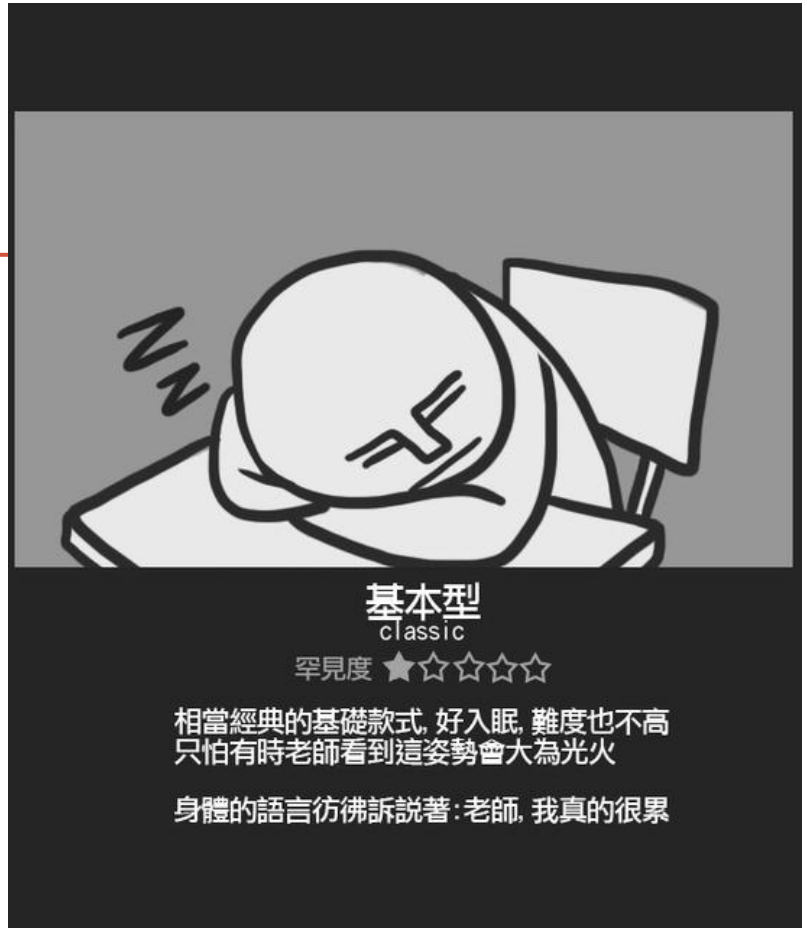
連載: 學生上課睡覺姿勢大全

http://www.wretch.cc/blog/chi771027/26489957

GRAPH 1

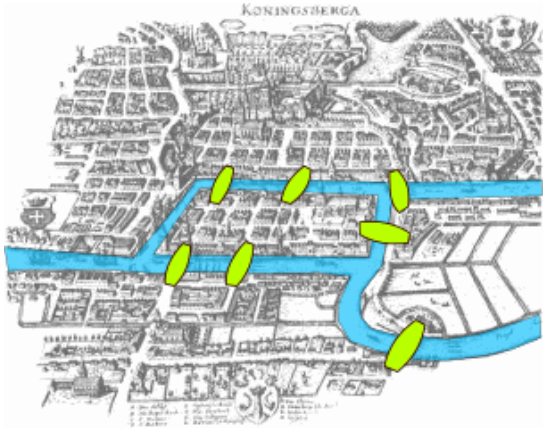
Michael Tsai

2013/4/2

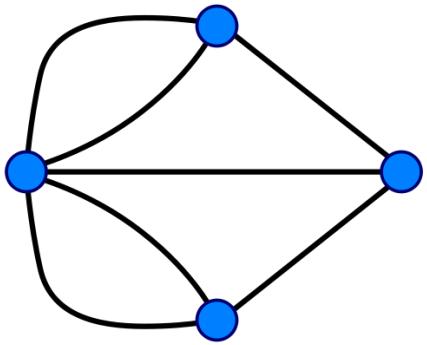


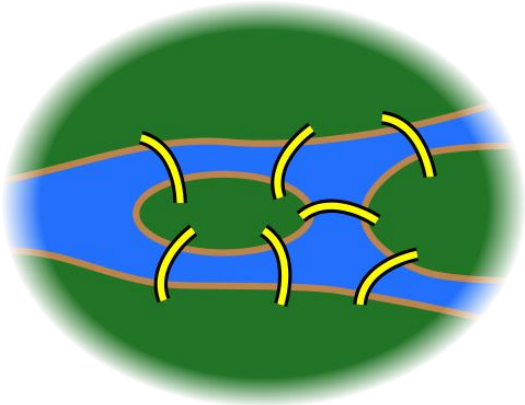
Königsberg Seven Bridge Problem

- 西元1736年, Euler嘗試著要解答這個問題:



- 右邊地圖中, 有沒有可能找出一條路徑, 使得每一條橋都走過一次之後又回到出發點?

