Notes on Domain Decomposition

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1 Forming Schur Complement Matrix

If there is a γ for each point on the interface, how to we determine the values?

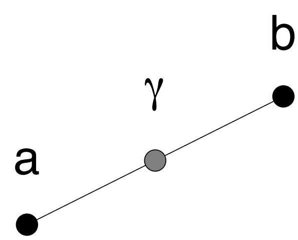

Figure 1: figure with l and r

We want the flux going out of ouf the l cell to match the flux going into the r cell. So we get the equation:

$$\gamma - l = r - \gamma \tag{1}$$

or:

$$2\gamma - (l + r) = 0 \tag{2}$$

Note that l and r are not constants, but are dependent on the value of γ. So, let's change them to the be functions of γ:

$$2\gamma - (L(\gamma) + R(\gamma)) = 0 \tag{3}$$

So now we have a function of γ:

$$F(\gamma) = 2\gamma - (L(\gamma) + R(\gamma)) \tag{4}$$

Therefore we have a function that we want to find the zero of this function:

$$F(\gamma) = 0 \tag{5}$$

If we have a single γ value that we are solving for, this function becomes a linear equation of the form:

$$F(\gamma) = a\gamma - b \tag{6}$$

we can find the coefficients by:

$$b = -F(0) \tag{7}$$

$$a = F(1) + b \tag{8}$$

With these coefficients, we can solve for γ

$$F(\gamma) = 0 \tag{9}$$

$$a\gamma - b = 0 \tag{10}$$

$$a\gamma = b \tag{11}$$

$$\gamma = \frac{b}{a} \tag{12}$$

When there are multiple interface points, then we have a system of linear equations:

$$F(\gamma) = A\gamma - b \tag{13}$$

the b vector is found by

$$b = -F(0) \tag{14}$$

each column of the matrix is found by

$$A(:, i) = F(e_i) + b \tag{15}$$

the γ vector is then found by

$$\gamma = A^{-1}b \tag{16}$$

2 Quick Formation of the Schur Complement Matrix

First, let's reconsider equations 14 and 15. If we are solving on a system where the rhs and lhs on each patch is zero, the b vector for the schur compliment matrix will be 0, since 0 is the correct solution for the interfaces. So equation 15 turns into

$$A(:, i) = F_{zero}(e_i) \tag{17}$$

Where F_{zero} is the same as equation 4 but with the rhs on each domain replaced with 0. So now we use can use this to reduce the amount of work needed to form the Schur compliment matrix.

Let's consider what happens when we solve for $F_{zero}(e_i)$. If a single interface value is set to 1, only the two adjacent domains will have non-zero dirichlet boundary conditions, meaning that only the two adjacent domains will have non-zero solutions. This means that when we are solving for $F_{zero}(e_i)$, we can go ahead and assume that solution on any domain that is not adjacent to the interface with the 1 is zero. In other words, we only have to do a solve on the two adjacent domains, rather than solving for all the domains.

Sparsity Let's look at figure 2. When a single 1 is set on the interface i_{main}, only 7 interfaces will end up having non-zero values: the main interface, i_{main}, and the 6 auxiliary interfaces, $i_{\text{left north}}$, $i_{\text{left south}}$, $i_{\text{left west}}$, $i_{\text{right north}}$, $i_{\text{right east}}$, and $i_{\text{right south}}$.

