Handout: Kronecker product and Vec-operator

Simultaneous equations and IV models
Exercise session

1 Kronecker product

Kronecker product, named after German mathematician Leopold Kronecker, is a special operator used in matrix algebra for multiplication of two matrices. This product, written as $\otimes$, gives the possibility to obtain a composite matrix of the elements of any pair of matrices. "any" stresses here that Kronecker product works without the assumptions on the size of composing matrices, as it is the case with ordinary matrix multiplication.

Definition 1 *Kronecker product of two matrices* $\mathbf{A} := (a_{ij})$ *and* $\mathbf{B} := (b_{st})$ *is defined as*

$$\underset{(m \times n)}{\mathbf{A}} \otimes \underset{(p \times q)}{\mathbf{B}} = \begin{pmatrix} a_{11}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{pmatrix}$$

The result of this product is a new matrix of order $(mp) \times (nq)$ composed of all possible $a_{ij}b_{st}$.

1.1 Some properties of Kronecker product

1. $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$

2. $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$

3. $(\mathbf{A} \otimes \mathbf{B})' = \mathbf{A}' \otimes \mathbf{B}'$

4. $(\mathbf{A} \otimes \mathbf{I}_n)(\mathbf{I}_m \otimes \mathbf{B}) = \mathbf{A} \otimes \mathbf{B} = (\mathbf{I}_m \otimes \mathbf{B})(\mathbf{A} \otimes \mathbf{I}_n)$

2 Vec-operator

Vec-operator, written as **vec**, is another linear algebra tool which is important for us in the multidimensional regression matrix representation. The mechanism of vec-operator is simple and can be applied to a matrix of any order. This operator transforms matrix into a vector, by stacking all the columns of this matrix one underneath the other.

Definition 2 *For any matrix* $\underset{m \times n}{\boldsymbol{A}}$ *with $\boldsymbol{i}$-th column defined as $\boldsymbol{a_i}$*

$$\boldsymbol{vec}(\mathbf{A}) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

2.1 Some properties of vec-operator

1. $\mathbf{vec}(\mathbf{A} + \mathbf{B}) = \mathbf{vec}(\mathbf{A}) + \mathbf{vec}(\mathbf{B})$

2. $\mathbf{vec}(\alpha\mathbf{A}) = \alpha \, \mathbf{vec} \, \mathbf{A}$

3. $\mathbf{vec} \, \mathbf{a}' = \mathbf{vec} \, \mathbf{a} = \mathbf{a}$, for any $\mathbf{a}$

2.2 Relationships of vec-operator and Kronecker product

1. $\mathbf{vec} \, \mathbf{ab}' = \mathbf{b} \otimes \mathbf{a}$ for any vectors $\mathbf{a}$ and $\mathbf{b}$

2. $\mathbf{vecABC} = (\mathbf{C}' \otimes \mathbf{A})\mathbf{vecB}$, whenever $\mathbf{ABC}$ is defined

3 Exercises

3.1 (Kronecker product)

Let
$$\mathbf{A} = \begin{pmatrix} 2 & 5 & 2 \\ 0 & 6 & 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 5 & 0 \end{pmatrix}, \mathbf{e}' = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

(a) Compute $\mathbf{I}_2 \otimes \mathbf{A}$ and $\mathbf{A} \otimes \mathbf{I}_2$
(b) Compute $\mathbf{A}' \otimes \mathbf{B}$
(c) Compute $\mathbf{A} \otimes \mathbf{e}$ and $\mathbf{A} \otimes \mathbf{e}'$

3.2 (Vec-operator)

Using matrices and vector in exercise above compute **vec A**, **vec A**$'$, **vec B**, **vec B**$'$, **vec e**, **vec e**$'$

3.3 (Relationship of vec-operator and Kronecker product)

Let $\mathbf{A}$ be an $(m \times n)$ matrix, $\mathbf{B}$ an $(n \times p)$ matrix, and $\mathbf{d}$ a $(p \times 1)$ vector. Show that

(a) $\mathbf{vecAB} = (\mathbf{B}' \otimes \mathbf{I}_m)\mathbf{vecA} = (\mathbf{B}' \otimes \mathbf{A})\mathbf{vecI}_n = (\mathbf{I}'_p \otimes \mathbf{A})\mathbf{vecB}$
(b) $\mathbf{ABd} = (\mathbf{d}' \otimes \mathbf{A})\mathbf{vecB} = (\mathbf{A} \otimes \mathbf{d}')\mathbf{vecB}'$
(c) $\mathbf{vecA} = (\mathbf{I}'_n \otimes \mathbf{A})\mathbf{vecI}_n = (\mathbf{A}' \otimes \mathbf{I}_m)\mathbf{vecI}_m$

References

[1] Chiang, A. (1984), Fundamental Methods of Mathematical Economics, McGraw-Hill, Inc.

[2] Abadir, K.M and Magnus J.R. (2005), Matrix Algebra, Cambridge University Press.