NuSTEC Neutrino Generator School

UNIVERSITY OF LIVERPOOL

Lecture T4

The nuclear ground state
and
Basics of many-body theory

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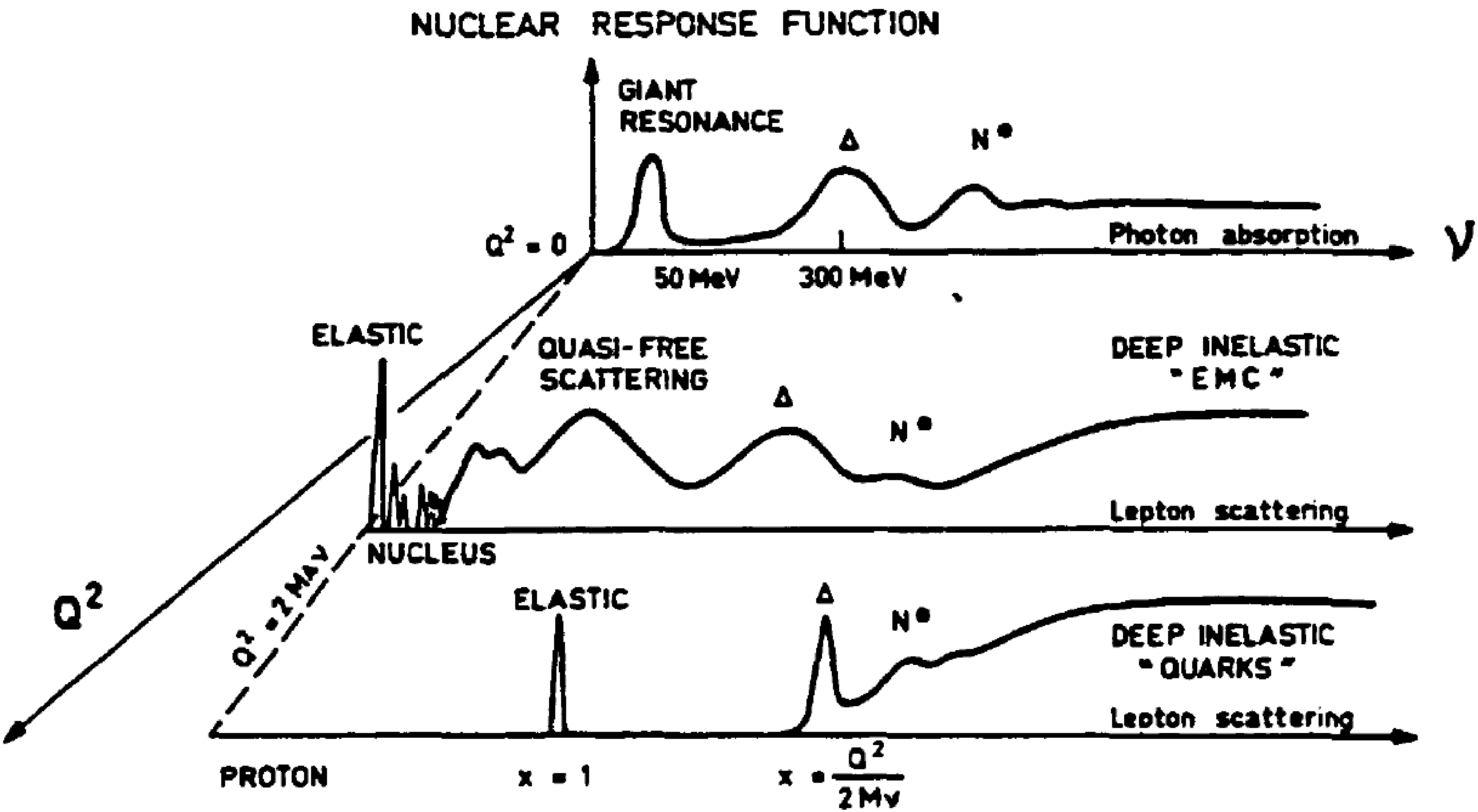
Outline

- Qualitative picture of the nuclear response
- The nucleon-nucleon interaction
- Independent particle models
  - Fermi Gas
  - Shell Model
- Nucleon propagator in the medium. Spectral functions

The nuclear response to EM probes

- Qualitative picture

B. Frois, NPA 434 (1985) 57c



The nucleon-nucleon interaction

- Constrained by
  - Deuteron properties
  - NN scattering data
- At low energies ⇒ non-relativistic potential $V = V(\vec{r}_i, \vec{p}_i, \vec{\sigma}_i, \vec{\tau}_i)\,, \quad i = 1,2$
- Symmetries:
  - Translational invariance: $V(\vec{r}_i, \ldots) = V(\vec{r} = \vec{r}_1 - \vec{r}_2, \ldots)$
  - Galilean invariance: $V(\vec{p}_i, \ldots) = V(\vec{p} = \vec{p}_1 - \vec{p}_2, \ldots)$
  - Parity invariance: $V(\vec{r}, \vec{p}, \vec{\sigma}_i, \vec{\tau}_i) = V(-\vec{r}, -\vec{p}, \vec{\sigma}_i, \vec{\tau}_i)$
  - Time reversal invariance: $V(\vec{r}, \vec{p}, \vec{\sigma}_i, \vec{\tau}_i) = V(\vec{r}, -\vec{p}, -\vec{\sigma}_i, \vec{\tau}_i)$
  - Isospin invariance: $V = V_0 + V_\tau(\vec{\tau}_1 \cdot \vec{\tau}_2)$

The nucleon-nucleon interaction

- Important terms:
  - Local central potential

$$V_C = V_0(r) + V_\sigma(r)(\vec{\sigma}_1 \cdot \vec{\sigma}_2) + V_\tau(r)(\vec{\tau}_1 \cdot \vec{\tau}_2) + V_{\sigma\tau}(r)(\vec{\sigma}_1 \cdot \vec{\sigma}_2)(\vec{\tau}_1 \cdot \vec{\tau}_2)$$

  - Tensor force
    - non central
    - explains the deuteron electric quadrupole moment

$$V_T = \left[V_{T_0}(r) + V_{T_\tau}(\vec{\tau}_1 \cdot \vec{\tau}_2)\right] S_{12}$$

