$ , or,
- the Riemann sum over that interval is convergent, or,
- $f(x)$  is bounded i.e., its image is a bounded set in  $\mathbb{R}$ .

A continuous function of one variable is guaranteed to be Riemann integrable over a subset  $B \in \mathbb{R}^2$  if,

- $B$  is bounded and,
- The boundary of  $\partial B$  is formed by a finite number of continuously differentiable curves.

## ▼ Riemann integrals of functions of one variable

For a function,  $f(x)$ , if it is Riemann integrable over the interval,  $[x_0, x_f]$ , then the Riemann integral/definite integral is written as,

$$\int_{x_0}^{x_f} f(x) dx = \left[ F(x) \right]_{x_0}^{x_f} = F(x_f) - F(x_0)$$

## ▼ Riemann integrals of functions of two variables

Consider a subset  $B \in \mathbb{R}^2$ , which satisfies the conditions:

(I)  $B$  is bounded, i.e.,  
 $B \subset [a_1, b_1] \times [a_2, b_2]$   
for some  $a_1, a_2, b_1, b_2 \in \mathbb{R}$

(II) The boundary  $\partial B$  of  $B$  is formed by a finite number  
of continuously differentiable curves

A function  $f : B \rightarrow \mathbb{R}$  is said to be Riemann integrable if for any  $\epsilon > 0$  there exists  $\delta > 0$  such that:

$$\left| \int_B f(x, y) d(x, y) - S_J \right| < \epsilon$$

for any Riemann sum  $S_J$  with  $\max_i \Delta x_i < \delta$ ,  $\max_j \Delta y_j < \delta$  and any  $\xi_{i,j} \in Q_{i,j}$ . The scalar  $\int_B f(x, y) d(x, y)$  is called the integral of  $f$  over  $B$ .

#### ▼ Integration by substitution

Using the following principle:

$$\frac{dg(x)}{dx} = g'(x) \Leftrightarrow dg(x) = g'(x) dx$$

We can arrive at the following formula:

$$\int f(g(x))g'(x) dx = \int f(t) + c dt, \text{ where } t = g(x)$$

#### ▼ Integration over a rectangle in $\mathbb{R}^2$

Considering a rectangle  $Q$  in  $\mathbb{R}^2$  of the form:

$$Q = \{(x, y) \in \mathbb{R}^2 | a_1 \leq x \leq b_1 \wedge a_2 \leq y \leq b_2\}$$

and let  $f : Q \rightarrow \mathbb{R}$  denote a continuous function. Then

$$\begin{aligned} \int_Q f(x, y) d(x, y) &= \int_{a_2}^{b_2} \left( \int_{a_1}^{b_1} f(x, y) dx \right) dy \\ &= \int_{a_1}^{b_1} \left( \int_{a_2}^{b_2} f(x, y) dy \right) dx \end{aligned}$$

#### ▼ Integration over a rectangle in $\mathbb{R}^n$

Consider a rectangle  $Q$  in  $\mathbb{R}^n$  of the form:

$$Q = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n | a_i \leq x_i \leq b_i \forall i = 1, 2, \dots, n\}$$

and let  $f : Q \rightarrow \mathbb{R}$  denote a continuous function. Then

$$\begin{aligned} &\int_Q f(x_1, x_2, \dots, x_n) d(x_1, x_2, \dots, x_n) \\ &= \int_{a_n}^{b_n} \dots \left( \int_{a_2}^{b_2} \left( \int_{a_1}^{b_1} f(x_1, x_2, \dots, x_n) dx_1 \right) dx_2 \right) \dots dx_n \end{aligned}$$

#### ▼ Integration over axis parallel regions (page 154)

1. Consider two continuously differentiable functions

$$\alpha_1 : [a, b] \rightarrow \mathbb{R}, \alpha_2 : [a, b] \rightarrow \mathbb{R}$$

and assume that  $\alpha_1(x) \leq \alpha_2(x)$  for all  $x \in [a, b]$ . Consider the subset  $B$  defined as

$$B = \left\{ (x, y) \in \mathbb{R}^2 | a \leq x \leq b \wedge \alpha_1(x) \leq y \leq \alpha_2(x) \right\}$$

