Preliminary work on a domain decomposition method for Poisson's equation (2D)

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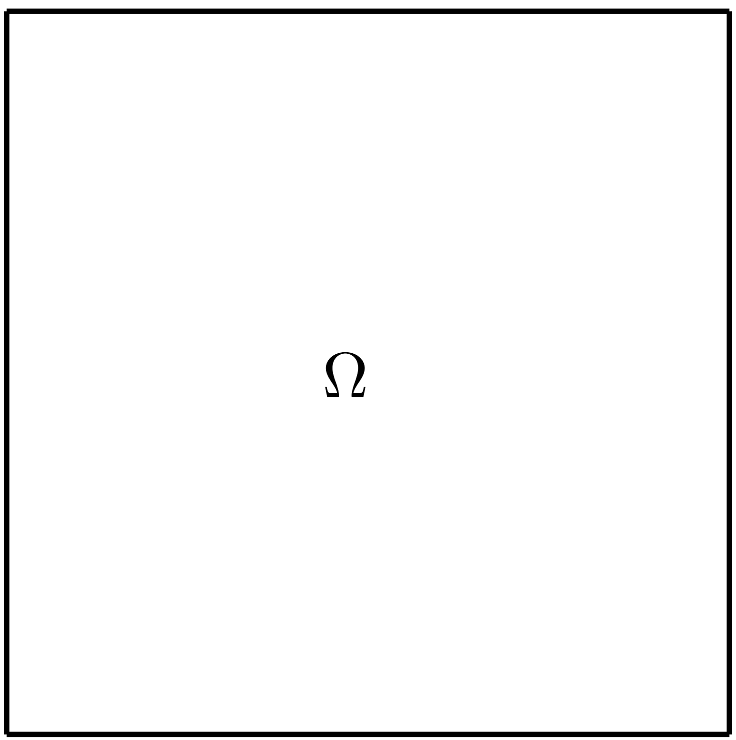
Problem

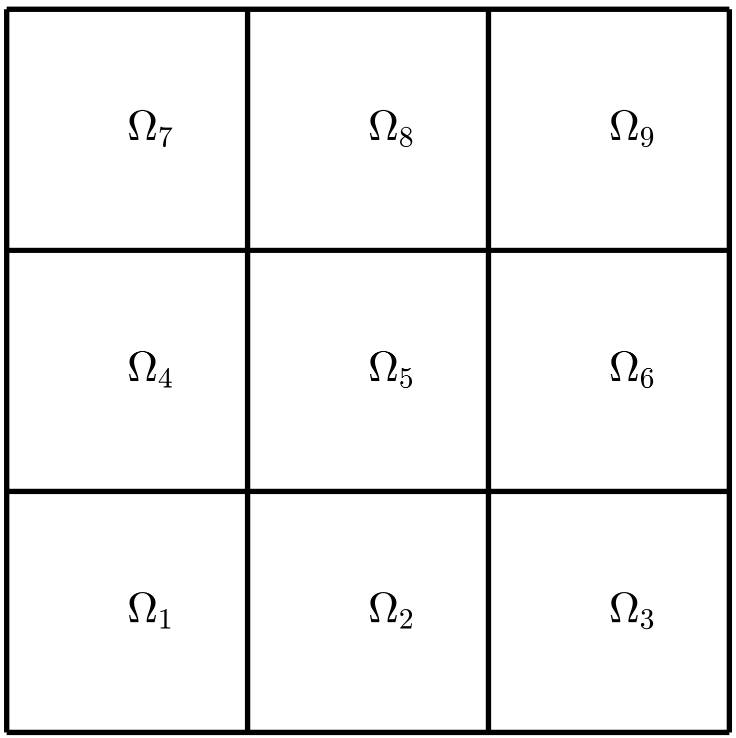
$$\begin{cases} \nabla^2 u = f & \text{on } \; \Omega = \left\{ (x,y) \middle| 0 \le x \le 1, \; 0 \le y \le 1 \right\} \\ u = 0 & \text{on } \; \partial\Omega \quad \text{(Non-homogeneous also possible with the code)} \end{cases}$$



Decompose the domain

$$\begin{cases} \nabla^2 u = f & \text{on } \ \Omega = \left\{(x,y) \middle| 0 \le x \le 1, \ 0 \le y \le 1\right\} \\ u = 0 & \text{on } \ \partial\Omega \quad \text{(Non-homogeneous also possible with the code)} \end{cases}$$

m-by-m decomposition, $M = m^2$



$$\Omega = \bigcup_{i=1}^{M} \Omega_i \qquad \Gamma_i = \text{Boundary of } \Omega_i$$

