Portfolio Optimization with linear and fixed transaction costs

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1 Introduction

In this paper, we consider the problem of optimal portfolio selection, with transaction costs and constraints on exposure to risk, short-sell constraints, etc. Under the assumption of linear transaction costs (which might not be completely realistic), bounds on the variance of the return, bounds on different shortfall probabilities, the optimization problem of maximizing the asset-wealth is convex which can be solved efficiently.

Portfolio optimization problems with transaction costs that include a fixed fee, or discount breakpoints, which are more realistic cannot be directly solved by convex optimization. We address this issue of non-convexity by solving a small number of related convex optimization problems which can be used to provide better bounds for the original optimization problem. Thus, this method only produces a sub-optimal solution with an upper bound on the original optimum. This method involves employing a heuristic which only yields a sub-optimal solution. However, the numerical results for several practical scenarios indicate that the gap between them is quite small.

The next section deals with the problem description in detail.

2 Problem description

2.1 Objective function

We consider an investment portfolio that consists of holdings in some or all of n assets. Transactions performed on these assets adjust the portfolios which are then held for a fixed period of time. The goal of the investor is the maximize the expected wealth at the end of the period, such that the portfolios held satisfy a set of constraints described below.

The current holds in the assets are given by the vector $w = [w_1, w_2, \dots, w_n]^T$. The vector $x = [x_1, x_2, \dots, x_n]^T$ denotes the transactions in each of these assets, with $x_i > 0$ for buying and $x_i < 0$ for selling. Thus the adjusted portfolio is given by $w + x$.

The portfolio $w + x$ is held for a fixed period of time. Return on each of the assets by the end of this period is given by a random variable (random vector) $a = [a_1, a_2, \dots, a_n]^T$. We assume that mean and variance of a are known.

$$\mathbb{E}(a) = \bar{a},\ \mathbb{E}(a - \bar{a})^T(a - \bar{a}) = \Sigma$$

The end of period wealth is a random variable, $W = a^T(w + x)$, with expected value and variance given by

$$\mathbb{E}(W) = \bar{a}^T(w + x),\ \mathbb{E}(W - \mathbb{E}(W))^2 = (w + x)^T\Sigma(w + x).$$

2.2 Constraints

2.2.1 Self-financing constraint

Let $\phi(x)$ denote the total transaction costs for the transactions x described above. We assume the portfolio is self-financing, i.e there is no exogenous infusion or withdrawal of money; the purchase of a new asset must be financed by the sale of an old one.[1] Mathematically,

$$\mathbb{1}^T x + \phi(x) = 0$$

We consider a inequality version of the above which is more appropriate (for example, if ϕ is convex) and justified (both of them yielding same results) in [2],

$$\mathbb{1}^T x + \phi(x) \leq 0 \tag{1}$$

Another motivation for considering (2) rather than (1) is that the latter needn't be convex.

We will describe the transaction costs in detail towards the end of this section with several variants.

2.2.2 Shortselling constraints

Shortselling constraints alead to linear inequalities. We impose individual bounds s_i on the maximum amount of shortselling permissible on asset i, i.e,

$$w_i + x_i \geq -s_i,\ i = 1, \ldots, n. \tag{2}$$

We can also handle other variants which are of more practical interest such as collaterization requirement along similar lines.[2]

2.2.3 Constraints on variance of wealth

The standard deviation of the end of period wealth W is constrained to be less than σ_{max} by the (convex) quadratic inequality

$$(w + x)^T \Sigma (w + x) \leq \sigma_{max}^2,$$

Equivalently,

$$\|\Sigma^{1/2}(w + x)\| \leq \sigma_{max} \tag{3}$$

The constraint (3) is a second-order conic constraint.

2.2.4 Shortfall risk constraints

We assume that the random vector for returns is Guassian, i.e $a \sim \mathcal{N}(\bar{a}, \Sigma)$. We impose that the end-period wealth W be greater than some undesired level W^{low} with a confidence level η (where $\eta \geq 0.5$),

$$\text{Prob}(W \geq W^{low}) \geq \eta$$

Since we know that $W = a^T(w + x)$, we have $W \sim \mathcal{N}(\mu, \sigma^2)$, which implies

$$\text{Prob}\left(\frac{W - \mu}{\sigma} \leq \frac{W^{low} - \mu}{\sigma}\right) \leq 1 - \eta,$$
$$\Rightarrow \frac{W^{low} - \mu}{\sigma} \leq \Phi^{-1}(1 - \eta),$$

where $\Phi(z)$ is the cumulative distributive function for standard guassian.
Using the fact that $\Phi^{-1}(1 - \eta) = \Phi^{-1}(\eta)$ and $\mu = \bar{a}^T(w + x),\ \sigma = \|\Sigma^{1/2}(w + x)\|$,

$$\Phi^{-1}(\eta)\|\Sigma^{1/2}(w + x)\| \leq \bar{a}^T(w + x) - W^{low} \tag{4}$$

For $\eta \geq 0.5,\ \Phi^{-1}(\eta) \geq 0$, and thus (4) is a convex quadratic constraint.