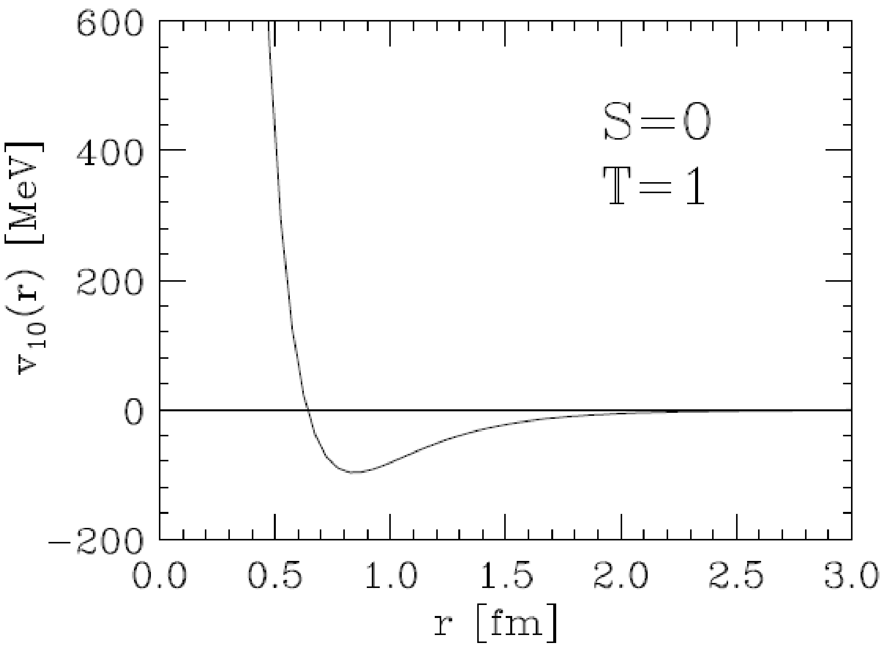
$$S_{12} = \frac{(\vec{r} \cdot \vec{\sigma}_1)(\vec{r} \cdot \vec{\sigma}_2)}{r^2} - \frac{1}{3}(\vec{\sigma}_1 \cdot \vec{\sigma}_1)$$

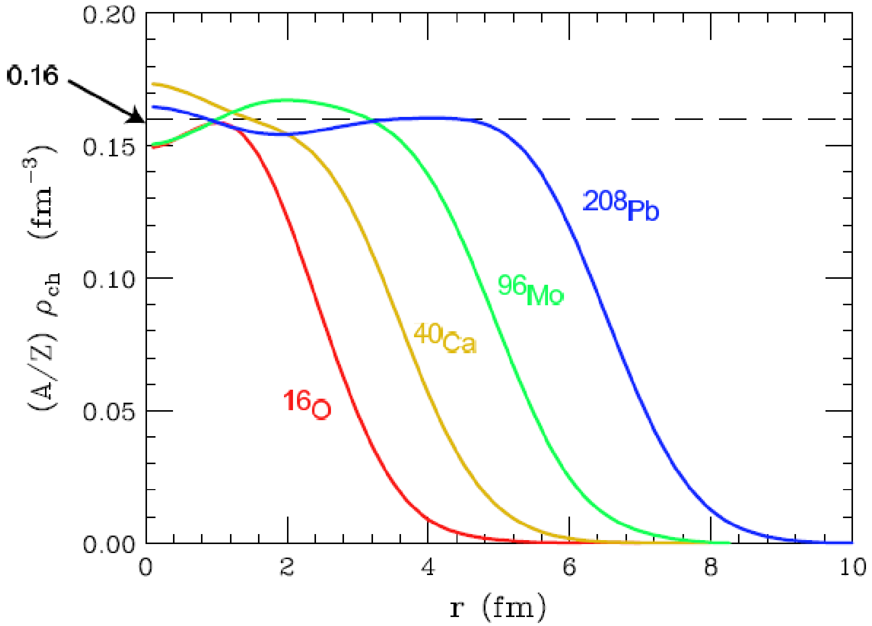
  - Spin-orbit force
    - most relevant nonlocal term
    - revealed in NN scattering through polarization observables
    - needed to obtain magic nuclei

$$V_{LS} = V_{LS}(r)\vec{L} \cdot (\vec{\sigma}_1 + \vec{\sigma}_2)\,, \;\; \vec{L} = (\vec{r} \times \vec{p})$$

The nucleon-nucleon interaction

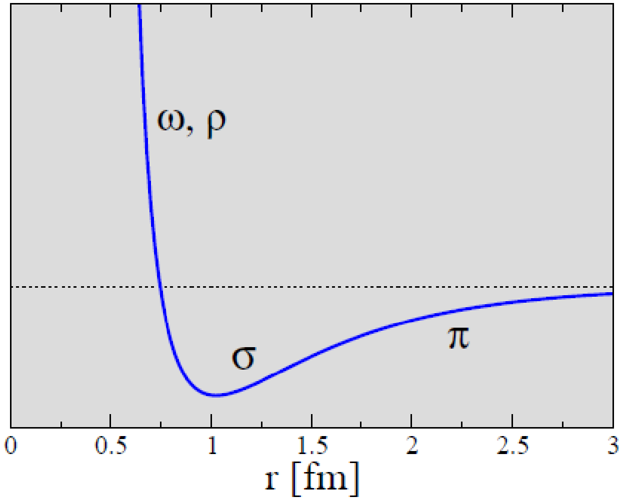
- Other properties:
  - Short range
  - Attraction at intermediate r
  - Strong repulsion at r < 0.5 fm
    - Consistent with saturation





The nucleon-nucleon interaction

- One Boson Exchange potentials (Bonn, Paris, ...)



$$V_\pi(\vec{r}) = \frac{1}{3}\frac{f^2}{4\pi}\left[\frac{e^{-m_\pi r}}{r} - \frac{4\pi}{m_\pi^2}\delta^{(3)}(\vec{r})\right]\vec{\sigma}_1\cdot\vec{\sigma}_2\ \vec{\tau}_1\cdot\vec{\tau}_2$$

$$+\underbrace{\frac{1}{3}\frac{f^2}{4\pi}\left(1+\frac{3}{m_\pi r}+\frac{3}{m_\pi^2 r^2}\right)\frac{e^{-m_\pi r}}{r}\hat{S}_{12}(\hat{r})\vec{\tau}_1\cdot\vec{\tau}_2}_{tensor\ term}$$