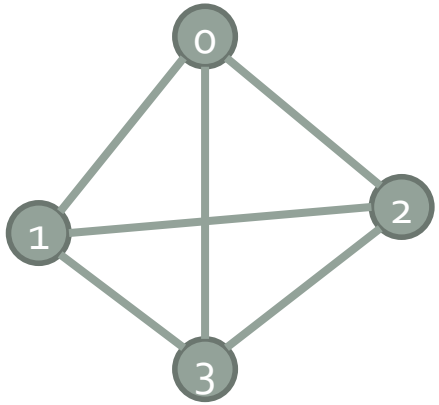




答案: Graph必須要是connected & 必須有0個或2個node有odd degree

A graph

- G: a graph, consists of two sets, V and E.
- V: a finite, nonempty set of vertices.
  - 單數: vertex; 複數: vertices
- E: a set of pairs of vertices.

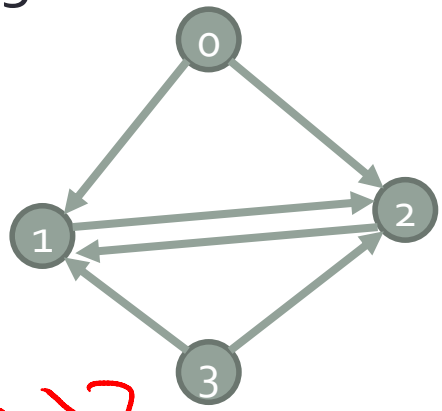


- G=(V,E)
- V={0,1,2,3}
- E={(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)}

Directed & undirected graph

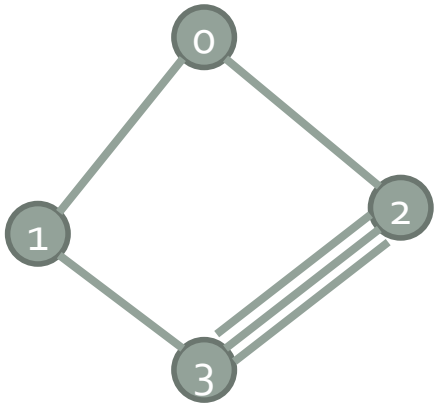
- Graph中, edge有方向的叫做directed graph, 沒方向的叫做undirected graph
- Directed graph 通常又叫digraph – edge是**ordered** pairs
- Undirected graph 通常就只叫做graph – edge是**unordered** pairs

- (1,2) 和 (2,1) 在undirected graph是一樣的edge
- <1,2>和<2,1>在digraph是不一樣的edge
- V={0,1,2,3}
- E=?
- E={<0,1>,<0,2>,<1,2>,<2,1>,<3,1>,<3,2>}

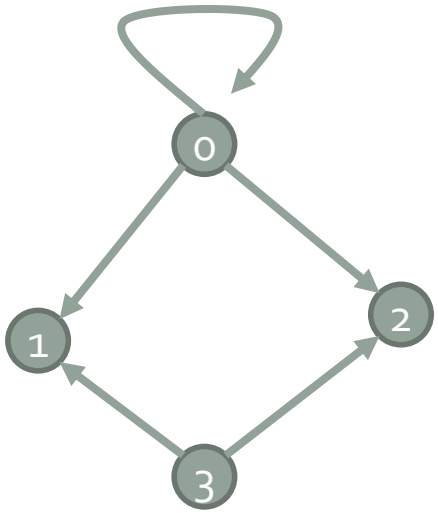


Self Edge & Multigraph

Multigraph



Graph with self-edges <v,v>

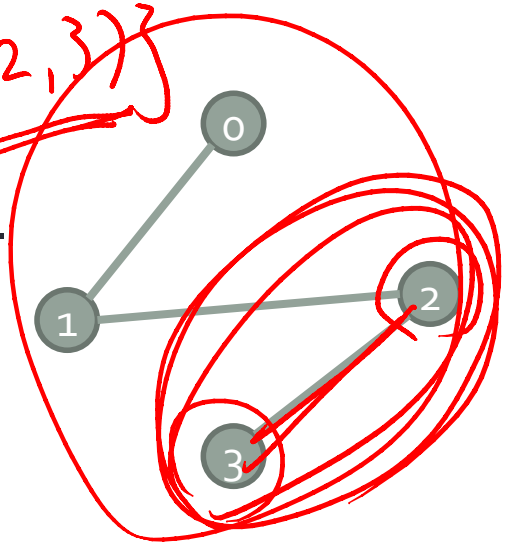


Maximum number of edges

- 一個有n vertices的graph, 最多有幾個edges?
- 一個有n vertices的digraph, 最多有幾個edges?

