

Random projection ensemble classification

Timothy I. Cannings and Richard J. Samworth
Statistical Laboratory, University of Cambridge

Abstract

We introduce a very general method for high-dimensional classification, based on careful combination of the results of applying an arbitrary base classifier to random projections of the feature vectors into a lower-dimensional space. In one special case that we study in detail, the random projections are divided into non-overlapping blocks, and within each block we select the projection yielding the smallest estimate of the test error. Our random projection ensemble classifier then aggregates the results of applying the base classifier on the selected projections, with a data-driven voting threshold to determine the final assignment. Our theoretical results elucidate the effect on performance of increasing the number of projections. Moreover, under a boundary condition implied by the sufficient dimension reduction assumption, we show that the test excess risk of the random projection ensemble classifier can be controlled by terms that do not depend on the original data dimension. The classifier is also compared empirically with several other popular high-dimensional classifiers via an extensive simulation study, which reveals its excellent finite-sample performance.

1 Introduction

Supervised classification concerns the task of assigning an object (or a number of objects) to one of two or more groups, based on a sample of labelled training data. The problem was first studied in generality in the famous work of Fisher (1936), where he introduced some of the ideas of Linear Discriminant Analysis (LDA), and applied them to his Iris Data set. Nowadays, classification problems arise in a plethora of applications, including spam filtering, fraud detection, medical diagnoses, market research, natural language processing and many others.

In fact, LDA is still widely used today, and underpins many other modern classifiers; see, for example, Friedman (1989) and Tibshirani et al. (2002). Alternative techniques include support vector machines (Cortes and Vapnik, 1995), tree classifiers (Breiman et al., 1984; Breiman, 2001), kernel methods (Hall and Kang, 2005) and nearest neighbour classifiers (Fix and Hodges, 1951). More substantial overviews and in-depth discussion of these techniques, and others, can be found in Devroye, Györfi and Lugosi (1996) and Hastie et al. (2009).

An increasing number of modern classification problems are *high-dimensional*, in the sense that the dimension p of the feature vectors may be comparable to or even greater than the number of training data points, n . In such settings, classical methods such as those mentioned in the previous paragraph tend to perform poorly (Bickel and Levina, 2004), and may even be intractable; for example, this is the case for LDA, where the problems are caused by the fact that the sample covariance matrix is not invertible when $p \geq n$.

Many methods proposed to overcome such problems assume that the optimal decision boundary between the classes is linear, e.g. Friedman (1989) and Hastie et al. (1995). Another common approach assumes that only a small subset of features are relevant for classification. Examples of works that impose such a sparsity condition include Fan and Fan (2008), where it is also assumed that the features are independent, as well as Tibshirani et al. (2003) and Guo, Hastie and Tibshirani (2007), where soft thresholding is used to obtain a sparse boundary. More recently, Witten and Tibshirani (2011) and Fan, Feng and Tong (2012) both solve an optimisation problem similar to Fisher’s linear discriminant, with the addition of an ℓ_1 penalty term to encourage sparsity.

In this paper we attempt to avoid the curse of dimensionality by projecting the feature vectors at random into a lower-dimensional space. The use of random projections in high-dimensional statistical problems is motivated by the celebrated Johnson–Lindenstrauss Lemma (e.g. Dasgupta and Gupta, 2002). This lemma states that, given $x_1, \dots, x_n \in \mathbb{R}^p$, $\epsilon \in (0, 1)$ and $d > \frac{8 \log n}{\epsilon^2}$, there exists a linear map $f : \mathbb{R}^p \rightarrow \mathbb{R}^d$ such that

$$(1 - \epsilon)\|x_i - x_j\|^2 \leq \|f(x_i) - f(x_j)\|^2 \leq (1 + \epsilon)\|x_i - x_j\|^2,$$

for all $i, j = 1, \dots, n$. In fact, the function f that nearly preserves the pairwise distances can be found in randomised polynomial time using random projections distributed according to Haar measure as described in Section 3 below. It is interesting to note that the lower bound on d in the Johnson–Lindenstrauss lemma does not depend on p . As a result, random projections have been used successfully as a computational time saver: when p is large compared to $\log n$, one may project the data at random into a lower-dimensional space and run the statistical procedure on the projected data, potentially making great computational savings, while achieving comparable or even improved statistical performance. As one example of the above strategy, Durrant and Kaban (2013) obtained Vapnik–Chervonenkis type bounds on the generalisation error of a linear classifier trained on a single random projection of the data. See also Dasgupta (1999), Ailon and Chazelle (2006) and McWilliams et al. (2014) for other instances.

Other works have sought to reap the benefits of aggregating over many random projections. For instance, Marzetta, Tucci and Simon (2011) considered estimating a $p \times p$ population inverse covariance (precision) matrix using $B^{-1} \sum_{b=1}^B A_b^T (A_b \hat{\Sigma} A_b^T)^{-1} A_b$, where $\hat{\Sigma}$ denotes the sample covariance matrix and A_1, \dots, A_B are random projections from \mathbb{R}^p to \mathbb{R}^d . Lopes, Jacob and Wainwright (2011) used this estimate when testing for a difference between two Gaussian population means in high dimensions, while Durrant and Kaban (2014) applied the same technique in Fisher’s linear discriminant for a high-dimensional classification problem.

Our proposed methodology for high-dimensional classification has some similarities with

the techniques described above, in the sense that we consider many random projections of the data, but is also closely related to *bagging* (Breiman, 1996), since the ultimate assignment of each test point is made by aggregation and a vote. Bagging has proved to be an effective tool for improving unstable classifiers; indeed, a bagged version of the (generally inconsistent) 1-nearest neighbour classifier is universally consistent as long as the resample size is carefully chosen; see Hall and Samworth (2005). More generally, bagging has been shown to be particularly effective in high-dimensional problems such as variable selection (Meinshausen and Bühlmann, 2010; Shah and Samworth, 2013). Another related approach to ours is Blaser and Fryzlewicz (2015), who consider ensembles of random rotations, as opposed to projections.

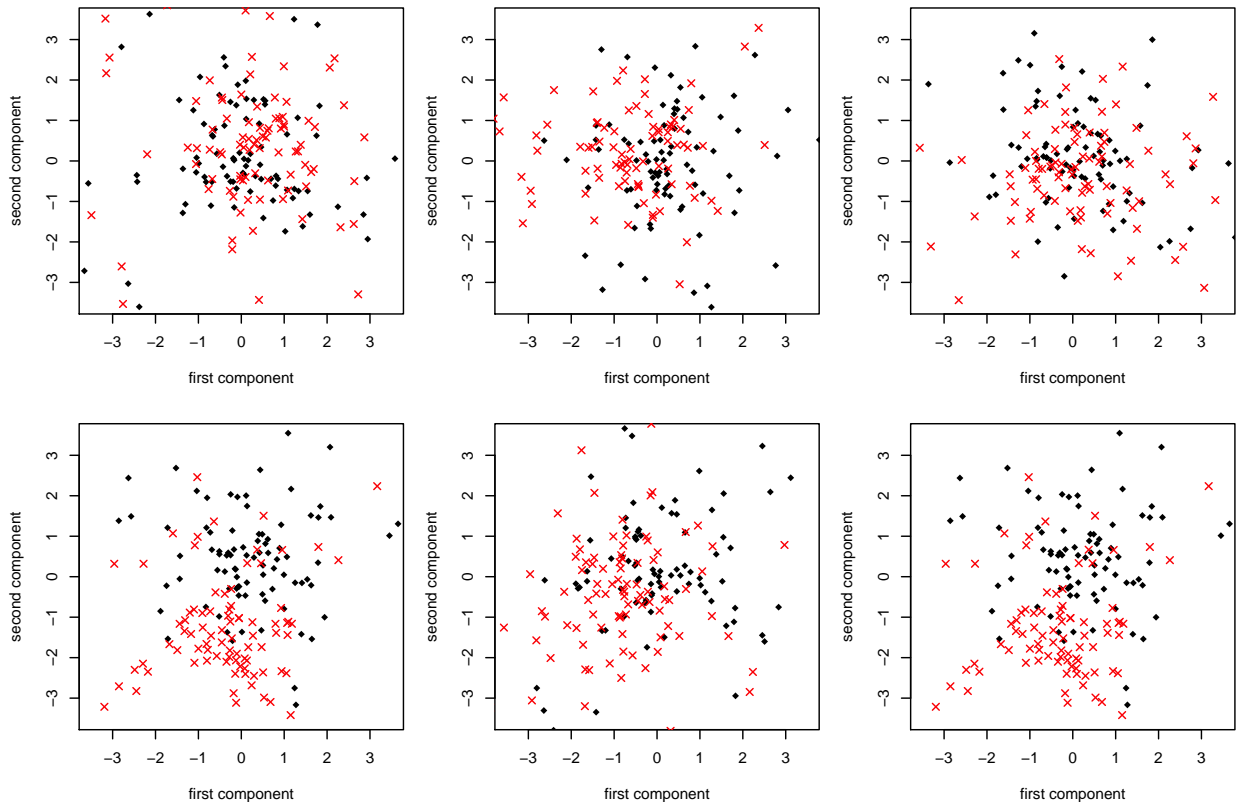


Figure 1: Different two-dimensional projections of a sample of size $n = 200$ from Model 2 in Section 6.1.2 with $p = 50$ dimensions and prior probability $\pi_1 = 1/2$. Top row: three projections drawn from Haar measure; bottom row: the projections with smallest estimate of test error out of 100 Haar projections with LDA (left), Quadratic Discriminant Analysis (middle) and k -nearest neighbours (right).

One of the basic but fundamental observations that underpins our proposal is the fact that aggregating the classifications of all random projections is not sensible, since most of these projections will typically destroy the class structure in the data; see the top row of Figure 1. For this reason, we advocate partitioning the projections into non-overlapping blocks, and within each block we retain only the projection yielding the smallest estimate of

the test error. The attraction of this strategy is illustrated in the bottom row of Figure 1, where we see a much clearer partition of the classes. Another key feature of our proposal is the realisation that a simple majority vote of the classifications based on the retained projections can be highly suboptimal; instead, we argue that the voting threshold should be chosen in a data-driven fashion in an attempt to minimise the test error of the infinite-simulation version of our random projection ensemble classifier. In fact, this estimate of the optimal threshold turns out to be remarkably effective in practice; see Section 5.1 for further details. We emphasise that our methodology can be used in conjunction with any base classifier, though we particularly have in mind classifiers designed for use in low-dimensional settings. The random projection ensemble classifier can therefore be regarded as a general technique for either extending the applicability of an existing classifier to high dimensions, or improving its performance.

Our theoretical results are divided into three parts. In the first, we consider a generic base classifier and a generic method for generating the random projections into \mathbb{R}^d and quantify the difference between the test error of the random projection ensemble classifier and its infinite-simulation counterpart as the number of projections increases. We then consider selecting random projections from non-overlapping blocks by initially drawing them according to Haar measure, and then within each block retaining the projection that minimises an estimate of the test error. Under a condition implied by the widely-used sufficient dimension reduction assumption (Li, 1991; Cook, 1998; Lee et al., 2013), we can then control the difference between the test error of the random projection classifier and the Bayes risk as a function of terms that depend on the performance of the base classifier based on projected data and our method for estimating the test error, as well as terms that become negligible as the number of projections increases. The final part of our theory gives risk bounds on the first two of these terms for specific choices of base classifier, namely Fisher’s linear discriminant and the k -nearest neighbour classifier. The key point here is that these bounds only depend on d , the sample size n and the number of projections, and not on the original data dimension p .

The remainder of the paper is organised as follows. Our methodology and general theory are developed in Sections 2 and 3. Specific choices of base classifier are discussed in Section 4, while Section 5 is devoted to a consideration of the practical issues of the choice of voting threshold, projected dimension and the number of projections used. In Section 6 we present results from an extensive empirical analysis on both simulated and real data where we compare the performance of the random projection ensemble classifier with several popular techniques for high-dimensional classification. The outcomes are extremely encouraging, and suggest that the random projection ensemble classifier has excellent finite-sample performance in a variety of different high-dimensional classification settings. We conclude with a discussion of various extensions and open problems. All proofs are deferred to the Appendix.

Finally in this section, we introduce the following general notation used throughout the paper. For a sufficiently smooth real-valued function g defined on a neighbourhood of $t \in \mathbb{R}$, let $\dot{g}(t)$ and $\ddot{g}(t)$ denote its first and second derivatives at t , and let $[t]$ and $\{t\} := t - [t]$ denote the integer and fractional part of t respectively.

2 A generic random projection ensemble classifier

We start by describing our setting and defining the relevant notation. Suppose that the pair (X, Y) takes values in $\mathbb{R}^p \times \{1, 2\}$, with joint distribution P , characterised by $\pi_1 := \mathbb{P}(Y = 1)$, and P_r , the conditional distribution of $X|Y = r$, for $r = 1, 2$. For convenience, we let $\pi_2 := \mathbb{P}(Y = 2) = 1 - \pi_1$. In the alternative characterisation of P , we let P_X denote the marginal distribution of X and write $\eta(x) := \mathbb{P}(Y = 1|X = x)$ for the regression function. Recall that a *classifier* on \mathbb{R}^p is a Borel measurable function $C : \mathbb{R}^p \rightarrow \{1, 2\}$, with the interpretation that we assign a point $x \in \mathbb{R}^p$ to class $C(x)$. We let \mathcal{C}_p denote the set of all such classifiers.

The misclassification rate, or *risk*, of a classifier C is $\mathcal{R}(C) := \mathbb{P}\{C(X) \neq Y\}$, and is minimised by the *Bayes* classifier

$$C^{\text{Bayes}}(x) = \begin{cases} 1 & \text{if } \eta(x) \geq 1/2; \\ 2 & \text{otherwise} \end{cases}$$

(e.g. Devroye, Györfi and Lugosi, 1996, p. 10). Its risk is $\mathcal{R}(C^{\text{Bayes}}) = \mathbb{E}[\min\{\eta(X), 1 - \eta(X)\}]$.

Of course, we cannot use the Bayes classifier in practice, since η is unknown. Nevertheless, we often have access to a sample of training data that we can use to construct an approximation to the Bayes classifier. Throughout this section and Section 3, it is convenient to consider the training sample $\mathcal{T}_n := \{(x_1, y_1), \dots, (x_n, y_n)\}$ to be fixed points in $\mathbb{R}^p \times \{1, 2\}$. Our methodology will be applied to a base classifier $\hat{C}_n = \hat{C}_{n, \mathcal{T}_n, d}$, which we assume can be constructed from an arbitrary training sample $\mathcal{T}_{n, d}$ of size n in $\mathbb{R}^d \times \{1, 2\}$; thus \hat{C}_n is a measurable function from $(\mathbb{R}^d \times \{1, 2\})^n$ to \mathcal{C}_d .

Now assume that $d \leq p$. We say a matrix $A \in \mathbb{R}^{d \times p}$ is a *projection* if $AA^T = I_{d \times d}$, the d -dimensional identity matrix. Let $\mathcal{A} = \mathcal{A}_{d \times p} := \{A \in \mathbb{R}^{d \times p} : AA^T = I_{d \times d}\}$ be the set of all such matrices. Given a projection $A \in \mathcal{A}$, define projected data $z_i^A := Ax_i$ and $y_i^A := y_i$ for $i = 1, \dots, n$, and let $\mathcal{T}_n^A := \{(z_1^A, y_1^A), \dots, (z_n^A, y_n^A)\}$. The projected data base classifier corresponding to \hat{C}_n is $\hat{C}_n^A : (\mathbb{R}^d \times \{1, 2\})^n \rightarrow \mathcal{C}_p$, given by

$$\hat{C}_n^A(x) = \hat{C}_{n, \mathcal{T}_n^A}^A(x) := \hat{C}_{n, \mathcal{T}_n^A}(Ax).$$

Note that although \hat{C}_n^A is a classifier on \mathbb{R}^p , the value of $\hat{C}_n^A(x)$ only depends on x through its d -dimensional projection Ax .

We now define a generic ensemble classifier based on random projections. For $B_1 \in \mathbb{N}$, let A_1, A_2, \dots, A_{B_1} denote independent and identically distributed projections in \mathcal{A} , independent of (X, Y) . The distribution on \mathcal{A} is left unspecified at this stage, and in fact our proposed method ultimately involves choosing this distribution depending on \mathcal{T}_n . Now set

$$\hat{\nu}_n^{B_1}(x) := \frac{1}{B_1} \sum_{b_1=1}^{B_1} \mathbb{1}_{\{\hat{C}_n^{A_{b_1}}(x)=1\}}. \quad (1)$$

For $\alpha \in (0, 1)$, the *random projection ensemble* classifier is defined to be

$$\hat{C}_n^{\text{RP}}(x) := \begin{cases} 1 & \text{if } \hat{\nu}_n^{B_1}(x) \geq \alpha; \\ 2 & \text{otherwise.} \end{cases} \quad (2)$$

We emphasise again here the additional flexibility afforded by not pre-specifying the voting threshold α to be $1/2$. Our analysis of the random projection ensemble classifier will require some further definitions. Let

$$\hat{\mu}_n(x) := \mathbb{E}\{\hat{\nu}_n^{B_1}(x)\} = \mathbb{P}\{\hat{C}_n^{A_1}(x) = 1\},$$

where the randomness here comes from the random projections. Let $G_{n,1}$ and $G_{n,2}$ denote the distribution functions of $\hat{\mu}_n(X)|\{Y = 1\}$ and $\hat{\mu}_n(X)|\{Y = 2\}$, respectively. We will make use of the following assumption:

(A.1) $G_{n,1}$ and $G_{n,2}$ are twice differentiable at α .