Then, for any continuous function  $f : B \rightarrow \mathbb{R}$ ,

$$\int_B f(x, y) d(x, y) = \int_a^b \left( \int_{\alpha_1(x)}^{\alpha_2(x)} f(x, y) dy \right) dx$$

2. Similarly, if the subset  $B$  can be described as:

$$B = \left\{ (x, y) \in \mathbb{R}^2 \mid c \leq y \leq d \wedge \beta_1(y) \leq x \leq \beta_2(y) \right\}$$

for some continuously differentiable functions  $\beta_1, \beta_2 : [c, d] \rightarrow \mathbb{R}$  with  $\beta_1(x) \leq \beta_2(x)$  for all  $y \in [c, d]$ , then for any continuous function  $f : B \rightarrow \mathbb{R}$ ,

Then, for any continuous function  $f : B \rightarrow \mathbb{R}$ ,

$$\int_B f(x, y) d(x, y) = \int_c^d \left( \int_{\beta_1(y)}^{\beta_2(y)} f(x, y) dx \right) dy$$

#### ▼ Jacobian determinant

Let  $A$  be a subset in  $\mathbb{R}^2$ . Consider a function  $r : A \rightarrow \mathbb{R}^2$ , i.e.,  $r(u, v) = (r_1(u, v), r_2(u, v))$  with coordinate functions  $r_i : A \rightarrow \mathbb{R}$ . Suppose  $r$  is a  $C^1$  vector function on  $A^\circ$ , i.e., the partial derivatives of the coordinate functions exist and are continuous on the interior of  $A$ . Let  $J_r(u, v) \in M_{2 \times 2}(\mathbb{R})$  denote the Jacobian matrix of  $r$  at  $(u, v) \in U$ . The determinant of  $J_r(u, v)$  is called the Jacobian determinant,  $\det(J_r(u, v))$  of  $r$  at  $(u, v) \in U$ .

#### ▼ Change of variables requirements

- The parametrization function  $\underline{r}$ , is injective on  $\Gamma^\circ$
- $\underline{r}$  is a  $C^1$  parametrization (continuous 1 time differentiable)
- The Jacobian function is non-zero (same thing as saying  $\underline{r}$  is regular)

#### ▼ Change of variables in $\mathbb{R}^2$

Let  $\Gamma$  be a bounded subset in  $\mathbb{R}^2$  with boundary formed by a finite number of continuously differentiable curves. Suppose that there exists a  $C^1$  vector function  $r : \Gamma \rightarrow \mathbb{R}^2$  that is injective on the interior  $\Gamma^\circ$  and such that

$$\det(J_r(u, v)) \neq 0 \text{ for all } (u, v) \in \Gamma^\circ$$

Then, for any continuous function  $f : r(\Gamma) \rightarrow \mathbb{R}$ ,

$$\int_{r(\Gamma)} f(x, y) d\underline{x} = \int_{\Gamma} f(r_1(u, v), r_2(u, v)) |\det(J_r(u, v))| du$$

where  $\underline{x} = (x, y)$  and  $\underline{u} = (u, v)$ .

#### ▼ Change of variables in $\mathbb{R}^n$

Let  $\Gamma$  be a bounded subset in  $\mathbb{R}^n$  having a smooth boundary. Suppose that there exists a  $C^1$  vector function  $r : \Gamma \rightarrow \mathbb{R}^n$  that is injective on the interior  $\Gamma^\circ$  and such that

$$\det(J_r(\underline{u})) \neq 0 \text{ for all } \underline{u} = (u_1, u_2, \dots, u_n) \in \Gamma^\circ$$