- 答案: graph: $\frac{n(n-1)}{2}$,
- digraph: $n(n-1)$

怎麼這麼多名詞XD



- 相鄰(adjacent):
  如果有edge (u,v), 那麼u, v兩vertices就是adjacent.
- 如果有edge <u,v> 那麼我們說
  v is adjacent to u.
  (有些課本用相反的定義T_T)
- 作用(incident): 如果有edge (u,v) 那麼u, v兩vertices就是incident (作用) on (在) u和 v上面
- <u,v> is incident **from** u and is incident **to** v.
- <u,v> **leaves** u and **enters** v.
- Subgraph: 如果G=(V,E), G'=(V',E')是G的subgraph, 則$V' \subseteq V$且$E' \subseteq E$

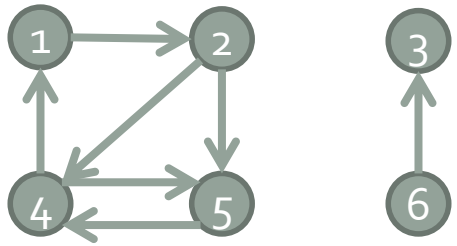
Degree of vertex

- Vertex的**degree**:
- 有幾個edge連在vertex上

- Digraph中
- 又可分為in-degree and out-degree
- in-degree: 進入vertex的edge數
- out-degree: 出去vertex的edge數
- degree=in-degree+out-degree

- 一個degree=0的vertex可稱為isolated

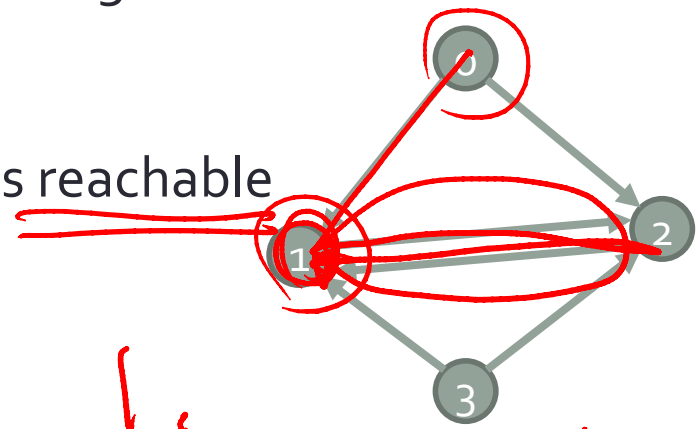
- Edge數和degree的關係: $e = \frac{\left(\sum_{i=0}^{n-1} d_i\right)}{2}$



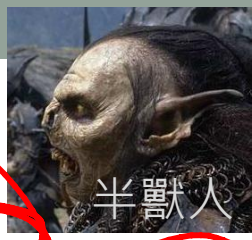
路徑（path）

- Path: 一條從u到v的path, 是一連串的vertices, $u, i_1, i_2, \ldots, i_k, v$, 而且$(u, i_1), (i_1, i_2), \ldots, (i_{k-1}, i_k), (i_k, v)$都是edge.

  (digraph的定義自行類推)
- 如果有一條path p從u到u', 則我們說u' is reachable from u via p.



- Path的length: 裡面有幾條edge
- Simple path: 除了u, v(起點和終點)以外, 其他的vertices都沒有重複過.
- Cycle: 一個u和v一樣的simple path
- Subpath:??
- 答案: 定義path的vertex sequence裡面的連續的一段.



有連接的(connected)

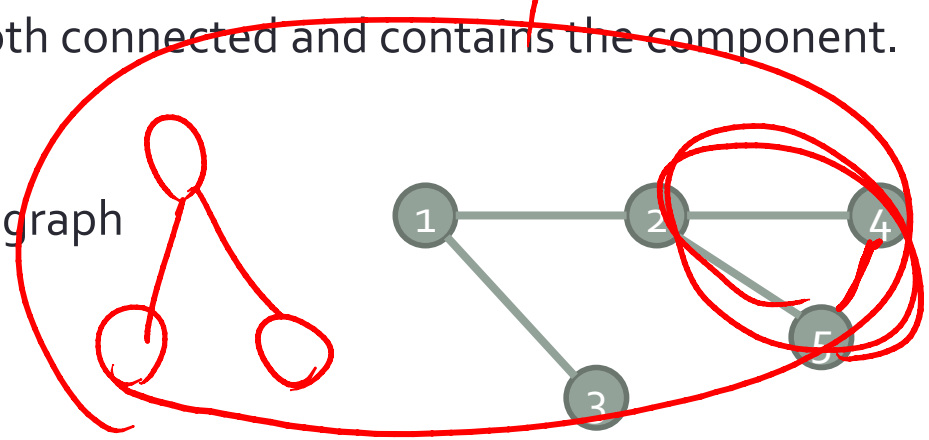
- **In undirected graph:**
- Vertices u and v are said to be **connected** iff
- there is a path from u to v. (graph & digraph)

