The Geodesic Shortest Path

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Abstract. In the geodesic shortest path problem we have to find the shortest path between two points on the boundary of a polyhedron. The shortest path must lead along the surface of the polyhedron. In the article, the description and solution of that problem is given. The case with forbidden areas on the surface is especially treated.

Keywords: the shortest path, geodesic, forbidden area, obstacle, triangulation, surface

1 Introduction

The shortest path problems belong to the fundamental problems studied in computational geometry and other areas including graph algorithms, geographical information systems (GIS), network optimization and robotics.

The geodesic shortest path problem: Given two points s and t on a triangulated surface, find the shortest path from s to t so that it lies on the surface. This article is concentrated especially on the surfaces with forbidden areas (obstacles - the shortest path has to avoid the forbidden area).

2 Previous work

As far as we know there is no previous work solving that problem which would concentrate on surfaces with the forbidden areas. There are more works which solve the general geodesic shortest path problem and some of them can also treat the surfaces with forbidden areas, but their algorithms are not much effective in such cases.

One of the first and exact algorithms solving the geodesic shortest path problem was given by Sharir and Schorr in 1984 [5]. They proved that:

1. The shortest path cannot go through any spherical vertex[1] (fig. 1)
2. The shortest path turns into the straight line after development to the plane.

The paths which satisfy these conditions are called (discrete) geodesics. Sharir and Schorr's algorithm finds the shortest path between two points on the surface of a convex polyhedron in time $O(n^3 \log n)$ where n denotes the number of edges in the triangulation.

[1] A spherical vertex is every vertex whose sum of adjacent angles is less than 2π.

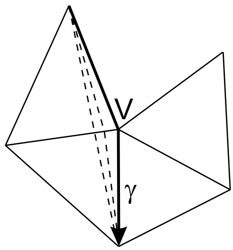

Figure 1: The geodesic shortest path cannot go through the spherical vertex.

Another kind of algorithms are the approximation solutions which convert the geometric problem to a graph problem and use Dijkstra's algorithm. Although the authors of these algorithms usually do not mention that their algorithm can work on surfaces with forbidden areas, the adjustment to such surfaces should not be difficult. As an example of a graph solution can be cited Kanai and Suzuki [3]. The complexity of their algorithm is of order $O(k \log k)$, where k is the number of edges passed by the shortest path, and alleged accuracy is within 0.4%.

One of the best, exact and already implemented algorithms has been proposed by Chen and Han [1] in 1990. This algorithm solves the single-source shortest path problem by unfolding the triangles into a plane within the time complexity $O(n^2)$. The algorithm was implemented in 2000 by Kaneva and O'Rourke [4] and the code is publicly available [2]. Unfortunately the algorithm cannot treat surfaces with forbidden areas.

In 2004 Deng and Zhou presented a new approach for the geodesic shortest path problem [2]. In preprocessing they first choose the region in which the global shortest path exists, afterward any other known algorithm can be applied.

Surazhsky et al. introduced an algorithm based on the development of the surface into the plane in 2005 [6]. The algorithm can also treat surfaces with forbidden areas: when the developed strip of the surface encounters the boundary of the forbidden area, a new pseudostartpoint is created and the development proceeds to the all directions from the beginning. The time complexity is $O(kn^2 \log n)$, where n is the number of the edges in the triangulation and k is the number of pseudostartpoints. Because the algorithm is not specially developed for surfaces with forbidden areas, the solution is not effective in such cases and k can grow arbitrarily high.

We will show how to treat surfaces with forbidden areas more effec-

[2]http://cs.smith.edu/~orourke/code.html.http://cs.smith.edu/~orourke/code.html.

tively. A scheme which can be added to any other known algorithm is presented. The scheme will not decrease the worst-case time complexity but it will lower the expected time complexity.

3 Geodesic shortest path with forbidden areas

3.1 Types of the problem

The geodesic shortest path problem can be categorized by the type of the surfaces and forbidden areas as follows

- Surface
  - convex or nonconvex
  - open (terrain) or closed
- Forbidden area
  - convex or nonconvex
  - connected or disconnected

Convex forbidden areas can be treated in the same way as the nonconvex forbidden area covered with the convex hull, on the assumption that none of two starting points lies inside the convex hull. In this article we will deal further only with a convex surface and a convex connected forbidden area.

