Divide and Conquer II

Michael Tsai

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矩陣相乘

- 問題:
- Input: A, B 都是 n x n 的Square Matrix
- Output: C, n x n 的square matrix, $C = A \cdot B$

矩陣相乘 – 基本法

n個乘法, n-1個加法,
產生了一個entry,
共有 n^2 個entries

$$\overbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}}^{n\text{個}}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

n個

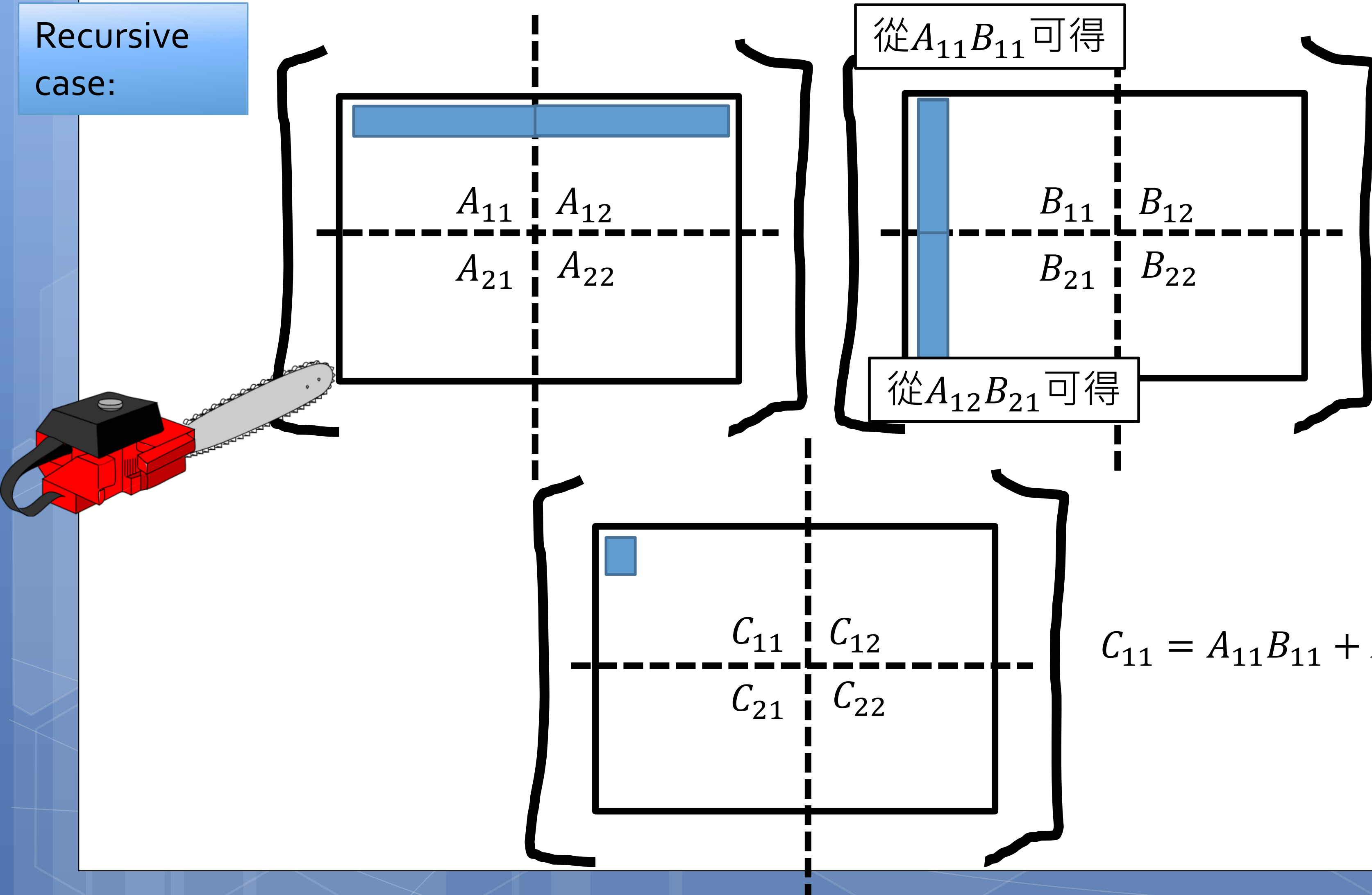
$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

Running time = ?

