Design and Analysis of Algorithms

Practice-sheet : **NP-completeness**

Date: *12th November, 2017*

1. **Polynomial reduction $\leq_P$**

   Let A and B be any two computational problems. Let χ be any algorithm for solving B. Problem A is said to be reducible to problem B in polynomial time if each instance I of A can be solved by

   - A polynomial number of executions of χ on instances (of B) each of which are also polynomial of size of I,
   - and, if required, basic computational steps (each taking $O(1)$ time) which are also polynomial in the size of I.

   Convince yourself that this definition of $\leq_P$ subsumes the definition of polynomial time reducibility discussed in the class.

2. **Application of $\leq_P$**

   Let problem A be defined as follows. Given any undirected graph and an integer k, determine if the graph has an independent set of size at least k.

   Let problem B be defined as follows. Given any undirected graph and an integer t, determine if the graph has a vertex cover of size k.

   Using the definition of $\leq_P$ given in the previous exercise, show that $A \leq_P B$.

3. **Resolving whether $P = NP$?**

   For each of the two questions below, decide whether the answer is (i)**yes**, (ii)**no**, (iii) **unknown**, because it would resolve the question of whether "P=NP". Give a brief explanation of your answer.

   (a) Let us define the decision version of the Interval Scheduling Problem (discussed under the topic of Greedy algorithms) as follows: Given a collection of Intervals on a time-line, and an integer k, does the collection contain a subset of nonoverlapping intervals of size at least k ?

   Question: Is it the case that *Interval Scheduling* $\leq_P$ *Vertex Cover* ?

   (b) Question: Is it the case that *Independent Set* $\leq_P$ *Interval Scheduling* ?

4. **Feedback set**

   Given an undirected graph $G = (V, E)$, a *feedback* set is a set $X \subseteq V$ with the property that $G - X$ has no cycle. The *Undirected Feedback Set Problem* asks: Given G and k, does there exist a feedback set of size at most k ? Prove that *Undirected Feedback Set Problem* is NP-complete.

5. **Subgraph Isomorphism**

   Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. G is said to be isomorphic to G' if we can obtain G' from G by renaming its vertices suitably. In formal words, it means the following.

   A 1-1 and onto function $f : V \to V'$ is said to be an isomorphism if for each pair of vertices $u, v \in V$, $(u, v) \in E$ if and only if $(f(u), f(v)) \in E'$.

   Subgraph-Isomorphism Problem is defined as follows. Given any two graphs $G = (V, E)$ and $G' = (V', E')$, does there exist any subgraph of G which is isomorphic to G'. Show that *Subgraph-Isomorphism Problem* is NP-complete.

6. **Clique Problem**

   A clique is a complete graph (edge exists between each pair of its vertices). Consider the following problem: Given an undirected graph $G = (V, E)$ and an integer k, does G contain a clique of size k ?

   Show that this problem is NP-complete.

   Hint: Use the fact that *Independent Set* is NP-complete.

7. **Approximation Algorithm for Vertex Cover**

   Recall the algorithm for computing vertex cover of a given graph as discussed in the class. Prove that the algorithm computes a vertex cover whose size is at most twice the size of minimum-size vertex cover.
   Hint: For each edge picked during the algorithm, at least one of its endpoints must be in the optimal vertex cover.

   In this course we discussed bipartite-matching problem. The notion of matching can be extended naturally to any arbitrary undirected graph. Based on the algorithm, what relationship can you draw between the matching of a graph and a vertex cover of the same graph?

Important Note: We discussed the area of approximation algorithm very briefly. So only simple/obvious exercises on approximation algorithms, if at all, may be expected in the exam.