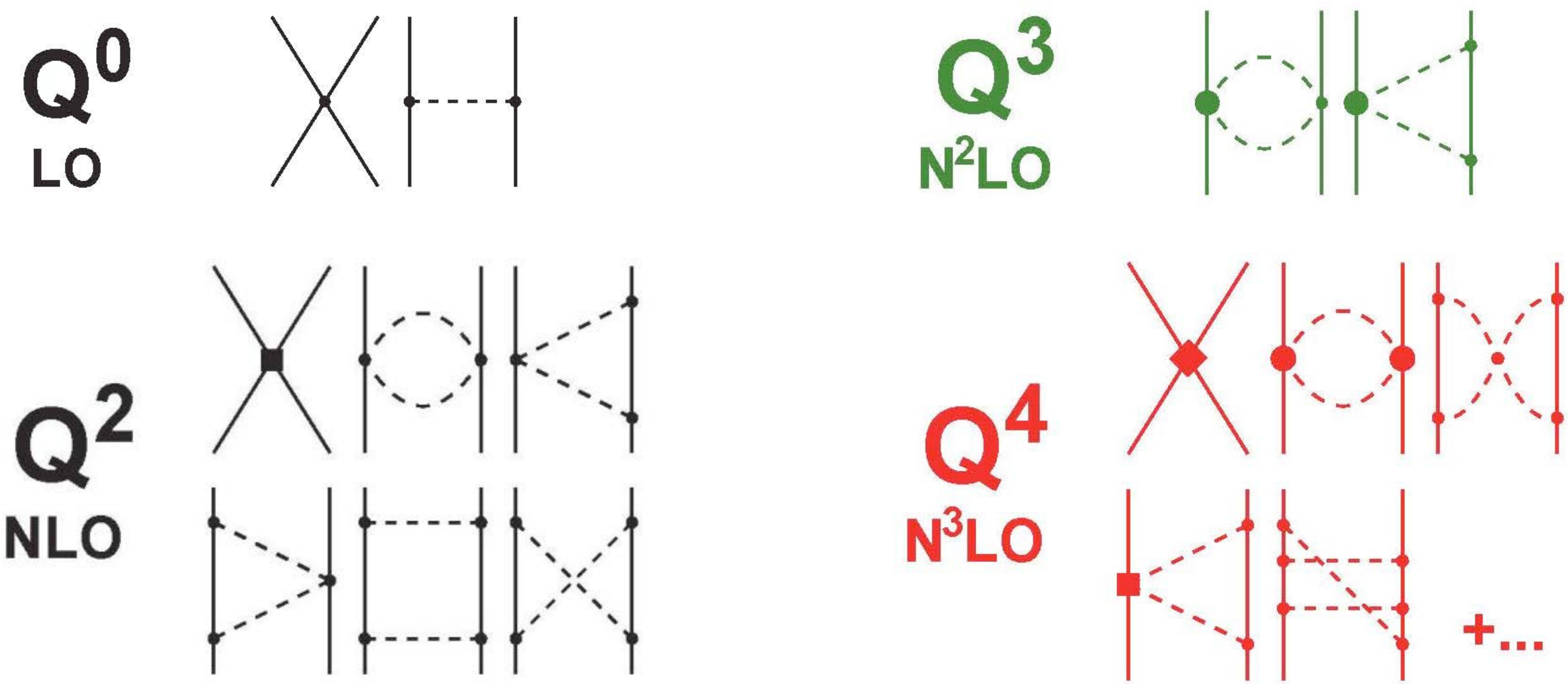
$$V_s(\vec{r}) = \underbrace{-\frac{g_s^2}{4\pi}\frac{e^{-m_s r}}{r}}_{\text{attractive central potential}} + \underbrace{\frac{g_s^2}{4\pi}\frac{1}{2M^2r^2}\frac{d}{dr}\left(\frac{e^{-m_s r}}{r}\right)\vec{L}\cdot\vec{S}}_{\text{spin-orbit potential}}$$

$$V_V = \underbrace{\frac{g_V^2}{4\pi}\frac{e^{-m_V r}}{r}}_{\text{repulsive}} + \frac{g_V^2}{4\pi}\left(3+4\frac{g_T}{g_V}\right)\frac{1}{2M^2r^2}\frac{d}{dr}\left(3+4\frac{g_T}{g_V}\right)\vec{L}\cdot\vec{S}$$

$$+\underbrace{\frac{g_V^2}{4\pi}\left(1+\frac{g_T}{g_V}\right)^2\frac{m_V^2}{4\pi}\left(\vec{\sigma}_1\times\vec{\nabla}\right)\left(\vec{\sigma}_2\times\vec{\nabla}\right)\frac{e^{-m_V r}}{r}}_{\text{tensor force, transversal coupling}}$$