$\Theta(n^3)$

矩陣相乘 – D&C嘗試一

Recursive case:



$$C_{11} = A_{11}B_{11} + A_{12}B_{21}$$

矩陣相乘 – D&C嘗試一

Base case:

- n=1
- 則$C = A \cdot B$直接算 (A,B,C各自為一個數)

矩陣相乘 – D&C嘗試一

- 以下面的等式可求得C的四個小分塊的解:
- $C_{11} = A_{11} \cdot B_{11} + A_{12} \cdot B_{21}$
- $C_{12} = A_{11} \cdot B_{12} + A_{12} \cdot B_{22}$
- $C_{21} = A_{21} \cdot B_{11} + A_{22} \cdot B_{21}$
- $C_{22} = A_{21} \cdot B_{12} + A_{22} \cdot B_{22}$

矩陣相乘 – D&C嘗試一

Pseudo code:

```
Square-Matrix-Multiply-Recursive(A,B)
n=A.rows
let C be a new n x n matrix
if n==1
```

$c_{11} = a_{11} \cdot b_{11}$

Base case $T(1) = \Theta(1)$

```
else partition the matrix into 4 n/2 x n/2 matrices
```

$\Theta(1)$

Recursive case

C_{11} =Square-Matrix-Multiply-Recursive(A_{11}, B_{11})
+Square-Matrix-Multiply-Recursive(A_{12}, B_{21})
C_{12} =Square-Matrix-Multiply-Recursive(A_{11}, B_{12})
+Square-Matrix-Multiply-Recursive(A_{12}, B_{22})
C_{21} =Square-Matrix-Multiply-Recursive(A_{21}, B_{11})
+Square-Matrix-Multiply-Recursive(A_{22}, B_{21})
C_{22} =Square-Matrix-Multiply-Recursive(A_{21}, B_{12})
+Square-Matrix-Multiply-Recursive(A_{22}, B_{22})

Combine $\Theta(n^2)$

$8T(\frac{n}{2})$

```
return C
```

$$T(n) = \Theta(1) + 8T\left(\frac{n}{2}\right) + \Theta(n^2) = 8T\left(\frac{n}{2}\right) + \Theta(n^2)$$

$$T(n) = \begin{cases} \Theta(1) & \text{,if } n = 1 \\ 8T\left(\frac{n}{2}\right) + \Theta(n^2) & \text{,if } n > 1 \end{cases}$$

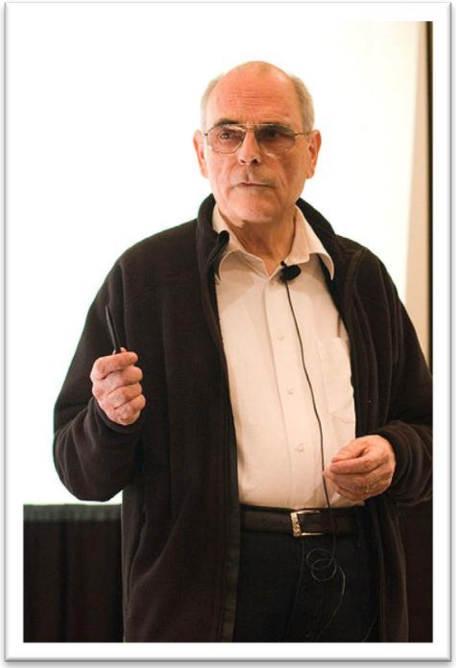
$$\boxed{T(n) = \Theta(n^3)}$$

矩陣相乘 – Strassen’s method

- 課本說: “Not at all obvious”
- 可達到$\Theta(n^{log7}) = \Theta(n^{2.8074})$

- Overview:

1. 將A, B及C都切成四塊n/2大小的matrix
2. 做出10個matrices, $S_1, S_2, \ldots, S_{10}$.這些matrix都是以步驟1中n/2大小的matrices加減後得到的結果.
3. 使用步驟1&2中得到的n/2大小的matrices做乘法後得到$P_1, P_2, \ldots, P_7$.
4. 以$P_1, P_2, \ldots, P_7$相加減後得到的結果產生$C_{11}, C_{12}, C_{21}, C_{22}$四個matrix



德國數學家
Volker Strassen
攝於2009年

- 細節先略過, 但我們可以計算running time了…

矩陣相乘 – Strassen’s method

$\Theta(1)$
1. 將A, B及C都切成四塊n/2大小的matrix

$\Theta(n^2)$
2. 做出10個matrices, $S_1, S_2, \dots, S_{10}$.這些matrix都是以步驟1中n/2大小的matrices加減後得到的結果.

$7T(\frac{n}{2})$
3. 使用步驟1&2中得到的n/2大小的matrices做乘法後得到$P_1, P_2, \dots, P_7$.

$\Theta(n^2)$
4. 以$P_1, P_2, \dots, P_7$相加減後得到的結果產生$C_{11}, C_{12}, C_{21}, C_{22}$四個matrix

$$T(n) = \Theta(1) + \Theta(n^2) + 7T\left(\frac{n}{2}\right) + \Theta(n^2) = 7T\left(\frac{n}{2}\right) + \Theta(n^2)$$

讚

$$\boxed{T(n) = \Theta\left(n^{\log 7}\right)}$$

矩陣相乘 – Strassen’s method

- $S_1 = B_{12} - B_{22}$
- $S_2 = A_{11} + A_{12}$
- $S_3 = A_{21} + A_{22}$
- $S_4 = B_{21} - B_{11}$
- $S_5 = A_{11} + A_{22}$
- $S_6 = B_{11} + B_{22}$
- $S_7 = A_{12} - A_{22}$
- $S_8 = B_{21} + B_{22}$
- $S_9 = A_{11} - A_{21}$
- $S_{10} = B_{11} + B_{12}$

- $P_1 = A_{11} \cdot S_1$
- $P_2 = S_2 \cdot B_{22}$
- $P_3 = S_3 \cdot B_{11}$
- $P_4 = A_{22} \cdot S_4$
- $P_5 = S_5 \cdot S_6$
- $P_6 = S_7 \cdot S_8$
- $P_7 = S_9 \cdot S_{10}$

Not at all obvious

矩陣相乘 – Strassen's method

- $P_5 = A_{11}B_{11} + A_{11}B_{22} + A_{22}B_{11} + A_{22}B_{22}$
- $P_4 = -A_{22}B_{11} + A_{22}B_{21}$
- $-P_2 = -A_{11}B_{22} - A_{12}B_{22}$
- $P_6 = A_{12}B_{21} - A_{22}B_{22} - A_{22}B_{21} + A_{12}B_{22}$

- $C_{11} = A_{11}B_{11} + A_{12}B_{21} = P_5 + P_4 - P_2 + P_6$

- C_{12}, C_{21}, C_{22}用類似的方法
- <Reading assignment> Textbook 4.2
- <Homework for yourself> Exercise 4.2-1

Not at all obvious

矩陣相乘 – Strassen’s method

- 實用嗎? 好用嗎? 讓我們來挑毛病:

1. $\Theta(n^{lg7})$的constant比$\Theta(n^3)$大
2. 那些sub-matrices要花額外的空間
3. 當是Sparse Matrix時, 特別為Sparse Matrix設計的方法比較快
4. Strassen’s method is not as numerically stable as 基本法.

- 1990年代的部分研究減輕了2&4的壞處

- 所以, Strassen’s method有什麼用呢?
- Key: find the crossover point and combine the two algorithms.