Solve:

$$\begin{cases} \nabla^2 u_i = f_i & \text{on } \ \Omega_i \\ u_i = \gamma_i & \text{on } \ \Gamma_i \end{cases}$$

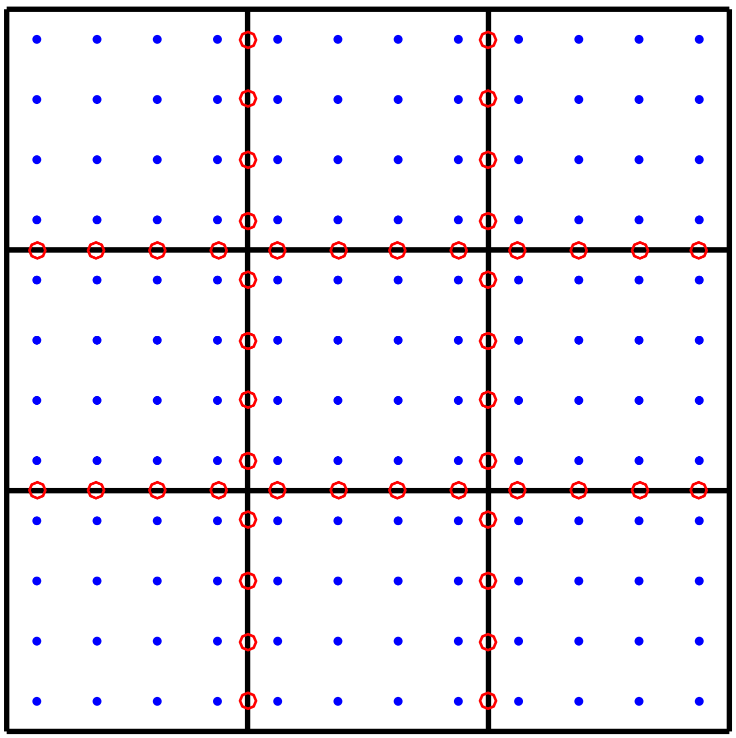
Issues:

1. γ_i are not known
2. u_i should be smooth across domains

Discretize the domain

$$\begin{cases} \nabla^2 u = f & \text{on } \Omega = \left\{(x,y) \middle| 0 \le x \le 1,\ 0 \le y \le 1\right\} \\ u = 0 & \text{on } \partial\Omega \quad \text{(Non-homogeneous also possible with the code)} \end{cases}$$

$n \times n$ grid on each Ω_i



Solve:

$$\begin{cases} \nabla^2 u_i = f_i & \text{on } \Omega_i \\ u_i = \gamma_i & \text{on } \Gamma_i \end{cases}$$

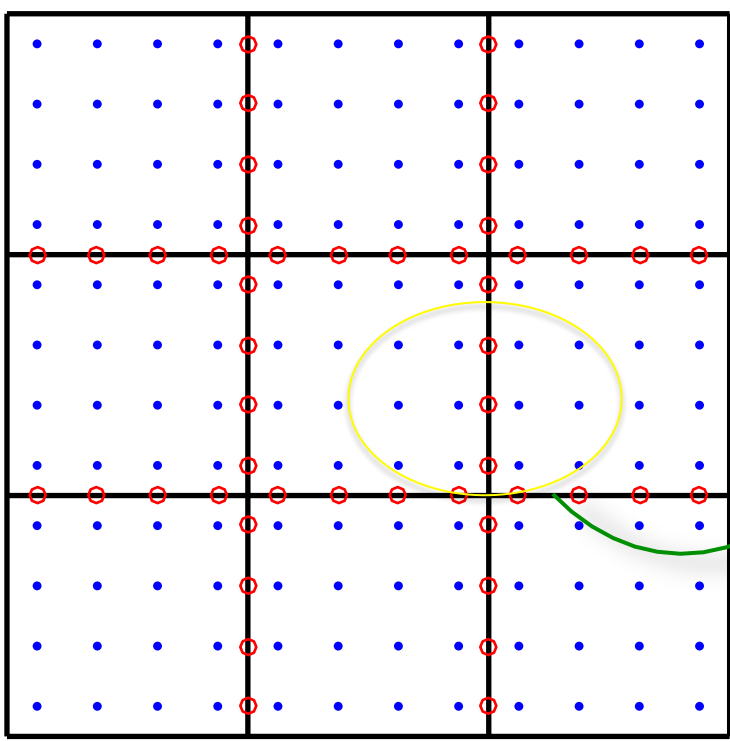
Interior to Ω_i:

$$-2(u_i)_{j,k} + (u_i)_{j-1,k} + (u_i)_{j+1,k}$$
$$-2(u_i)_{j,k} + (u_i)_{j,k-1} + (u_i)_{j,k+1} = h^2 (f_i)_{j,k}$$

