Name: Rollno:

CSE340: Theory of Computation (Final Exam)

18th November, 2017

Total Number of Pages: 6 Total Points 100

Question	Points	Score
1	10	
2	12	
3	6	
4	6	
5	7	
6	10	
7	14	
8	6	
9	18	
10	11	
Total:	100	

Instructions

1. Read these instructions carefully.
2. Write you name and roll number on all the pages.
3. Cheating or resorting to unfair means will be severely penalized.
4. Do not exchange question books or change the seat after obtaining question paper.
5. Superfluous and irrelevant writing will result in negative marking.
6. Using pens (blue/black ink) and not pencils. Do not use red pens for answering.

Helpful hints

1. It is advisable to solve a problem first before writing down the solution.
2. The questions are *not* arranged according to the increasing order of difficulty.
3. For each question, you are given more space than what is needed for writing your solutions.

Useful Information

1. $\mathsf{TR} = \{L \mid L \text{ is Turing recognizable}\}$
2. $\mathsf{coTR} = \{L \mid \overline{L} \in \mathsf{TR}\}$

SOLUTIONS

Question 1. (10 points) For each of the following languages encircle the smallest class in which the problem is known to be contained.

(2 marks for correct answer, 0.5 mark for leaving the question blank and 0 mark for incorrect answer)

(a) $\{a^i b^j c^i \mid i, j \geq 0\}$
A. Regular **B. Context-free** C. Decidable D. Undecidable

(b) $\{\langle M \rangle \mid M \text{ is a TM and } L(M) = \Sigma^*\}$
A. Decidable B. TR C. coTR **D. Neither in TR nor in coTR**

(c) $\{\langle M \rangle \mid M \text{ is a TM and } L(M) \in \mathsf{NP}\}$
A. P B. NP C. Decidable **D. Undecidable**

(d) $\{\langle G, k \rangle \mid G \text{ has a clique of size } k\}$
A. P B. EXP **C. PSPACE** D. NEXP

(e) $\{\langle \phi \rangle \mid \phi$ is a Boolean formula that evaluates to true on every truth assignment$\}$

A. P B. NP C. NP-complete **D.** NEXP

Question 2. (12 points) For each of the following statements, state whether it is *True*, *False* or *Open* based on our current knowledge.

(2 marks for correct answer, 0.5 mark for leaving the question blank and 0 mark for incorrect answer)

(a) $\mathsf{NL} \subseteq \mathsf{P}$ **True**

(b) $\mathsf{P} \subseteq \mathsf{TIME}(n^{10})$ **False**

(c) $\mathsf{Clique} \leq_p \overline{\mathsf{Clique}}$ **Open**

(d) $\mathsf{TR} = \mathsf{coTR}$ **False**

(e) $\mathsf{PSPACE} \subseteq \mathsf{EXP}$ **True**

(f) $\mathsf{SAT} \in \mathsf{L}$ **Open**

Question 3. Consider the following property of languages. For all $L \subseteq \Sigma^*$,

$$P(L) = \begin{cases} 1 & \text{if } \exists \text{ a TM } M \text{ with an even no. of states such that } L = L(M) \\ 0 & \text{otherwise} \end{cases}$$

(a) (2 points) P is a non-trivial property of languages of TMs (True or False). **False**

(b) (2 points) Give reason for your answer in part (a).

Solution: For every language L such that $L = L(M)$ for some TM M, there exists a TM with an even no. of states (if the no. of states in M is odd, add a dummy state).

(c) (2 points) Is the language $L = \{\langle M \rangle \mid M$ has an even no. of states$\}$ decidable? Give reason for your answer.

Solution: Yes. Just count the no. of states in M and answer accordingly.

Question 4. (6 points) Let p and q be two positive integers. Give a CFG for the following language, having only one variable (say S)

$$L = \{a^i b^j \mid i, j \geq 0,\ pi = qj\}.$$

Solution:

$$S \longrightarrow a^{\frac{q}{\gcd(p,q)}} S b^{\frac{p}{\gcd(p,q)}} \mid \epsilon$$

So if you assume p and q are coprime the solution is

$$S \longrightarrow a^q S b^p \mid \epsilon$$

Question 5. Consider the language

$$L = \{\langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are two TMs and } L(M_1) \subseteq L(M_2)\}.$$

(a) (1 point) Is L decidable? **No**

(b) (6 points) Prove your answer.

Solution: Consider the undecidable language

$$\mathsf{E_{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}.$$

Claim 1. $\mathsf{E_{TM}} \leq_m L$.
We will construct a computable function f that takes as input $\langle M\rangle$ and produces an output $\langle M_1, M_2\rangle$ such that $L(M) = \emptyset \iff L(M_1) \subseteq L(M_2)$.

The reduction function f

Input: $\langle M\rangle$

1. Set $M_1 := M$.
2. Set M_2 to be a TM that rejects all strings ($L(M_2) = \emptyset$).

Output: $\langle M_1, M_2\rangle$
Note that $\overline{L(M_2)} = \Sigma^*$.