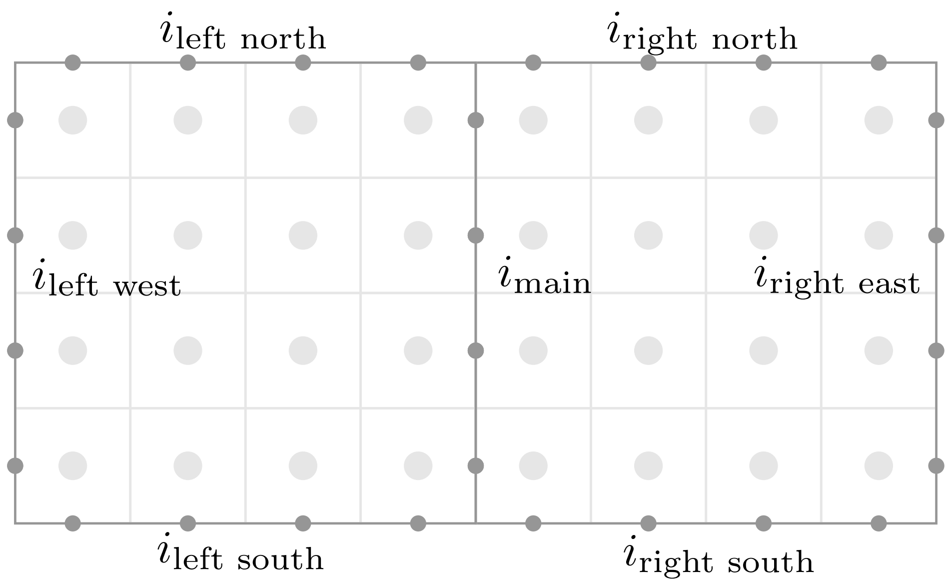

Figure 2: Two domain example

Block Structure For each 1 that is set along i_{main}, the same 7 interfaces will have non-zero values. This means that for each interface, there will be 7 blocks of size $n \times n$.

Splitting up the work If we look at equation 4, we can split up the work and process the two domains at different times.

$$F_{left}(\gamma) = \gamma - L(\gamma) \tag{18}$$

$$F_{right}(\gamma) = \gamma - R(\gamma) \tag{19}$$

So, when we process the left and right domains, we use equations 18 and 19, respectively, for the diagonal blocks (i_{main}). When we are forming the matrix, we will insert the diagonal block twice, summing one block into the other.

3 Handling Refinement

We want the flux going out of the coarse cell to match the fluxes going into the fine cells.

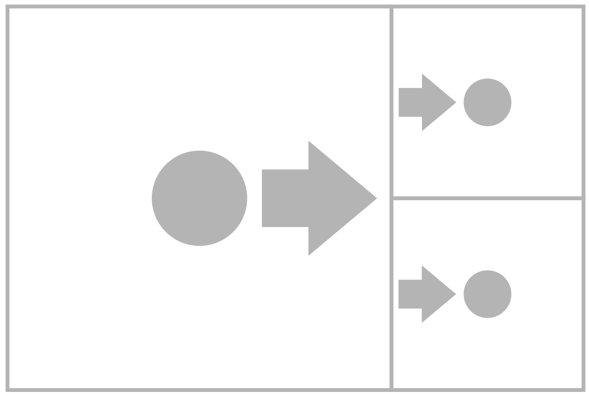

Figure 3: flux

We can represent this with the equation:

$$\Phi_c = \Phi_{f_1} + \Phi_{f_2} \tag{20}$$

Coming up with a stencil

Lets say we want to find the ghost values for the coarse cell, the first fine cell, and the second fine cell. Labeled g_c,g_{f_1},and g_{f_2}, respectively.

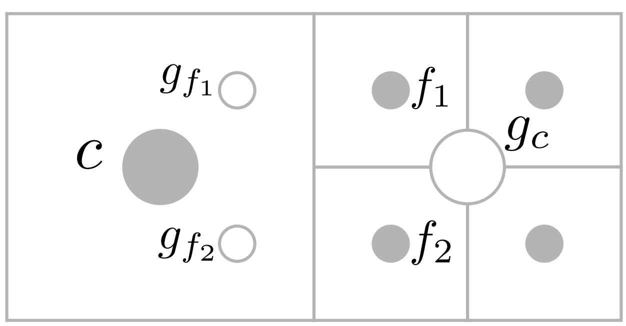

Figure 4: ghost points

We can enforce flux conservation by interpolating to the fine ghost points, and then using equation 20 to find the ghost point for the coarse cell.