- 目前所知, the most asymptotically efficient algorithm has a running time of $O(n^{2.376})$. (Coppersmith and Winograd)

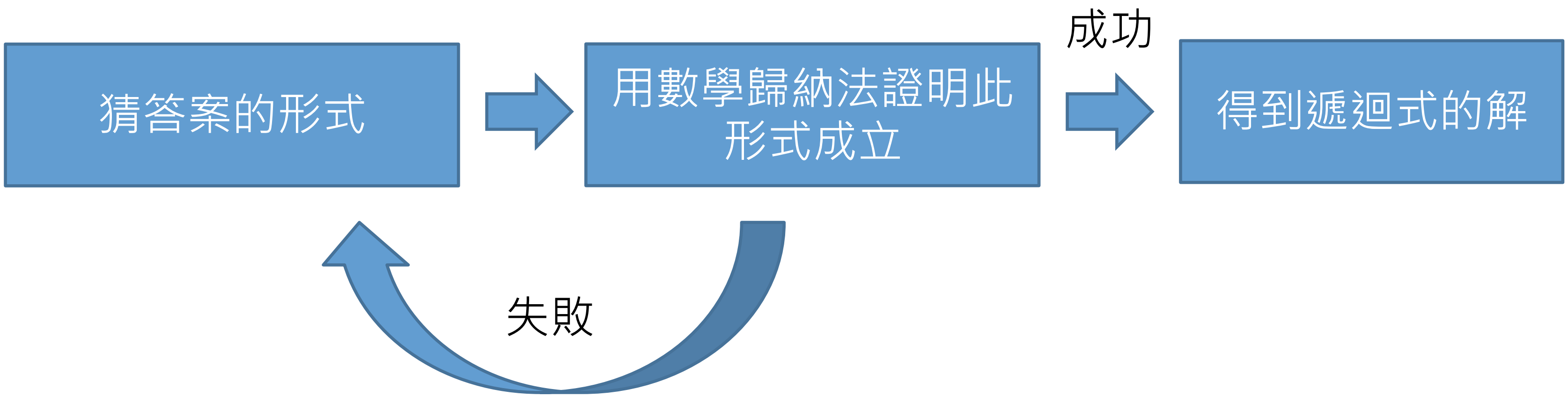
接下來...

- 問題: 我們要怎麼解遞迴式?

- 取代法
- 遞迴樹法
- 大師定理法

http://www.origin-zero.com/senzi/JOKE1.jpg

取代法



取代法-例子

- 問題: $T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n$
- 猜測: $T(n) = O(n \log n)$
- 用歸納法證明 $T(n) \leq \mathrm{c\ n\ log\ n}$, for a constant $c > 0$
- 假設上面的bound在m<n時成立
- 則$m = \left\lfloor \frac{n}{2} \right\rfloor$時亦成立.
- 也就是說$T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \leq \mathrm{c} \left\lfloor \frac{n}{2} \right\rfloor \log\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$

取代法-例子

- $$\begin{aligned} T(n) &= 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n \\ &\le 2\left(c\left\lfloor \frac{n}{2} \right\rfloor \log\left(\left\lfloor \frac{n}{2} \right\rfloor\right)\right) + n \\ &\le cn \log\left(\frac{n}{2}\right) + n \\ &= cn \log n - cn \log 2 + n \\ &= cn \log n - cn + n \\ &\le cn \log n \end{aligned}$$

as long as $c \geq 1$

取代法-例子

- 接著必須也證明邊界條件成立.
- 有時候需要多一點努力....
- 假設$T(1) = 1$.
- 我們必須證明$T(n) \leq cn\log n\,, n = 1$
- 天不從人願: $n = 1$時, $cn\log n = 0$
- $\mathrm{T}(1) = 1 > 0 = cn\log n$
- 娃~

取代法-例子

- 事實上, 我們只須證明, 當$n > n_0$時, $T(n) \leq cn \log n$即可. (把不聽話的$n = 1$拔掉)
- 從原本的遞迴式, 我們可以得到$T(2) = 4, T(3) = 5$.
- 接著設$n_0 = 2$. 我們發現:
- $T(2) \leq c\,2 \log 2$ & $T(3) \leq c\,3 \log 3$
- (只要$c \geq 2$) 至此可以使邊界條件成立.
- 喔耶.

取代法-怎麼猜?

- 靠經驗.