The nucleon-nucleon interaction

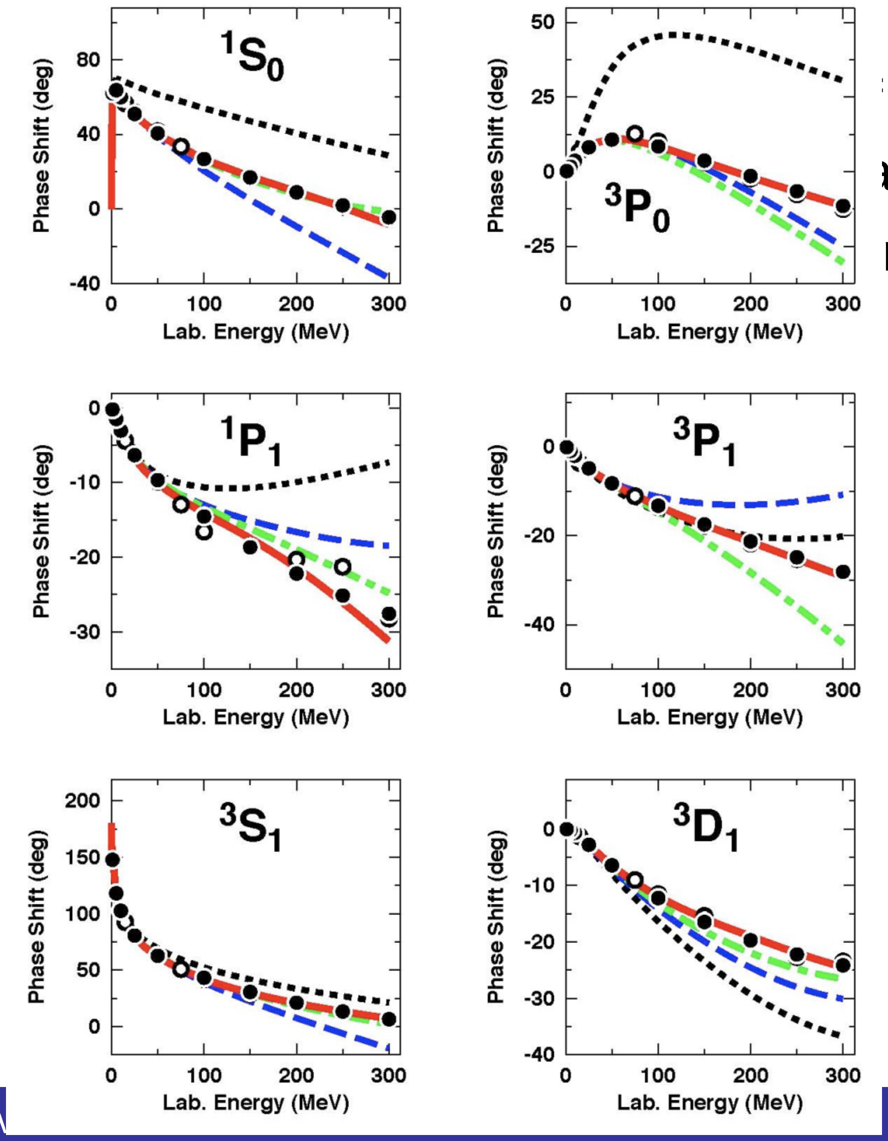
- Effective Field Theory of NN (and NNN) interactions
  - Obeys the symmetries of QCD
  - Systematic expansion in powers of (small) momenta
  - In terms of π, N and contact interactions (fitted to experiment)
  - Parameters directly connected to (nonperturbative) QCD



R. Machleidt , NTSE 2013

The nucleon-nucleon interaction

- Effective Field Theory of NN (and NNN) interactions



(small) momenta

actions (fitted to experiment)

nonperturbative) QCD

R. Machleidt , NTSE 2013

Nuclear Many-Body Theory

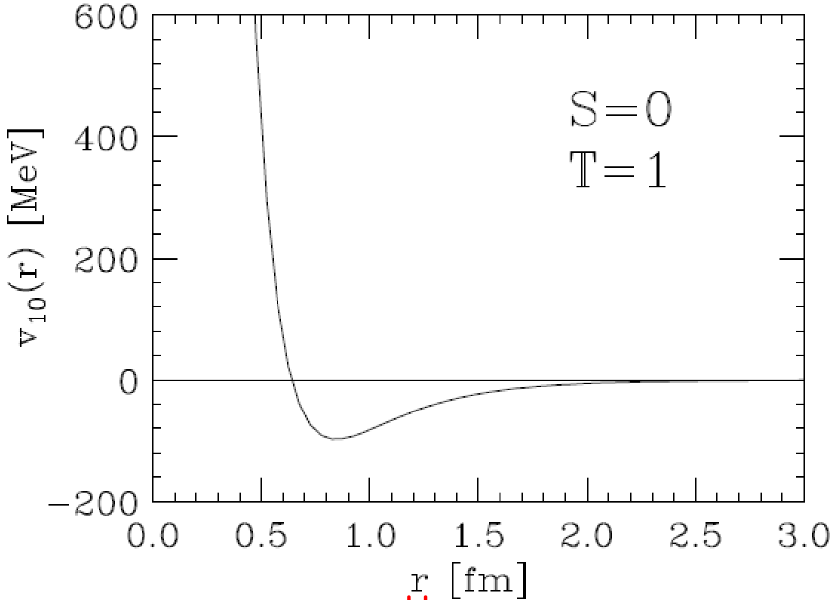
- Hamiltonian:

$$H = \sum_{i}^{A} T_i + \sum_{j>i}^{A} V_{ij} + \sum_{k>j>i}^{A} V_{ijk}$$

- Ab initio methods:
  - Properties of H fixed at A $\leq$ 3
  - Computationally demanding
  - Ground and low lying exited states up to A=12 obtained
  - Inclusive EW scattering investigated:
    - Non-relativistic framework
    - Only below the Δ(1232)
    - Explicit calculation of EM response functions for ^{2}H
    - Sum rules for NC response on ^{12}C Lobato et al., arxiv:1401.2605
  - Benchmark for neutrino cross section models

Independent particle models

- Saturation density ρ_0 = 0.16 fm^{-3} $\Rightarrow$ $r_{12} \sim$ 2 fm
- At $r_{12} \sim$ 2 fm, the NN interaction is "weak" !



$\Downarrow$

- Nucleons in nuclei follow single particle orbits in a mean-field potential created by all nucleons

$$\sum_{j>i}^{A} V_{ij} + \sum_{k>j>i}^{A} V_{ijk} \approx \sum_{i}^{A} \tilde{V}(i)$$ $\leftarrow$ Hartree-Fock approximation

Independent particle models

- Fermi gas model (of nuclear matter)

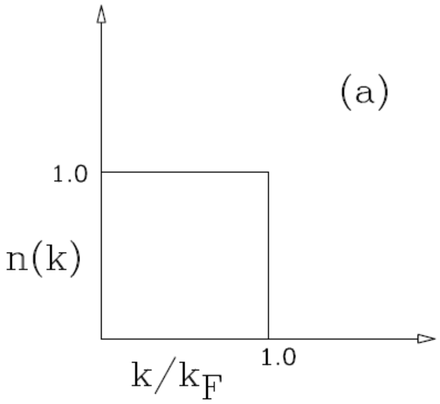
$$H = \sum_i^A T_i = \sum_i^A \frac{\vec{p}_i^{\,2}}{2M} \quad \text{or} \quad H_{\rm rel} = \sum_i^A T_i = \sum_i^A \sqrt{\vec{p}_i^{\,2} + M^2}$$