The fluxes for each cell will be:

$$\Phi_c = g_c - c \tag{21}$$
$$\Phi_{f_1} = f_1 - g_{f_1} \tag{22}$$
$$\Phi_{f_2} = f_2 - g_{f_2} \tag{23}$$

We can then solve for the value of g_c:

$$\Phi_c = \Phi_{f_1} + \Phi_{f_2} \tag{24}$$
$$g_c - c = f_1 - g_{f_1} + f_2 - g_{f_2} \tag{25}$$
$$g_c = c + f_1 - g_{f_1} + f_2 - g_{f_2} \tag{26}$$

Bilinear interpolation

Bilinear interpolation for the fine ghost points works, but error is not continuous.

TODO: Explain this and show an example

Quadratic interpolation

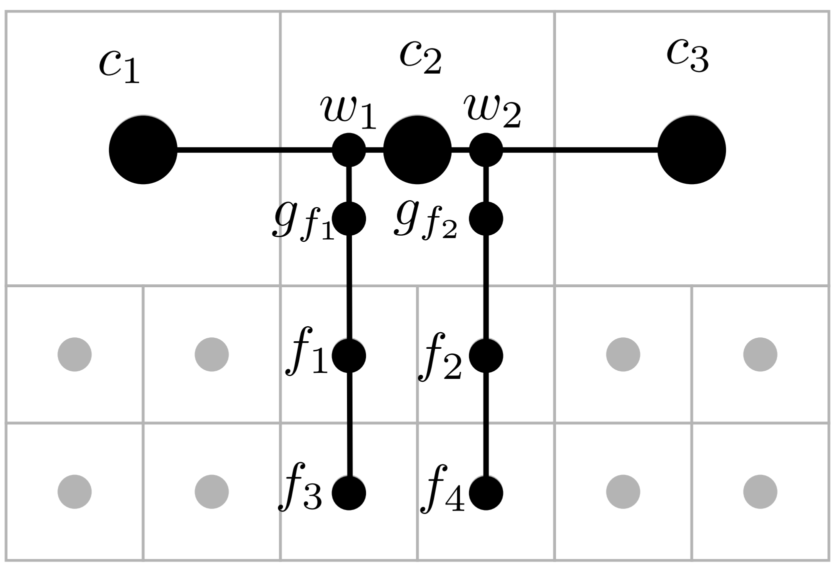

Figure 5: flux

To find the value of g_{f_1}, we first use quadratic interpolation with the points c_1, c_2, and c_3 to interpolate to w_1:

$$w_1 = \frac{5}{32}c_1 + \frac{15}{16}c_2 - \frac{3}{32}c_3$$

We then use quadratic interpolation with the points w_1, f_1, and f_3 to interpolate to g_{f_1}:

$$g_{f_1} = \frac{8}{15}w_1 + \frac{2}{3}f_1 - \frac{1}{5}f_3$$

Plug in the value for w_2, and we get the final equation for g_{f_1}:

$$g_{f_1} = \frac{1}{12}c_1 + \frac{1}{2}c_2 - \frac{1}{20}c_3 + \frac{2}{3}f_1 - \frac{1}{5}f_3$$

The equation for g_{f_2} is similar:

$$g_{f_2} = -\frac{1}{20}c_1 + \frac{1}{2}c_2 + \frac{1}{12}c_3 + \frac{2}{3}f_2 - \frac{1}{5}f_4$$

Now that we have g_{f_1} and g_{f_2}, we can use Eq. 26 to get the value of the ghost point for the coarse cell, g_{c_2}:

$$g_{c_2} = c_2 + f_1 - g_{f_1} + f_2 - g_{f_2}$$
$$g_{c_2} = c_2 + f_1 - \left(\frac{1}{12}c_1 + \frac{1}{2}c_2 - \frac{1}{20}c_3 + \frac{2}{3}f_1 - \frac{1}{5}f_3\right) + f_2 - \left(-\frac{1}{20}c_1 + \frac{1}{2}c_2 + \frac{1}{12}c_3 + \frac{2}{3}f_2 - \frac{1}{5}f_4\right)$$
$$g_{c_2} = -\frac{1}{30}c_1 - \frac{1}{30}c_3 + \frac{1}{3}f_1 + \frac{1}{3}f_2 + \frac{1}{5}f_3 + \frac{1}{5}f_4$$