- 跟沒講一樣.
- 一些小方法:

1. 根據以前看過類似的遞迴式來猜測
2. 使用等一下要介紹的遞迴樹
3. 證明比較鬆的upper bound或lower bound來慢慢接近tight bound

$$T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor + 17\right) + n$$

取代法-小技巧1

- 題目: $\mathrm{T}(n) = T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + T\left(\left\lceil\frac{n}{2}\right\rceil\right) + 1$
- 猜測: $T(n) = O(n)$
- 歸納法證明:
- $T(n) \le c\left(\left\lfloor\frac{n}{2}\right\rfloor + \left\lceil\frac{n}{2}\right\rceil\right) + 1 = cn + 1 \nleq cn$
- 爛掉了…

取代法-小技巧1

- 方法: 改使用$T(n) \leq cn - d, d \geq 0$
- **<u>(減掉一個order較低的term)</u>**
- $T(n) \leq \left(c\left\lfloor\frac{n}{2}\right\rfloor - d\right) + \left(c\left\lceil\frac{n}{2}\right\rceil - d\right) + 1$
  $= cn - 2d + 1$
  $\leq cn - d$
- (as long as $d \geq 1$)
- (然後繼續選c, 使得boundary condition成立, 在此省略)

取代法–小技巧2

- 看起來挺嚇人的: $T(n) = 2T\left(\left\lfloor\sqrt{n}\right\rfloor\right) + \log n$
- 替換變數: $m = \log n$
- $T(2^m) = 2T\left(2^{\frac{m}{2}}\right) + m$

暫時不管flooring

- 定義: $S(m) = T(2^m)$
- $S(m) = 2S\left(\frac{m}{2}\right) + m$
- →$S(m) = O(m \log m)$
- →$T(n) = T(2^m) = S(m) = O(m \log m) = O(\log n \log \log n)$