- **Connected graph**:
- iff every pair of distinct vertices u and v in V(G) is connected

- **Connected component** (也有人直接叫component):
- A maximal connected subgraph

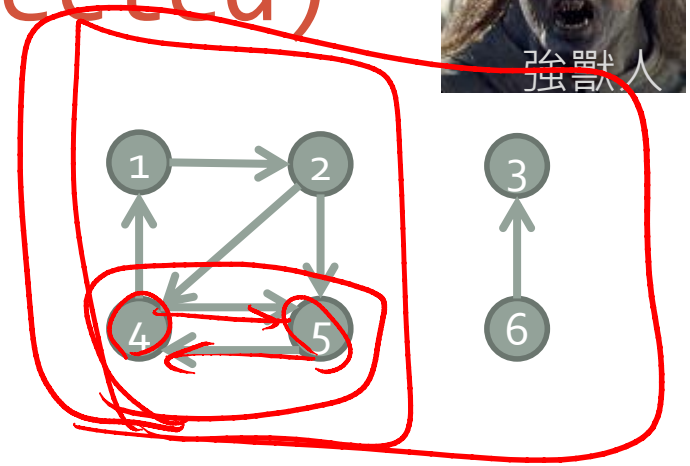
- **Maximal**: no other subgraph in G is both connected and contains the component.

- 問: Tree是一個怎麼樣的graph?
- 答: connected and acyclic (沒有cycle) graph



- 問: Forest是一個怎麼樣的graph?
- 答: acyclic graph

強連接(strongly connected)



- **In digraph:**

- **Strongly connected**:
- G is strongly connected iff
- for every pair of distinct vertices u and v in V
- there is a directed path from u to v and also a directed path from v to u.

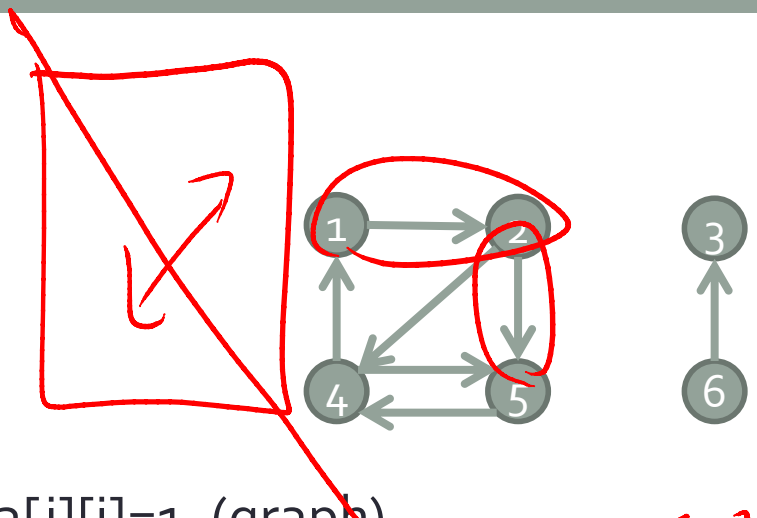
- **Strongly connected component**:
- A maximal subgraph which is strongly connected.

要怎麼表示一個graph呢?

- 主流表示法:
- Adjacency matrix (用array)
- Adjacency lists (用linked list)