Proof of correctness

Now,

$$L(M) = \emptyset \iff L(M_1) = \emptyset \iff L(M_1) \cap \overline{L(M_2)} = \emptyset \iff L(M_1) \subseteq L(M_2)$$

Therefore, $\mathsf{E_{TM}} \leq_m L$. This proves that L is undecidable.

Question 6. Recall that

$$\mathsf{E_{TM}} = \{\langle M\rangle \mid L(M) = \emptyset\}.$$

Consider two languages L_1 and L_2 defined as follows:

$$L_1 = \{\langle N\rangle \mid N \text{ is a TM and } L(N) = \mathsf{E_{TM}}\}$$
$$L_2 = \{\langle N\rangle \mid N \text{ is a TM and } L(N) = \overline{\mathsf{E_{TM}}}\}.$$

One of the above two languages is decidable and the other is not.

(a) (2 points) The language $\underline{\quad L_1 \quad}$ is decidable.

(b) (3 points) Why is the language in part (a) decidable?

Solution: We have show in class that $\mathsf{A_{TM}}$ is undecidable but in TR and $\overline{\mathsf{A_{TM}}} \leq_m \mathsf{E_{TM}}$. Therefore $\mathsf{E_{TM}}$ is in coTR and not in TR. Hence there does not exist any TM M such that $L(M) = \mathsf{E_{TM}}$. Hence L_1 is empty set and thus decidable.

(c) (3 points) Why is the other language undecidable? (You may use Rice's Theorem for this part)

Solution: Given a TM M, we can nondeterministically guess a string x and verify whether M accepts x. If $L(M) \neq \emptyset$ then there exists a string for which this is true. Hence $\overline{\mathsf{E_{TM}}}$ is in TR. Therefore there exists a TM N_1 such that $L(N_1) = \overline{\mathsf{E_{TM}}}$. Also there exists a TM N_2 such that $L(N_2) \neq \overline{\mathsf{E_{TM}}}$ (say N_2 is the TM that rejects all strings). Therefore by Rice's Theorem L_2 is undecidable.

(d) (2 points) What can we say about L_1 and L_2 if we replace $\mathsf{E_{TM}}$ with the language FIN? (Recall that $\mathsf{FIN} = \{\langle M \rangle \mid L(M) \text{ is finite}\}$.)

- L_1 is **decidable**.
- L_2 is **decidable**.

Question 7. (a) (6 points) Show that for every $k \geq 0$, $\mathsf{NTIME}(n^k) \subsetneq \mathsf{PSPACE}$.

Solution: Let $k \geq 0$. Then,

$$\begin{aligned} \mathsf{NTIME}(n^k) &\subseteq \mathsf{NSPACE}(n^k) & \\ &\subseteq \mathsf{SPACE}(n^{2k}) & \text{by Savitch's Theorem} \\ &\subsetneq \mathsf{SPACE}(n^{2k+1}) & \text{by Space Hierarchy Theorem} \\ &\subseteq \mathsf{PSPACE}. & \end{aligned}$$

(b) (4 points) Since $\mathsf{NP} = \cup_{k\geq 0}\mathsf{NTIME}(n^k)$, does this show that $\mathsf{NP} \subsetneq \mathsf{PSPACE}$? Why?

Solution: No. Because part (a) shows that for every k there is a problem in PSPACE that is not in $\mathsf{NTIME}(n^k)$, but the problem can very well be in $\mathsf{NTIME}(n^l)$ for some $l > k$. However to show that $\mathsf{NP} \subsetneq \mathsf{PSPACE}$ one needs to show that there is a problem in PSPACE that is not $\mathsf{NTIME}(n^k)$ for every k.

(c) (4 points) Show that $\bigcup_{k\geq 0} \mathsf{NSPACE}(\log^k n) = \bigcup_{k\geq 0} \mathsf{SPACE}(\log^k n)$.

Solution: By definition, $\bigcup_{k\geq 0} \mathsf{SPACE}(\log^k n) \subseteq \bigcup_{k\geq 0} \mathsf{NSPACE}(\log^k n)$.

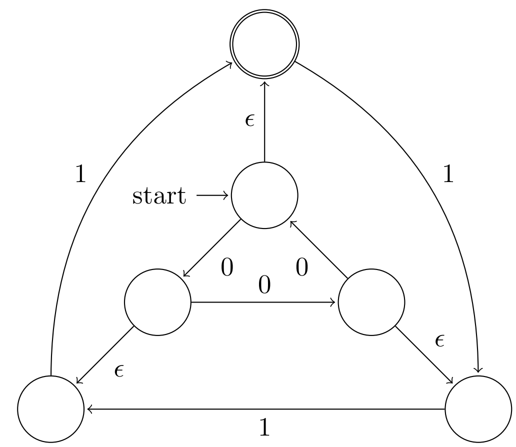
Let $L \in \mathsf{NSPACE}(\log^k n)$ for some k. Then by Savitch's Theorem, $L \in \mathsf{SPACE}(\log^{2k} n)$. Hence $\bigcup_{k\geq 0} \mathsf{NSPACE}(\log^k n) \subseteq \bigcup_{k\geq 0} \mathsf{SPACE}(\log^k n)$ as well.