Then, for any continuous function  $f : r(\Gamma) \rightarrow \mathbb{R}$ ,

$$\int_{r(\Gamma)} f(\underline{x}) d\underline{x} = \int_{\Gamma} f(r(\underline{u})) |\det(J_r(\underline{u}))| d\underline{u}$$

where  $\underline{x} = (x_1, x_2, \dots, x_n)$  and  $\underline{u} = (u_1, u_2, \dots, u_n)$ .

that is,

$$\begin{aligned} \int_Q f(x_1, x_2, \dots, x_n) d(x_1, x_2, \dots, x_n) &= \\ \int_{\Gamma} f(r(u_1, \dots, u_n)) |\det(J_r(u_1, \dots, u_n))| d(u_1, \dots, u_n) \end{aligned}$$

#### ▼ Integration in polar coordinates

Consider a subset  $B \in \mathbb{R}^2$  of the form:

$$B = \{(x, y) : (r \cos(\theta), r \sin(\theta)) | \alpha \leq \theta \leq \beta \wedge \varphi_1(\theta) \leq r \leq \varphi_2(\theta)\}$$

Then for any continuous function  $f : B \rightarrow \mathbb{R}$ ,

$$\int_B f(x, y) d(x, y) = \int_{\alpha}^{\beta} \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(r \cos(\theta), r \sin(\theta)) r dr d\theta$$

▼ General integration formula of  $k$ -dimension geometries in  $m$ -dimension

$m \geq k$ , is always true

$$\int_{a_k}^{b_k} \dots \int_{a_1}^{b_1} f(\underline{x}) \sqrt{\det(\underline{\underline{J}}^T \underline{\underline{J}})} du_1 \dots du_k$$

$\underline{\underline{J}}$ , denotes the  $m \times k$  Jacobian matrix

▼ Area of subset

Consider a subset  $B \in \mathbb{R}$ , which satisfies the conditions:

$$\begin{aligned} & \text{(I) } B \text{ is bounded, i.e.,} \\ & B \subset [a_1, b_1] \times [a_2, b_2] \\ & \text{for some } a_1, a_2, b_1, b_2 \in \mathbb{R} \end{aligned}$$

(II) The boundary  $\partial B$  of  $B$  is formed by a finite number of continuously differentiable curves

Then the area of  $B$  is defined as

$$\text{area}(B) = \int_B 1 dX$$

▼ Center of mass

Let  $f : B \rightarrow \mathbb{R}$  be a continuous function on a bounded, connected set  $B \subset \mathbb{R}^n$  describing the density ( $kg m^{-n}$ ). In other words, we consider a “solid”  $B \subset \mathbb{R}^n$ , whose mass density distribution is continuous with density  $f(\underline{x})$  for  $\underline{x} \in B$ . Typically  $n = 3$ , but we can easily consider the general case. The total mass  $M$  [kg] of  $B$  is given by

$$M = \int_B f(\underline{x}) d\underline{x}$$

By similar considerations, the center of mass  $\underline{x}^{CM}$  is the integral of the vector-values function  $\underline{x} \rightarrow \underline{x}f(\underline{x})$ ,  $B \rightarrow \mathbb{R}^n$ :

$$\underline{x}_i^{CM} = \frac{1}{M} \int_B x_i f(\underline{x}) d\underline{x}$$

for each  $i = 1, \dots, n$ . The unit of each coordinate  $x_i^{CM}$  is m.

▼ Integrals over  $m$ -folds

Let  $r : \Gamma \rightarrow \mathbb{R}^n$ ,  $\Gamma \subset \mathbb{R}^m$ , be a regular parametrization that is injective on  $\Gamma^\circ$ . Suppose that  $f : A \rightarrow \mathbb{R}$ ,  $A \subseteq \mathbb{R}^n$  is a scalar, continuous function on the piecewise smooth  $m$ -fold  $\mathcal{D} := r(\Gamma)$ . Then the Riemann integral of  $f$  over  $\mathcal{D}$  is defined by

$$\int_{\mathcal{D}} f(\underline{x}) dS := \int_{\Gamma} f(r(\underline{u})) \sqrt{\det((\mathbf{J}_r(\underline{u}))^T \mathbf{J}_r(\underline{u}))} d\underline{u}$$