For the above shortfall constraint setting, we can also impose constraint on a merely bad return, with some modest confidence, as well as a constraint on a truly disastrous return, with much greater confidence.

2.2.5 Transaction costs

We assume that transaction cost is separable, i.e

$$\phi(x) = \sum_{i=1}^{n} \phi_i(x_i)$$

where ϕ_i is the transaction cost associated with trading asset i.

The two realistic variants of transaction costs that consider are linear transaction costs and fixed transaction costs. First consider the linear case:

$$\phi_i(x_i) = \begin{cases} \alpha_i^+ x_i^+ & \text{if } x \geq 0 \\ \alpha_i^- x_i^- & \text{if } x \leq 0 \end{cases}$$

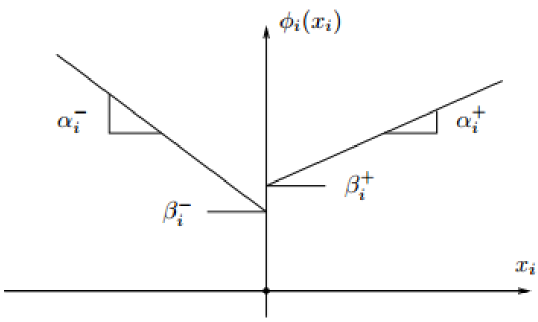
where $x_i = x_i^+ - x_i^-,\ x_i^+ = \max\{x_i, 0\},\ x_i^- = \max\{-x_i, 0\}$. Here $\alpha_i^+, \alpha_i^- \geq 0$ are the cost rates associated with buying and selling asset i. More succinctly,

$$\phi_i(x_i) = \alpha_i^+ x_i^+ + \alpha_i^- x_i^-.$$

However in practice, transaction costs are not convex functions of the amount traded. Indeed, the costs for either buying or selling are likely to be concave[2].
We will consider a simple model that includes fixed plus linear costs. Let β_i^+ and β_i^- are the costs associated with selling and buying the asset i respectively. Then the fixed-plus-linear transaction cost function is given by:

$$\phi_i(x_i) = \begin{cases} \beta_i^+ + \alpha_i^+ x_i^+ & : x > 0 \\ \beta_i^- + \alpha_i^- x_i^- & : x < 0 \\ 0 & : x = 0 \end{cases}$$

Clearly, $\phi_i(x_i)$ is not a convex function and poses a problem to the convexity of the optimization problem.



3 Optimization problem

We pose the optimization problem of maximizing the wealth $W = a^T(w + x)$ subject to the constraints discussed above.

3.1 Linear transaction costs

We consider the transaction costs to be linear in this case. The corresponding optimization problem with short-fall risk constraints, short-selling constraints, self-financing constraint being included is given by:

$$\begin{array}{ll} \underset{x}{\text{maximize}} & \bar{a}^T(w + x^+ - x^-) \\ \text{subject to} & \mathbb{1}^T x + \sum_{i=1}^{n}(\alpha_i^+ x_i^+ + \alpha_i^- x_i^-) \leq 0 \\ & w_i + x_i^+ - x_i^- \leq s_i, i = 1, 2, \ldots, n. \\ & \Phi^{-1}(\eta_j)\|\Sigma^{1/2}(w + x)\| \leq \bar{a}^T(w + x) - W_j^{low},\ j = 1, 2 \\ & x_i^+ \geq 0,\ x_i^- \geq 0,\ i = 1, 2, \ldots, n \end{array} \tag{5}$$

It is obvious that (5) is a convex optimization problem since each of the constraints as well as objective function are convex.

3.2 Fixed transaction costs

The general transaction cost function is given by:

$$\phi(x) = \sum_{i=1}^{n} \phi_i(x_i)$$

with

$$\phi_i(x_i) = \begin{cases} \beta_i + \alpha_i|x_i| & : x > 0 \\ 0 & : x = 0 \end{cases}$$

Costs such as above lead to a hard combinatorial optimization problem.

If the β_i are very small, this may lead to an acceptable approximation. In general, however, it will generate inefficient solutions with too many transactions.[2]

On the other hand, by considering the fixed costs, we discourage trading small amounts of a large number of assets. Thus, we obtain a sparse vector of trades; i.e., one that has many zero entries. This means most of the trading will be concentrated in a few assets, which is a desirable property.

We will consider a heuristic for this fixed transaction costs case where the problem is non-convex. This procedure though yields only a sub-optimal solution, it is found to be good in practice with respect to the actual solution. This

References

[1] Wikipedia

[2] Portfolio optimization with linear and fixed transaction costs, ...