- $\bullet$ Unknown u_i values (total $m^2 n^2$)
- $\circ$ Unknown γ_i values (total $2m(m-1)n$)

Discretize the domains

$$\begin{cases} \nabla^2 u = f & \text{on } \ \Omega = \left\{ (x,y) \middle| 0 \le x \le 1,\ 0 \le y \le 1 \right\} \\ u = 0 & \text{on } \ \partial\Omega \quad \text{(Non-homogeneous also possible with the code)} \end{cases}$$

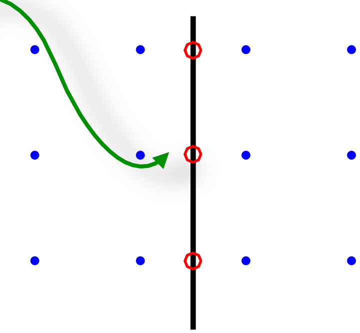


- Unknown u_i values (total $m^2 n^2$)
- Unknown γ_i values (total $2m(m-1)n$)

Solve:

$$\begin{cases} \nabla^2 u_i = f_i & \text{on } \ \Omega_i \\ u_i = \gamma_i & \text{on } \ \Gamma_i \end{cases}$$

Equation for γ_i:



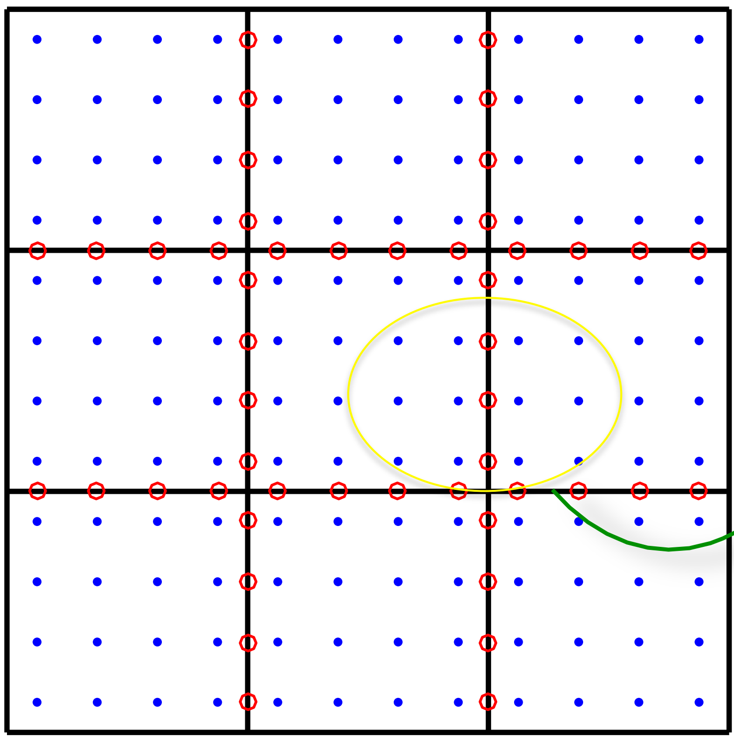
$$\frac{(u_i)_{j,n} + (u_{i+1})_{j,1}}{2} - (\gamma_i)_{j,n+\frac{1}{2}} = 0$$

Discretize the domains

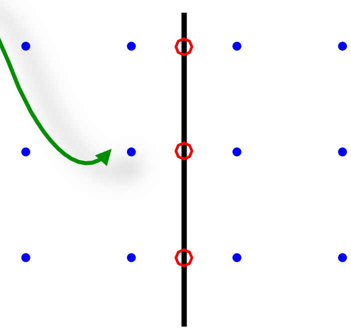
$$\begin{cases} \nabla^2 u = f & \text{on} \;\; \Omega = \left\{ (x,y) \middle| 0 \le x \le 1, \; 0 \le y \le 1 \right\} \\ u = 0 & \text{on} \;\; \partial\Omega \quad \text{(Non-homogeneous also possible with the code)} \end{cases}$$