Note: This integral will always be the same independent of parametrization Remark 7.2.1 of the book)

▼ Line integrals (Integral over 1-fold)

Let  $m = 1$  and  $\Gamma = [a, b]$ . The “infinitesimal element”  $dS$  for curves is written  $ds$ . Let us denote the curve  $\mathcal{C} := r(\Gamma)$ . The line integral (also called path or curve integral) can be written:

$$\int_{\mathcal{C}} f(\underline{x}) ds = \int_a^b f(r(u)) \|r'(u)\| du$$

The length of the curve  $\mathcal{C}$  is found by setting  $f(\underline{x}) = 1 \forall \underline{x} \in \text{dom}(f)$ :

$$\text{length}(\mathcal{C}) = \int_{\mathcal{C}} 1 \, ds = \int_a^b \|r'(u)\| \, du$$

### ▼ Surface integrals (Integral over 2-folds)

Let  $m = 2$  and  $\Gamma \subset \mathbb{R}^2$ , and let us denote the surface by  $\mathcal{F} := r(\Gamma)$ , and denote the vectors in  $\Gamma$  by  $\underline{u} = (u_1, u_2)$ . The surface integral can be written:

$$\int_{\mathcal{F}} f(\underline{x}) \, dS = \int_{\Gamma} f(r(\underline{u})) \|r_{u_1}'(\underline{u}) \times r_{u_2}'(\underline{u})\| \, d(u_1, u_2)$$

The area of the surface  $\mathcal{F}$  is found by setting  $f(\underline{x}) = 1 \forall \underline{x} \in \text{dom}(f)$ :

$$\text{area}(\mathcal{F}) = \int_{\mathcal{F}} 1 \, dS = \int_{\Gamma} \|r_{u_1}'(\underline{u}) \times r_{u_2}'(\underline{u})\| \, d(u_1, u_2)$$

### ▼ Parametrization of spheres

For a solid sphere the parametrization is known as,

$$\mathbf{r}(u, v, w) = \begin{bmatrix} w \cos(u) \cos(v) \\ w \sin(u) \cos(v) \\ w \sin(v) \end{bmatrix}$$

Where  $u \in [0, 2\pi[, v \in [0, \pi[, w \in [0, R[$

For

### ▼ Vector fields

A vector field is just a vector function of the form:

$$\mathbf{V} : U \rightarrow \mathbb{R}^n$$

Where the domain  $U$  is an open subset of  $\mathbb{R}^n$ . The vector field expressed in terms of its coordinate function is:

$$\mathbf{V}(x_1, x_2, \dots, x_n) = \begin{bmatrix} V_1(x_1, x_2, \dots, x_n) \\ V_2(x_1, x_2, \dots, x_n) \\ \vdots \\ V_n(x_1, x_2, \dots, x_n) \end{bmatrix}$$

$$(x_1, x_2, \dots, x_n) \in U$$

Where each coordinate function  $V_i : U \rightarrow \mathbb{R}, i = 1, \dots, n$ , is a scalar function of  $n$  variables.

### ▼ Gradient field and anti-derivative

A continuous vector field  $\mathbf{V} : U \rightarrow \mathbb{R}^n$ , where  $U \subseteq \mathbb{R}^n$  is an open set, is called a gradient field if there exists a  $C^1$  function  $f : U \rightarrow \mathbb{R}$  such that  $\mathbf{V} = \nabla f$ , i.e.,

$$\mathbf{V}(\underline{x}) = (V_1(\underline{x}), \dots, V_n(\underline{x})) = \left( \frac{\partial f}{\partial x_1}(\underline{x}), \dots, \frac{\partial f}{\partial x_n}(\underline{x}) \right)$$

for all  $\underline{x} \in U$ . The function  $f$  is called the antiderivative of the vector field  $\mathbf{V}$ .