遞迴樹法

畫出遞迴樹

↓

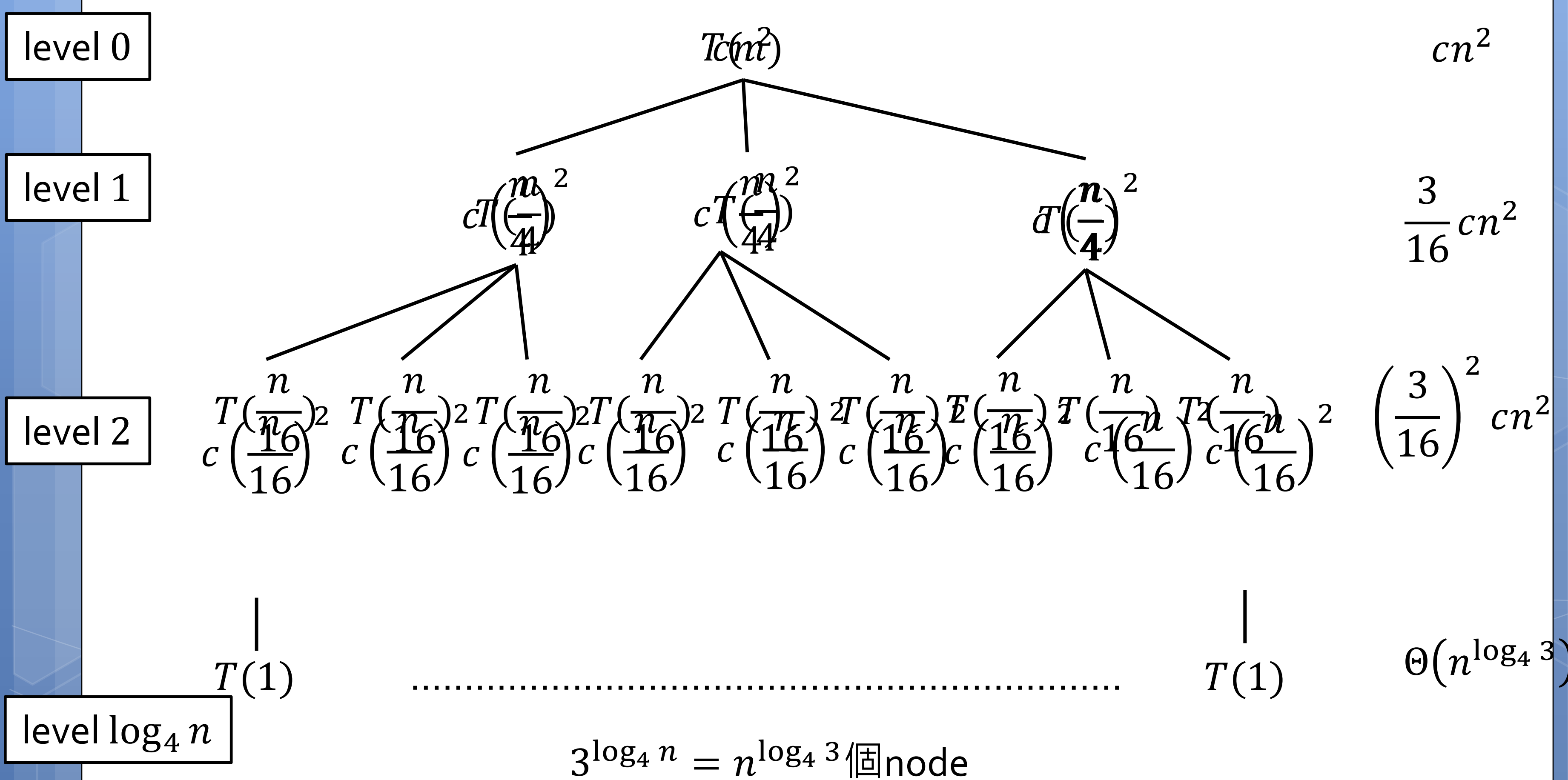
用比較不嚴謹的方法加總得到解

↓

用數學歸納法證明此解成立

遞迴樹法–例子1

- 例子: $T(n) = 3T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + \Theta(n^2)$



遞迴樹法-例子1

- $T(n) = cn^2 + \frac{3}{16}cn^2 + \left(\frac{3}{16}\right)^2 cn^2 + \cdots + \left(\frac{3}{16}\right)^{\log_4 n-1} cn^2 + \Theta\left(n^{\log_4 3}\right)$

$$= \sum_{i=0}^{\log_4 n-1} \left(\frac{3}{16}\right)^i cn^2 + \Theta\left(n^{\log_4 3}\right)$$

$$\leq \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i cn^2 + \Theta\left(n^{\log_4 3}\right)$$

$$= \frac{1}{1 - \frac{3}{16}} cn^2 + \Theta\left(n^{\log_4 3}\right)$$

$$= \frac{16}{13} cn^2 + \Theta\left(n^{\log_4 3}\right)$$

$$= O(n^2)$$

遞迴樹法–例子1

- 用歸納法證明: $T(n) \le dn^2$ for some $d > 0$.
- $T(n) \le 3T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + cn^2$

$$
\begin{aligned}
&\le 3d\left(\left\lfloor \frac{n}{4} \right\rfloor\right)^2 + cn^2 \\
&\le 3d\left(\frac{n}{4}\right)^2 + cn^2 \\
&= \frac{3}{16}dn^2 + cn^2 \\
&\le dn^2
\end{aligned}
$$

as long as $d \ge \left(\frac{16}{13}\right)c$

遞迴樹法–例子2

- 例子: $T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + O(n)$
- 請一位同學上來畫遞迴樹 (有點跛腳的遞迴樹)

遞迴樹法–例子2

- 歸納法證明: $T(n) \le dn \log n$
- $T(n) \le T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + cn$

$$\le d\left(\frac{n}{3}\right)\log\frac{n}{3} + d\left(\frac{2n}{3}\right)\log\frac{2n}{3} + cn$$

$$= d\left(\frac{n}{3}\right)\log n + d\left(\frac{2n}{3}\right)\log n - d\left(\frac{n}{3}\right)\log 3 - d\left(\frac{2n}{3}\right)\log\left(\frac{3}{2}\right) + cn$$

$$= dn\log n - d\left(\left(\frac{n}{3}\right)\log 3 + \left(\frac{2n}{3}\right)\log 3 - \left(\frac{2n}{3}\right)\log 2\right) + \mathrm{cn}$$

$$= dn\log n - dn\left(\log 3 - \frac{2}{3}\right) + cn$$

$$\le dn\log n$$

as long as $d \ge \frac{c}{\log 3 - \frac{2}{3}}$

大師定理

- Master Theorem:

Let $a \geq 1$ and $b \geq 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where we interpret $\frac{n}{b}$ to mean either $\left\lfloor \frac{n}{b} \right\rfloor$ or $\left\lceil \frac{n}{b} \right\rceil$. Then $T(n)$ has the following asymptotic bounds:

1. if $f(n) = O\left(n^{\log_b a - \epsilon}\right)$ for some constant $\epsilon > 0$, then $T(n) = \Theta\left(n^{\log_b a}\right)$
2. if $f(n) = \Theta\left(n^{\log_b a}\right)$, then $T(n) = \Theta\left(n^{\log_b a} \log n\right)$
3. if $f(n) = \Omega\left(n^{\log_\mathrm{b} \mathrm{a} + \epsilon}\right)$ for some constant $\epsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta\left(f(n)\right)$

大師定理

$n^{\log_b a}$ is larger

1. if $f(n) = O\left(n^{\log_b a-\epsilon}\right)$ for some constant $\epsilon > 0$, then $T(n) = \Theta\left(n^{\log_b a}\right)$
2. if $f(n) = \Theta\left(n^{\log_b a}\right)$, then $T(n) = \Theta\left(n^{\log_b a} \log n\right)$

The same order

3. if $f(n) = \Omega\left(n^{\log_b a+\epsilon}\right)$ for some constant $\epsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta\left(f(n)\right)$

$f(n)$ is larger

Which one is polynomially larger/smaller?

$$\boxed{f(n)} \qquad \boxed{n^{\log_b a}}$$

Note: not all possibilities for $f(n)$ can be covered by these 3 cases!

大師定理-例子1

- $T(n) = 9T\left(\frac{n}{3}\right) + n$
- a=9, b=3, f(n)=n
- $\log_b a = \log_3 9 = 2$
- satisfies case 1: $f(n) = O\left(n^{\log_b a-\epsilon}\right), \epsilon = 1$
- so $T(n) = \Theta\left(n^{\log_b a}\right) = \Theta(n^2)$

For your reference: $T(n) = aT\left(\frac{n}{b}\right) + f(n)$

$f(n)$ | $n^{\log_b a}$

大師定理-例子2

- $T(n) = T\left(\frac{2n}{3}\right) + 1$
- a=1, b=3/2, f(n)=1
- $n^{\log_b a} = \mathrm{n}^{\log_{\frac{3}{2}} 1} = \mathrm{n}^0 = 1$
- satisfies case 2: $f(n) = \Theta\left(n^{\log_b a} \log n\right) = \Theta(\log n)$

For your reference: $T(n) = aT\left(\frac{n}{b}\right) + f(n)$

$\boxed{f(n)}$ $\boxed{n^{\log_b a}}$

大師定理-更多例子

上台解題時間...

- $T(n) = 3T\left(\frac{n}{4}\right) + n\log n$
- $T(n) = 2T\left(\frac{n}{2}\right) + n\log n$
- $T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$
- $T(n) = 8T\left(\frac{n}{2}\right) + \Theta(n^2)$
- $T(n) = 7T\left(\frac{n}{2}\right) + \Theta(n^2)$

取數問題

- Selection Problem
- 問題:
- Input: n個數字之集合
- Output: 取出此n個數字之中位數

- 中位數之定義: n個數字中第k=$\left\lceil \frac{n}{2} \right\rceil$小的數字

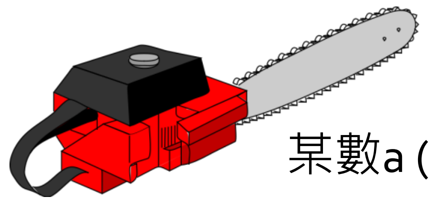
取數問題

- 菜瓜布解法:

- 先把n個數sort好
- 從最小的數過去算到第k小的數字即為答案
- running time=?
- $\Omega(n \log n)$
- Can we do a better job?