The first derivatives of $G_{n,1}$ and $G_{n,2}$, when they exist, are denoted as $g_{n,1}$ and $g_{n,2}$ respectively; under **(A.1)**, these derivatives are well-defined in a neighbourhood of α . Our first main result below gives an asymptotic expansion for the test error $\mathcal{L}(\hat{C}_n^{\text{RP}}) := \mathbb{P}\{\hat{C}_n^{\text{RP}}(X) \neq Y\}$ of our generic random projection ensemble classifier as the number of projections increases. In particular, we show that this test error can be well approximated by the test error of the infinite-simulation random projection classifier

$$\hat{C}_n^{\text{RP}*}(x) := \begin{cases} 1 & \text{if } \hat{\mu}_n(x) \geq \alpha; \\ 2 & \text{otherwise.} \end{cases}$$

This infinite-simulation classifier turns out to be easier to analyse in subsequent results. Note that under **(A.1)**,

$$\mathcal{L}(\hat{C}_n^{\text{RP}*}) := \mathbb{P}\{\hat{C}_n^{\text{RP}*}(X) \neq Y\} = \pi_1 G_{n,1}(\alpha) + \pi_2 \{1 - G_{n,2}(\alpha)\}. \quad (3)$$

Theorem 1. *Assume (A.1). Then*

$$\mathcal{L}(\hat{C}_n^{\text{RP}}) - \mathcal{L}(\hat{C}_n^{\text{RP}*}) = \frac{\gamma_n(\alpha)}{B_1} + o\left(\frac{1}{B_1}\right)$$

as $B_1 \rightarrow \infty$, where

$$\gamma_n(\alpha) := (1 - \alpha - \llbracket B_1 \alpha \rrbracket) \{ \pi_1 g_{n,1}(\alpha) - \pi_2 g_{n,2}(\alpha) \} + \frac{\alpha(1 - \alpha)}{2} \{ \pi_1 \dot{g}_{n,1}(\alpha) - \pi_2 \dot{g}_{n,2}(\alpha) \}.$$

The proof of Theorem 1 in the Appendix is lengthy, and involves a one-term Edgeworth approximation to the distribution function of a standardised Binomial random variable. One of the technical challenges is to show that the error in this approximation holds uniformly in the binomial proportion.

Define the test error of \hat{C}_n^A by¹

$$\mathcal{L}_n^A := \int_{\mathbb{R}^p \times \{1,2\}} \mathbb{1}_{\{\hat{C}_n^A(x) \neq y\}} dP(x, y).$$

Our next result controls the test excess risk, i.e. the difference between the test error and the Bayes risk, of the infinite-simulation random projection classifier in terms of the expected test excess risk of the classifier based on a single random projection. An attractive feature of this result is its generality: no assumptions are placed on the configuration of the training data \mathcal{T}_n , the distribution P of the test point (X, Y) or on the distribution of the individual projections.

Theorem 2. *We have*

$$\mathcal{L}(\hat{C}_n^{\text{RP}^*}) - \mathcal{R}(C^{\text{Bayes}}) \leq \frac{1}{\min(\alpha, 1 - \alpha)} \{\mathbb{E}(\mathcal{L}_n^{A_1}) - \mathcal{R}(C^{\text{Bayes}})\}.$$

3 Choosing good random projections

In this section, we study a special case of the generic random projection ensemble classifier introduced in Section 2, where we propose a screening method for choosing the random projections. Let \hat{L}_n^A be an estimator of \mathcal{L}_n^A , based on $\{(z_1^A, y_1^A), \dots, (z_n^A, y_n^A)\}$, that takes values in the set $\{0, 1/n, \dots, 1\}$. Examples of such estimators include resubstitution and leave-one-out estimates; we discuss these choices in greater detail in Section 4. For $B_1, B_2 \in \mathbb{N}$, let $\{A_{b_1, b_2} : b_1 = 1, \dots, B_1, b_2 = 1, \dots, B_2\}$ denote independent projections, independent of (X, Y) , distributed according to Haar measure on \mathcal{A} . One way to simulate from Haar measure on the set \mathcal{A} is to first generate a matrix $R \in \mathbb{R}^{d \times p}$, where each entry is drawn independently from a standard normal distribution, and then take A^T to be the matrix of left singular vectors in the singular value decomposition of R^T . For $b_1 = 1, \dots, B_1$, let

$$b_2^*(b_1) := \underset{b_2 \in \{1, \dots, B_2\}}{\text{sargmin}} \hat{L}_n^{A_{b_1, b_2}}, \quad (4)$$

where sargmin denotes the smallest index where the minimum is attained in the case of a tie. We now set $A_{b_1} := A_{b_1, b_2^*(b_1)}$, and consider the random projection ensemble classifier from Section 2 constructed using the independent projections A_1, \dots, A_{B_1} .

Let

$$\hat{L}_n^* := \min_{A \in \mathcal{A}} \hat{L}_n^A$$

denote the optimal test error estimate over all projections. The minimum is attained here, since \hat{L}_n^A takes only finitely many values. For $j = 0, 1, \dots, \lfloor n(1 - \hat{L}_n^*) \rfloor$, let

$$\beta_n(j) := \mathbb{P}(\hat{L}_n^A \leq \hat{L}_n^* + j/n),$$

where A is distributed according to Haar measure on \mathcal{A} . We assume the following:

¹We define \mathcal{L}_n^A through an integral rather than defining $\mathcal{L}_n^A := \mathbb{P}\{\hat{C}_n^A(X) \neq Y\}$ to make it clear that when A is a random projection, it should be conditioned on when computing \mathcal{L}_n^A .

(A.2) There exist $\beta_0 \in (0, 1)$ and $\beta, \rho > 0$ such that

$$\beta_n(j) \geq \beta_0 + \frac{\beta j^\rho}{n^\rho}$$

for $j \in \{0, 1, \dots, \lfloor n(\frac{\log^2 B_2}{\beta B_2})^{1/\rho} \rfloor + 1\}$.

Condition **(A.2)** asks for a certain growth rate of the distribution function of \hat{L}_n^A close to its minimum value \hat{L}_n^* ; observe that the strength of the condition decreases as B_2 increases. Under this condition, the following result is a starting point for controlling the expected test excess risk of the classifier based on a single projection chosen according to the scheme described above.

Proposition 3. *Assume (A.2). Then*

$$\begin{aligned} \mathbb{E}(\mathcal{L}_n^{A_1}) - \mathcal{R}(C^{\text{Bayes}}) &\leq \hat{L}_n^* - \mathcal{R}(C^{\text{Bayes}}) + \epsilon_n \\ &\quad + (1 - \beta_0)^{B_2} \left\{ \frac{1}{n} + \frac{(1 - \beta_0)^{1/\rho} \Gamma(1 + 1/\rho)}{B_2^{1/\rho} \beta^{1/\rho}} + \exp\left(-\frac{\log^2 B_2}{1 - \beta_0}\right) \right\}, \end{aligned}$$

where $\epsilon_n = \epsilon_n^{(B_2)} := \mathbb{E}(\mathcal{L}_n^{A_1} - \hat{L}_n^{A_1})$.

The form of the bound in Proposition 3 motivates us to seek to control $\hat{L}_n^* - \mathcal{R}(C^{\text{Bayes}})$ in terms of the test excess risk of a classifier based on the projected data, in the hope that we will be able to show this does not depend on p . To this end, define the regression function on \mathbb{R}^d induced by the projection $A \in \mathcal{A}$ to be $\eta^A(z) := \mathbb{P}(Y = 1 | AX = z)$. The corresponding induced Bayes classifier, which is the optimal classifier knowing only the distribution of (AX, Y) , is given by

$$C^{A\text{-Bayes}}(z) := \begin{cases} 1 & \text{if } \eta^A(z) \geq 1/2; \\ 2 & \text{otherwise.} \end{cases}$$

Its risk is

$$\mathcal{R}^{A\text{-Bayes}} := \int_{\mathbb{R}^p \times \{1, 2\}} \mathbb{1}_{\{C^{A\text{-Bayes}}(Ax) \neq y\}} dP(x, y).$$

In order to ensure that \hat{L}_n^* will be close to the Bayes risk, we will invoke an additional assumption on the form of the Bayes classifier:

(A.3) There exists a projection $A^* \in \mathcal{A}$ such that

$$P_X(\{x \in \mathbb{R}^p : \eta(x) \geq 1/2\} \triangle \{x \in \mathbb{R}^p : \eta^{A^*}(A^*x) \geq 1/2\}) = 0,$$

where $B \triangle C := (B \cap C^c) \cup (B^c \cap C)$ denotes the symmetric difference of two sets B and C .

Condition **(A.3)** requires that the set of points $x \in \mathbb{R}^p$ assigned by the Bayes classifier to class 1 can be expressed as a function of a d -dimensional projection of x . Note that if the Bayes decision boundary is a hyperplane, then **(A.3)** holds with $d = 1$. Moreover, Proposition 4 below shows that, in fact, **(A.3)** holds under the sufficient dimension reduction condition, which states that Y is independent of X given A^*X ; see Cook (1998) for many statistical settings where such an assumption is natural.

Proposition 4. *If Y is independent of X given A^*X , then (A.3) holds.*

Finally, then, we are in a position to control the test excess risk of our random projection ensemble classifier in terms of the test excess risk of a classifier based on d -dimensional data, as well as terms that reflect our ability to estimate the test error of classifiers based on projected data and terms that depend on B_1 and B_2 .

Theorem 5. *Assume (A.1), (A.2) and (A.3). Then*

$$\begin{aligned} \mathcal{L}(\hat{C}_n^{\text{RP}}) - \mathcal{R}(C^{\text{Bayes}}) &\leq \frac{\mathcal{L}_n^{A^*} - \mathcal{R}^{A^*-\text{Bayes}}}{\min(\alpha, 1 - \alpha)} + \frac{\epsilon_n - \epsilon_n^{A^*}}{\min(\alpha, 1 - \alpha)} + \frac{\gamma_n(\alpha)}{B_1} \{1 + o(1)\} \\ &\quad + \frac{(1 - \beta_0)^{B_2}}{\min(\alpha, 1 - \alpha)} \left\{ \frac{1}{n} + \frac{(1 - \beta_0)^{1/\rho} \Gamma(1 + 1/\rho)}{B_2^{1/\rho} \beta^{1/\rho}} + \exp\left(-\frac{\log^2 B_2}{1 - \beta_0}\right) \right\} \end{aligned}$$

as $B_1 \rightarrow \infty$, where $\gamma_n(\alpha)$ is defined in Theorem 1, ϵ_n is defined in Proposition 3 and $\epsilon_n^{A^*} := \mathcal{L}_n^{A^*} - \hat{\mathcal{L}}_n^{A^*}$.

Regarding the bound in Theorem 5 as a sum of four terms, we see that the last two of these can be seen as the price we have to pay for the fact that we do not have access to an infinite sample of random projections. These terms can be made negligible by choosing B_1 and B_2 to be sufficiently large, but it should be noted that ϵ_n may increase with B_2 . This is a reflection of the fact that minimising an estimate of test error may lead to overfitting. The behaviour of this term, together with that of $\mathcal{L}_n^{A^*} - \mathcal{R}^{A^*-\text{Bayes}}$ and $\epsilon_n^{A^*}$, depends on the choice of base classifier, but in the next section below we describe settings where these terms can be bounded by expressions that do not depend on p .

4 Possible choices of the base classifier

In this section, we change our previous perspective and regard the training data $\mathcal{T}_n := \{(X_1, Y_1), \dots, (X_n, Y_n)\}$ as independent random pairs with distribution P , so our earlier statements are interpreted conditionally on \mathcal{T}_n . We consider particular choices of base classifier, and study the first two terms in the bound in Theorem 5.

4.1 Linear Discriminant Analysis

Linear Discriminant Analysis (LDA), introduced by Fisher (1936), is arguably the simplest classification technique. Recall that in the special case where $X|Y = r \sim N_p(\mu_r, \Sigma)$, we have

$$\text{sgn}\{\eta(x) - 1/2\} = \text{sgn}\left\{\log \frac{\pi_1}{\pi_2} + \left(x - \frac{\mu_1 + \mu_2}{2}\right)^T \Sigma^{-1}(\mu_1 - \mu_2)\right\},$$

so (A.3) holds with $d = 1$ and $A^* = \frac{(\mu_1 - \mu_2)^T \Sigma^{-1}}{\|\Sigma^{-1}(\mu_1 - \mu_2)\|}$, a $1 \times p$ matrix. In LDA, π_r , μ_r and Σ are estimated by their sample versions, using a pooled estimate of Σ . Although LDA cannot be

applied directly when $p \geq n$ since the sample covariance matrix is singular, we can still use it as the base classifier for a random projection ensemble, provided that $d < n$. Indeed, noting that for any $A \in \mathcal{A}$, we have $AX|Y = r \sim N_d(\mu_r^A, \Sigma^A)$, where $\mu_r^A := A\mu_r$ and $\Sigma^A := A\Sigma A^T$, we can define

$$\hat{C}_n^A(x) = \hat{C}_n^{A\text{-LDA}}(x) := \begin{cases} 1 & \text{if } \log \frac{\hat{\pi}_1^A}{\hat{\pi}_2^A} + (Ax - \frac{\hat{\mu}_1^A + \hat{\mu}_2^A}{2})^T \hat{\Omega}^A (\hat{\mu}_1^A - \hat{\mu}_2^A) \geq 0; \\ 2 & \text{otherwise.} \end{cases} \quad (5)$$

Here, $\hat{\pi}_r^A := n^{-1} \sum_{i=1}^n \mathbb{1}_{\{Y_i^A=r\}}$, $\hat{\mu}_r^A := n^{-1} \sum_{i=1}^n AX_i \mathbb{1}_{\{Y_i^A=r\}}$,

$$\hat{\Sigma}^A := \frac{1}{n-2} \sum_{i=1}^n \sum_{r=1}^2 (AX_i - \hat{\mu}_r^A)(AX_i - \hat{\mu}_r^A)^T \mathbb{1}_{\{Y_i^A=r\}}$$

and $\hat{\Omega}^A := (\hat{\Sigma}^A)^{-1}$.

Write Φ for the standard normal distribution function. Under the normal model specified above, the test error of the LDA classifier can be written as

$$\mathcal{L}_n^A = \pi_1 \Phi \left(\frac{\log \frac{\hat{\pi}_2^A}{\hat{\pi}_1^A} - (\hat{\delta}^A)^T \hat{\Omega}^A (\bar{\mu}^A - \mu_1^A)}{\sqrt{(\hat{\delta}^A)^T \hat{\Omega}^A \Sigma^A \hat{\Omega}^A \hat{\delta}^A}} \right) + \pi_2 \Phi \left(\frac{\log \frac{\hat{\pi}_1^A}{\hat{\pi}_2^A} + (\hat{\delta}^A)^T \hat{\Omega}^A (\bar{\mu}^A - \mu_2^A)}{\sqrt{(\hat{\delta}^A)^T \hat{\Omega}^A \Sigma^A \hat{\Omega}^A \hat{\delta}^A}} \right),$$

where $\hat{\delta}^A := \hat{\mu}_2^A - \hat{\mu}_1^A$ and $\bar{\mu}^A := (\hat{\mu}_1^A + \hat{\mu}_2^A)/2$. Okamoto (1963) studied the excess risk of the LDA classifier in an asymptotic regime in which d is fixed as n diverges. In fact, he considered a very slightly different data generating model, in which the training sample sizes from each population are assumed to be known in advance, so that without loss of generality, we may assume that $Y_1 = \dots = Y_{n_1} = 1$ and $Y_{n_1+1} = \dots = Y_n = 2$, while $X_i|Y_i = r \sim N_p(\mu_r, \Sigma)$, as before. Specialising his results for simplicity to the case where n is even and $n_1 = n_2$, Okamoto (1963) showed that using the LDA classifier (5) with $A = A^*$, $\hat{\pi}_1^{A^*} = n_1/n$ and $\hat{\pi}_2^{A^*} = n_2/n$ yields

$$\mathbb{E}(\mathcal{L}_n^{A^*}) - \mathcal{R}^{A^*-\text{Bayes}} = \mathcal{R}(\hat{C}_n^{A^*}) - \mathcal{R}^{A^*-\text{Bayes}} = \frac{d}{n} \phi \left(-\frac{\Delta}{2} \right) \left\{ \frac{\Delta}{4} + \frac{d-1}{d\Delta} \right\} \{1 + O(n^{-1})\} \quad (6)$$

as $n \rightarrow \infty$, where $\Delta := \|\Sigma^{-1/2}(\mu_1 - \mu_2)\| = \|(\Sigma^{A^*})^{-1/2}(\mu_1^{A^*} - \mu_2^{A^*})\|$.