- Free fermions (nucleons) in a box of volume V ($\to \infty$) at T=0
- Number of occupied states:

$$N = 2V \int \frac{d^3p}{(2\pi)^3} n(|\vec{p}|) \qquad n(p) = \theta(p - p_F)$$

n(p) $\leftarrow$ occupation number

$P_F \leftarrow$ Fermi momentum



$$\rho = \frac{N}{V} = \frac{1}{3\pi^2} p_F^3$$ for protons or neutrons separately

$$\rho = \frac{N}{V} = \frac{2}{3\pi^2} p_F^3$$ for isospin symmetric nuclear matter

$\rho = \rho_0 = 0.16$ fm^{-3} $\Rightarrow$ $p_F = 263$ MeV

Independent particle models

- Fermi gas model for nuclei
  - Global FG:
    - p_F = const. for a given nucleus
    - Fit parameter in (e,e') scattering
  - Local FG:

$$\rho_{p,n} = \rho_{p,n}(r) \qquad p_F^{p,n} = p_F^{p,n}(r) = \sqrt[3]{\frac{3}{2}\pi^2 \rho_{p,n}(r)}$$

ρ_p(r) $\leftarrow$ from experiment

ρ_n(r) $\leftarrow$ from realistic calculations of the ground state

  - space-momentum correlations absent in the Global FG

Independent particle models

- Global FG with constant binding energy
  - Vector potential in the nucleus rest frame: $V^\mu = (V^0 = -E_B, \vec{0})$

$$H = \sqrt{\vec{p}^2 + M^2} - E_B$$

  - Effective mass:

$$\sqrt{\vec{p}^2 + M^2} - E_B = \sqrt{\vec{p}^2 + M^{*2}}$$

  - Scalar potential U:

$$H = \sqrt{\vec{p}^2 + (M+U)^2}$$

  - U and V are equivalent when

$$U^2 + 2MU - E_B^2 + 2E_B\sqrt{\vec{p}^2 + M^2} = 0$$

Independent particle models

- Shell Model
- Motivation:
  - Successful atomic shell model
  - Magic numbers: Z or A-Z = 2, 8, 20, 28, 50, 82, 126
  - Doubly magic nuclei ($^{2}He_{2}$, $^{16}O_{8}$, $^{40}Ca_{20}$, $^{48}Ca_{28}$, $^{208}Pb_{126}$) are exceptionally stable
- Assumption:
  - Instead of the external Coulomb field: mean field created by nucleons
  - Inner core + valence nucleons
- Schrödinger eq. in the mean field potential

Independent particle models

- Shell Model
- Schrödinger eq. in the mean field potential
- Harmonic oscillator

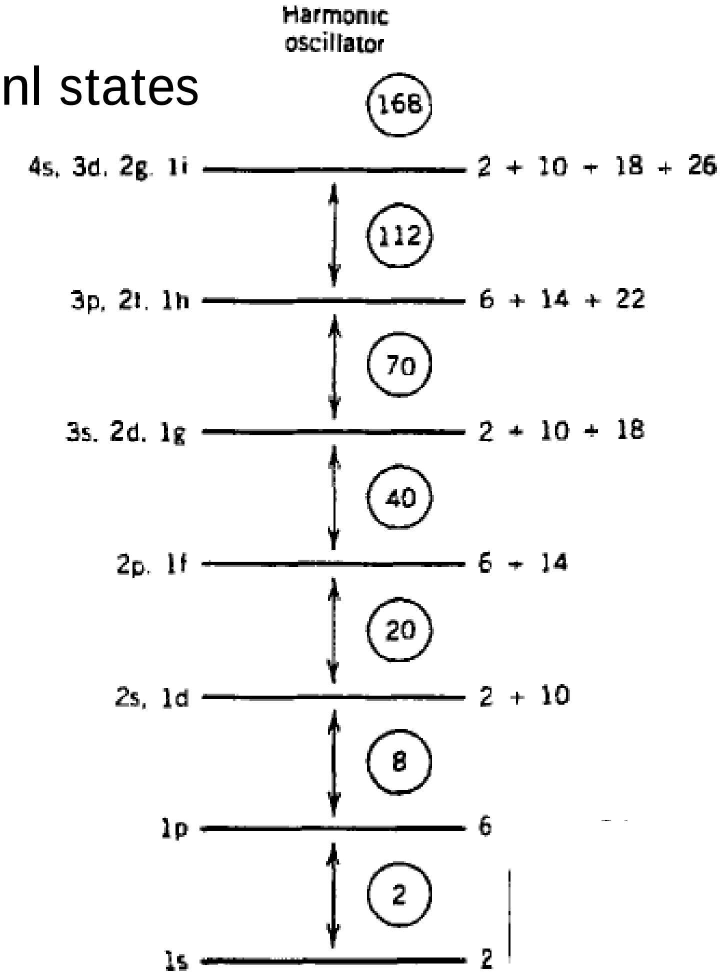
$$V(r) = \frac{1}{2} M \omega^2 r^2$$

$$E = \omega \left(N + \frac{3}{2} \right)$$

$$N = 2(n-1) + l$$

- 2(2l+1) degenerated states
- Only 2,8, 20 magic numbers emerge

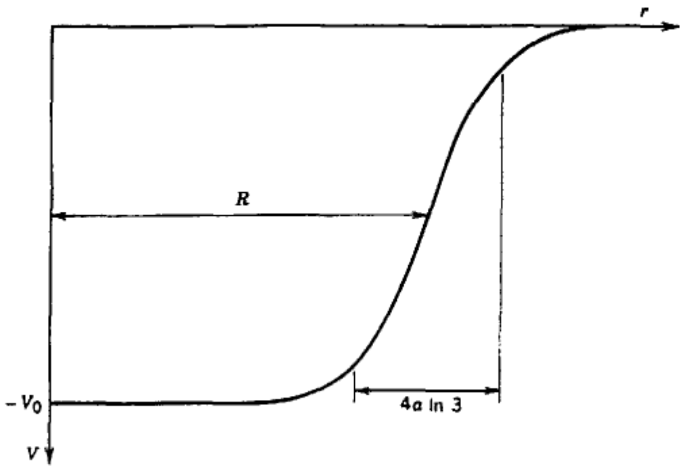
nl states



Independent particle models

- Shell Model
- Schrödinger eq. in the mean field potential
- Woods-Saxon potential

$$V(r) = -\frac{V_0}{1 + \exp\left[(r - R)/a\right]}$$



- Radius (R) and skin thickness [4 a ln(3)] are fitted to density distributions
- V_0: adjusted to the separation energies
- Typical values:
  - $R = 1.2\ A^{1/3}$ fm
  - A=0.524 fm
  - $V_0 \approx 50$ MeV