Solve:

$$\begin{cases} \nabla^2 u_i = f_i & \text{on} \;\; \Omega_i \\ u_i = \gamma_i & \text{on} \;\; \Gamma_i \end{cases}$$



Next to boundary of Ω_i:



• Unknown u_i values

○ Unknown γ_i values

$$-2(u_i)_{j,n} + (u_i)_{j-1,n} + (u_i)_{j+1,n}$$
$$-3(u_i)_{j,n} + (u_i)_{j,n-1} + 2(\gamma_i)_{j,n+\frac{1}{2}} = h^2(f_i)_{j,n}$$

Linear system

$$
\left[\begin{array}{cccc|c}
A_{11} & & & & A_{1\Gamma} \\
& A_{22} & & & A_{2\Gamma} \\
& & \ddots & & \vdots \\
& & & A_{MM} & A_{M\Gamma} \\
\hline
& & A_{\Gamma 1} & & \\
& & A_{\Gamma 2} & & \\
& & \vdots & & A_{\Gamma\Gamma} \\
& & A_{\Gamma 1} & &
\end{array}\right]
\left[\begin{array}{c}
u_1 \\ u_2 \\ \vdots \\ u_M \\ \hline \\ \gamma \\ \\
\end{array}\right]
=
\left[\begin{array}{c}
f_1 \\ f_2 \\ \vdots \\ f_M \\ \hline \\ 0 \\ \\
\end{array}\right]
$$

A_{ii} = Discretization of the Laplacian on Ω_i

$A_{i\Gamma}$ = Stencils involving γ_i near the boundary

$A_{\Gamma i}$ = Averaging operators for determining γ_i

Schur Complement

$$\left[\begin{array}{cccc|c} A_{11} & & & & A_{1\Gamma} \\ & A_{22} & & & A_{2\Gamma} \\ & & \ddots & & \vdots \\ & & & A_{MM} & A_{M\Gamma} \\ \hline & & A_{\Gamma 1} & & \\ & & A_{\Gamma 2} & & \\ & & \vdots & & A_{\Gamma\Gamma} \\ & & A_{\Gamma 1} & & \end{array}\right] \left[\begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_M \\ \hline \\ \gamma \\ \\ \end{array}\right] = \left[\begin{array}{c} f_1 \\ f_2 \\ \vdots \\ f_M \\ \hline \\ 0 \\ \\ \end{array}\right]$$

We can compute γ by solving

$$S\gamma = b$$

where

$$S = A_{\Gamma\Gamma} - \sum_{i=1}^{M} A_{\Gamma i} A_{ii}^{-1} A_{i\Gamma} \text{ and } b = -\sum_{i=1}^{M} A_{\Gamma i} A_{ii}^{-1} f_i$$

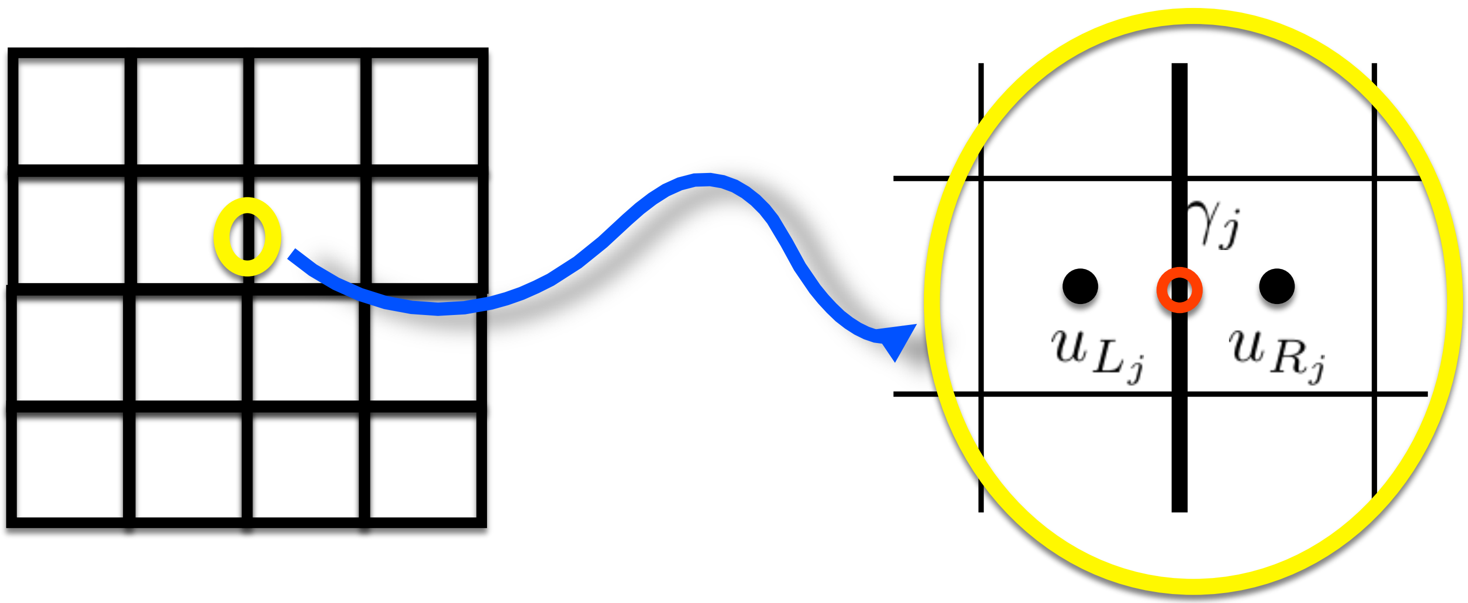
Once γ is known we can solve for all u_i (in parallel).