Question 8. (6 points) Design an NFA with at most six states that accepts the following language

$$L = \{0^{n+3k}1^n \mid n \geq 0, k \in \mathbb{Z}\}.$$

Solution: Observe that, we can rephrase L as set of strings of the form 0^i1^j such that i and j leave the same remainder modulo 3.

Here is an NFA for L.



Question 9. Let Σ be some fixed alphabet. For any string $w \in \Sigma^*$, let w^R denote the reverse of w. Let A be a regular language. Consider the following two languages

$$L_1 = \{ww^R \mid w \in A\}$$
$$L_2 = \{w \mid ww^R \in A\}.$$

One of the above two languages is necessarily regular, and one is not. Which is which? Give a proof and a counterexample.

(a) (1 point) $\underline{\quad L_2 \quad}$ is regular.

(b) (1 point) $\underline{\quad L_1 \quad}$ is not regular.

(c) (8 points) Prove that the language in part (a) is regular.

Solution: Let $D = (Q, \Sigma, \delta, q_0, F)$ be a DFA for A. We will construct an NFA N for L_2 that will simultaneously simulate the input string w from the starting state in forward direction, and from the accepting states in reverse direction. If at the end of reading w, both the simulations reach the same state we accept otherwise reject.
Define $N = (Q^2 \cup \{s\}, \Sigma, \delta_N, s, F_N)$.
From the start state s of N, we add transitions of the form

$$\delta_N(s, \epsilon) = \{(q_0, f) \mid f \in F\}.$$

The other transitions of N are defined as

$$\delta_N((p_1, q_1), a) = \{(p_2, q_2) \mid \delta(p_1, a) = p_2 \text{ and } \delta(q_2, a) = q_1\}.$$

The set of accepting states of N are

$$F_N = \{(p, p) \mid p \in Q\}.$$

Then $L_2 = L(N)$.

(d) (8 points) Prove that the language in part (b) is not regular by giving a suitable counter example and showing why it is not regular.

Solution: Let $A = \{0+1\}^*$ which is clearly regular. Then $L_1 = \{ww^R \mid w \in \{0+1\}^*\}$. We prove L_1 is not regular using the Pumping Lemma for regular languages.

Given $n \geq 1$, pick $w = 0^n 110^n$ which is a string in L_1. Now for any partition of $w = xyz$ such that $|xy| \leq n$ and $|y| > 0$, y will be of the form 0^i, where $0 < i \leq n$. Therefore the string $xy^0z = 0^{n-i}110^n$ and is clearly not in L_1. Therefore L_1 is not regular.

Question 10. A *walk* in a graph $G = (V, E)$ is a sequence of vertices $P = (v_0, v_1, \ldots, v_k)$ such that $(v_{i-1}, v_i) \in E$ for $1 \leq i \leq k$. Given an edge weight function $w : E \longrightarrow \mathbb{Z}$, weight of the above walk P, $w(P) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$.

$$\begin{aligned}\mathsf{TSP} = \{\langle G, w, k\rangle \mid & G \text{ is a directed graph, } w \text{ is an edge weight function,}\\ & \exists \text{ a walk } P \text{ in } G \text{ s.t. } w(P) \leq k \text{ and } P \text{ visits every vertex at least once}\}\end{aligned}$$

In this question we will show that TSP is NP-complete.

(a) (3 points) Show that $\mathsf{TSP} \in \mathsf{NP}$.

Solution: Certificate: A sequence of vertices $v_1, \ldots, v_m$.
Verifier's Algorithm
Input: $\langle G, w, k, v_1, \ldots, v_m\rangle$

1. Check if (v_i, v_{i+1}) is an edge for all $1 \leq i \leq m-1$. If not then REJECT.
2. Check if $\sum_{i=1}^{m-1} w(v_i, v_{i+1}) \leq k$. If not then REJECT.
3. Check if $v_1, \ldots, v_m$ covers all the vertices in G. If so then ACCEPT else REJECT.

(b) (1 point) Choosing a known NP-complete problem: HamPath

(c) (4 points) Give the reduction

Solution: Let $\langle G = (V, E)\rangle$ be an instance of HamPath. We will construct an instance of TSP, $\langle G' = (V', E'), w, k\rangle$ as defined below:

$$\begin{aligned} V' &= V \\ E' &= E \\ w(e) &= 1 \quad \forall e \in E' \\ k &= |V| - 1 \end{aligned}$$

(d) (1 point) Time complexity of the reduction

Solution: The construction of $\langle G', w, k\rangle$ can be done in linear time in the size of G.

(e) (2 points) Proof of correctness of the reduction.

Solution: If G has a Hamiltonian path P then $w(P) = |V| - 1$ hence G' has a walk of weight at most k that visits all vertices (the path P itself).

If G' has a walk of weight at most k that visits all vertices then the walk must be a path since $k = |V| - 1$. Hence G has a Hamiltonian path.