取數問題

n個數



某數a (pivot)

L_2 小於等於a的

L_1 大於a的

if $k > |L_2|$, 找L_1中第$k-|L_2|$小的

if $k \leq |L_2|$, 找L_2中第k小的

取數問題

- 下一個問題: 怎麼選a?
- 選不好的話...$L_1 = n, L_2 = 0$

- 最好是可以平均分成兩分.
- 那就是選中位數.
- 咦, 我們不是就要找中位數嗎?

- 能不能花少一點時間, 找個"差不多"的中位數

取數問題

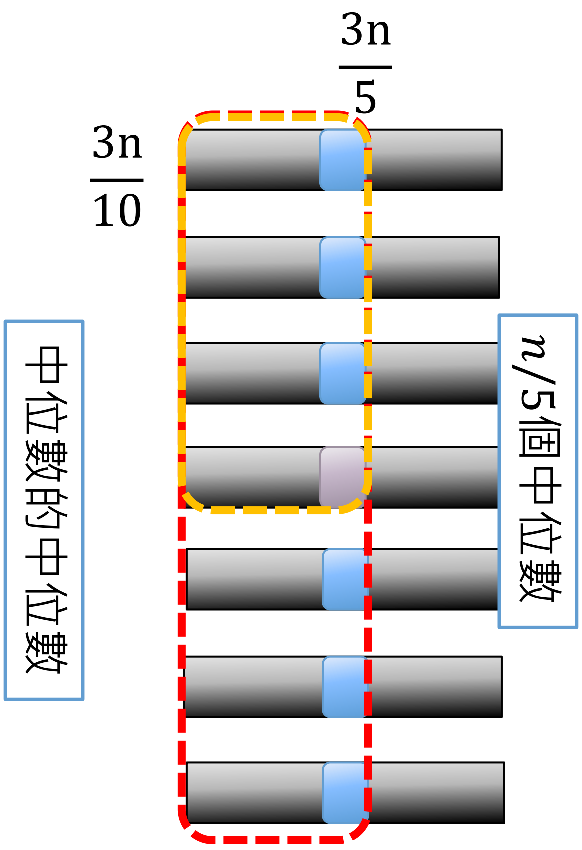
- 差不多的中位數:

1. 把n個數分成很多大小為5個的sub list (大約共有n/5個sub list
2. 這些sub list中各自找中位數
3. 找出n/5個中位數中的中位數

→此為差不多的中位數

取數問題

- 有多差不多呢?
- 比"差不多中位數"小的至少有$\frac{3n}{10}$
- $|L_2| \geq \frac{3n}{10}$
- $|L_1| \leq \frac{7n}{10}$



取數問題

- Algorithm:

1. if $n \leq 5$ then 直接找出其中位數 $\Theta(1)$
2. else
3. 把數列拆成$\frac{n}{5}$個大小為5的小數列 $\Theta(n)$
4. 每個小數列找出其中位數 $\Theta(n)$
5. 找出$\frac{n}{5}$個中位數的中位數m $\mathrm{T}(\frac{n}{5})$
6. 用此中位數把原本的數列拆成兩部分: 比m大(L_1)及不比m大的(L_2) $\Theta(n)$
7. if $k > |L_2|$, 找L_1中第$k - |L_2|$小的
8. if $k \leq |L_2|$, 找L_2中第k小的

Max: $\mathrm{T}(\frac{7n}{10})$

$$T(n) = \Theta(n) + \Theta(n) + T\left(\frac{n}{5}\right) + \Theta(n) + T\left(\frac{7n}{10}\right)$$

$$= T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + \Theta(n)$$

$$\boxed{T(n) = \Theta(n)}$$

Today's Reading Assignment

- Cormen ch 4.2 – 4.5