It remains to control the errors ϵ_n and $\epsilon_n^{A^*}$ in Theorem 5. For the LDA classifier, we consider the resubstitution estimator

$$\hat{L}_n^A := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{\hat{C}_n^{A\text{-LDA}}(X_i) \neq Y_i\}}. \quad (7)$$

Devroye and Wagner (1976) provided a Vapnik–Chervonenkis bound for \hat{L}_n^A under no assumptions on the underlying data generating mechanism: for every $n \in \mathbb{N}$ and $\epsilon > 0$,

$$\sup_{A \in \mathcal{A}} \mathbb{P}(|\mathcal{L}_n^A - \hat{L}_n^A| > \epsilon) \leq 8n^d e^{-n\epsilon^2/32}; \quad (8)$$

see also Devroye et al. (1996, Theorem 23.1). We can then conclude that

$$\mathbb{E}(|\epsilon_n^{A^*}|) \leq \mathbb{E}\{(\mathcal{L}_n^{A^*} - \hat{L}_n^{A^*})^2\}^{1/2} \leq \inf_{\epsilon_0 \in (0,1)} \left\{ \epsilon_0 + 8n^d \int_{\epsilon_0}^1 e^{-ns/32} ds \right\}^{1/2} \leq 8\sqrt{\frac{d \log n + 3 \log 2 + 1}{2n}}. \quad (9)$$

The more difficult term to deal with is $\mathbb{E}(|\epsilon_n|) = \mathbb{E}\{|\mathbb{E}(\mathcal{L}_n^{A_1} - \hat{L}_n^{A_1} | \mathcal{T}_n)|\} \leq \mathbb{E}|\mathcal{L}_n^{A_1} - \hat{L}_n^{A_1}|$. In this case, the bound (8) cannot be applied directly, because $(X_1, Y_1), \dots, (X_n, Y_n)$ are no longer independent conditional on A_1 . Nevertheless, since $A_{1,1}, \dots, A_{1,B_2}$ are independent of \mathcal{T}_n , we still have that

$$\begin{aligned} \mathbb{P}\left\{\max_{b_2=1, \dots, B_2} |\mathcal{L}_n^{A_{1,b_2}} - \hat{L}_n^{A_{1,b_2}}| > \epsilon \mid A_{1,1}, \dots, A_{1,B_2}\right\} &\leq \sum_{b_2=1}^{B_2} \mathbb{P}\{|\mathcal{L}_n^{A_{1,b_2}} - \hat{L}_n^{A_{1,b_2}}| > \epsilon \mid A_{1,b_2}\} \\ &\leq 8n^d B_2 e^{-n\epsilon^2/32}. \end{aligned}$$

We can therefore conclude by almost the same argument as that leading to (9) that

$$\mathbb{E}(|\epsilon_n|) \leq \mathbb{E}\left\{\max_{b_2=1, \dots, B_2} (\mathcal{L}_n^{A_{1,b_2}} - \hat{L}_n^{A_{1,b_2}})^2\right\}^{1/2} \leq 8\sqrt{\frac{d \log n + 3 \log 2 + \log B_2 + 1}{2n}}. \quad (10)$$

Note that none of the bounds (6), (9) and (10) depend on the original data dimension p . Moreover, substituting the bound (10) into Theorem 5 reveals a trade-off in the choice of B_2 when using LDA as the base classifier. Choosing B_2 to be large gives us a good chance of finding a projection with a small estimate of test error, but we may incur a small overfitting penalty as reflected by (10).

4.2 Quadratic Discriminant Analysis

Quadratic Discriminant Analysis (QDA) is designed to handle situations where the class-conditional covariance matrices are unequal. Recall that when $X|Y = r \sim N_p(\mu_r, \Sigma_r)$, and $\pi_r := \mathbb{P}(Y = r)$, for $r = 1, 2$, the Bayes decision boundary is given by $\{x \in \mathbb{R}^p : \Delta(x; \pi_1, \mu_1, \mu_2, \Sigma_1, \Sigma_2) = 0\}$, where

$$\begin{aligned} \Delta(x; \pi_1, \mu_1, \mu_2, \Sigma_1, \Sigma_2) &:= \log \frac{\pi_1}{\pi_2} - \frac{1}{2} \log \left(\frac{\det \Sigma_1}{\det \Sigma_2} \right) - \frac{1}{2} x^T (\Sigma_1^{-1} - \Sigma_2^{-1}) x \\ &\quad + x^T (\Sigma_1^{-1} \mu_1 - \Sigma_2^{-1} \mu_2) - \frac{1}{2} \mu_1^T \Sigma_1^{-1} \mu_1 + \frac{1}{2} \mu_2^T \Sigma_2^{-1} \mu_2. \end{aligned}$$

In QDA, π_r , μ_r and Σ_r are estimated by their sample versions. If $p \geq \min(n_1, n_2)$, where $n_r := \sum_{i=1}^n \mathbb{1}_{\{Y_i=r\}}$ is the number of training sample observations from the r th class, then at least one of the sample covariance matrix estimates is singular, and QDA cannot be used directly. Nevertheless, we can still choose $d < \min\{n_1, n_2\}$ and use QDA as the base classifier in a random projection ensemble. Specifically, we can set

$$\hat{C}_n^A(x) = \hat{C}_n^{A-\text{QDA}}(x) := \begin{cases} 1 & \text{if } \Delta(x; \hat{\pi}_1^A, \hat{\mu}_1^A, \hat{\mu}_2^A, \hat{\Sigma}_1^A, \hat{\Sigma}_2^A) \geq 0; \\ 2 & \text{otherwise,} \end{cases}$$

where $\hat{\pi}_r^A$, $\hat{\Sigma}_r^A$ and $\hat{\mu}_r^A$ were defined in Section 4.1, and where

$$\hat{\Sigma}_r^A := \frac{1}{n_r - 1} \sum_{i: Y_i^A = r} (AX_i - \hat{\mu}_r^A)(AX_i - \hat{\mu}_r^A)^T$$

for $r = 1, 2$. Unfortunately, analogous theory to that presented in Section 4.1 does not appear to exist for the QDA classifier (unlike for LDA, the risk does not have a closed form). Nevertheless, we found in our simulations presented in Section 6 that the QDA random projection ensemble classifier can perform very well in practice. In this case, we estimate the test errors using the leave-one-out method given by

$$\hat{L}_n^A := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{\hat{C}_{n,i}^A(X_i) \neq Y_i\}}, \quad (11)$$

where $\hat{C}_{n,i}^A$ denotes the classifier \hat{C}_n^A , trained without the i th pair, i.e. based on $\mathcal{T}_n^A \setminus \{X_i^A, Y_i^A\}$. For a method like QDA that involves estimating more parameters than LDA, we found that the leave-one-out estimator was less susceptible to overfitting than the resubstitution estimator.

4.3 The k -nearest neighbour classifier

The k -nearest neighbour classifier (knn), first proposed by Fix and Hodges (1951), is a non-parametric method that classifies the test point $x \in \mathbb{R}^p$ according to a majority vote over the classes of the k nearest training data points to x . The enormous popularity of the knn classifier can be attributed partly due to its simplicity and intuitive appeal; however, it also has the attractive property of being universally consistent: for every distribution P , as long as $k \rightarrow \infty$ and $k/n \rightarrow 0$, the risk of the knn classifier converges to the Bayes risk (Devroye et al., 1996, Theorem 6.4).

Hall, Park and Samworth (2008) derived the rate of convergence of the excess risk of the k -nearest neighbour classifier. Under regularity conditions, the optimal choice of k , in terms of minimising the excess risk, is $O(n^{4/(p+4)})$, and the rate of convergence of the excess risk with this choice is $O(n^{-4/(p+4)})$. Thus, in common with other nonparametric methods, there is a ‘curse of dimensionality’ effect that means the classifier typically performs poorly in high dimensions. Samworth (2012a,b) found the optimal way of assigning decreasing weights to increasingly distant neighbours, and quantified the asymptotic improvement in risk over the unweighted version, but the rate of convergence remains the same.

We can use the knn classifier as the base classifier for a random projection ensemble as follows: let $\mathcal{T}_n^A := \{(Z_1^A, Y_1^A), \dots, (Z_n^A, Y_n^A)\}$, where $Z_i^A := AX_i$ and $Y_i^A := Y_i$. Given $z \in \mathbb{R}^d$, let $(Z_{(1)}^A, Y_{(1)}^A), \dots, (Z_{(n)}^A, Y_{(n)}^A)$ be a re-ordering of the training data such that $\|Z_{(1)}^A - z\| \leq \dots \leq \|Z_{(n)}^A - z\|$, with ties split at random. Now define

$$\hat{C}_n^A(x) = \hat{C}_n^{A-knn}(x) := \begin{cases} 1 & \text{if } \hat{S}_n^A(Ax) \geq 1/2; \\ 2 & \text{otherwise,} \end{cases}$$

where $\hat{S}_n^A(z) := k^{-1} \sum_{i=1}^k \mathbb{1}_{\{Y_{(i)}^A=1\}}$. The theory described in the previous paragraph can be applied to show that, under regularity conditions, $\mathbb{E}(\mathcal{L}_n^{A*}) - \mathcal{R}(C^{A*-\text{Bayes}}) = O(n^{-4/(d+4)})$.

Once again, a natural estimate of the test error in this case is the leave-one-out estimator defined in (11), where we use the same value of k on the leave-one-out datasets as for the base classifier, and where distance ties are split in the same way as for the base classifier. For this estimator, Devroye and Wagner (1979) showed that for every $n \in \mathbb{N}$,

$$\sup_{A \in \mathcal{A}} \mathbb{E}\{(\hat{L}_n^A - \mathcal{L}_n^A)^2\} \leq \frac{1}{n} + \frac{24k^{1/2}}{n\sqrt{2\pi}};$$

see also Devroye et al. (1996, Chapter 24). It follows that

$$\mathbb{E}(|\epsilon_n^{A*}|) \leq \left(\frac{1}{n} + \frac{24k^{1/2}}{n\sqrt{2\pi}}\right)^{1/2} \leq \frac{1}{n^{1/2}} + \frac{2\sqrt{3}k^{1/4}}{n^{1/2}\sqrt{\pi}}.$$

Devroye and Wagner (1979) also provided a tail bound analogous to (8) for the leave-one-out estimator. Arguing very similarly to Section 4.1, we can deduce that under no conditions on the data generating mechanism,

$$\mathbb{E}(|\epsilon_n|) \leq 3\{4(3^d + 1)\}^{1/3} \left\{ \frac{k(1 + \log B_2 + 3 \log 2)}{n} \right\}^{1/3}.$$

5 Practical considerations

5.1 Choice of α

We now discuss the choice of the voting threshold α in (2). The expression for the test error of the infinite-simulation random projection ensemble classifier given in (3) suggests the ‘oracle’ choice

$$\alpha^* \in \operatorname{argmin}_{\alpha' \in [0,1]} [\pi_1 G_{n,1}(\alpha') + \pi_2 \{1 - G_{n,2}(\alpha')\}]. \quad (12)$$

Note that, if assumption **(A.1)** holds and $\alpha^* \in (0, 1)$ then $\pi_1 g_{n,1}(\alpha^*) = \pi_2 g_{n,2}(\alpha^*)$ and in Theorem 1, we have

$$\gamma_n(\alpha^*) = \frac{1}{2} \alpha^* (1 - \alpha^*) \{ \pi_1 \dot{g}_{n,1}(\alpha^*) - \pi_2 \dot{g}_{n,2}(\alpha^*) \}.$$

Of course, α^* cannot be used directly, because we do not know $G_{n,1}$ and $G_{n,2}$ (and we may not know π_1 and π_2 either). Nevertheless, for the LDA base classifier we can estimate $G_{n,r}$ using

$$\hat{G}_{n,r}(t) := \frac{1}{n_r} \sum_{i: Y_i=r} \mathbb{1}_{\{\hat{\nu}_n(X_i) < t\}}$$

for $r = 1, 2$. For the QDA and k -nearest neighbour base classifiers, we use the leave-one-out-based estimate $\tilde{\nu}_n(X_i) := B_1^{-1} \sum_{b_1=1}^{B_1} \mathbb{1}_{\{\hat{C}_{n,i}^{A_{b_1}}(X_i)=1\}}$ in place of $\hat{\nu}_n(X_i)$. We also estimate

π_r by $\hat{\pi}_r := n^{-1} \sum_{i=1}^n \mathbb{1}_{\{Y_i=r\}}$, and then set the cut-off in (2) as

$$\hat{\alpha} \in \operatorname{argmin}_{\alpha' \in [0,1]} [\hat{\pi}_1 \hat{G}_{n,1}(\alpha') + \hat{\pi}_2 \{1 - \hat{G}_{n,2}(\alpha')\}]. \quad (13)$$

Since empirical distribution functions are piecewise constant, the objective function in (13) does not have a unique minimum, so we choose $\hat{\alpha}$ to be the average of the smallest and largest minimisers. An attractive feature of the method is that $\{\hat{\nu}_n(X_i) : i = 1, \dots, n\}$ (or $\{\tilde{\nu}_n(X_i) : i = 1, \dots, n\}$ in the case of QDA and knn) are already calculated in order to choose the projections, so calculating $\hat{G}_{n,1}$ and $\hat{G}_{n,2}$ carries negligible extra computational cost.

Figures 2 and 3 illustrate $\hat{\pi}_1 \hat{G}_{n,1}(\alpha') + \hat{\pi}_2 \{1 - \hat{G}_{n,2}(\alpha')\}$ as an estimator of $\pi_1 G_{n,1}(\alpha') + \pi_2 \{1 - G_{n,2}(\alpha')\}$, for different base classifiers as well as different values of n and π_1 . Here, a very good approximation to the estimand was obtained using an independent data set of size 5000. Unsurprisingly, the performance of the estimator improves as n increases, but the most notable feature of these plots is the fact that for all classifiers and even for small sample sizes, $\hat{\alpha}$ is an excellent estimator of α^* , and may be a substantial improvement on the naive choice $\hat{\alpha} = 1/2$ (which may result in a classifier that assigns every point to a single class).

5.2 Choice of d

We want to choose d as small as possible in order to obtain the best possible performance bounds as described in Section 4 above. This also reduces the computational cost. However, the performance bounds rely on assumption **(A.3)**, whose strength decreases as d increases, so we want to choose d large enough that this condition holds (at least approximately).

In Section 6 we see that the random ensemble projection method is quite robust to the choice of d . Nevertheless, in some circumstances it may be desirable to have an automatic choice. As one way to do this, suppose that we wish to choose d from a set $\mathcal{D} \subseteq \{1, \dots, p\}$. For each $d \in \mathcal{D}$, generate independent and identically distributed projections $\{A_{d,b_1,b_2} : b_1 = 1, \dots, B_1, b_2 = 1, \dots, B_2\}$ according to Haar measure on $\mathcal{A}_{d \times p}$. For each $d \in \mathcal{D}$ and for $b = 1, \dots, B_1$, we can then set

$$A_{d,b_1} := A_{d,b_1,b_2^*(b_1)},$$

where $b_2^*(b_1) := \operatorname{sargmin}_{b_2 \in \{1, \dots, B_2\}} \hat{L}_n^{A_{d,b_1,b_2}}$. Finally, we can select

$$\hat{d} := \operatorname{sargmin}_{d \in \mathcal{D}} \frac{1}{B_1} \sum_{b_1=1}^{B_1} \hat{L}_n^{A_{d,b_1}}.$$

In Figures 4 and 5 we present the empirical distribution functions of $\{\hat{L}_n^{A_{d,b_1}}\}_{b_1=1}^{B_1}$, where $d \in \{2, 3, 4, 5\}$, for one training dataset from Model 1 (described in Section 6.1.1), and Model 3 (described in Section 6.1.3). In each case, we set $\pi_1 = 1/2$, $n = 100$, $p = 50$ and $B_1 = B_2 = 100$.

Figures 4 and 5 do not suggest great differences in performance for different choices of d , especially for the QDA and knn base classifiers. For the LDA classifier, it appears,

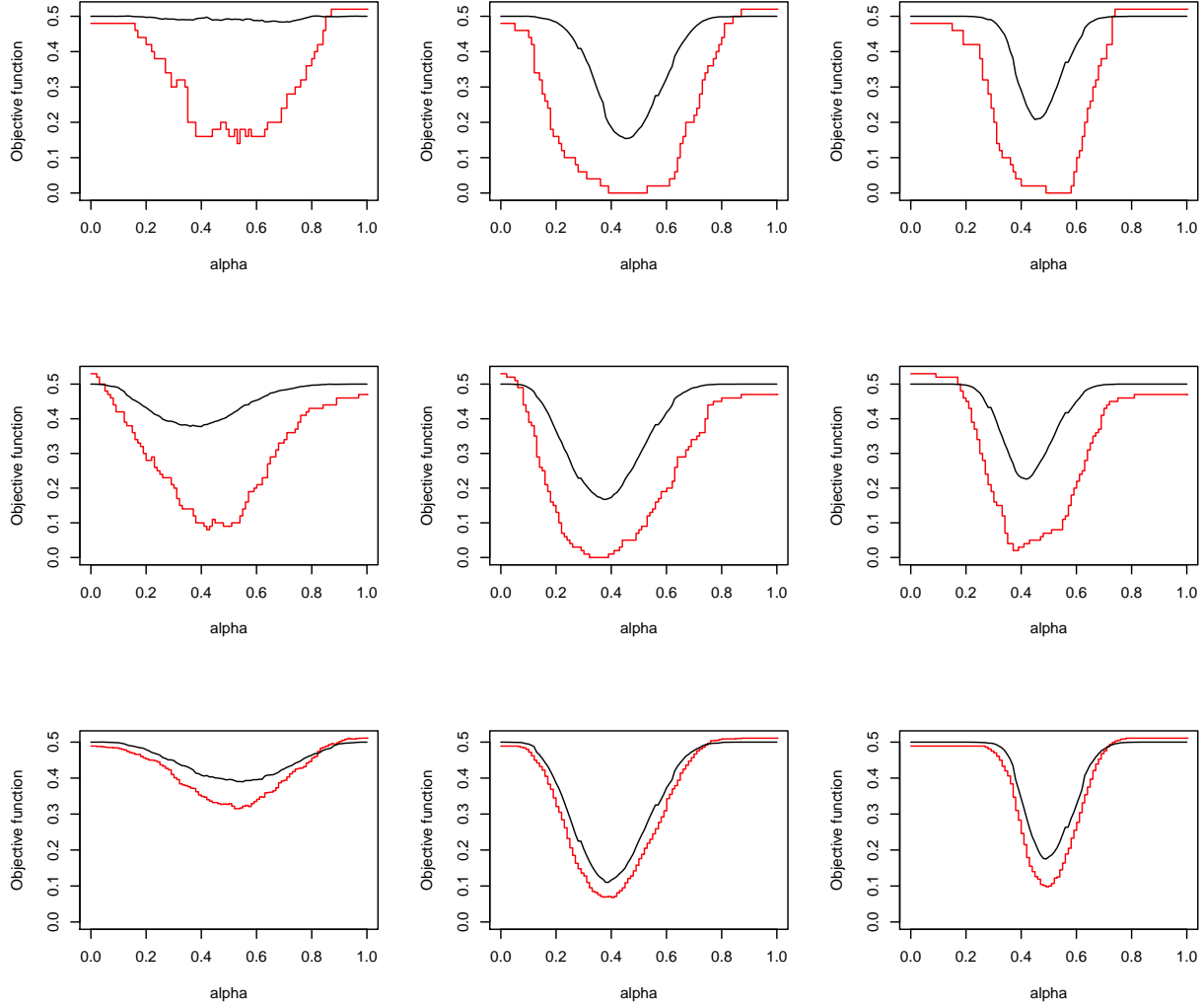


Figure 2: $\pi_1 G_{n,1}(\alpha') + \pi_2 \{1 - G_{n,2}(\alpha')\}$ in (12) (black) and $\hat{\pi}_1 \hat{G}_{n,1}(\alpha') + \hat{\pi}_2 \{1 - \hat{G}_{n,2}(\alpha')\}$ (red) for the LDA (left), QDA (middle) and knn (right) base classifiers after projecting for one training data set of size $n = 50$ (top), 100 (middle) and 1000 (bottom) from Model 1. Here, $\pi_1 = 0.5$, $p = 50$ and $d = 2$.

particularly for Model 2, that projecting into a slightly larger dimensional space is preferable, and indeed this appears to be the case from the relevant entry of Table 2 below.