- 兩者都可以表示directed & undirected graph

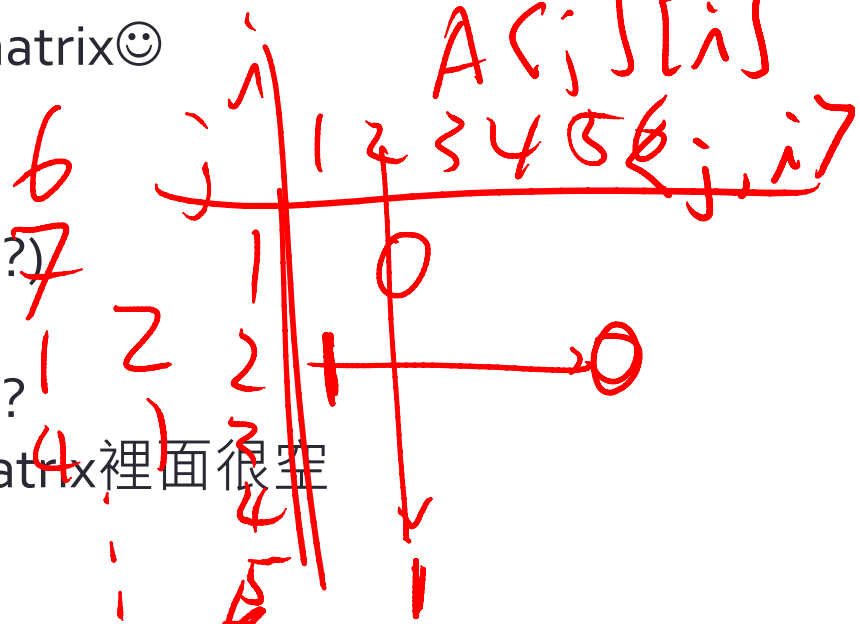
表示法I 用Array



- 本方法叫做adjacency matrix
- index當作vertex號碼
- edge用matrix的值來表示:
- 如果有(i,j)這條edge, 則a[i][j]=1, a[j][i]=1. (graph)
- 如果有<i,j>這條edge, 則a[i][j]=1. (digraph)
- 對undirected graph, $a = a^T$ (也就是對對角線對稱)

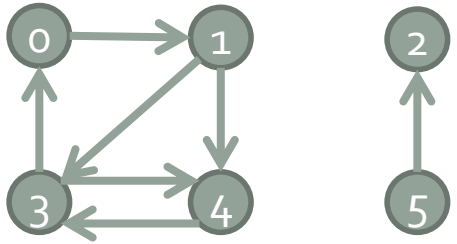
- 請同學解釋要怎麼建adjacency matrix☺

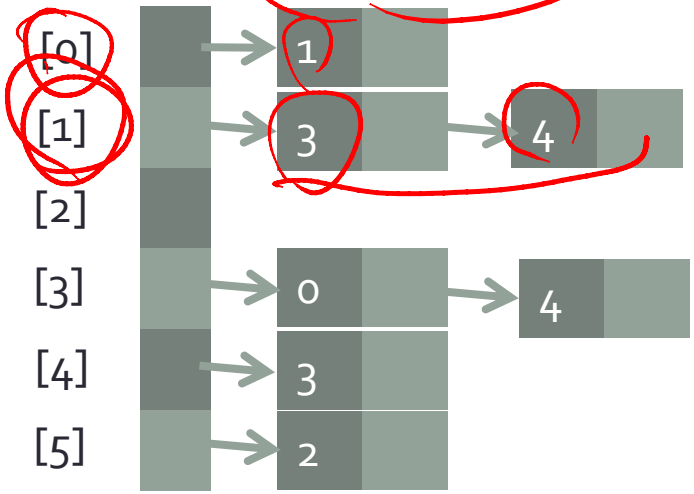
- 粉簡單. 那麼來看看好不好用:
- 如果要看總共有多少條edge? O(??)
- 答: $O(|V|^2)$
- 有沒有可能跟edge數(e)成正比呢?
- 通常$|E| \ll |V|^2$ 所以adjacency matrix裡面很空

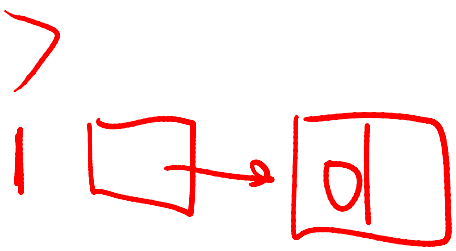


表示法II 用linked list

- 本方法叫做adjacency lists
- 建立一個list array a[n] (n為vertex數目)
- 每個list裡面紀錄通往別的vertex的edge
- 問: 怎麼算in-degree?



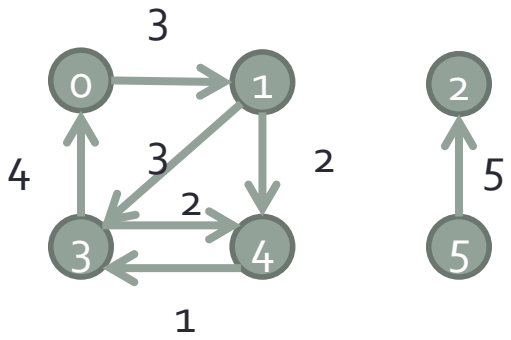




- 答: 如果要比較容易的話, 要另外建"inverse adjacency lists"
- 請同學告訴我怎麼畫☺

Weighted Edge

- Edge可以有"weight"
- 表示長度, 或者是需要花費
- 一個edge有weight的graph
- 又叫做network



- 想想看: 用剛剛的representation要怎麼儲存weight?
- 答:
- adjacency matrix可以用array的值來存
- adjacency list可以在list node裡面多開一個欄位存

兩者比較

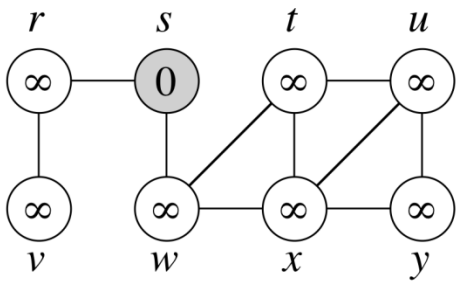
- Adjacency lists:
- 用來表示sparse graphs時使用比較少空間
- 使用空間: $O(|V| + |E|)$

