CSE340: Theory of Computation (Problem Set – 1)

Question 1. Construct DFAs for the following languages.

1. $L_1 = \{w \in \{0,1\}^* | \#_0(w) \text{ is even and } \#_1(w) \text{ is odd}\}$
2. $L_2 = \{w \in \{0\}^* || w| \text{ is divisible by 2 or 7}\}$
3. $L_3 = \{w \in \{0,1\}^* | w \text{ is divisible by 5}\}$

Remark. $\#_0(w)$ denotes the number of occurrences of 0 in w. Similarly $\#_1(w)$.

Question 2. Consider the following language

$$L = \{w \in \{0,1\}^* \mid \text{the 3rd last symbol of } w \text{ is } 1\}$$

Construct a DFA for the above language. What can you say about the size (i.e. no. of states) of the DFA compared to the NFA? Consider the language

$$L_k = \{w \in \{0,1\}^* \mid \text{the } k\text{-th last symbol of } w \text{ is } 1\}$$

What is the smallest sized NFA that can accept L_k (as a function of k)? What about the smallest sized DFA?

Question 3. Solve problem 1.5 from chapter 1 in the textbook.

Question 4. For a language $L \subseteq \Sigma^*$, define

$$\text{SecondHalves}(L) = \{y \mid \exists x \text{ such that } |x| = |y|, xy \in L\}.$$

Prove that if L is regular, $\text{SecondHalves}(L)$ is also regular.

Question 5. For a language L, let

$$\text{MiddleThirds}(L) = \{y \mid \exists x, z \text{ and } |x| = |y| = |z| \text{ and } xyz \in L\}$$

For example, $\text{MiddleThirds}(\{\epsilon, a, ab, bab, bbab, aabbab\}) = \{\epsilon, a, bb\}$.
Prove that if L is regular, $\text{MiddleThirds}(L)$ is also regular.

Question 6. Given $L \subseteq \{0,1\}^*$, define

$$L' = \{xy \mid x1y \in L\}.$$

Show that if L is regular then L' is also regular.

Question 7. For a language A, let

$$A'' = \{xz \mid \exists y \text{ and } |x| = |y| = |z| \text{ and } xyz \in A\}$$

Show that even if A is regular, A'' is not necessarily regular.

Question 8. Show that the following languages are not regular.

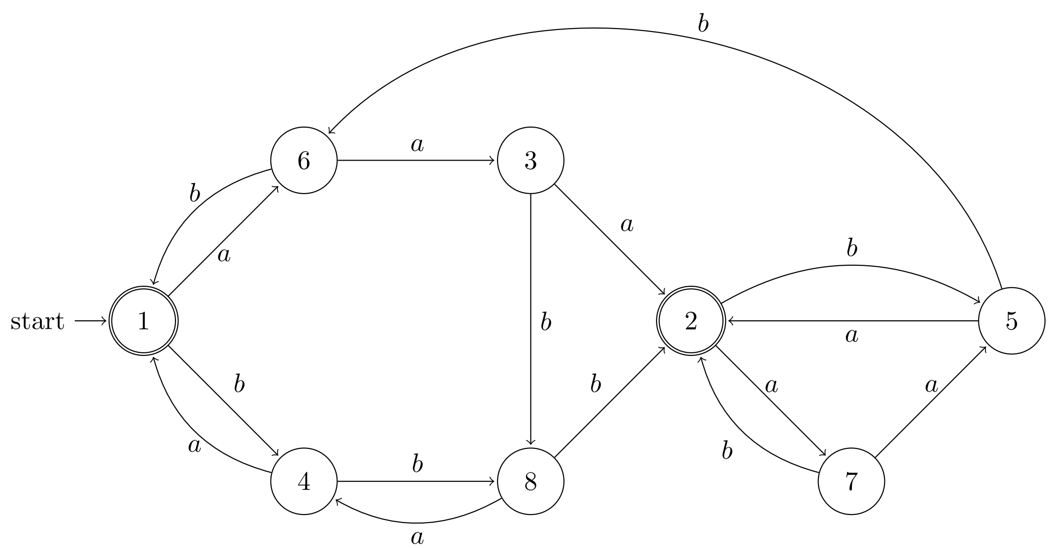
1. $\{0^{n^2}1^n \mid n \geq 0\}$
2. $\{0^n 1^m \mid n > m\}$
3. $\{ww \mid w \in \{0,1\}^*\}$
4. $\{a^i b^j c^k \mid i \neq 2 \text{ or } j = k\}$

Question 9. Verify that $\approx$ (defined in lecture 7) is an equivalence relation.

Question 10. Show that the δ' (define in lecture 7) is well defined. In other words, if $[p] = [q]$, then $[\delta(p,a)] = [\delta(q,a)]$ for all $a \in \Sigma$.

Question 11. Can you collapse the quotient DFA any further? What happens if you try to do so?

Question 12. Minimize the following DFA.