The ideas presented here may also be used to decide between two different base classifiers. For example, comparing the green lines across different panels of Figure 4, we see that for Model 1 and $d = 5$, we might expect the best results with the QDA base classifier, and indeed this is confirmed by the simulation results in Table 1 below.

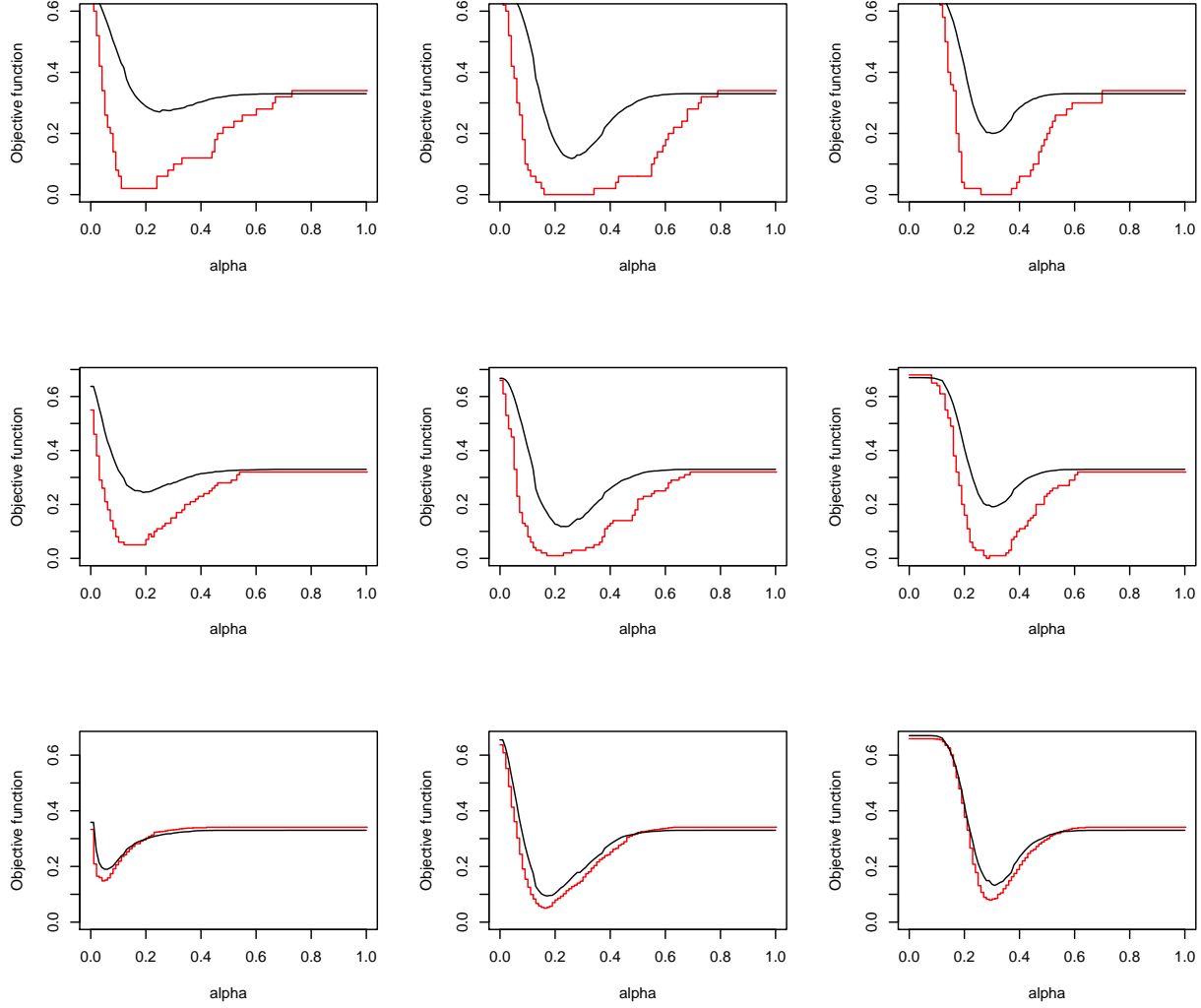


Figure 3: $\pi_1 G_{n,1}(\alpha') + \pi_2 \{1 - G_{n,2}(\alpha')\}$ in (12) (black) and $\hat{\pi}_1 \hat{G}_{n,1}(\alpha') + \hat{\pi}_2 \{1 - \hat{G}_{n,2}(\alpha')\}$ (red) for the LDA (left), QDA (middle) and k nn (right) base classifiers after projecting for one training data set of size $n = 50$ (top), 100 (middle) and 1000 (bottom) from Model 1. Here, $\pi_1 = 0.33$, $p = 50$ and $d = 2$.

5.3 Choice of B_1 and B_2

In order to minimise the third term in the bound in Theorem 5, we should choose B_1 to be as large as possible. The constraint, of course, is that the computational cost of the random projection classifier scales linearly with B_1 . The choice of B_2 is more subtle; while the fourth term in the bound in Theorem 5 decreases as B_2 increases, we saw in Section 4 that upper bounds on $\mathbb{E}(|\epsilon_n|)$ may increase with B_2 . In principle, we could try to use the expressions given in Theorem 5 and Section 4 to choose B_2 to minimise the overall upper bound on $\mathcal{L}(\hat{C}_n^{\text{RP}}) - \mathcal{R}(C^{\text{Bayes}})$. Although β_0, β and ρ are unknown to the practitioner,

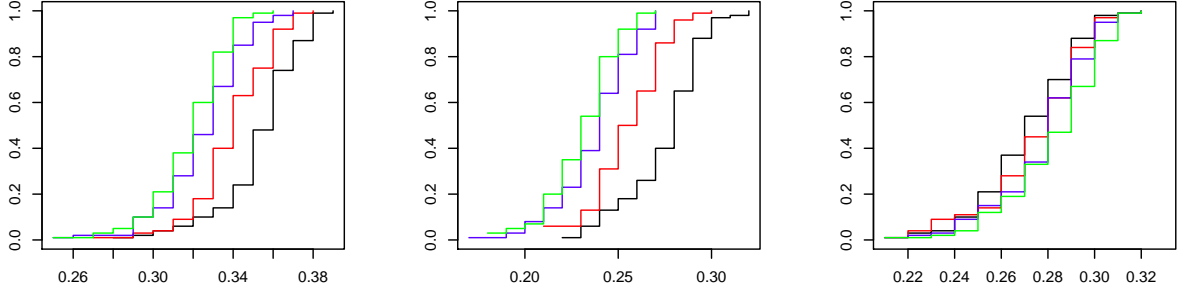


Figure 4: Empirical distribution functions of the test error estimates $\{\hat{L}_n^{A_{d,b_1}}\}_{b_1=1}^{B_1}$ for the LDA (left), QDA (middle) and knn (right) base classifiers after projecting for Model 1, $\pi_1 = 1/2$, $n = 100$, $p = 50$, and $d = 2$ (black), 3 (red), 4 (blue) and 5 (green).

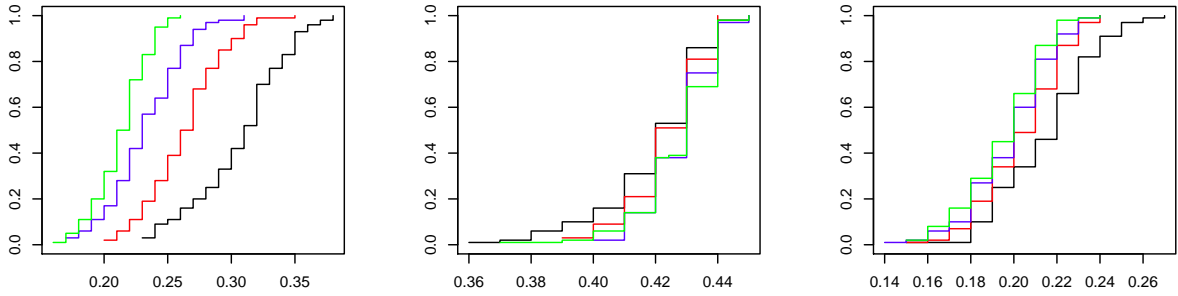


Figure 5: Empirical distribution functions of the test error estimates $\{\hat{L}_n^{A_{d,b_1}}\}_{b_1=1}^{B_1}$ for the LDA (left), QDA (middle) and knn (right) base classifiers after projecting for Model 2, $\pi_1 = 1/2$, $n = 100$, $p = 50$, and $d = 2$ (black), 3 (red), 4 (blue) and 5 (green).

these could be estimated based on the empirical distribution of $\{\hat{L}_n^{A_b}\}_{b=1}^B$, where $\{A_b\}_{b=1}^B$ are independent projections drawn according to Haar measure. In practice, however, we found that an involved approach such as this was unnecessary, and that the ensemble method was robust to the choice of B_1 and B_2 . In all of our simulations, we set $B_1 = B_2 = 100$.

6 Empirical analysis

In this section, we assess the empirical performance of the random projection ensemble classifier. Throughout this section, $RP-knn_d$, $RP-LDA_d$ and $RP-QDA_d$ denote the random projection classifier with LDA, QDA and knn base classifiers, respectively; the subscript d refers to the dimension of the image space of the projections. For comparison, we also present

results for several state-of-the-art methods for high-dimensional classification, namely Penalized LDA (Witten and Tibshirani, 2011), Nearest Shrunk Centroids (Tibshirani et al., 2003), Shrunk Centroids Regularized Discriminant Analysis (Guo, Hastie and Tibshirani, 2007), and Independence Rules (IR) (Bickel and Levina, 2004), as well as for the base classifier applied in the original space. For the standard knn classifier, we chose k via leave-one-out cross validation. The tuning parameters for the other methods were chosen using the default settings in the corresponding R packages `PenLDA` (Witten, 2011), `NSC` (Hastie et al., 2015) and `SCRDA` (Islam and Mcleod, 2015), namely 6-fold, 10-fold and 10-fold cross validation, respectively.

In each of the simulated examples below, we used $n \in \{50, 100, 200\}$, $p = 50$, two different values of the prior probability π_1 and $B_1 = B_2 = 100$. Each experiment was repeated 100 times and the risk was estimated on an independent test set of size 1000.

6.1 Simulated examples

6.1.1 Independent features

Model 1: Here, P_1 is the distribution of p independent components, each with a standard Laplace distribution, while $P_2 = N_p(\mu, I_{p \times p})$, with $\mu = \frac{1}{8}(1, \dots, 1)^T$.

Table 1: Misclassification rates multiplied by 100 (with standard errors as subscripts) for Model 1, with smallest in bold. *N/A: not available due to singular covariance estimates.

n	$\pi_1 = 0.5$, Bayes risk = 4.91			$\pi_1 = 0.33$, Bayes risk = 4.09		
	50	100	200	50	100	200
RP-LDA ₂	43.99 _{0.64}	41.99 _{0.58}	41.14 _{0.55}	29.83 _{0.51}	26.70 _{0.33}	23.76 _{0.25}
RP-LDA ₅	43.95 _{0.44}	42.10 _{0.39}	41.15 _{0.37}	33.11 _{0.52}	30.03 _{0.36}	26.55 _{0.24}
LDA	N/A	44.17 _{0.27}	41.88 _{0.20}	N/A	37.64 _{0.31}	33.75 _{0.20}
RP-QDA ₂	19.74 _{0.43}	15.65 _{0.24}	13.60 _{0.15}	17.64 _{0.50}	13.37 _{0.27}	11.88 _{0.21}
RP-QDA ₅	19.05 _{0.42}	14.05 _{0.24}	11.75 _{0.14}	18.06 _{0.46}	12.86 _{0.31}	10.64 _{0.16}
QDA	N/A	N/A	39.93 _{0.29}	N/A	N/A	33.05 _{0.16}
RP- knn_2	26.19 _{0.37}	20.96 _{0.23}	18.56 _{0.14}	23.58 _{0.34}	20.02 _{0.22}	16.58 _{0.21}
RP- knn_5	27.41 _{0.35}	21.30 _{0.24}	17.48 _{0.15}	24.64 _{0.31}	19.33 _{0.28}	16.15 _{0.20}
knn	48.81 _{0.18}	48.68 _{0.19}	48.29 _{0.18}	32.69 _{0.17}	32.50 _{0.16}	32.64 _{0.15}
PenLDA	44.40 _{0.27}	42.60 _{0.25}	41.05 _{0.20}	33.19 _{0.33}	32.61 _{0.25}	31.31 _{0.17}
NSC	46.51 _{0.33}	44.60 _{0.39}	43.03 _{0.39}	31.76 _{0.21}	31.13 _{0.17}	31.65 _{0.18}
SCRDA	46.76 _{0.31}	44.55 _{0.38}	42.55 _{0.37}	33.56 _{0.37}	32.52 _{0.23}	31.94 _{0.18}
IR	43.87 _{0.24}	42.25 _{0.25}	40.55 _{0.18}	35.04 _{0.34}	36.26 _{0.28}	36.48 _{0.23}

In Model 1, the class boundaries are non-linear and, in fact, assumption (A.3) is not satisfied for any $d < p$. Nevertheless, in Table 1 we see that the random projection versions outperform their respective vanilla counterparts (when these are tractable) as well as all of the other methods implemented. The RP-QDA classifiers perform especially well; in particular, they are able to cope better with the non-linearity of the class boundaries than

the RP-LDA classifiers. There is little difference between the performance of the $d = 2$ and $d = 5$ versions of the random projection classifiers.

6.1.2 t -distributed features

Model 2: Here, $X|\{Y = r\} = \mu_r + \frac{Z}{\sqrt{U/\nu_r}}$, where $Z \sim N_p(0, \Sigma_r)$ independent of $U \sim \chi_{\nu_r}^2$. Thus, P_r is the multivariate t -distribution centred at μ_r , with ν_r degrees of freedom and shape parameter Σ_r . We set $\mu_1 = 0$, $\mu_2 = 2(1, \dots, 1, 0, \dots, 0)^T$, where μ_2 has 5 non-zero components, $\nu_1 = 1$, $\nu_2 = 2$, $\Sigma_1 = I_{p \times p}$ and $\Sigma_2 = (\Sigma_{j,k})$, where $\Sigma_{j,j} = 1$, $\Sigma_{j,k} = 0.5$ if $\max(j, k) \leq 5$ and $j \neq k$, $\Sigma_{j,k} = 0$ otherwise.

Table 2: Misclassification rates for Model 2.

n	$\pi_1 = 0.5$, Bayes risk = 10.07			$\pi_1 = 0.75$, Bayes risk = 6.67		
	50	100	200	50	100	200
RP-LDA ₂	23.86 _{0.91}	21.74 _{0.81}	22.90 _{1.06}	29.43 _{0.48}	26.69 _{0.34}	25.48 _{0.19}
RP-LDA ₅	21.14 _{0.52}	18.23 _{0.35}	17.50 _{0.43}	31.29 _{0.45}	30.26 _{0.43}	27.26 _{0.31}
LDA	N/A	26.49 _{0.28}	21.74 _{0.22}	N/A	23.35 _{0.26}	20.15 _{0.23}
RP-QDA ₂	32.97 _{0.87}	33.41 _{1.03}	38.89 _{0.71}	18.97 _{0.79}	22.95 _{1.08}	27.34 _{1.07}
RP-QDA ₅	34.91 _{0.69}	37.35 _{0.75}	40.90 _{0.40}	20.25 _{0.74}	28.78 _{1.09}	40.56 _{0.96}
QDA	N/A	N/A	39.95 _{0.23}	N/A	N/A	N/A
RP-knn ₂	17.16 _{0.31}	15.02 _{0.20}	13.31 _{0.14}	13.24 _{0.37}	11.08 _{0.23}	9.51 _{0.13}
RP-knn ₅	16.86 _{0.27}	15.37 _{0.22}	13.56 _{0.13}	12.17 _{0.31}	9.99 _{0.17}	9.02 _{0.12}
knn	20.26 _{0.37}	18.33 _{0.18}	16.79 _{0.14}	15.64 _{0.33}	13.71 _{0.25}	12.18 _{0.13}
PenLDA	38.69 _{1.08}	35.07 _{1.24}	35.19 _{1.23}	28.96 _{1.19}	25.69 _{0.33}	26.11 _{0.19}
NSC	40.03 _{1.19}	38.48 _{1.24}	39.29 _{1.19}	26.18 _{0.21}	25.25 _{0.14}	25.32 _{0.13}
SCRDA	21.95 _{0.59}	18.35 _{0.28}	16.98 _{0.20}	20.34 _{0.44}	18.35 _{0.32}	16.98 _{0.22}
IR	39.24 _{1.04}	37.02 _{1.15}	38.06 _{1.08}	48.45 _{2.08}	49.10 _{2.02}	50.93 _{1.98}

Model 2 explores the effect of heavy tails and the presence of correlation between the features. Again, assumption **(A.3)** is not satisfied for any $d < p$. We see in Table 2 that the RP-knn classifiers are the most successful among all of the methods in the comparison, and there is little difference between the $d = 2$ and $d = 5$ versions. The RP-QDA classifiers perform poorly here because the heavy-tailed distributions mean that the mean and covariance matrix estimates are poor.