(Lemma 7.3.2) If  $\mathbf{V}$  is a gradient field, then  $\mathbf{J}_{\mathbf{V}}(\underline{x})$  is symmetric for all  $\underline{x} \in U$ , i.e.,

$$\frac{\partial V_i}{\partial x_j}(\underline{x}) = \frac{\partial V_j}{\partial x_i}(\underline{x}), \forall i, j = 1, 2, \dots, n$$

(Lemma 7.3.3) Let  $U \subseteq \mathbb{R}^n$  be an open and simply connected subset, and let  $\mathbf{V} : U \rightarrow \mathbb{R}^n$  be a  $C^1$  vector field. If  $\mathbf{J}_{\mathbf{V}}(\underline{x})$  is symmetric for all  $\underline{x} \in U$  then  $\mathbf{V}$  is gradient vector field.

### ▼ Line integral of a vector field

Let  $U \subset \mathbb{R}^n$  be an open set, and let  $\mathbf{V} : U \rightarrow \mathbb{R}^n$  be a continuous vector field. Let  $r : [a, b] \rightarrow \mathbb{R}^n$  denote a regular, piecewise  $C^1$  parametrization of the curve  $\mathcal{C} := r([a, b])$ . The line integral of the vector field is defined as,

$$\int_{\mathcal{C}} \mathbf{V} \cdot d\mathbf{s} = \int_a^b \langle \mathbf{V}(r(u)), r'(u) \rangle du$$

Suppose  $\mathbf{V}$  has an anti-derivative  $f : U \rightarrow \mathbb{R}$ , i.e. suppose  $\mathbf{V}$  is a gradient field  $\mathbf{V} = \nabla f$ . Then

$$f(r(b)) - f(r(a)) = \int_{\mathcal{C}} \mathbf{V} \cdot d\mathbf{s}$$

▼ Surface integral of a vector field (Flux)

Let  $U \subset \mathbb{R}^3$  be an open set, and let  $\mathbf{V} : U \rightarrow \mathbb{R}^3$  be a continuous vector field. Let  $r : \Gamma \rightarrow \mathbb{R}^3$  denote a regular, piecewise  $C^1$  parametrization of the surface  $\mathcal{F} := r(\Gamma)$ . The surface integral of the vector field is defined as,

$$\int_{\mathcal{F}} \mathbf{V} \cdot dS = \int_{\Gamma} \langle \mathbf{V}(r(u_1, u_2)), n_{\mathcal{F}}(u_1, u_2) \rangle du$$

Where  $n_{\mathcal{F}}(u_1, u_2)$  is defined as,

$$n_{\mathcal{F}}(u_1, u_2) = r'_{u_1} \times r'_{u_2}$$

Where the cross product  $\times$  of 2 vectors can be defined as,

$$\underline{a} \times \underline{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

where  $\underline{a}, \underline{b}$  are defined as,

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

▼ Anti-derivative of a vector field

Suppose  $\mathbf{V} : U \rightarrow \mathbb{R}^n$  is a continuous vector field, and that it's a gradient field, and the anti-derivative  $f : U \rightarrow \mathbb{R}$  that satisfies  $f(\underline{x}_0) = 0$  is given by

$$f(\underline{x}) = \int_{\mathcal{C}} \mathbf{V} \cdot d\mathbf{s}, \forall \underline{x} \in U$$

All anti-derivatives will be of the form  $f(\underline{x}) + c$ , for some constant  $c \in \mathbb{R}$ .

▼ The circulation theorem

Let  $U \subset \mathbb{R}^n$  be an open and connected set. suppose  $\mathbf{V} : U \rightarrow \mathbb{R}^n$  is a continuous vector field. Then the following assertions are equivalent:

1. The vector field  $\mathbf{V}$  is a gradient field, i.e., it has an anti-derivative.
2. For any closed piecewise  $C^1$  curve  $\mathcal{C}$  in  $U$  it holds,

$$\int_{\mathcal{C}} \mathbf{V} \cdot d\mathbf{s} = 0$$