Question 13.

$$\begin{aligned} S &\longrightarrow ASB \mid \epsilon \\ A &\longrightarrow a \\ B &\longrightarrow bb \end{aligned}$$

The language generated by the above grammar is

$$L = \{a^n b^{2n} \mid n \geq 0\}$$

which is not regular. What happens if we add the production rule

$$B \longrightarrow \epsilon$$

to the above grammar?

Question 14. Prove Theorem 4 from lecture 8.

Question 15. Give an example of an unambiguous grammar that has at least 2 derivations for some string.

Question 16. Solve problem 2.14 from textbook.

Question 17. Prove that the following languages are not context-free.

1. $L_1 = \{a^n b^m c^n d^m \mid n, m \geq 0\}$
2. $L_2 = \{0^n 1^{n^2} \mid n \geq 0\}$
3. $L_3 = \{0^n \mid n \text{ is prime}\}$

Question 18. Construct PDA for the following languages

(i) $L_1 = \{w \in \{0,1\}^* \mid \#_0(w) = \#_1(w)\}$

(ii) $L_2 = \{0^{2n} 1^{3n} \mid n \geq 0\}$

Question 19. Construct PDA for the following languages

(i) $L_1 = \{a^i b^j c^k \mid j \leq i + k \leq 2j\}$

(ii) $L_2 = \{a^i b^j \mid i \neq j\}$

(iii) $L_3 = L(a^* b^* c^*) \setminus \{a^n b^n c^n \mid n \geq 0\}$

(iv) $L_4 = \overline{L}$, where $L = \{ww \mid w \in \{a,b\}^*\}$

Question 20. Show that CFLs are closed under homomorphism and inverse inverse homomorphism. (*Hint*: For homomorphism start with a CFG and for inverse homomorphism start with a PDA.)

Question 21. Construct a DPDA for the language $L_1 = \{0^n 1^n \mid n \geq 0\}$.

Question 22. Show that there is a CFL that is not a DCFL and has an unambiguous grammar.

Question 23. Construct a DFA and a RE for the language

$$L = \{w \in \{0,1\}^* \mid \text{every } 1 \text{ in } w \text{ is immediately preceded and followed by a } 0\}.$$

Example: The strings 00 and 0010100010 are in L whereas, 0110 and 1010010 are not in L.

Question 24. Give REs for the following languages

(a) $L = \{w \in \{0,1\}^* \mid w \text{ does not contain 01 as a substring}\}$

(b) $L = \{w \in \{0,1\}^* \mid w \text{ does not contain 101 as a substring}\}$

Question 25. Give DFAs equivalent to the following REs

(a) $(000)^*1 + (00)^*1$

(b) $(00 + 11)^*(01 + 10)(00 + 11)^*$

Question 26. For a set $A \subseteq \mathbb{N}$, binary(A) is the set of binary representations of all numbers in A and unary(A) is the set of unary representations of all numbers in A. For example, if $A = \{3, 5, 8\}$ then binary(A) $= \{11, 101, 1000\}$ and unary(A) $= \{000, 00000, 00000000\}$. Consider the following two statements

1. For all A, if unary A is regular then binary(A) is also regular.
2. For all A, if binary A is regular then unary(A) is also regular.

Show that one of the above two statements is true and the other is false.

Question 27. Which of the following languages are regular? Prove your answer.

(a) $\{x\#x \mid x \in \{0,1\}^*\}$

(b) $\{x\#y \mid x, y \in \{0,1\}^*\}$

(c) $\{x \in \{0,1\}^* \mid \#_0(x) = 2 \cdot \#_1(x)\}$

(d) $\{x \in \{0,1\}^* \mid \#_0(x) \cdot \#_1(x) \text{ is even}\}$

(e) $\{x \in \{0,1\}^* \mid \#_0(x) + \#_1(x) \text{ is even}\}$

Question 28. Hamming distance between two strings, $w_1, w_2 \in \{0,1\}^n$ is said to be k if w_1 and w_2 differ in exactly k positions. This is denoted as $H(w_1, w_2)$. For example, the strings 1010010 and 1100011 have Hamming distance 3. If two strings have unequal length, we say their Hamming distance is infinite.

For a language $L \subseteq \{0,1\}^*$, define

$$H_k(L) = \{w \in \{0,1\}^* \mid \exists x \in L,\ H(w,x) \leq k\}.$$

(a) Show that if L is regular, then $H_2(L)$ is regular.

(b) For any $k > 2$, show that if L is regular, then $H_k(L)$ is regular.

Question 29. For a language $A \subseteq \{0,1\}^*$ define $\min(L)$ as

$$\min(L) = \{w \in L \mid \text{no proper prefix of } w \text{ is in } L\}.$$

Prove that if L is regular, then $\min(L)$ is regular.

Question 30. Let $L \subseteq \{a\}^*$. Show that L^* is regular.

Question 31. Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be a function such that for some fixed $n_0 \in \mathbb{N}$,

$$f(n+1) - f(n) \geq n + 1, \qquad \text{for all } n \geq n_0.$$

Consider the unary language

$$L = \{a^{f(n)} \mid n \geq 1\}.$$

Is L regular? Is it context-free?