6.1.3 Multi-modal features

Model 3: Here, $X|\{Y = 1\} \sim \frac{1}{2}N_p(\mu_1, \Sigma_1) + \frac{1}{2}N_p(-\mu_1, \Sigma_1)$ and $X|\{Y = 2\}$ has p independent components, the first five of which are standard Cauchy and the remaining $p - 5$ of which are standard normal. We set $\mu_1 = (1, \dots, 1, 0, \dots, 0)^T$, where μ_1 has 5 non-zero components, and $\Sigma_1 = I_{p \times p}$.

Model 3 is chosen to investigate a setting in which one class is multi-modal. Note that assumption **(A.3)** holds with $d = 5$; indeed, for example, the first five rows of A^* may be

Table 3: Misclassification rates for Model 3.

n	$\pi_1 = 0.5$, Bayes risk = 11.59			$\pi_1 = 0.75$, Bayes risk = 13.12		
	50	100	200	50	100	200
RP-LDA ₂	43.95 _{0.63}	42.57 _{0.61}	41.60 _{0.70}	23.41 _{0.57}	22.24 _{0.34}	22.31 _{0.22}
RP-LDA ₅	45.29 _{0.49}	44.57 _{0.43}	43.78 _{0.49}	27.15 _{0.68}	23.67 _{0.43}	22.48 _{0.24}
LDA	N/A	49.39 _{0.19}	49.37 _{0.16}	N/A	35.33 _{0.43}	30.57 _{0.29}
RP-QDA ₂	29.78 _{0.57}	26.34 _{0.42}	23.47 _{0.29}	19.36 _{0.70}	17.19 _{0.44}	15.36 _{0.30}
RP-QDA ₅	29.93 _{0.64}	24.83 _{0.39}	22.20 _{0.26}	20.04 _{0.86}	16.50 _{0.45}	14.23 _{0.28}
QDA	N/A	N/A	27.58 _{0.34}	N/A	N/A	N/A
RP-knn ₂	29.43 _{0.39}	26.48 _{0.61}	23.57 _{0.23}	19.31 _{0.37}	16.64 _{0.26}	14.61 _{0.20}
RP-knn ₅	29.36 _{0.38}	26.29 _{0.29}	23.38 _{0.19}	19.93 _{0.40}	18.82 _{0.36}	14.74 _{0.18}
knn	34.46 _{0.32}	31.26 _{0.23}	28.88 _{0.21}	22.46 _{0.20}	19.85 _{0.19}	17.70 _{0.16}
PenLDA	48.34 _{0.22}	48.97 _{0.37}	49.21 _{0.17}	27.54 _{0.50}	27.05 _{0.42}	26.26 _{0.29}
NSC	47.67 _{0.42}	47.69 _{0.37}	47.83 _{0.32}	23.03 _{0.16}	23.12 _{0.15}	23.63 _{0.14}
SCRDA	45.44 _{0.52}	44.86 _{0.55}	43.27 _{0.51}	23.65 _{0.42}	21.89 _{0.27}	21.63 _{0.19}
IR	48.40 _{0.21}	48.91 _{0.17}	49.29 _{0.16}	32.26 _{0.59}	35.35 _{0.53}	38.54 _{0.37}

taken to be the first five standard Euclidean basis vectors. In Table 3, we see that the random projection ensembles for each of the three base classifiers outperform their standard counterparts. The RP-knn and RP-QDA classifiers are particularly effective here. Even though the Bayes decision boundary here is sparse – it only depends on the first 5 features – the PenLDA, NSC, SCRDA and IR classifiers perform poorly because the Bayes decision boundary is non-linear.

6.1.4 Rotated Sparse Normal

Model 4: Here, $X|\{Y = 1\} \sim N_p(R\mu_1, R\Sigma_1R^T)$ and $X|\{Y = 2\} \sim N_p(R\mu_2, R\Sigma_2R^T)$ where R is a $p \times p$ rotation matrix that was sampled once according to Haar measure, and remained fixed thereafter, and we set $\mu_1 = (0, \dots, 0)^T$, $\mu_2 = (1, 1, 1, 0, \dots, 0)^T$. Moreover, Σ_1 and Σ_2 are block diagonal, with blocks $\Sigma_{r,1}$, and $\Sigma_{r,2}$, for $r = 1, 2$, where $\Sigma_{1,1}$ is a 3×3 matrix with diagonal entries equal to 1 and off-diagonal entries equal to $1/2$, and $\Sigma_{2,1} = \Sigma_{1,1} + I_{3 \times 3}$. In both cases $\Sigma_{r,2}$ is a $(p-3) \times (p-3)$ matrix, with diagonal entries equal to 1 and off-diagonal entries equal to $1/2$.

The setting in Model 4 is chosen so that assumption **(A.3)** holds with $d = 3$; in fact, A^* can be taken to be the first three rows of R^T . Note that, if $R = I_{p \times p}$, then the model would be sparse and we would expect the PenLDA, NSC, and SCRDA methods to perform very well. However, for a generic R , the model is not sparse and the random projection ensemble methods, which are invariant to the choice of coordinate system, typically perform as well or better (especially RP-LDA). Classification is difficult in this setting, and the risks of all of the classifiers are considerably greater than the Bayes risk.

Table 4: Misclassification rates for Model 4.

n	$\pi_1 = 0.5$, Bayes risk = 9.84			$\pi_1 = 0.75$, Bayes risk = 6.22		
	50	100	200	50	100	200
RP-LDA ₂	32.65 _{0.37}	30.34 _{0.22}	28.33 _{0.17}	26.83 _{0.45}	24.43 _{0.26}	22.94 _{0.18}
RP-LDA ₅	32.63 _{0.32}	30.17 _{0.19}	28.24 _{0.16}	26.51 _{0.37}	23.86 _{0.21}	22.60 _{0.17}
LDA	N/A	38.47 _{0.25}	32.86 _{0.17}	N/A	32.96 _{0.33}	26.95 _{0.21}
RP-QDA ₂	33.35 _{0.37}	30.37 _{0.21}	28.41 _{0.18}	28.33 _{0.50}	25.54 _{0.33}	23.76 _{0.23}
RP-QDA ₅	33.87 _{0.31}	31.02 _{0.21}	28.60 _{0.16}	28.70 _{0.46}	25.39 _{0.32}	23.56 _{0.19}
QDA	N/A	N/A	44.25 _{0.21}	N/A	N/A	N/A
RP-knn ₂	34.36 _{0.39}	31.12 _{0.23}	28.99 _{0.17}	27.35 _{0.34}	25.25 _{0.28}	23.69 _{0.20}
RP-knn ₅	34.33 _{0.38}	31.44 _{0.23}	29.07 _{0.17}	26.59 _{0.35}	24.71 _{0.25}	23.11 _{0.17}
knn	39.53 _{0.32}	38.16 _{0.23}	37.06 _{0.24}	28.70 _{0.39}	27.68 _{0.25}	27.36 _{0.21}
PenLDA	34.73 _{0.50}	31.87 _{0.37}	29.27 _{0.21}	30.06 _{0.55}	27.74 _{0.45}	26.13 _{0.34}
NSC	37.80 _{0.58}	34.28 _{0.44}	30.76 _{0.27}	26.05 _{0.37}	24.83 _{0.43}	23.81 _{0.21}
SCRDA	35.74 _{0.56}	31.44 _{0.37}	28.18 _{0.17}	25.80 _{0.33}	24.43 _{0.20}	23.20 _{0.19}
IR	34.95 _{0.52}	32.06 _{0.39}	29.39 _{0.23}	32.95 _{0.57}	31.03 _{0.46}	29.78 _{0.36}

6.2 Real data example

The Ozone dataset, available from the UCI machine learning repository, classifies days as either ‘normal days’ (class 1) or ‘ozone days’ (class 2), and also records $p = 72$ potentially relevant features. After removing the missing values, there are 1719 observations in class 1 and 128 observations in class 2. To avoid the situation where classifiers assign every point to class 1, we subsampled 384 of the class 1 observations to yield a prevalence ratio of 3:1. We then sampled a training set of size $n \in \{50, 100, 200\}$ and used the remaining $512 - n$ pairs as the test set. The means and standard errors of the risk, estimated over 100 repetitions of the training set sampling, are presented in Table 5. In each case we took $B_1 = B_2 = 100$.

Table 5: Misclassification rates for the Ozone data set.

n	50	100	200
RP-LDA ₂	24.34 _{0.30}	22.20 _{0.25}	20.71 _{0.18}
LDA	N/A	34.47 _{0.38}	24.46 _{0.25}
RP-QDA ₂	26.12 _{0.34}	23.03 _{0.23}	21.56 _{0.23}
QDA	N/A	N/A	N/A
RP-knn ₂	26.82 _{0.39}	23.75 _{0.26}	22.14 _{0.19}
knn	26.94 _{0.28}	25.93 _{0.19}	24.52 _{0.18}
PenLDA	29.98 _{0.35}	29.71 _{0.31}	29.66 _{0.27}
NSC	26.55 _{0.38}	25.51 _{0.32}	24.30 _{0.21}
SCRDA	25.60 _{0.32}	23.39 _{0.26}	19.66 _{0.25}
IR	30.70 _{0.31}	30.63 _{0.26}	30.67 _{0.22}

In this example, all of the random projection ensemble classifiers performed very well,

though SCRDA was also competitive, particularly for the largest sample size.

7 Discussion and extensions

We have introduced a general framework for high-dimensional classification via the combination of the results of applying a base classifier on carefully selected low-dimensional random projections of the data. One of its attractive features is its generality: the approach can be used in conjunction with any base classifier. Moreover, although we explored in detail one method for combining the random projections (partly because it facilitates rigorous statistical analysis), there are many other options available here. For instance, instead of only retaining the projection within each block yielding the smallest estimate of test error, one might give weights to the different projections, where the weights decrease as the estimate of test error increases. Another interesting avenue to explore would be alternative methods for estimating the test error, such as sample splitting. The idea here would be to split the sample \mathcal{T}_n into $\mathcal{T}_{n,1}$ and $\mathcal{T}_{n,2}$, say, where $|\mathcal{T}_{n,1}| = n^{(1)}$ and $|\mathcal{T}_{n,2}| = n^{(2)}$. We then use

$$\hat{L}_{n^{(1)},n^{(2)}}^A := \frac{1}{n^{(2)}} \sum_{(X_i, Y_i) \in \mathcal{T}_{n,2}} \mathbb{1}_{\{\hat{C}_{n^{(1)},\mathcal{T}_{n,1}}^A(X_i) \neq Y_i\}}$$

to estimate the test error $\mathcal{L}_{n^{(1)},1}^A$ based on the training data $\mathcal{T}_{n,1}$. Since $\mathcal{T}_{n,1}$ and $\mathcal{T}_{n,2}$ are independent, we can apply Hoeffding's inequality to deduce that

$$\sup_{A \in \mathcal{A}} \mathbb{P}\{|\mathcal{L}_{n^{(1)},1}^A - \hat{L}_{n^{(1)},n^{(2)}}^A| \geq \epsilon \mid \mathcal{T}_{n,1}\} \leq 2e^{-2n^{(2)}\epsilon^2}.$$

It then follows by very similar arguments to those given in Section 4.1 that

$$\begin{aligned} \mathbb{E}(|\mathcal{L}_{n^{(1)}}^{A^*} - \hat{L}_{n^{(1)},n^{(2)}}^{A^*}| \mid \mathcal{T}_{n,1}) &\leq \left(\frac{1 + \log 2}{2n^{(2)}}\right)^{1/2}, \\ \mathbb{E}(|\mathcal{L}_{n^{(1)}}^{A_1} - \hat{L}_{n^{(1)},n^{(2)}}^{A_1}| \mid \mathcal{T}_{n,1}) &\leq \left(\frac{1 + \log 2 + \log B_2}{2n^{(2)}}\right)^{1/2}. \end{aligned} \quad (14)$$

The advantages of this approach are twofold: first, the bounds hold for any choice of base classifier (and still without any assumptions on the data generating mechanism); second, the bounds on the terms in (14) merely rely on Hoeffding's inequality as opposed to Vapnik–Chervonenkis theory, so are typically sharper. The disadvantage is that the other terms in the bound in Theorem 5 will tend to be larger due to the reduced effective sample size. The choice of $n^{(1)}$ and $n^{(2)}$ in such an approach therefore becomes an interesting question.

Many practical classification problems involve $K > 2$ classes. The main issue in extending our methodology to such settings is the definition of \hat{C}_n^{RP} analogous to (2). To outline one approach, let

$$\hat{\nu}_{n,r}^{B_1}(x) := \frac{1}{B_1} \sum_{b_1=1}^{B_1} \mathbb{1}_{\{\hat{C}_n^{A_{b_1}}(x)=r\}}$$

for $r = 1, \dots, K$. Given $\alpha_1, \dots, \alpha_K > 0$ with $\sum_{r=1}^K \alpha_r = 1$, we can then define

$$\hat{C}_n^{\text{RP}}(x) := \operatorname{sargmax}_{r=1, \dots, K} \{\alpha_r \hat{\nu}_{n,r}^{B_1}(x)\}.$$

The choice of $\alpha_1, \dots, \alpha_K$ is analogous to the choice of α in the case $K = 2$. It is therefore natural to seek to minimise the test error of the corresponding infinite-simulation random projection ensemble classifier as before.

In other situations, it may be advantageous to consider alternative types of projection, perhaps because of additional structure in the problem. One particularly interesting issue concerns ultrahigh-dimensional settings, say p in the thousands. Here, it may be too time-consuming to generate enough random projections to explore adequately the space $\mathcal{A}_{d \times p}$. As a mathematical quantification of this, the cardinality of an ϵ -net in the Euclidean norm of the surface of the Euclidean ball in \mathbb{R}^p increases exponentially in p (e.g. Vershynin, 2012; Kim and Samworth, 2014). In such challenging problems, one might restrict the projections A to be axis-aligned, so that each row of A consists of a single non-zero component, equal to 1, and $p - 1$ zero components. There are then only $\binom{p}{d} \leq p^d/d!$ choices for the projections, and if d is small, it may be feasible even to carry out an exhaustive search. Of course, this approach loses one of the attractive features of our original proposal, namely the fact that it is equivariant to orthogonal transformations. Nevertheless, corresponding theory can be obtained provided that the projection A^* in **(A.3)** is axis-aligned. This is a much stronger requirement, but it seems that imposing greater structure is inevitable to obtain good classification in such settings.

8 Appendix

Proof of Theorem 1. Recall that the training data $\mathcal{T}_n = \{(x_1, y_1), \dots, (x_n, y_n)\}$ are fixed and the projections A_1, A_2, \dots , are independent and identically distributed in \mathcal{A} , independent of the pair (X, Y) . The test error of the random projection ensemble classifier has the following representation:

$$\begin{aligned} \mathcal{L}(\hat{C}_n^{\text{RP}}) &= \mathbb{P}\{\hat{C}_n^{\text{RP}}(X) \neq Y\} = \pi_1 \mathbb{P}\{\hat{C}_n^{\text{RP}}(X) = 2 | Y = 1\} + \pi_2 \mathbb{P}\{\hat{C}_n^{\text{RP}}(X) = 1 | Y = 2\} \\ &= \pi_1 \mathbb{P}\{\hat{\nu}_n^{B_1}(X) < \alpha | Y = 1\} + \pi_2 \mathbb{P}\{\hat{\nu}_n^{B_1}(X) \geq \alpha | Y = 2\}, \end{aligned}$$

where $\hat{\nu}_n^{B_1}(x)$ is defined in (1).

Let $U_{b_1} := \mathbb{1}_{\{\hat{C}_n^{A_{b_1}}(X)=1\}}$, for $b_1 = 1, \dots, B_1$. Then, conditional on $\hat{\mu}_n(X) = \theta \in [0, 1]$, the random variables U_1, \dots, U_{B_1} are independent, each having a Bernoulli(θ) distribution. Recall that $G_{n,1}$ and $G_{n,2}$ are the distribution functions of $\hat{\mu}_n(X) | \{Y = 1\}$ and $\hat{\mu}_n(X) | \{Y = 2\}$, respectively. We can therefore write

$$\begin{aligned} \mathbb{P}\{\hat{\nu}_n^{B_1}(X) < \alpha | Y = 1\} &= \int_0^1 \mathbb{P}\left\{\frac{1}{B_1} \sum_{b_1=1}^{B_1} U_{b_1} < \alpha \middle| \hat{\mu}_n(X) = \theta\right\} dG_{n,1}(\theta) \\ &= \int_0^1 \mathbb{P}(T < B_1 \alpha) dG_{n,1}(\theta), \end{aligned}$$

where here and throughout the proof, T denotes a $\text{Bin}(B_1, \theta)$ random variable. Similarly,

$$\mathbb{P}\{\hat{\nu}_n^{B_1}(X) \geq \alpha | Y = 2\} = 1 - \int_0^1 \mathbb{P}(T < B_1 \alpha) dG_{n,2}(\theta).$$

It follows that

$$\mathcal{L}(\hat{C}_n^{\text{RP}}) = \pi_2 + \int_0^1 \mathbb{P}(T < B_1 \alpha) dG_n^\circ(\theta),$$

where $G_n^\circ := \pi_1 G_{n,1} - \pi_2 G_{n,2}$. Writing $g_n^\circ := \pi_1 g_{n,1} - \pi_2 g_{n,2}$, we now show that

$$\int_0^1 \{\mathbb{P}(T < B_1 \alpha) - \mathbb{1}_{\{\theta < \alpha\}}\} dG_n^\circ(\theta) = \frac{1 - \alpha - \llbracket B_1 \alpha \rrbracket}{B_1} g_n^\circ(\alpha) + \frac{\alpha(1 - \alpha)}{2B_1} \dot{g}_n^\circ(\alpha) + o\left(\frac{1}{B_1}\right) \quad (15)$$

as $B_1 \rightarrow \infty$. Our proof involves a one-term Edgeworth expansion to the binomial distribution function in (15), where the error term is controlled uniformly in the parameter. The expansion relies on the following version of Esseen's smoothing lemma.