- Adjacency matrix:
- 要找某個edge (u,v)有沒有在graph裡面比較快
- 一個entry只需要1 bit (unweighted graph)
- 簡單容易, graph小的時候用adjacency matrix比較方便
- 使用空間: $O(|V|^2)$

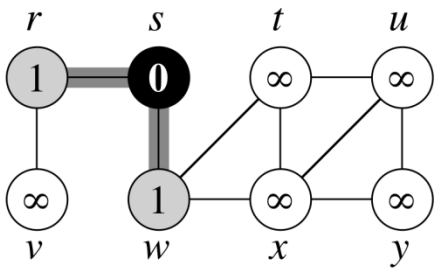
Breadth-First Search (BFS)

- 給一個graph G=(V,E)及一個source vertex s
- 找出所有從s reachable的vertices
- 計算從s到每一個reachable的vertex的最少edge數目
- 產生breadth-first tree, s為root, 而其他reachable的vertices都在樹裡面
- Directed graph & undirected graph皆可

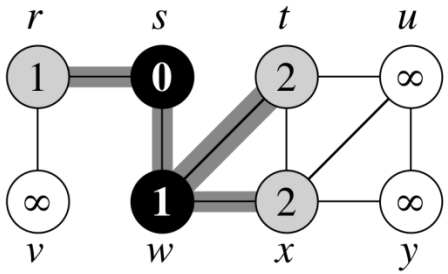
- 此方法會先找到所有距離s distance為k的vertex, 然後再繼續找距離為k+1的vertex



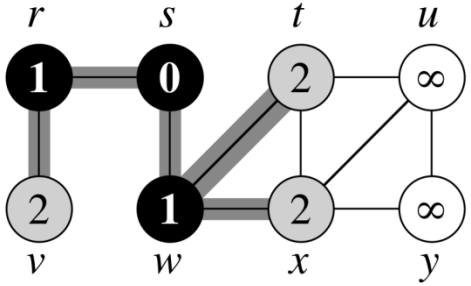
Q | s |
0



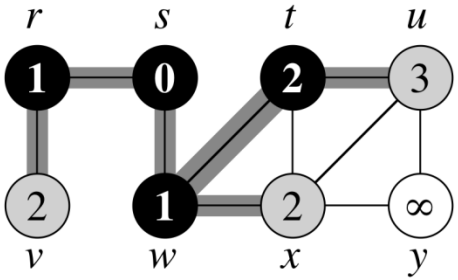
Q | w | r |
1 1



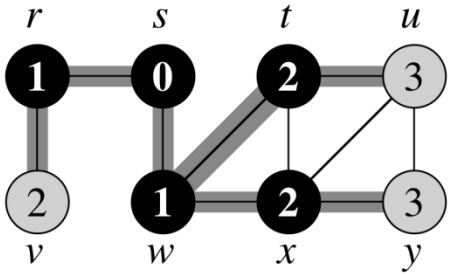
Q | r | t | x |
1 2 2



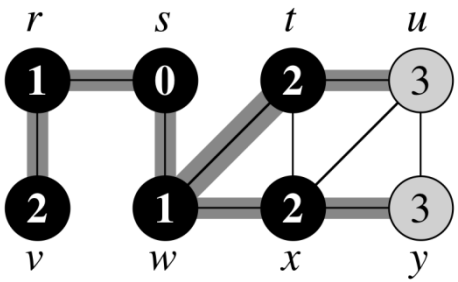
Q | t | x | v |
2 2 2



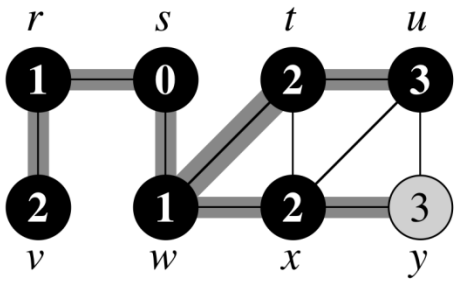
Q | x | v | u |
2 2 3



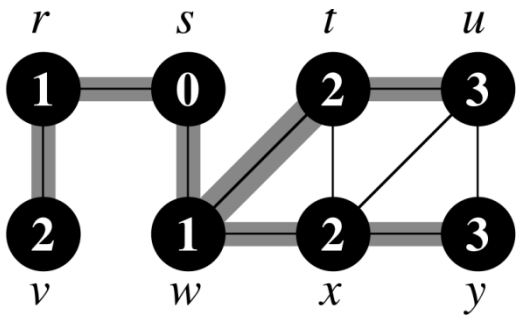
Q | v | u | y |
2 3 3



Q | u | y |
3 3



Q | y |
3



Q $\emptyset$

BFS Pseudo-code

```
BFS(G,s)
for each vertex u ∈ G.V − {s}
        u.color=WHITE
        u.d=∞
        u.pi=NIL
s.color=GRAY
s.d=0
s.pi=NIL
Q={}
ENQUEUE(Q,s)
while Q!={}
        u=DEQUEUE(Q)
        for each v ∈G.Adj[u]
                if v.color==WHITE
                        v.color=GRAY
                        v.d=u.d+1
                        v.pi=u
                        ENQUEUE(Q,v)
        u.color=BLACK
```