Solving the Schur complement system

Basic idea : Suppose we knew the exact solution to the boundary value problem $\nabla^2 \mathbf{u} = \mathbf{f}$. And, suppose we know the Dirichlet conditions γ for each block boundary. Then for each interface value γ_j, we could satisfy

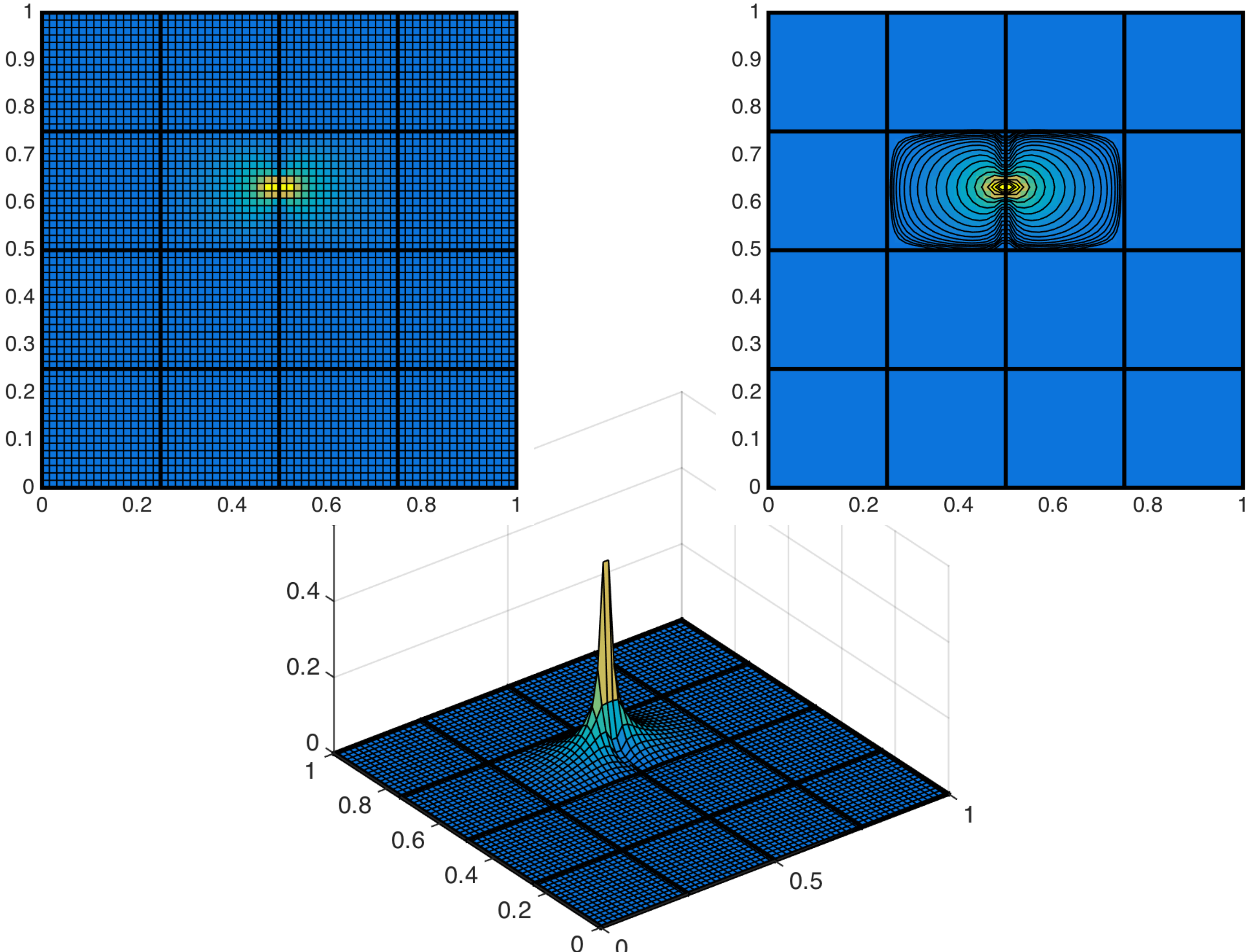
$$\frac{u_{L_j} + u_{R_j}}{2} - \gamma_j = 0$$