Theorem 6. (*Esseen, 1945, Chapter 2, Theorem 2b*) Let $c_1, C_1, S > 0$, let $F : \mathbb{R} \rightarrow [0, \infty)$ be a non-decreasing function and let $G : \mathbb{R} \rightarrow \mathbb{R}$ be a function of bounded variation. Let $F^*(s) := \int_{-\infty}^{\infty} \exp(ist) dF(t)$ and $G^*(s) := \int_{-\infty}^{\infty} \exp(ist) dG(t)$ be the Fourier-Stieltjes transforms of F and G , respectively. Suppose that

- $\lim_{t \rightarrow -\infty} F(t) = \lim_{t \rightarrow -\infty} G(t) = 0$ and $\lim_{t \rightarrow \infty} F(t) = \lim_{t \rightarrow \infty} G(t)$;
- $\int_{-\infty}^{\infty} |F(t) - G(t)| dt < \infty$;
- The set of discontinuities of F and G is contained in $\{t_i : i \in \mathbb{Z}\}$, where (t_i) is a strictly increasing sequence with $\inf_i \{t_{i+1} - t_i\} \geq c_1$; moreover F is constant on the intervals $[t_i, t_{i+1})$ for all $i \in \mathbb{Z}$;
- $|\dot{G}(t)| \leq C_1$ for all $t \notin \{t_i : i \in \mathbb{Z}\}$.

Then there exist constants $c_2, C_2 > 0$ such that

$$\sup_{t \in \mathbb{R}} |F(t) - G(t)| \leq \frac{1}{\pi} \int_{-S}^S \left| \frac{F^*(s) - G^*(s)}{s} \right| ds + \frac{C_1 C_2}{S},$$

provided that $Sc_1 \geq c_2$.

Let $\sigma^2 := \theta(1 - \theta)$, and let Φ and ϕ denote the standard normal distribution and density functions, respectively. Moreover, for $t \in \mathbb{R}$, let

$$p(t) = p(t, \theta) := \frac{(1 - t^2)(1 - 2\theta)}{6\sigma},$$

and

$$q(t) = q(t, B_1, \theta) := \frac{1/2 - \llbracket B_1 \theta + B_1^{1/2} \sigma t \rrbracket}{\sigma}.$$

In Proposition 7 below we apply Theorem 6 to the following functions:

$$F_{B_1}(t) = F_{B_1}(t, \theta) := \mathbb{P}\left(\frac{T - B_1\theta}{B_1^{1/2}\sigma} < t\right), \quad (16)$$

and

$$G_{B_1}(t) = G_{B_1}(t, \theta) := \Phi(t) + \phi(t) \frac{p(t, \theta) + q(t, B_1, \theta)}{B_1^{1/2}}. \quad (17)$$

Proposition 7. *Let F_{B_1} and G_{B_1} be as in (16) and (17). There exists a constant $C > 0$ such that, for all $B_1 \in \mathbb{N}$,*

$$\sup_{\theta \in (0,1)} \sup_{t \in \mathbb{R}} \sigma^3 |F_{B_1}(t, \theta) - G_{B_1}(t, \theta)| \leq \frac{C}{B_1}.$$

Proposition 7, whose proof is given after the proof of Theorem 5, bounds uniformly in θ the error in the one-term Edgeworth expansion G_{B_1} of the distribution function F_{B_1} . Returning to the proof of Theorem 1, we will argue that the dominant contribution to the integral in (15) arises from the interval $(\max\{0, \alpha - \epsilon_1\}, \min\{\alpha + \epsilon_1, 1\})$, where $\epsilon_1 := B_1^{-1/2} \log B_1$. For the remainder of the proof we assume B_1 is large enough that $[\alpha - \epsilon_1, \alpha + \epsilon_1] \subseteq (0, 1)$.

For the region $|\theta - \alpha| \geq \epsilon_1$, by Hoeffding's inequality, we have that

$$\sup_{|\theta - \alpha| \geq \epsilon_1} |\mathbb{P}(T < B_1\alpha) - \mathbb{1}_{\{\theta < \alpha\}}| \leq \sup_{|\theta - \alpha| \geq \epsilon_1} \exp\{-2B_1(\theta - \alpha)^2\} \leq e^{-2\log^2 B_1} = O(B_1^{-M}),$$

for each $M > 0$, as $B_1 \rightarrow \infty$. It follows that

$$\int_0^1 \{\mathbb{P}(T < B_1\alpha) - \mathbb{1}_{\{\theta < \alpha\}}\} dG_n^\circ(\theta) = \int_{\alpha - \epsilon_1}^{\alpha + \epsilon_1} \{\mathbb{P}(T < B_1\alpha) - \mathbb{1}_{\{\theta < \alpha\}}\} dG_n^\circ(\theta) + O(B_1^{-M}), \quad (18)$$

for each $M > 0$, as $B_1 \rightarrow \infty$.

For the region $|\theta - \alpha| < \epsilon_1$, by Proposition 7, there exists $C' > 0$ such that, for all B_1 sufficiently large,

$$\sup_{|\theta - \alpha| < \epsilon_1} \left| \mathbb{P}(T < B_1\alpha) - \Phi\left(\frac{B_1^{1/2}(\alpha - \theta)}{\sigma}\right) - \frac{1}{B_1^{1/2}} \phi\left(\frac{B_1^{1/2}(\alpha - \theta)}{\sigma}\right) r\left(\frac{B_1^{1/2}(\alpha - \theta)}{\sigma}\right) \right| \leq \frac{C'}{B_1},$$

where $r(t) := p(t) + q(t)$. Hence, using the fact that for large B_1 , $\sup_{|\theta - \alpha| < \epsilon_1} |g_n^\circ(\theta)| \leq |g_n^\circ(\alpha)| + 1 < \infty$ under **(A.1)**, we have

$$\begin{aligned} & \int_{\alpha - \epsilon_1}^{\alpha + \epsilon_1} \{\mathbb{P}(T < B_1\alpha) - \mathbb{1}_{\{\theta < \alpha\}}\} dG_n^\circ(\theta) \\ &= \int_{\alpha - \epsilon_1}^{\alpha + \epsilon_1} \left\{ \Phi\left(\frac{B_1^{1/2}(\alpha - \theta)}{\sigma}\right) - \mathbb{1}_{\{\theta < \alpha\}} \right\} dG_n^\circ(\theta) \\ & \quad + \frac{1}{B_1^{1/2}} \int_{\alpha - \epsilon_1}^{\alpha + \epsilon_1} \phi\left(\frac{B_1^{1/2}(\alpha - \theta)}{\sigma}\right) r\left(\frac{B_1^{1/2}(\alpha - \theta)}{\sigma}\right) dG_n^\circ(\theta) + o\left(\frac{1}{B_1}\right), \quad (19) \end{aligned}$$

as $B_1 \rightarrow \infty$. To aid exposition, we will henceforth concentrate on the dominant terms in our expansions, denoting the remainder terms as R_1, R_2, \dots . These remainders are then controlled at the end of the argument. For the first term in (19), we write

$$\begin{aligned} & \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \left\{ \Phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sigma}\right) - \mathbb{1}_{\{\theta < \alpha\}} \right\} dG_n^\circ(\theta) \\ &= \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \left\{ \Phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) - \mathbb{1}_{\{\theta < \alpha\}} \right\} dG_n^\circ(\theta) \\ & \quad + \frac{(1-2\alpha)B_1^{1/2}}{2\{\alpha(1-\alpha)\}^{3/2}} \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} (\alpha-\theta)^2 \phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) dG_n^\circ(\theta) + R_1. \end{aligned} \quad (20)$$

Now, for the first term in (20),

$$\begin{aligned} & \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \left\{ \Phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) - \mathbb{1}_{\{\theta < \alpha\}} \right\} dG_n^\circ(\theta) \\ &= \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \left\{ \Phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) - \mathbb{1}_{\{\theta < \alpha\}} \right\} \{g_n^\circ(\alpha) + (\theta-\alpha)\dot{g}_n^\circ(\alpha)\} d\theta + R_2 \\ &= \frac{\sqrt{\alpha(1-\alpha)}}{B_1^{1/2}} \int_{-\infty}^{\infty} \{\Phi(-u) - \mathbb{1}_{\{u < 0\}}\} \left\{ g_n^\circ(\alpha) + \frac{\sqrt{\alpha(1-\alpha)}}{B_1^{1/2}} u \dot{g}_n^\circ(\alpha) \right\} du + R_2 + R_3 \\ &= \frac{\alpha(1-\alpha)}{2B_1} \dot{g}_n^\circ(\alpha) + R_2 + R_3. \end{aligned} \quad (21)$$

For the second term in (20), write

$$\begin{aligned} & \frac{(1-2\alpha)B_1^{1/2}}{2\{\alpha(1-\alpha)\}^{3/2}} \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} (\alpha-\theta)^2 \phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) dG_n^\circ(\theta) \\ &= \frac{(1-2\alpha)B_1^{1/2}}{2\{\alpha(1-\alpha)\}^{3/2}} g_n^\circ(\alpha) \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} (\alpha-\theta)^2 \phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) d\theta + R_4 \\ &= \frac{1/2-\alpha}{B_1} g_n^\circ(\alpha) \int_{-\infty}^{\infty} u^2 \phi(-u) du + R_4 + R_5 = \frac{1/2-\alpha}{B_1} g_n^\circ(\alpha) + R_4 + R_5. \end{aligned} \quad (22)$$

Returning to the second term in (19), observe that

$$\begin{aligned}
& \frac{1}{B_1^{1/2}} \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sigma}\right) r\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sigma}\right) dG_n^\circ(\theta) \\
&= \frac{1/2 - \llbracket B_1 \alpha \rrbracket}{B_1^{1/2}} \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \frac{1}{\sigma} \phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sigma}\right) dG_n^\circ(\theta) \\
&\quad + \frac{1}{6B_1^{1/2}} \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \frac{(1-2\theta)}{\sigma} \left\{1 - \frac{B_1(\alpha-\theta)^2}{\sigma^2}\right\} \phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sigma}\right) dG_n^\circ(\theta) \\
&= \frac{1/2 - \llbracket B_1 \alpha \rrbracket}{B_1^{1/2}} \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \frac{1}{\sigma} \phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sigma}\right) dG_n^\circ(\theta) + R_6 \\
&= \frac{1/2 - \llbracket B_1 \alpha \rrbracket}{B_1^{1/2} \sqrt{\alpha(1-\alpha)}} g_n^\circ(\alpha) \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) d\theta + R_6 + R_7 \\
&= \frac{1/2 - \llbracket B_1 \alpha \rrbracket}{B_1} g_n^\circ(\alpha) + R_6 + R_7 + R_8. \tag{23}
\end{aligned}$$

The claim (15) will now follow from (18), (19), (20), (21), (22) and (23), once we have shown that

$$\sum_{j=1}^8 |R_j| = o(B_1^{-1}) \tag{24}$$

as $B_1 \rightarrow \infty$.

To bound R_1 : For $\zeta \in (0, 1)$, let $h_\theta(\zeta) := \Phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\zeta(1-\zeta)}}\right)$. Observe that, by a Taylor expansion about $\zeta = \alpha$, there exists $B_0 \in \mathbb{N}$, such that, for all $B_1 > B_0$ and all $\theta, \zeta \in (\alpha - \epsilon_1, \alpha + \epsilon_1)$,

$$\begin{aligned}
& \left| \Phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\zeta(1-\zeta)}}\right) - \Phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) + (\zeta - \alpha) \frac{(1-2\alpha)B_1^{1/2}(\alpha-\theta)}{2\{\alpha(1-\alpha)\}^{3/2}} \phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) \right| \\
&= |h_\theta(\zeta) - h_\theta(\alpha) - (\zeta - \alpha) \dot{h}_\theta(\alpha)| \\
&\leq \frac{(\zeta - \alpha)^2}{2} \sup_{\zeta' \in [\alpha-\zeta, \alpha+\zeta]} |\ddot{h}_\theta(\zeta')| \leq (\zeta - \alpha)^2 \frac{\log^3 B_1}{2\sqrt{2\pi}\{\alpha(1-\alpha)\}^{7/2}}.
\end{aligned}$$

Using this bound with $\zeta = \theta$, we deduce that, for all B_1 sufficiently large,

$$\begin{aligned}
|R_1| &= \left| \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \left\{ \Phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sigma}\right) - \Phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) - \frac{(1-2\alpha)B_1^{1/2}(\alpha-\theta)^2}{2\{\alpha(1-\alpha)\}^{3/2}} \phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) \right\} dG_n^\circ(\theta) \right| \\
&\leq \frac{\log^3 B_1}{2\sqrt{2\pi}\{\alpha(1-\alpha)\}^{7/2}} \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} (\theta - \alpha)^2 |g_n^\circ(\theta)| d\theta \\
&\leq \frac{\log^6 B_1}{3\sqrt{2\pi}B_1^{3/2}\{\alpha(1-\alpha)\}^{7/2}} \sup_{|\theta-\alpha| \leq \epsilon_1} |g_n^\circ(\theta)| = o\left(\frac{1}{B_1}\right)
\end{aligned}$$

as $B_1 \rightarrow \infty$.

To bound R_2 : Since g_n° is differentiable at α , given $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$|g_n^\circ(\theta) - g_n^\circ(\alpha) - (\theta - \alpha)\dot{g}_n^\circ(\alpha)| < \epsilon|\theta - \alpha|,$$

for all $|\theta - \alpha| < \delta_\epsilon$. It follows that, for all B_1 sufficiently large,

$$\begin{aligned} |R_2| &= \left| \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \left\{ \Phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) - \mathbb{1}_{\{\theta < \alpha\}} \right\} dG_n^\circ(\theta) \right. \\ &\quad \left. - \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \left\{ \Phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) - \mathbb{1}_{\{\theta < \alpha\}} \right\} \{g_n^\circ(\alpha) + (\theta - \alpha)\dot{g}_n^\circ(\alpha)\} d\theta \right| \\ &\leq \epsilon \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \left| \Phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) - \mathbb{1}_{\{\theta < \alpha\}} \right| |\theta - \alpha| d\theta \\ &\leq \frac{\epsilon\alpha(1-\alpha)}{B_1} \int_{-\log B_1/\sqrt{\alpha(1-\alpha)}}^{\log B_1/\sqrt{\alpha(1-\alpha)}} |\Phi(-u) - \mathbb{1}_{\{u < 0\}}| |u| du \\ &\leq \frac{2\epsilon\alpha(1-\alpha)}{B_1} \int_0^\infty u\Phi(-u) du = \frac{\epsilon\alpha(1-\alpha)}{2B_1}. \end{aligned}$$

We deduce that $|R_2| = o(B_1^{-1})$ as $B_1 \rightarrow \infty$.

To bound R_3 : For large B_1 , we have

$$\begin{aligned} |R_3| &= \left| \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \left\{ \Phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) - \mathbb{1}_{\{\theta < \alpha\}} \right\} \{g_n^\circ(\alpha) + (\theta - \alpha)\dot{g}_n^\circ(\alpha)\} d\theta \right. \\ &\quad \left. - \frac{\sqrt{\alpha(1-\alpha)}}{B_1^{1/2}} \int_{-\infty}^\infty \{\Phi(-u) - \mathbb{1}_{\{u < 0\}}\} \left\{ g_n^\circ(\alpha) + \frac{\sqrt{\alpha(1-\alpha)}}{B_1^{1/2}} u \dot{g}_n^\circ(\alpha) \right\} du \right| \\ &= \frac{2\alpha(1-\alpha)}{B_1} |\dot{g}_n^\circ(\alpha)| \int_{\epsilon_1 B_1^{1/2}/\{\alpha(1-\alpha)\}^{1/2}}^\infty u\Phi(-u) du \\ &\leq \frac{2\{\alpha(1-\alpha)\}^{3/2}}{B_1 \log B_1} |\dot{g}_n^\circ(\alpha)| \int_0^\infty u^2\Phi(-u) du = \frac{2\sqrt{2}\{\alpha(1-\alpha)\}^{3/2}}{3\sqrt{\pi}B_1 \log B_1} |\dot{g}_n^\circ(\alpha)| = o(B_1^{-1}) \end{aligned}$$

as $B_1 \rightarrow \infty$.