初始所有vertex的值

初始開始search的vertex s的值

對每一個和u相連vertex

v.color: 用顏色來區別discover的狀況
WHITE: 還沒discovered
GRAY: discovered了, 但是和該vertex相連的鄰居還沒有都discovered
BLACK: discovered了且和該vertex相連的鄰居都已discovered

v.d: 和root的距離

v.pi: 祖先 (predecessor)

BFS Pseudo-code

```
BFS(G,s)
for each vertex u ∈ G.V − {s}
        u.color=WHITE
        u.d=∞
        u.pi=NIL
s.color=GRAY
s.d=0
s.pi=NIL
Q={}
ENQUEUE(Q,s)
while Q!={}
        u=DEQUEUE(Q)
        for each v ∈G.Adj[u]
                if v.color==WHITE
                        v.color=GRAY
                        v.d=u.d+1
                        v.pi=u
                        ENQUEUE(Q,v)
        u.color=BLACK
```

$O(V)$

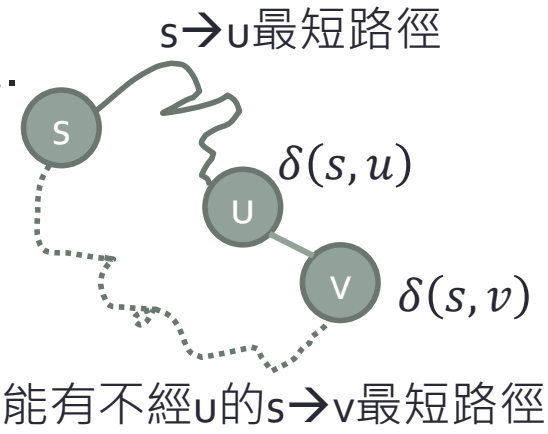
$O(V+E)$

$O(E)$

證明BFS可以找到最短路徑

- 定義: $\delta(s, v)$: s到v的最短路徑的長度(邊的數目)

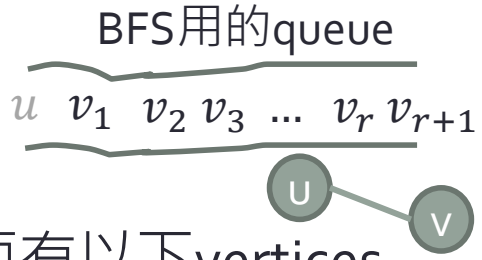
- Lemma 22.1: G=(V,E)是一個directed或undirected graph. s是一個任意vertex. 則對任何edge $(u, v) \in E$, $\delta(s, v) \leq \delta(s, u) + 1$.
- 證明:
- 如果從s開始, u是reachable, 那麼v也是.
- 這個狀況下, s→v的最短路徑只可能比$\delta(s, u) + 1$短 (最短路徑可能不經由u過來).不等式成立.
- 如果u不是reachable的話, 那麼不等式一定成立.



證明BFS可以找到最短路徑

- Lemma 22.2: G=(V,E)是一個directed或undirected graph. 對G及vertex s跑BFS. 則結束的時候, 每一個$v \in V$, BFS計算的$v.d \geq \delta(s,v)$.
- 證明:
- 使用歸納法證明. 假設為"每一個$v \in V$, BFS計算的$v.d \geq \delta(s,v)$".
- 一開始把s丟進queue的時候, 成立. $s.d = \delta(s,s) = 0$. 而其他的vertex $v.d = \infty > \delta(s,v)$. (起始條件)
- 當找到由edge (u,v)找到vertex v時, 我們可以假設$u.d \geq \delta(s,u)$. (k的時候成立)
- 且我們知道$v.d = u.d + 1 \geq \delta(s,u) + 1 \geq \delta(s,v)$ Lemma 22.1 (證出k+1的時候成立)
- 每個v都只做以上步驟(更改v.d值)一次, 歸納法證明完成.