Independent particle models

- Shell Model
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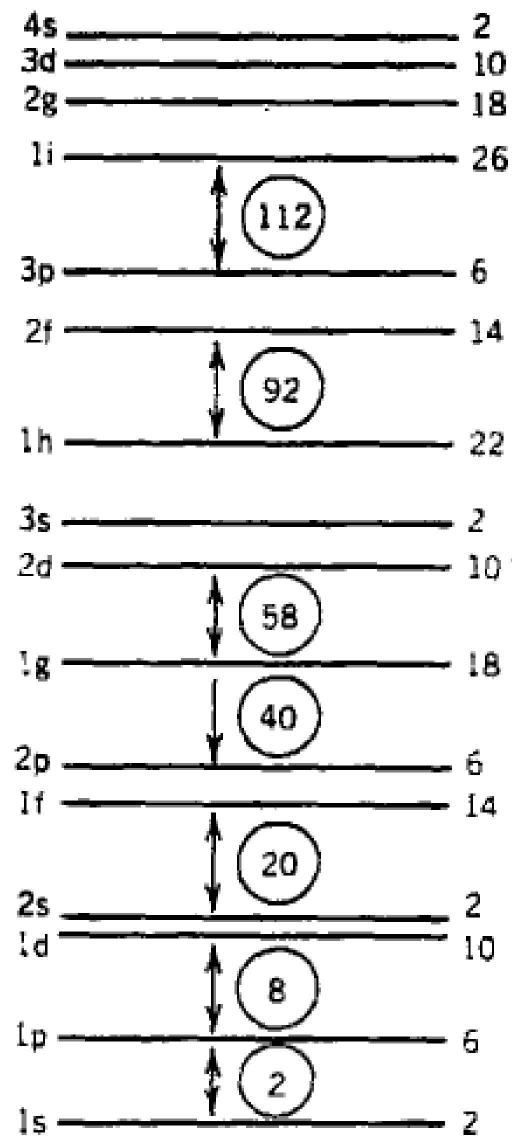


Independent particle models

- Shell Model
- Spin-Orbit potential

$$V_{LS} = V_{LS}(r)\vec{l} \cdot \vec{s}$$

$$\vec{l} \cdot \vec{s} = \frac{1}{2}\left(\vec{j}^2 - \vec{l}^2 - \vec{s}^2\right)$$

$$\langle \vec{l} \cdot \vec{s} \rangle = \frac{1}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]$$

- LS interaction splits $nl_{l-1/2}$ and $nl_{l+1/2}$ by $\Delta E \sim \frac{1}{2}(2l+1)$

