CS340: Theory of Computation Sem I 2017-18

# Lecture Notes 10: Non Context-Free Languages

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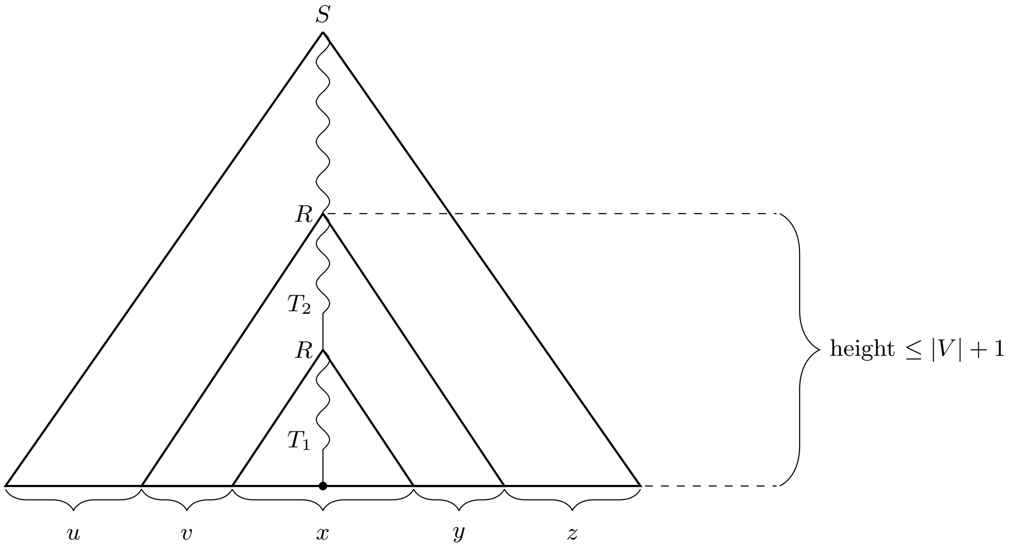
## 1 Pumping Lemma for Context-free Languages

We will prove a pumping lemma for context-free languages. Let $L$ be a CFL and $G = (V, \Sigma, P, S)$ be a CFG such that $L = L(G)$. Let $w$ be a string in $L$. Consider a smallest parse tree of $w$ with respect to $G$ (say $T_{G,w}$). Few observations:

- A path from the root to a leaf in $T_{G,w}$ is a sequence of variables ending with a terminal/$\epsilon$.
- The height of a tree is the maximum number of edges on a path from the root to a leaf node.
- Let $d$ be the maximum degree of a node in $T_{G,w}$. If the height of the tree is $h$, then $|w| \leq d^h$.
- Recall that $w$ is the concatenation of the terminal symbols at the leaves of $T_{G,w}$, from left to right.

If $|w| \geq d^{|V|+1}$, then height of $T_{G,w}$ is at least $|V| + 1$ (no. of nodes is at least $|V| + 2$) and there exists a path in $T_{G,w}$ from root to a leaf on which it has at least $|V| + 1$ variables. Consider the lowest $|V| + 1$ variables on that path. By pigeon hole principle there exists a variable $R$ which appears twice on that portion of the path. We define a partition of $w = uvxyz$ as illustrated in the figure below.

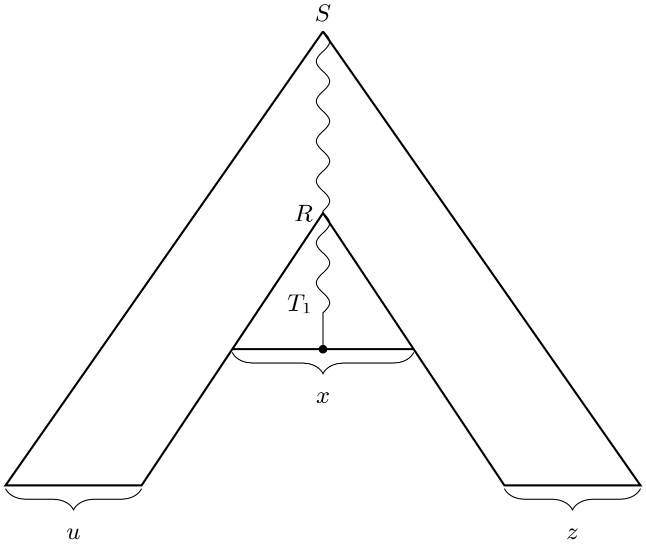


In the above parse tree for $w$, $T_1$ is the subtree rooted at the bottom $R$ and it generates the string $x$ and $T_2$ is the subtree rooted at the top $R$ and it generates the string $vxy$.