To bound R_4 : By the bound in (25), we have that, given $\epsilon > 0$, for all B_1 sufficiently large,

$$\begin{aligned} |R_4| &= \left| \frac{(1-2\alpha)B_1^{1/2}}{2\{\alpha(1-\alpha)\}^{3/2}} \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} (\alpha-\theta)^2 \phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) \{g_n^\circ(\theta) - g_n^\circ(\alpha)\} d\theta \right| \\ &\leq \frac{\epsilon|1-2\alpha|}{2B_1} \int_{-\infty}^\infty u^2\phi(-u) du = \frac{\epsilon|1-2\alpha|}{2B_1}. \end{aligned}$$

To bound R_5 : For all B_1 sufficiently large,

$$\begin{aligned} |R_5| &= \frac{|1-2\alpha|}{B_1} |g_n^\circ(\alpha)| \int_{\log B_1 / \sqrt{\alpha(1-\alpha)}}^{\infty} u^2 \phi(-u) du \\ &\leq \frac{\sqrt{\alpha(1-\alpha)}}{B_1 \log B_1} |g_n^\circ(\alpha)| \int_0^{\infty} u^3 \phi(-u) du = \frac{\sqrt{2\alpha(1-\alpha)}}{\sqrt{\pi} B_1 \log B_1} |g_n^\circ(\alpha)| = o\left(\frac{1}{B_1}\right) \end{aligned}$$

as $B_1 \rightarrow \infty$.

To bound R_6 : We write $R_6 = R_{61} + R_{62}$, where

$$R_{61} := \frac{(1-2\alpha)}{6B_1^{1/2} \sqrt{\alpha(1-\alpha)}} \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \left\{ 1 - \frac{B_1(\alpha-\theta)^2}{\alpha(1-\alpha)} \right\} \phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) dG_n^\circ(\theta)$$

and

$$\begin{aligned} R_{62} &:= \frac{1}{6B_1^{1/2}} \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \frac{(1-2\theta)}{\sigma} \left\{ 1 - \frac{B_1(\alpha-\theta)^2}{\sigma^2} \right\} \phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sigma}\right) dG_n^\circ(\theta) \\ &\quad - \frac{(1-2\alpha)}{6B_1^{1/2} \sqrt{\alpha(1-\alpha)}} \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \left\{ 1 - \frac{B_1(\alpha-\theta)^2}{\alpha(1-\alpha)} \right\} \phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) dG_n^\circ(\theta). \end{aligned}$$

Since g_n° is continuous at α , given $\epsilon > 0$, there exists $B'_0 \in \mathbb{N}$ such that, for all $B_1 > B'_0$,

$$\sup_{|\theta-\alpha| \leq \epsilon_1} |g_n^\circ(\theta) - g_n^\circ(\alpha)| < \epsilon. \quad (25)$$

It follows that, for $B_1 > B'_0$,

$$\begin{aligned} |R_{61}| &\leq \frac{|1-2\alpha|}{6B_1^{1/2} \sqrt{\alpha(1-\alpha)}} |g_n^\circ(\alpha)| \left| \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \left\{ 1 - \frac{B_1(\alpha-\theta)^2}{\alpha(1-\alpha)} \right\} \phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) d\theta \right| \\ &\quad + \epsilon \frac{|1-2\alpha|}{6B_1^{1/2} \sqrt{\alpha(1-\alpha)}} \int_{\alpha-\epsilon_1}^{\alpha+\epsilon_1} \left| 1 - \frac{B_1(\alpha-\theta)^2}{\alpha(1-\alpha)} \right| \phi\left(\frac{B_1^{1/2}(\alpha-\theta)}{\sqrt{\alpha(1-\alpha)}}\right) d\theta. \\ &\leq \frac{|1-2\alpha|}{6B_1} |g_n^\circ(\alpha)| \left| \int_{-\log B_1 / \sqrt{\alpha(1-\alpha)}}^{\log B_1 / \sqrt{\alpha(1-\alpha)}} (1-u^2) \phi(-u) du \right| \\ &\quad + \epsilon \frac{|1-2\alpha|}{6B_1} \int_{-\infty}^{\infty} (1+u^2) \phi(-u) du \leq \frac{\epsilon}{B_1} \end{aligned}$$

for all sufficiently large B_1 . We deduce that $R_{61} = o(B_1^{-1})$ as $B_1 \rightarrow \infty$.

To control R_{62} , by the mean value theorem, we have that for all B_1 sufficiently large

and all $\zeta \in [\alpha - \epsilon_1, \alpha + \epsilon_1]$,

$$\begin{aligned} \sup_{|\theta - \alpha| < \epsilon_1} & \left| \frac{(1 - 2\zeta)}{\sqrt{\zeta(1 - \zeta)}} \left\{ 1 - \frac{B_1(\alpha - \theta)^2}{\zeta(1 - \zeta)} \right\} \phi \left(\frac{B_1^{1/2}(\alpha - \theta)}{\sqrt{\zeta(1 - \zeta)}} \right) \right. \\ & \quad \left. - \frac{(1 - 2\alpha)}{\sqrt{\alpha(1 - \alpha)}} \left\{ 1 - \frac{B_1(\alpha - \theta)^2}{\alpha(1 - \alpha)} \right\} \phi \left(\frac{B_1^{1/2}(\alpha - \theta)}{\sqrt{\alpha(1 - \alpha)}} \right) \right| \\ & \leq \frac{\log^4 B_1}{\sqrt{2\pi} \{\alpha(1 - \alpha)\}^{7/2}} |\zeta - \alpha|. \end{aligned}$$

Thus, for large B_1 ,

$$\begin{aligned} |R_{62}| & \leq \frac{\log^4 B_1}{6\sqrt{2\pi} B_1^{1/2} \{\alpha(1 - \alpha)\}^{7/2}} \sup_{|\theta - \alpha| \leq \epsilon_1} |g_n^\circ(\theta)| \int_{\alpha - \epsilon_1}^{\alpha + \epsilon_1} |\theta - \alpha| d\theta \\ & \leq \frac{\log^6 B_1 \{1 + |g_n^\circ(\alpha)|\}}{6\sqrt{2\pi} B_1^{3/2} \{\alpha(1 - \alpha)\}^{7/2}} = o\left(\frac{1}{B_1}\right). \end{aligned}$$

We deduce that $|R_6| = o(B_1^{-1})$ as $B_1 \rightarrow \infty$.

To bound R_7 : write $R_7 = R_{71} + R_{72}$, where

$$R_{71} := \frac{1/2 - \llbracket B_1 \alpha \rrbracket}{B_1^{1/2} \sqrt{\alpha(1 - \alpha)}} \int_{\alpha - \epsilon_1}^{\alpha + \epsilon_1} \phi \left(\frac{B_1^{1/2}(\alpha - \theta)}{\sqrt{\alpha(1 - \alpha)}} \right) \{g_n^\circ(\theta) - g_n^\circ(\alpha)\} d\theta,$$

and

$$R_{72} := \frac{1/2 - \llbracket B_1 \alpha \rrbracket}{B_1^{1/2}} \int_{\alpha - \epsilon_1}^{\alpha + \epsilon_1} \left\{ \frac{1}{\sigma} \phi \left(\frac{B_1^{1/2}(\alpha - \theta)}{\sigma} \right) - \frac{1}{\sqrt{\alpha(1 - \alpha)}} \phi \left(\frac{B_1^{1/2}(\alpha - \theta)}{\sqrt{\alpha(1 - \alpha)}} \right) \right\} dG_n^\circ(\theta).$$

By the bound in (25), given $\epsilon > 0$, for all B_1 sufficiently large,

$$|R_{71}| \leq \frac{\epsilon}{2B_1^{1/2} \sqrt{\alpha(1 - \alpha)}} \int_{-\infty}^{\infty} \phi \left(\frac{B_1^{1/2}(\alpha - \theta)}{\sqrt{\alpha(1 - \alpha)}} \right) d\theta = \frac{\epsilon}{2B_1}.$$

Moreover, by the mean value theorem, for all B_1 sufficiently large and all $|\zeta - \alpha| \leq \epsilon_1$,

$$\begin{aligned} \sup_{|\theta - \alpha| < \epsilon_1} & \left| \frac{1}{\sqrt{\zeta(1 - \zeta)}} \phi \left(\frac{B_1^{1/2}(\alpha - \theta)}{\sqrt{\zeta(1 - \zeta)}} \right) - \frac{1}{\sqrt{\alpha(1 - \alpha)}} \phi \left(\frac{B_1^{1/2}(\alpha - \theta)}{\sqrt{\alpha(1 - \alpha)}} \right) \right| \\ & \leq \frac{\log^2 B_1}{\sqrt{2\pi} \{\alpha(1 - \alpha)\}^{5/2}} |\zeta - \alpha|. \end{aligned}$$

It follows that, for all B_1 sufficiently large,

$$\begin{aligned} |R_{72}| & \leq \frac{\log^2 B_1}{2\sqrt{2\pi} B_1^{1/2} \{\alpha(1 - \alpha)\}^{5/2}} \sup_{|\theta - \alpha| \leq \epsilon_1} |g_n^\circ(\theta)| \int_{\alpha - \epsilon_1}^{\alpha + \epsilon_1} |\theta - \alpha| d\theta \\ & \leq \frac{\log^4 B_1 \{1 + |g_n^\circ(\alpha)|\}}{2\sqrt{2\pi} B_1^{3/2} \{\alpha(1 - \alpha)\}^{5/2}}. \end{aligned}$$

We deduce that $|R_7| = o(B_1^{-1})$ as $B_1 \rightarrow \infty$.

To bound R_8 : We have

$$|R_8| = \frac{2(1/2 - \llbracket B_1 \alpha \rrbracket)}{B_1} |g_n^\circ(\alpha)| \int_{\epsilon_1 B_1^{1/2}/\{\alpha(1-\alpha)\}^{1/2}}^{\infty} \phi(-u) du = o\left(\frac{1}{B_1}\right)$$

as $B_1 \rightarrow \infty$.

We have now established the claim at (24), and the result follows. \square

Proof of Theorem 2. We have

$$\begin{aligned} \mathcal{L}(\hat{C}_n^{\text{RP}^*}) - \mathcal{R}(C^{\text{Bayes}}) &= \mathbb{E}[\mathbb{P}\{\hat{C}_n^{\text{RP}^*}(X) \neq Y|X\} - \mathbb{P}\{C^{\text{Bayes}}(X) \neq Y|X\}] \\ &= \mathbb{E}[\eta(X)(\mathbb{1}_{\{\hat{C}_n^{\text{RP}^*}(X)=2\}} - \mathbb{1}_{\{C^{\text{Bayes}}(X)=2\}}) + \{1 - \eta(X)\}(\mathbb{1}_{\{\hat{C}_n^{\text{RP}^*}(X)=1\}} - \mathbb{1}_{\{C^{\text{Bayes}}(X)=1\}})] \\ &= \mathbb{E}\{|2\eta(X) - 1| |\mathbb{1}_{\{\hat{\mu}_n^{B_2}(X) < \alpha\}} - \mathbb{1}_{\{\eta(X) < 1/2\}}|\} \\ &= \mathbb{E}\{|2\eta(X) - 1| \mathbb{1}_{\{\hat{\mu}_n^{B_2}(X) \geq \alpha\}} \mathbb{1}_{\{\eta(X) < 1/2\}}\} + \mathbb{E}\{|2\eta(X) - 1| \mathbb{1}_{\{\hat{\mu}_n^{B_2}(X) < \alpha\}} \mathbb{1}_{\{\eta(X) \geq 1/2\}}\} \\ &\leq \frac{1}{\alpha} \mathbb{E}\{|2\eta(X) - 1| \hat{\mu}_n^{B_2}(X) \mathbb{1}_{\{\eta(X) < 1/2\}}\} + \frac{1}{1 - \alpha} \mathbb{E}[|2\eta(X) - 1| \{1 - \hat{\mu}_n^{B_2}(X)\} \mathbb{1}_{\{\eta(X) \geq 1/2\}}] \\ &\leq \frac{1}{\min(\alpha, 1 - \alpha)} \mathbb{E}\{|2\eta(X) - 1| |1 - \hat{\mu}_n^{B_2}(X) - \mathbb{1}_{\{\eta(X) < 1/2\}}|\}. \end{aligned}$$

Now, for each $x \in \mathbb{R}^p$,

$$|1 - \hat{\mu}_n^{B_2}(x) - \mathbb{1}_{\{\eta(x) < 1/2\}}| = |\mathbb{P}\{\hat{C}_n^{A_1}(x) = 2\} - \mathbb{1}_{\{\eta(x) < 1/2\}}| = \mathbb{E}|\mathbb{1}_{\{\hat{C}_n^{A_1}(x)=2\}} - \mathbb{1}_{\{\eta(x) < 1/2\}}|.$$

We deduce that

$$\begin{aligned} \mathbb{E}\{|2\eta(X) - 1| |1 - \hat{\mu}_n^{B_2}(X) - \mathbb{1}_{\{\eta(X) < 1/2\}}|\} &= \mathbb{E}\left[\mathbb{E}\{|2\eta(X) - 1| |\mathbb{1}_{\{\hat{C}_n^{A_1}(X)=2\}} - \mathbb{1}_{\{\eta(X) < 1/2\}}| |X\}\right] \\ &= \mathbb{E}\left[\mathbb{E}\{|2\eta(X) - 1| |\mathbb{1}_{\{\hat{C}_n^{A_1}(X)=2\}} - \mathbb{1}_{\{\eta(X) < 1/2\}}| |A_1\}\right] \\ &= \mathbb{E}(\mathcal{L}_n^{A_1}) - \mathcal{R}(C^{\text{Bayes}}). \end{aligned}$$

The result follows. \square

Proof of Proposition 3. First write

$$\mathbb{E}(\mathcal{L}_n^{A_1}) - \mathcal{R}(C^{\text{Bayes}}) = \mathbb{E}(\hat{L}_n^{A_1}) - \mathcal{R}(C^{\text{Bayes}}) + \epsilon_n.$$

Using (A.2), we have that

$$\begin{aligned} \mathbb{E}(\hat{L}_n^{A_1}) &= \hat{L}_n^* + \frac{1}{n} \sum_{j=0}^{\lfloor n(1-\hat{L}_n^*) \rfloor - 1} \{1 - \beta_n(j)\}^{B_2} \\ &\leq \hat{L}_n^* + \frac{1}{n} \sum_{j=0}^J \left(1 - \beta_0 - \beta \frac{j^\rho}{n^\rho}\right)^{B_2} + \frac{1}{n} \sum_{j=J+1}^{\lfloor n(1-\hat{L}_n^*) \rfloor - 1} \{1 - \beta_n(j)\}^{B_2}, \end{aligned}$$

where $J := \lfloor n(\frac{\log^2 B_2}{\beta B_2})^{1/\rho} \rfloor$. Now,

$$\begin{aligned} \frac{1}{n} \sum_{j=0}^J \left(1 - \beta_0 - \beta \frac{j^\rho}{n^\rho}\right)^{B_2} &\leq \frac{(1 - \beta_0)^{B_2}}{n} + \int_0^{J/n} (1 - \beta_0 - \beta x^\rho)^{B_2} dx \\ &\leq (1 - \beta_0)^{B_2} \left\{ \frac{1}{n} + \int_0^{J/n} \exp\left(-\frac{B_2 \beta x^\rho}{1 - \beta_0}\right) dx \right\} \\ &\leq (1 - \beta_0)^{B_2} \left\{ \frac{1}{n} + \frac{(1 - \beta_0)^{1/\rho} \Gamma(1 + 1/\rho)}{B_2^{1/\rho} \beta^{1/\rho}} \right\}. \end{aligned}$$

Moreover, since $(J + 1)^\rho/n^\rho \geq \log^2 B_2/(\beta B_2)$, we have

$$\frac{1}{n} \sum_{j=J+1}^{\lfloor n(1 - \hat{L}_n^*) \rfloor - 1} \{1 - \beta_n(j)\}^{B_2} \leq \left(1 - \beta_0 - \frac{\log^2 B_2}{B_2}\right)^{B_2} \leq (1 - \beta_0)^{B_2} \exp\left(-\frac{\log^2 B_2}{1 - \beta_0}\right).$$

The result follows. \square

Proof of Proposition 4. For a Borel set $C \subseteq \mathbb{R}^d$, let $P_{A^*X}(C) := \int_{\{x: A^*x \in C\}} dP_X(x)$, so that P_{A^*X} is the marginal distribution of A^*X . Further, for $z \in \mathbb{R}^d$, write $P_{X|A^*X=z}$ for the conditional distribution of X given $A^*X = z$. If Y is independent of X given A^*X , and if B is a Borel subset of \mathbb{R}^p , then

$$\begin{aligned} \int_B \eta^{A^*}(A^*x) dP_X(x) &= \int_{\mathbb{R}^d} \int_{B \cap \{x: A^*x=z\}} \eta^{A^*}(A^*x) dP_{X|A^*X=z}(w) dP_{A^*X}(z) \\ &= \int_{\mathbb{R}^d} \eta^{A^*}(z) \mathbb{P}(X \in B | A^*X = z) dP_{A^*X}(z) \\ &= \int_{\mathbb{R}^d} \mathbb{P}(Y = 1, X \in B | A^*X = z) dP_{A^*X}(z) \\ &= \mathbb{P}(Y = 1, X \in B) = \int_B \eta(x) dP_X(x). \end{aligned}$$

We deduce that $P_X(\{x \in \mathbb{R}^p : \eta(x) \neq \eta^{A^*}(A^*x)\}) = 0$; in particular, **(A.3)** holds, as required. \square

Proof of Theorem 5. By the definitions of \hat{L}_n^* and $\epsilon_n^{A^*}$, we have $\hat{L}_n^* \leq \hat{L}_n^{A^*} = \mathcal{L}_n^{A^*} - \epsilon_n^{A^*}$. Moreover,

$$\begin{aligned} \mathcal{R}^{A^*-\text{Bayes}} &= \int_{\mathbb{R}^p \times \{1,2\}} \mathbb{1}_{\{C^{A^*-\text{Bayes}}(A^*x) \neq y\}} dP(x, y) \\ &= \int_{\mathbb{R}^p} \eta(x) \mathbb{1}_{\{\eta^{A^*}(A^*x) < 1/2\}} dP_X(x) + \int_{\mathbb{R}^p} \{1 - \eta(x)\} \mathbb{1}_{\{\eta^{A^*}(A^*x) \geq 1/2\}} dP_X(x) \\ &= \int_{\mathbb{R}^p} \eta(x) \mathbb{1}_{\{\eta(x) < 1/2\}} dP_X(x) + \int_{\mathbb{R}^p} \{1 - \eta(x)\} \mathbb{1}_{\{\eta(x) \geq 1/2\}} dP_X(x) = \mathcal{R}(C^{\text{Bayes}}). \end{aligned}$$