Independent particle models

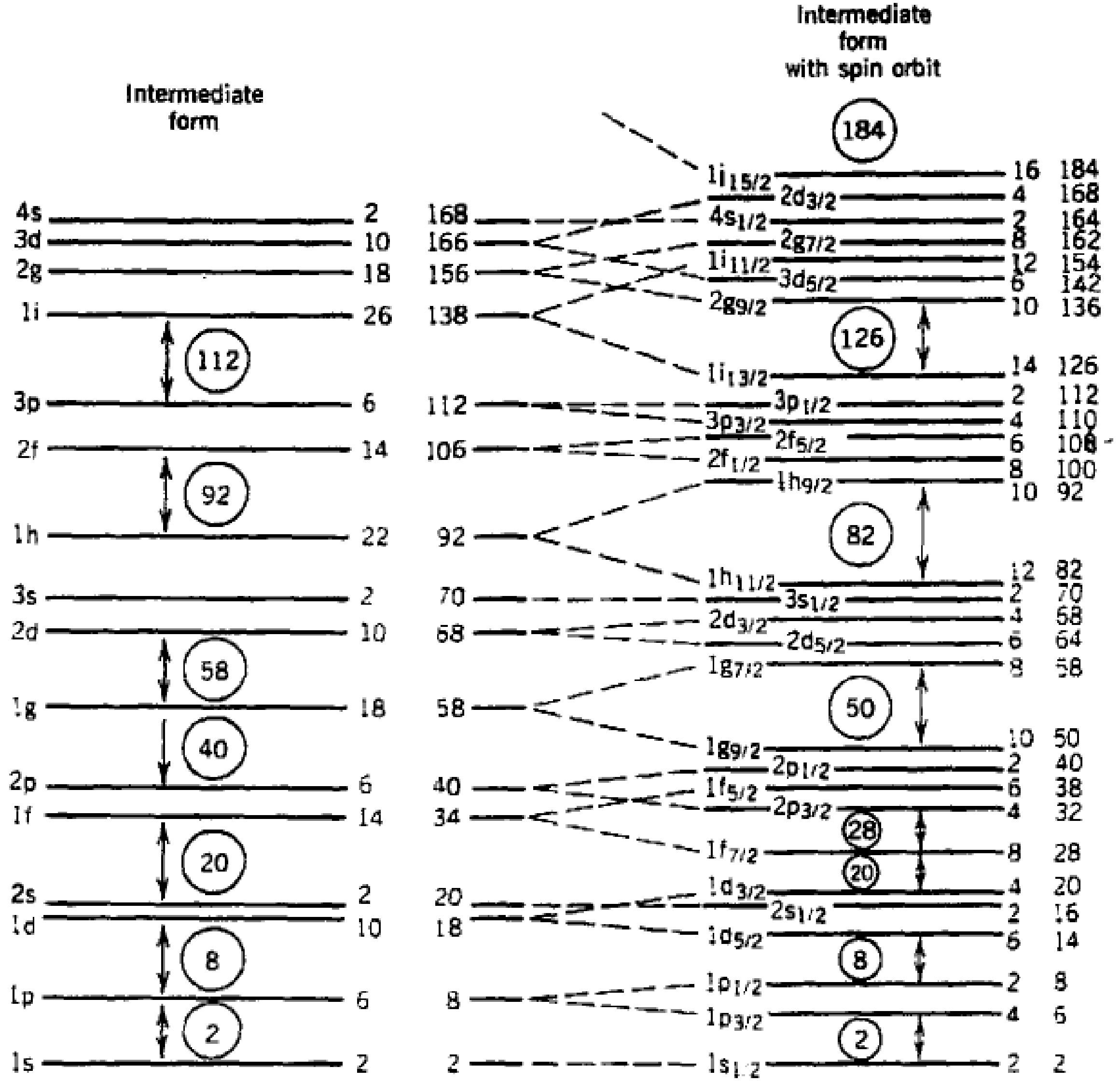
- Shell Model
- Spin-Orbit potential

$$V_{LS} = V_{LS}(r)\vec{l}\cdot\vec{s}$$

$$\vec{l}\cdot\vec{s} = \frac{1}{2}\left(\vec{j}^2 - \vec{l}^2 - \vec{s}^2\right)$$

$$\langle\vec{l}\cdot\vec{s}\rangle = \frac{1}{2}\left[j(j+1) - l(l+\right.$$

- LS interaction splits $nl_{l-1/2}$ a
- Magic numbers explained!



Independent particle models

- Shell Model
- Explains spin and parity of many nuclei
- Fair description of magnetic dipole and electric quadrupole moments
- Can be extended to deformed nuclei
- Relativistic extensions have been developed

Nucleon propagator in the medium

- Green's function:

$$iG(x,x') = \frac{\left\langle \phi_0 \left| T\left[\psi(x)\psi^\dagger(x')\right]\right| \phi_0 \right\rangle}{\langle \phi_0 | \phi_0 \rangle}$$

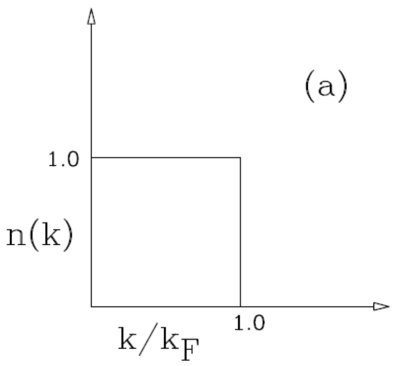
- $\phi_0 \leftarrow$ ground state of the system: $H|\phi_0\rangle = E|\phi_0\rangle$

- Free nucleon propagator in the medium
- ϕ_0 : system of non-interacting nucleons $\Leftrightarrow$ Fermi gas

$$\begin{aligned} D(p) &= (\not{p} + M) G_0(p) \\ G_0(p) &= \frac{1}{p^2 - M^2 + i\epsilon} + 2\pi i \delta(p^2 - M^2)\theta(p^0) n(\vec{p}) \\ &= \frac{n(\vec{p})\theta(p^0)}{p^2 - M^2 - i\epsilon} + \frac{1 - n(\vec{p})\theta(p^0)}{p^2 - M^2 + i\epsilon} \\ &= \frac{1}{p^0 + E_p - i\epsilon}\left[\frac{n(\vec{p})}{p^0 - E_p - i\epsilon} + \frac{1 - n(\vec{p})}{p^0 - E_p + i\epsilon}\right] \end{aligned}$$

hole particle

$$n(p) = \theta(p - p_F)$$



$$E_p = \sqrt{\vec{p}^2 + M^2}$$

Nucleon propagator in the medium

- Full nucleon propagator in the medium
- Selfenergy: $G = G_0 + G_0 \Sigma G_0$
- In terms of the proper selfenergy: $\Sigma = \Sigma_0 + \Sigma_0 G_0 \Sigma_0 + \ldots$
- Dyson equation:

$$
\begin{aligned}
G &= G_0 \Sigma_0 G_0 + G_0 \Sigma_0 G_0 \Sigma_0 G_0 + \ldots \\
&= G_0 + G_0 \Sigma_0 (G_0 + G_0 \Sigma_0 G_0 + \ldots) \\
G &= G_0 + G_0 \Sigma_0 G \\
G &= G_0 (1 - \Sigma_0 G_0)^{-1}
\end{aligned}
$$

  - For particles and holes separately:

$$G_0 = \frac{1}{p^2 - M^2} \Rightarrow G = \frac{1}{p^2 - M^2 - \Sigma_0}$$

Σ_0 is calculated "perturbatively"

Spectral functions

- Full nucleon propagator in the medium
- Lehmann representation:

$$\begin{aligned} D(p) &= (\not{p} + M) G(p) \\ G(p) &= \frac{1}{p^0 + E_p - i\epsilon} \left[\int_{-\infty}^{\mu} \frac{\mathcal{A}_h(\omega, \vec{p})}{p^0 - \omega - i\epsilon} d\omega + \int_{\mu}^{\infty} \frac{\mathcal{A}_p(\omega, \vec{p})}{p^0 - \omega + i\epsilon} d\omega \right] \end{aligned}$$

$$\mu^2 = \vec{p}_F^{\,2} + M^2 + \mathrm{Re}\Sigma(\mu, p_F)$$

$$\mathcal{A}_{p,h}(p) = \mp \frac{1}{\pi} \frac{\mathrm{Im}\Sigma(p)}{[p^2 - M^2 - \mathrm{Re}\Sigma(p)]^2 + [\mathrm{Im}\Sigma(p)]^2}$$

- The hole (particle) spectral function $A_{h(p)}(p^0,\mathbf{p})$ represents the probability of removing (adding) a nucleon of momentum $|\mathbf{p}|$ changing the energy of the system by p^0
- Occupation number: $n(\vec{p}) = \int dp_0 (2p_0) \mathcal{A}_h(p^0, \vec{p})$

Spectral functions

- Full nucleon propagator in the medium
- Lehmann representation:

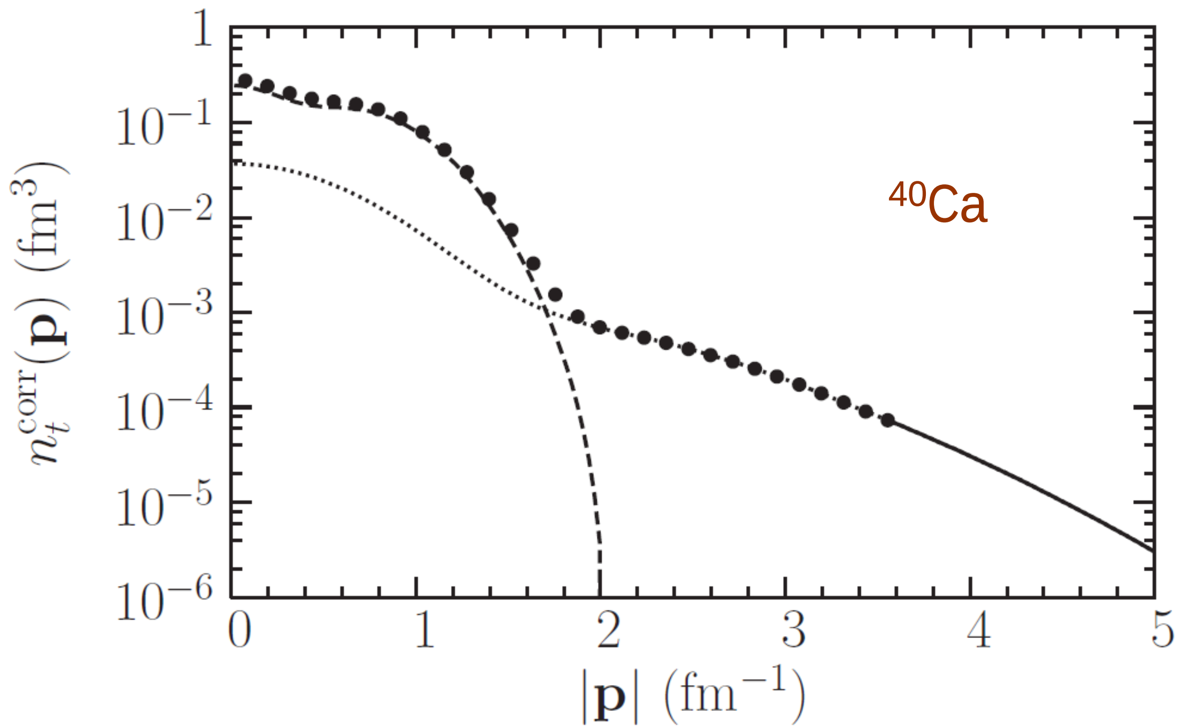
$$
\begin{aligned}
D(p) &= (\not{p} + M)G(p) \\
G(p) &= \frac{1}{p^0 + E_p - i\epsilon}\left[\int_{-\infty}^{\mu} \frac{\mathcal{A}_h(\omega,\vec{p})}{p^0 - \omega - i\epsilon} d\omega + \int_{\mu}^{\infty} \frac{\mathcal{A}_p(\omega,\vec{p})}{p^0 - \omega + i\epsilon} d\omega\right]
\end{aligned}
$$

$$
\mathcal{A}_{p,h}(p) = \mp\frac{1}{\pi}\frac{\mathrm{Im}\Sigma(p)}{[p^2 - M^2 - \mathrm{Re}\Sigma(p)]^2 + [\mathrm{Im}\Sigma(p)]^2}
$$

- $\Sigma \to 0$: G $\to$ G_0
- ImΣ = 0 $\Rightarrow$ mean-field approximation: $p^2 - M^2 - \mathrm{Re}\Sigma(p) = 0$
- In particular, if ReΣ = 2 M U + U^2: $p^0 = \sqrt{\vec{p}^{\,2} + [M + U(p)]^2}$

Spectral functions

- Ingredients of a realistic spectral function:
  - Mean field part (80-90 %)
  - Correlated part (from NN interactions)



Ankowski, Sobczyk. PRC77(2008)

Spectral functions

- Ingredients of a realistic spectral function:
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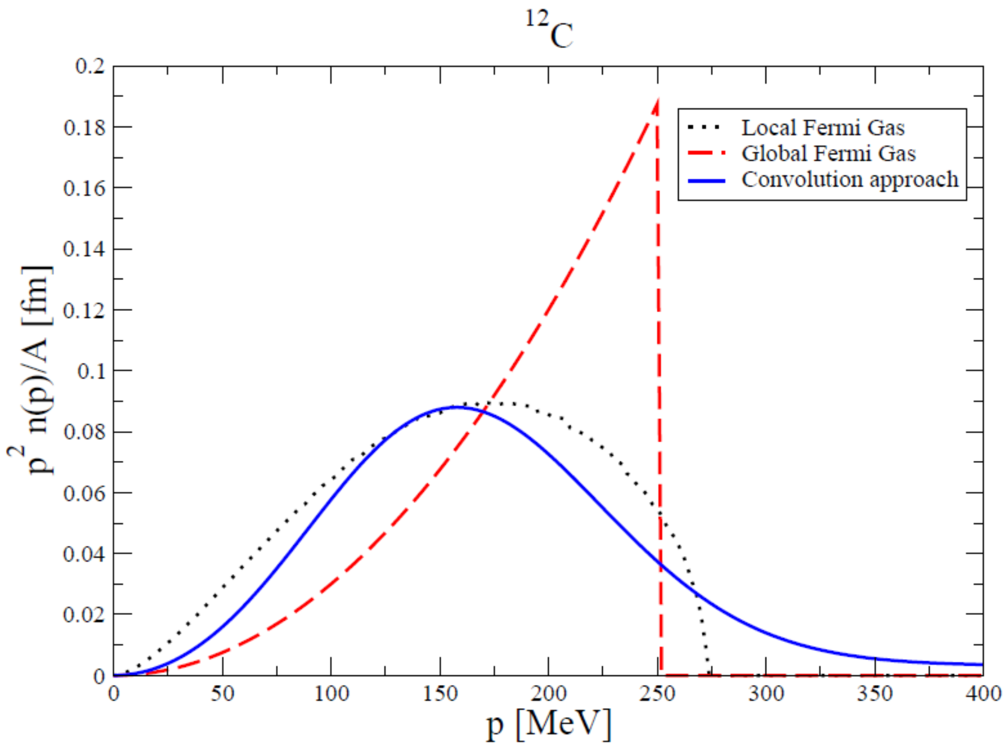
  - Correlated part (from NN interactions)

- Local FG has a more realistic momentum distribution than Global FG



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