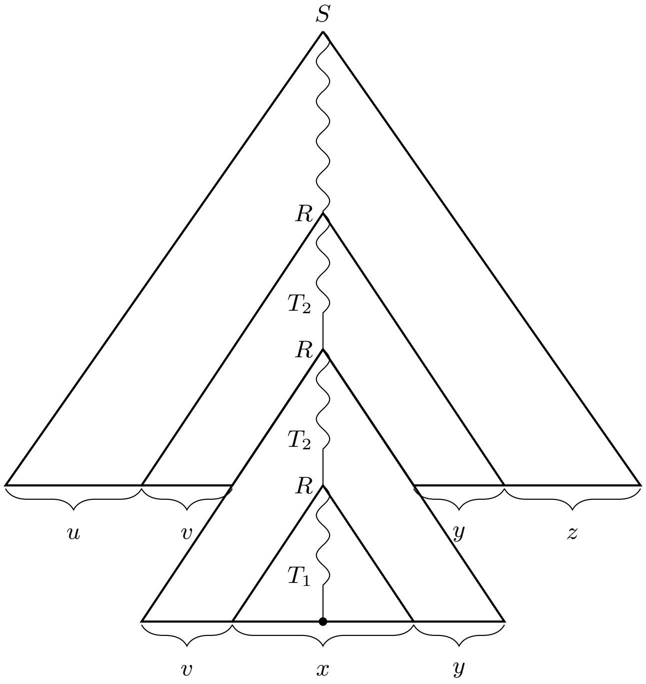
**Observation 1.** *Suppose there are two internal nodes in a parse tree labelled with the same variable say $A$, and say $T_1^A$ and $T_2^A$ are the subtrees rooted at these two nodes respectively. If we replace $T_1^A$ with $T_2^A$ or vice versa then we will still get a parse tree for* some *string in the language of the grammar (essentially the string formed by concatenating the leaves from left to right).*

- Since height of $T_2$ is at most $|V|+1$, therefore $|vxy| \leq d^{|V|+1}$.
- Moreover since $T_{G,w}$ is the smallest parse tree of $w$ with respect to $G$, therefore $T_1$ cannot be substituted for $T_2$ to get the same string $w$. This implies that both $v$ and $y$ cannot be the empty string. Therefore $|vy| > 0$.

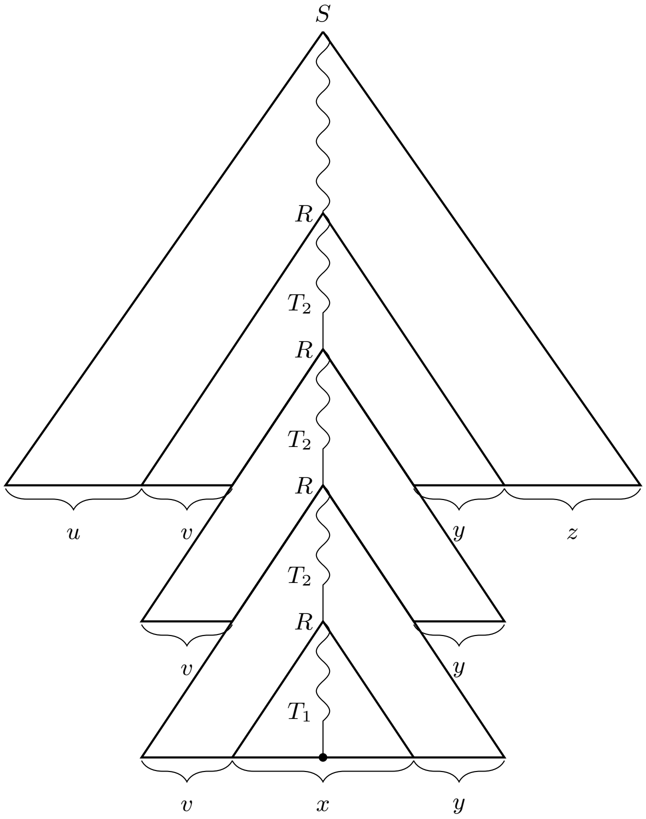
By Obversation 1 if we replace $T_2$ with $T_1$ we get the parse tree of the string $uxz$ and hence this string is in $L$. The parse tree is shown below.



Similarly if we replace the $T_1$ with $T_2$ we get the parse tree of the string $uv^2xy^2z$ and hence this string is in $L$. The parse tree is shown below.