證明BFS可以找到最短路徑



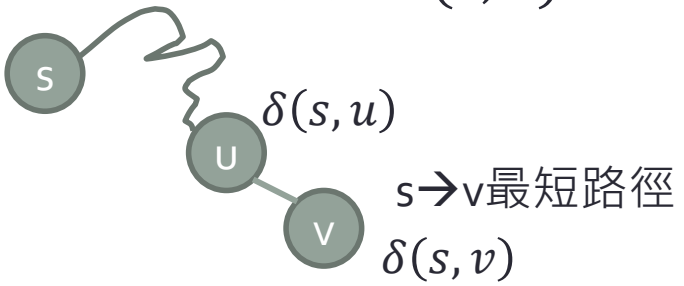
- Lemma 22.3: 假設BFS執行在G=(V,E)上, queue裡面有以下vertices $\langle v_1, v_2, \ldots, v_r \rangle$, v_1是queue的頭, 而v_r是queue的尾. 則$v_r.d \leq v_1.d + 1$ and $v_i.d \leq v_{i+1}.d$ for $i = 1,2, \ldots, r-1$.
- 證明:
- 使用歸納法.
- 當queue一開始裡面只有s的時候成立. (起始條件)
- 假設某個時候queue裡面的東西是符合條件的. (k的時候)
- 現在我們必須證明每次dequeue或enqueue的時候, 都還是成立.
- (k+1的時候)
- (1) dequeue的時候, v_2變成新的queue頭. 但$v_1.d \leq v_2.d$. 且 $v_r.d \leq v_1.d + 1 \leq v_2.d + 1$.其他不等式都不變. 因此成立.
- (2) enqueue的時候, 新加入的vertex v變成v_{r+1}.
- 此時我們剛剛把u拿掉(當時是queue的頭). 所以應該$u.d \leq v_1.d$. 所以$v_{r+1}.d = v.d = u.d + 1 \leq v_1.d + 1$.
- 且$v_r.d \leq u.d + 1$, so $v_r.d \leq u.d + 1 = v.d = v_{r+1}.d$.
- 其他的不等式都不變, 因此成立.

證明BFS可以找到最短路徑

- Corollary 22.4: vertices v_i 和 v_j在執行BFS時被enqueue且v_i在v_j之前被enqueue. 則當v_j被enqueue的時候$v_i.d \leq v_j.d$.
- 證明: 直接從Lemma 22.3就可以得到. 因為對任一vertex v來說, v.d只會被指定值一次(enqueue之前).

證明BFS可以找到最短路徑

- Theorem 22.5: 證明BFS正確性. BFS執行在G=(V,E)上, 從$s \in V$開始. BFS執行的時候會找出所有從s reachable的vertex $v \in V$. 結束的時候, 每個$v.d = \delta(s,v), \forall v \in V$.
- 證明:
- (反證法)假設有一些v.d不是$\delta(s,v)$. 讓v是其中$\delta(s,v)$最小的一個.
- Lemma 22.2說 $v.d \geq \delta(s,v)$, 所以現在$v.d > \delta(s,v)$.
- 此時v一定是從s reachable, 不然$\delta(s,v) = \infty \geq \text{v.d}$.
- 假設u是s→v最短路徑上v的前一個vertex, 則$\delta(s,v) = \delta(s,u) + 1$
- (隱含意思: s→v上最短路徑上s→u的部分也是s→u的最短路徑)
- 因為$\delta(s,u) < \delta(s,v)$, 所以$u.d = \delta(s,u)$ (已經假設v是其中$\delta(s,v)$最小的一個).
- $v.d > \delta(s,v) = \delta(s,u) + 1 = u.d + 1$



證明BFS可以找到最短路徑

- 上頁得到$v.d > \delta(s,v) = \delta(s,u) + 1 = u.d + 1$
- 考慮BFS從Queue裡面把u dequeue出來的時候.
- u的鄰居v們, 可能是WHITE, GRAY, 或BLACK
- 如果是WHITE, 則會設$v.d = u.d + 1$, 矛盾.
- 如果是BLACK, 則它之前已經被dequeue過. Corollary 22.4說$v.d \leq u.d$, 矛盾.
- 如果是GRAY, 則它是剛剛dequeue某個vertex w的時候被改成GRAY的(比u dequeue的時間早). 所以$v.d = w.d + 1$. 但Corollary 22.4說$w.d \leq u.d$, 因此$v.d = w.d + 1 \leq u.d + 1$, 矛盾.
- 因此原假設不成立.證明完畢.

下課休息


無聊型
bor i ng
新奇度
通常此姿勢在聲音單調乏味的老師課堂上相當常見, 學生聽了整節課無聊到都快把橡皮擦啃完了
身體姿勢彷彿說著:你的課實在無聊.

下課休息


不屑型
disdain
新奇度
此姿勢常見於功課優秀, 不太需要聽課之學生
連課都懶的聽, 自信似乎充滿了全身
身體姿勢彷彿說著:你講的我都會啦.

Today's Reading Assignment

- Cormen B.4 (Appendix) and 22.1-22.2