Note that we have used **(A.3)** to obtain the penultimate equality. The result now follows immediately from these facts, together with Theorem 1, Theorem 2 and Proposition 3. \square

Proof of Proposition 7. Recall that $\sigma^2 := \theta(1 - \theta)$. Let

$$F_{B_1}^*(s) = F_{B_1}^*(s, \theta) := \int_{-\infty}^{\infty} e^{ist} dF_{B_1}(t) = \left\{ (1 - \theta) \exp\left(-\frac{is\theta}{B_1^{1/2}\sigma}\right) + \theta \exp\left(\frac{is(1 - \theta)}{B_1^{1/2}\sigma}\right) \right\}^{B_1}.$$

Moreover, let $P(t) := \frac{\phi(t)p(t)}{B_1^{1/2}}$ and $Q(t) := \frac{\phi(t)q(t)}{B_1^{1/2}}$. By, for example, Gnedenko and Kolmogorov (1954, Chapter 8, Section 43), we have

$$\Phi^*(s) := \int_{-\infty}^{\infty} \exp(ist) d\Phi(t) = \exp(-s^2/2),$$

$$P^*(s) := \int_{-\infty}^{\infty} \exp(ist) dP(t) = -\frac{1 - 2\theta}{6B_1^{1/2}\sigma} is^3 \exp(-s^2/2)$$

and

$$Q^*(s) := \int_{-\infty}^{\infty} \exp(ist) dQ(t) = -\frac{s}{2\pi B_1^{1/2}\sigma} \sum_{l \in \mathbb{Z} \setminus \{0\}} \frac{\exp(i2\pi B_1 l \theta)}{l} \exp\left\{-\frac{1}{2}(s + 2\pi B_1^{1/2}\sigma l)^2\right\}.$$

Thus

$$\begin{aligned} G_{B_1}^*(s) &= G_{B_1}^*(s, \theta) := \int_{-\infty}^{\infty} \exp(ist) dG_{B_1}(t) = \Phi^*(s) + P^*(s) + Q^*(s) \\ &= \exp(-s^2/2) - \frac{1 - 2\theta}{6B_1^{1/2}\sigma} is^3 \exp(-s^2/2) \\ &\quad - \frac{s}{2\pi B_1^{1/2}\sigma} \sum_{l \in \mathbb{Z} \setminus \{0\}} \frac{\exp(i2\pi B_1 l \theta)}{l} \exp\left\{-\frac{1}{2}(s + 2\pi B_1^{1/2}\sigma l)^2\right\}. \end{aligned}$$

Letting $c_2 > 0$ be the constant given in the statement of Theorem 6 (in fact we assume without loss of generality that $c_2 > \pi$), we show that there exists a constant $C' > 0$ such that, for all $B_1 \in \mathbb{N}$,

$$\sup_{\theta \in (0,1)} \sigma^3 \int_{-c_2 B_1^{1/2}\sigma}^{c_2 B_1^{1/2}\sigma} \left| \frac{F_{B_1}^*(s, \theta) - G_{B_1}^*(s, \theta)}{s} \right| ds \leq \frac{C'}{B_1}. \quad (26)$$

To show (26), write

$$\begin{aligned} \int_{-c_2 B_1^{1/2}\sigma}^{c_2 B_1^{1/2}\sigma} \left| \frac{F_{B_1}^*(s) - G_{B_1}^*(s)}{s} \right| ds &= \int_{-S_1}^{S_1} \left| \frac{F_{B_1}^*(s) - G_{B_1}^*(s)}{s} \right| ds \\ &\quad + \int_{S_1 \leq |s| \leq S_2} \left| \frac{F_{B_1}^*(s) - G_{B_1}^*(s)}{s} \right| ds + \int_{S_2 \leq |s| \leq c_2 B_1^{1/2}\sigma} \left| \frac{F_{B_1}^*(s) - G_{B_1}^*(s)}{s} \right| ds, \end{aligned} \quad (27)$$

where $S_1 := \frac{B_1^{1/2}\sigma^{3/2}}{32(3\theta^2 - 3\theta + 1)^{3/4}}$ and $S_2 := \pi B_1^{1/2}\sigma$. Note that $S_1 \leq S_2/2$ for all $\theta \in (0, 1)$.

We bound each term in (27) in turn. By Gnedenko and Kolmogorov (1954, Theorem 1, Section 41), there exists a universal constant $C_3 > 0$, such that, for all $|s| \leq S_1$,

$$|F_{B_1}^*(s, \theta) - \Phi^*(s) - P^*(s)| \leq \frac{C_3}{B_1 \sigma^3} (s^4 + s^6) \exp(-s^2/4).$$

Thus

$$\int_{-S_1}^{S_1} \left| \frac{F_{B_1}^*(s) - \Phi^*(s) - P^*(s)}{s} \right| ds \leq \frac{C_3}{B_1 \sigma^3} \int_{-\infty}^{\infty} (|s|^3 + |s|^5) \exp(-s^2/4) ds = \frac{144C_3}{B_1 \sigma^3}. \quad (28)$$

Moreover, observe that $(s + 2\pi B_1^{1/2} \sigma l)^2 \geq s^2 + 2\pi^2 B_1 \sigma^2 l^2$ for all $|s| \leq S_1$. Thus, for $|s| \leq S_1$,

$$\begin{aligned} \left| \frac{Q^*(s)}{s} \right| &\leq \frac{1}{2\pi B_1^{1/2} \sigma} \left| \sum_{l \in \mathbb{Z} \setminus \{0\}} \frac{\exp(i2\pi B_1 l \theta)}{l} \exp\left\{-\frac{1}{2}(s + 2\pi B_1^{1/2} \sigma l)^2\right\} \right| \\ &\leq \frac{\phi(s)}{\sqrt{2\pi} B_1^{1/2} \sigma} \int_{-\infty}^{\infty} \exp(-\pi^2 B_1 \sigma^2 u^2) du = \frac{\phi(s)}{\sqrt{2\pi} B_1 \sigma^2}. \end{aligned}$$

It follows that

$$\int_{-S_1}^{S_1} \left| \frac{Q^*(s)}{s} \right| ds \leq \frac{1}{\sqrt{2\pi} B_1 \sigma^2}. \quad (29)$$

For $|s| \in [S_1, S_2]$, observe that

$$|F_{B_1}^*(s)| = \left[1 - 2\sigma^2 \left\{ 1 - \cos\left(\frac{s}{B_1^{1/2} \sigma}\right) \right\} \right]^{B_1/2} \leq \exp(-s^2/8).$$

Thus

$$\int_{S_1 \leq |s| \leq S_2} \left| \frac{F_{B_1}^*(s)}{s} \right| ds \leq \frac{2}{S_1^2} \int_{S_1}^{S_2} s \exp(-s^2/8) ds \leq \frac{2^{13}}{B_1 \sigma^3}. \quad (30)$$

Now,

$$\int_{S_1 \leq |s| \leq S_2} \left| \frac{\Phi^*(s)}{s} \right| ds \leq \frac{2}{S_1^2} \int_0^{\infty} s \exp(-s^2/2) ds \leq \frac{2^{11}}{B_1 \sigma^3}, \quad (31)$$

and

$$\int_{S_1 \leq |s| \leq S_2} \left| \frac{P^*(s)}{s} \right| ds \leq \frac{1}{3S_1 B_1^{1/2} \sigma} \int_0^{\infty} s^3 \exp(-s^2/2) ds \leq \frac{2^6}{3\sqrt{2} B_1 \sigma^3}. \quad (32)$$

To bound the final term, observe that, for all $|s| \in [S_1, S_2]$, since $(a + b)^2 \geq (a^2 + b^2)/5$ for all $|a| \leq |b|/2$, we have

$$\int_{S_1 \leq |s| \leq S_2} \left| \frac{Q^*(s)}{s} \right| ds \leq \frac{1}{2\pi B_1^{1/2} \sigma} \int_{S_1 \leq |s| \leq S_2} e^{-s^2/10} \int_{-\infty}^{\infty} e^{-2\pi^2 B_1 \sigma^2 u^2/5} du ds \leq \frac{5}{2\sqrt{2\pi} B_1 \sigma^3}. \quad (33)$$

Finally, for $|s| \in [S_2, c_2 B_1^{1/2} \sigma]$, note that

$$\begin{aligned} \int_{S_2 \leq |s| \leq c_2 B_1^{1/2} \sigma} \left| \frac{\Phi^*(s) + P^*(s)}{s} \right| ds &\leq \frac{2}{S_2^2} \int_0^\infty s e^{-s^2/2} ds + \frac{1}{3S_2 B_1^{1/2} \sigma} \int_0^\infty s^3 e^{-s^2/2} ds \\ &\leq \frac{1}{\pi^2 B_1 \sigma^3} \left(1 + \frac{\pi}{3} \right). \end{aligned} \quad (34)$$

To bound the remaining terms, by substituting $s = B_1^{1/2} \sigma u$, we see that

$$\begin{aligned} \int_{S_2}^{c_2 B_1^{1/2} \sigma} \left| \frac{F_{B_1}^*(s) - Q_{B_1}^*(s)}{s} \right| ds &= \int_\pi^{c_2} \left| \frac{F_{B_1}^*(B_1^{1/2} \sigma u) - Q_{B_1}^*(B_1^{1/2} \sigma u)}{u} \right| du \\ &= \sum_{j=1}^J \int_{\pi(2j-1)}^{\pi(2j+1)} \left| \frac{F_{B_1}^*(B_1^{1/2} \sigma u) - Q_{B_1}^*(B_1^{1/2} \sigma u)}{u} \right| du \\ &\quad + \int_{\pi(2J+1)}^{c_2} \left| \frac{F_{B_1}^*(B_1^{1/2} \sigma u) - Q_{B_1}^*(B_1^{1/2} \sigma u)}{u} \right| du, \end{aligned} \quad (35)$$

where $J := \lfloor \frac{c_2 - \pi}{2\pi} \rfloor$. Let

$$\begin{aligned} I_j &:= \int_{\pi(2j-1)}^{\pi(2j+1)} \left| \frac{F_{B_1}^*(B_1^{1/2} \sigma u) - Q_{B_1}^*(B_1^{1/2} \sigma u)}{u} \right| du \\ &= \int_{-\pi}^\pi \left| \frac{F_{B_1}^*(B_1^{1/2} \sigma(v + 2\pi j)) - Q_{B_1}^*(B_1^{1/2} \sigma(v + 2\pi j))}{v + 2\pi j} \right| dv. \end{aligned} \quad (36)$$

Observe that

$$\begin{aligned} F_{B_1}^*(B_1^{1/2} \sigma(v + 2\pi j)) &= \left[(1 - \theta) \exp\{-i(v + 2\pi j)\theta\} + \theta \exp\{i(v + 2\pi j)(1 - \theta)\} \right]^{B_1} \\ &= \exp(-i2\pi B_1 j \theta) \left[(1 - \theta) \exp(-iv\theta) + \theta \exp\{iv(1 - \theta)\} \right]^{B_1} \\ &= \exp(-i2\pi B_1 j \theta) F_{B_1}^*(B_1^{1/2} \sigma v). \end{aligned}$$

Similarly,

$$\begin{aligned} Q_{B_1}^*(B_1^{1/2} \sigma(v + 2\pi j)) &= -\frac{(v + 2\pi j)}{2\pi} \sum_{l \in \mathbb{Z} \setminus \{0\}} \frac{\exp(i2\pi B_1 l \theta)}{l} \exp\left\{-\frac{B_1 \sigma^2}{2} (v + 2\pi j + 2\pi l)^2\right\} \\ &= \frac{(v + 2\pi j) \exp(-i2\pi B_1 j \theta)}{2\pi j} \exp\left(-\frac{B_1 \sigma^2 v^2}{2}\right) \\ &\quad - \frac{(v + 2\pi j)}{2\pi} \sum_{l \in \mathbb{Z} \setminus \{0, -j\}} \frac{\exp(i2\pi B_1 l \theta)}{l} \exp\left\{-\frac{B_1 \sigma^2}{2} (v + 2\pi j + 2\pi l)^2\right\}. \end{aligned}$$

But, for $v \in [-\pi, \pi]$,

$$\begin{aligned} \left| \frac{1}{2\pi} \sum_{l \in \mathbb{Z} \setminus \{0, -j\}} \frac{e^{i2\pi B_1 l \theta}}{l} \exp\left\{-\frac{B_1 \sigma^2}{2} (v + 2\pi j + 2\pi l)^2\right\} \right| &\leq \frac{1}{2\pi} \sum_{m \in \mathbb{Z} \setminus \{0\}} e^{-\frac{B_1 \sigma^2}{2} (v + 2\pi m)^2} \\ &\leq \frac{e^{-B_1 \sigma^2 v^2/10}}{2\pi} \sum_{m \in \mathbb{Z} \setminus \{0\}} e^{-2\pi^2 B_1 \sigma^2 m^2/5} \leq \frac{e^{-B_1 \sigma^2 v^2/10}}{\pi(e^{2\pi^2 B_1 \sigma^2/5} - 1)} \leq \frac{5e^{-B_1 \sigma^2 v^2/10}}{2\pi^3 B_1 \sigma^2}. \end{aligned}$$

It follows that

$$I_j \leq \int_{-\pi}^{\pi} \left| \frac{F_{B_1}^*(B_1^{1/2}\sigma v) - \left(\frac{v}{2\pi j} + 1\right) \exp\left(-\frac{B_1\sigma^2 v^2}{2}\right)}{v + 2\pi j} \right| dv + \frac{5\sqrt{5}}{\sqrt{2}\pi^{5/2}B_1^{3/2}\sigma^3}. \quad (37)$$

Now

$$\begin{aligned} \int_{-\pi}^{\pi} \left| \frac{F_{B_1}^*(B_1^{1/2}\sigma v) - \exp\left(-\frac{B_1\sigma^2 v^2}{2}\right)}{v + 2\pi j} \right| dv &\leq \frac{1}{\pi j B_1^{1/2}\sigma} \int_{-\pi B_1^{1/2}\sigma}^{\pi B_1^{1/2}\sigma} |F_{B_1}^*(u) - e^{-u^2/2}| du \\ &= \frac{1}{\pi j B_1^{1/2}\sigma} \int_{-S_3}^{S_3} |F_{B_1}^*(u) - e^{-u^2/2}| du + \frac{1}{\pi j B_1^{1/2}\sigma} \int_{S_3 \leq |u| \leq \pi B_1^{1/2}\sigma} |F_{B_1}^*(u) - e^{-u^2/2}| du, \end{aligned} \quad (38)$$

where $S_3 := \frac{B_1^{1/2}\sigma}{5(2\theta^2 - 2\theta + 1)} \geq S_1$. By Gnedenko and Kolmogorov (1954, Theorem 2, Section 40), we have that

$$\frac{1}{\pi j B_1^{1/2}\sigma} \int_{-S_3}^{S_3} |F_{B_1}^*(u) - e^{-u^2/2}| du \leq \frac{7}{6\pi j B_1\sigma^2} \int_{-S_3}^{S_3} |u|^3 e^{-u^2/4} du \leq \frac{56}{3\pi j B_1\sigma^2}. \quad (39)$$

Moreover,

$$\begin{aligned} \frac{1}{\pi j B_1^{1/2}\sigma} \int_{S_3 \leq |u| \leq \pi B_1^{1/2}\sigma} |F_{B_1}^*(u) - e^{-u^2/2}| du &\leq \frac{2}{\pi j S_3 B_1^{1/2}\sigma} \int_0^{\infty} u(e^{-u^2/8} + e^{-u^2/2}) du \\ &\leq \frac{50}{\pi j B_1\sigma^2}. \end{aligned} \quad (40)$$

Finally,

$$\frac{1}{2\pi j} \int_{-\pi}^{\pi} \frac{|v|}{|v| + 2\pi j} \exp\left(-\frac{B_1\sigma^2 v^2}{2}\right) dv \leq \frac{1}{2\pi^2 j^2} \int_0^{\pi} v \exp\left(-\frac{B_1\sigma^2 v^2}{2}\right) dv \leq \frac{1}{2\pi^2 j^2 B_1\sigma^2}. \quad (41)$$

By (35), (36), (37), (38), (39), (40) and (41), it follows that

$$\begin{aligned} \int_{S_2 \leq |s| \leq c_2 B_1^{1/2}\sigma} \left| \frac{F_{B_1}^*(s) - Q_{B_1}^*(s)}{s} \right| ds &\leq \frac{10\sqrt{5}(J+1)}{\sqrt{2}\pi^{5/2}B_1^{3/2}\sigma^3} + \frac{140}{\pi B_1\sigma^2} \sum_{j=1}^{J+1} \frac{1}{j} \\ &= \frac{10\sqrt{5}(J+1)}{\sqrt{2}\pi^{5/2}B_1^{3/2}\sigma^3} + \frac{140}{\pi B_1\sigma^2} \{1 + \log(J+1)\}. \end{aligned} \quad (42)$$

By (27), (28), (29), (30), (31), (32), (33), (34) and (42), we conclude that (26) holds. The result now follows from Theorem 6, by taking $c_1 = \frac{1}{B_1^{1/2}\sigma}$, $C_1 = \frac{1}{3B_1^{1/2}\sigma}$ and $S = c_2 B_1^{1/2}\sigma$ in that result. \square

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