Once again if we replace the $T_1$ with $T_2$ in the above parse tree we get the parse tree of the string $uv^3xy^3z$ and hence this string is in $L$ as well. The parse tree is shown below.



We can generalize and extend the above argument to show that for all $i \geq 0$, $uv^ixy^iz \in L$. Now by setting $p = d^{|V|+1}$ we get the following theorem.

**Theorem 2** (Pumping Lemma for Context-free Languages)**.** *Let $L$ be a context-free language. Then there exists an integer $p > 0$, such that for all $w \in L$ of length at least $p$, there exists a partition of $w = uvxyz$ such that $|vxy| \leq p$, $|vy| > 0$, and for all $i \geq 0$, $uv^ixy^iz \in L$.*

*Remark.* The choice of $p$ for a CFL $L$ is solely dependent on the CFG that we choose for $L$. Recall that $p = d^{|V|+1}$. Here $d$ is the maximum number of symbols in the right hand side of a substitution rule in the CFG and $V$ is of course the variable set of the CFG. Hence a different grammar for the same language might give a different $p$.

To prove that languages are not context-free, the pumping lemma will be used in its contrapositive form.

**Theorem 3** (Contrapositive form of Pumping Lemma for CFLs)**.** *Let $L$ be a language. If*

- *$\forall p \geq 0$,* *(opponent's move)*
- *$\exists w \in L$ with $|w| \geq p$, such that,* *(your move)*
- *$\forall$ possible partitions of $w$ as $w = uvxyz$, satisfying* *(opponent's move)*
    - *$|vxy| \leq p$, and*
    - *$|vy| > 0$,*
- *$\exists i \geq 0$ such that $uv^ixy^iz \notin L$,* *(your move)*

*then $L$ is not context-free.*

## 2 Examples of Non Context-free Languages

1.

$$L_1 = \{a^n b^n c^n \mid n \geq 0\}$$

Given $p$, choose $w = a^p b^p c^p$. Now for any partition $w = uvxyz$, set $i = 2$. We show below that $w' = uv^2xy^2z$ is not in $L_1$.

Consider the string $vxy$. Since $|vxy| \leq p$, therefore $vxy$ cannot contain all three symbols. More specifically, it does not contain either $a$ or $c$. Assume that it does not contain $c$'s. Also since $v$ and $y$ cannot both be empty, therefore $w'$ will have more number of either $a$'s or $b$'s than the number of $c$'s. Hence $w' \notin L_1$. The case when $w'$ does not contain $a$'s is analogous.

2.

$$L_2 = \{ww \mid w \in \{a, b\}^*\}$$

Given $p$, choose $w = a^p b^p a^p b^p$. Clearly $w \in L_2$ and has length at least $p$. Now for any partition $w = uvxyz$, consider the following cases.

**Case 1: $vxy$ has only $a$'s or only $b$'s.** We set $i = 2$ and let $w' = uv^2xy^2z$. Assume $vxy$ lies in the first block of $a$'s. Let $|vy| = k$. Now $0 < k \leq p$. As a result the first half of $w'$ is $a^{p+k}b^{p-k/2}$ and second half of $w'$ is $b^{k/2}a^p b^p$. Clearly the strings are not equal and hence $w' \notin L_2$.

If $vxy$ lies in any other block, the argument is analogous.

**Case 2: $vxy$ has both $a$'s and $b$'s.** We set $i = 0$ and let $w' = uxz$. Assume $vxy$ straddles the first boundary between $a$'s and $b$'s. Let $vy = a^{k_1}b^{k_2}$. Note that Now $0 < k_1 + k_2 \leq p$. Then $w' = a^{p-k_1}b^{p-k_2}a^p b^p$. Then the first half of $w'$ is $a^{p-k_1}b^{p-k_2}a^{\frac{k_1+k_2}{2}}$ and the second half is $a^{p-\frac{k_1+k_2}{2}}b^p$. Clearly the strings are not equal and hence $w' \notin L_2$.

If $vxy$ straddles any other boundary, the argument is analogous.

*Remark.* Note that in the above proof we could have fixed $i = 0$ or $i = 2$ for both the cases. But that would make the argument a little more tedious. Also the above proof illustrates the fact that $i$ can vary on a case by case basis.

**Exercise 1.** Prove that the following languages are not context-free.

(a) $L_1 = \{a^n b^m c^n d^m \mid n, m \geq 0\}$

(b) $L_2 = \{0^n 1^{n^2} \mid n \geq 0\}$

(c) $L_3 = \{0^n \mid n \text{ is prime}\}$