We write this expression as a function $d = F(\gamma)$ and solve $F(\gamma) = 0$ to get γ.



```
% Construct Schur complement system
g = zeros(number_of_interface_values,1);

b = F(g);    % Inhomogeneous part

for j = 1:number_of_interface_values
    g(j) = 1;
    S(:,j) = F(g) - b;
    g(j) = 0;
end

g = -S\b      % Solve Sg + b = 0
```

```
function d = F(g)     % F(g) = Sg + b

for k = 1:number_of_blocks
    Solve Au_k = f_k + b_k g
end

% Get difference
for j = 1:number_of_interface_values
    d(j) = (u_left(j) + u_right(j))/2 - g(j)
end

end
```



0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
0
0.2
0.4
0.5
0.6
0.8
1

Software Used

Serial version:

- Eigen - for matrix and vector classes, and iterative solvers
- fftw – for fast direct solve on each subdomain
- boost - for the graph coloring algorithm

Parallel version:

- Epetra from Trilinos, matrix and vector data structures
- Belos2 from Trilinos, iterative solvers (PCG specifically)
- fftw – for fast direct solve on each subdomain

Numerical results

Exact solution: $u(x, y) = \sin(\pi x)\cos(2\pi y)$

Error

CELLS \ DOMAINS	1	2	4	8	16
16	1.45e-02	3.63e-03	9.14e-04	2.28e-04	5.71e-05
32	3.65e-03	9.14e-04	2.28e-04	5.71e-05	1.42e-05
64	9.14e-04	2.28e-04	5.71e-05	1.42e-05	3.56e-06
128	2.28e-04	5.71e-05	1.42e-05	3.56e-06	8.92e-07

Residual

CELLS \ DOMAINS	1	2	4	8	16
16	1.98e-17	1.29e-17	6.73e-18	3.38e-18	1.68e-18
32	8.87e-18	6.29e-18	3.28e-18	1.67e-18	8.41e-19
64	5.92e-18	3.24e-18	1.68e-18	8.73e-19	4.32e-19
128	2.71e-18	1.69e-18	8.97e-19	4.45e-19	2.20e-19

Numerical results

Exact solution: $u(x, y) = \sin(\pi x)\cos(2\pi y)$

Conjugate Gradient Iterations

CELLS \ DOMAINS	2	4	8	16
16	15	27	41	64
32	23	40	60	91
64	35	58	85	130
128	51	84	121	185

With Preconditoner

CELLS \ DOMAINS	2	4	8	16
16	9	14	22	39
32	10	15	25	43
64	11	17	28	48
128	12	19	31	52

Condition number of Schur complement

CELLS \ DOMAINS	2	4	8	16
16	5.37e01	1.97e02	7.74e02	3.08e03
32	1.09e02	3.96e02	1.55e03	6.16e03
64	2.19e02	7.95e02	3.10e03	1.23e04
128	4.39e02	1.59e03	6.20e03	2.46e04

What’s next?

- Neumann boundary conditions
- Non-rectangular domains (still polygonal->rectilinear curves)
- Adaptively refined mesh
- Parallel performance on Kestrel
- How do we solve the Schur complement system in Parallel
- Look at a better preconditioner and different iterative solvers
- Extend to 3D
- Cut cell domains?