3.2 Basic scheme

If the shortest path exists and if it has some intersections with the forbidden area, then:

- Between the intersections the shortest path is formed by geodesics (in the admissible areas).
- For the intersection points the following conditions hold:
  1. The intersection point cannot lie on an edge of the triangulation of forbidden area (fig. 2). This implies that the intersection point can coincide only with a vertex in a triangulation. But it cannot be an arbitrary vertex.
  2. The intersection point can coincide only with such vertex in the triangulation on the boundary of the forbidden area where the shortest path bends more than π. In other words there must exist an angle φ_B which satisfies:

$$\varphi_B = \varphi - \pi - \varphi_D, \tag{1}$$

     where φ is the total vertex angle around the vertex and φ_D is the vertex angle inside the forbidden area (fig. 3). Because φ_B

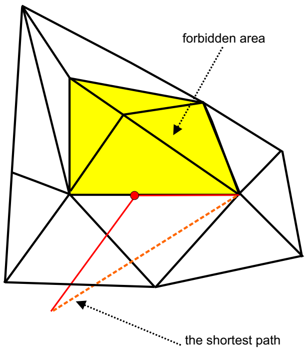

Figure 2: The intersection point cannot lay on the edge of the border of forbidden area - such paths can be shortened.

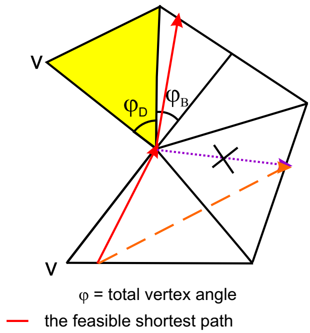

Figure 3: The shortest path at the convex vertex on the border of forbidden area must bend more than π.

must be non-negative: $\varphi_B \geq 0$, then the shortest path will go through the vertex only if it satisfies

$$\varphi - \varphi_D \geq \pi. \tag{2}$$

Vertices which fulfil the condition 2 will be denoted as admissible vertices. It is obvious that the shortest path cannot enter these vertices in arbitrarily direction; it must bend there more than π. The set of admissible vertices can be easy computed (see fig. 4).

If this scheme had been used in Surazhsky's algorithm, the number of pseudostartpoints would have been limited and reduced (in the time complexity $O(kn^2 \log n)$, $k = 6$ for the fig. 4). The scheme is independent of a

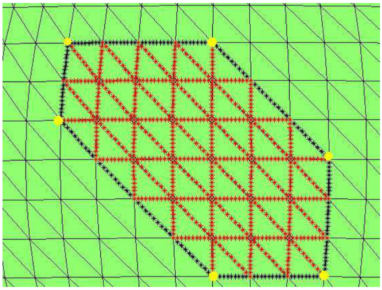

Figure 4: The feasible vertices.

concrete algorithm and can be inserted to any other algorithm solving the geodesic shortest path problem on the surface with or without forbidden areas.

3.3 Algorithm

In the last part we will present an algorithm for the geodesic shortest path problem which concentrates on the surfaces with forbidden areas and which uses our scheme. The approach uses the Zabransky's algorithm [7].

Zabransky's algorithm finds the shortest path between two points on a surface using the rule that there exists exactly one geodesic with the given direction emanating from the given point. It searches for all geodesics emanating from the starting point in various directions and chooses the shortest one going through the end point.

Our algorithm works in the following steps:

1. For all the directions coming out from the starting point find the unique geodesic. For all the geodesics:
   (a) If the geodesic meets an edge of the forbidden area or inadmissible vertex, break (go on with the next direction).
   (b) If the geodesic meets an admissible vertex from an admissible angle, then save its length if it is the first one, or compare its length with the length of the last saved geodesic in the vertex and if the new geodesic is shorter, save it, its length and the possible right and left angle of proceeding.
   (c) If the geodesic goes through the final point, compare its length with the length of the last saved geodesic and if the new one is shorter, save the geodesic and its length.
2. For all the feasible vertices v do:
   (a) Search for the geodesics in all feasible directions emanating from v and for all of them repeat 1(a)-1(c)

3. Create a graph whose nodes are the feasible vertices and the starting and final point, and the edges are the shortest found geodesics between them. Using a graph algorithm we find the shortest path from the starting point to the final point.

4 Conclusion

A scheme which helps to solve the geodesic shortest path problem with focus on surfaces with forbidden areas has been given. The scheme can be implemented into any known algorithm that solves the geodesic shortest path problem; the algorithm mentioned above is only one of the examples how the scheme can be used. The scheme does not decrease the worst-time complexity but it essentially decreases the average complexity.

In future work we would like to suggest a similar scheme for the non-convex surfaces and the disconnected forbidden areas.

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