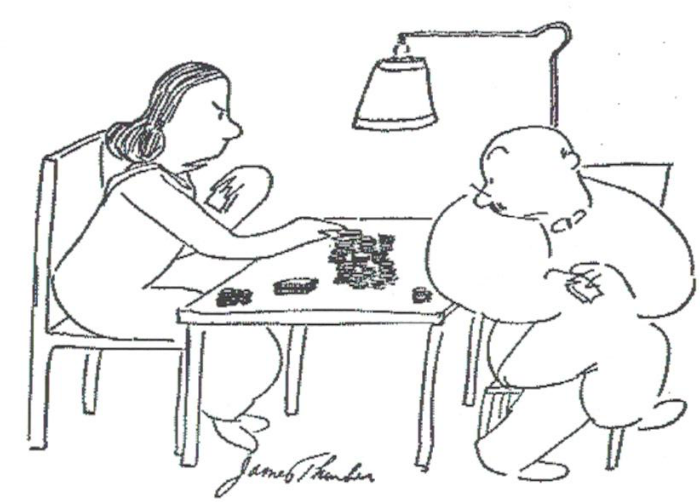
Resolving Conflicts with *Mathematica*: Algorithms for two-person games

Chapter 1: Non-Cooperative Games



Why do you keep raising me when you know I'm bluffing?

James Thurber's cartoon has a lot to do with the subject of this book. It deals not only with a so-called *non-cooperative game* (in this case poker), not only with the *strategies* involved therein (raising, bluffing), not only with the *information state* of the players (she knows that he knows that she's bluffing) and not only with the typical *nonzero-sum* character of the eternal battle of the sexes (see below). Above all it has to do with *rationality*, for it is the woman's delightful lack of it that makes us laugh. Game theory is all about finding rational solutions for people involved in conflict situations, whether these be family poker games or things considerably more serious.

The theory of games can be traced back to Zermelo (1913), who examined what are now referred to as games of perfect information. He showed that chess, for example, is uninteresting because it has a strategy in which either White or Black can force a win or a draw. (Fortunately, Zermelo wasn’t able to specify that strategy further, so that chess is still played avidly by men, women and machines.) But the true birth of the discipline coincided with the appearance of the *Theory of Games and Economic Behavior* by von Neumann and Morgenstern (1947). That book was an inspiration to mathematicians, economists and social and political scientists all over the world. Together with Nash’s proposal of strategic equilibrium as a prerequisite for rational behavior (Nash 1951), it set the foundations of game theory. A half-century of research thereafter culminated with the awarding of the Nobel Prize for Economics in 1994 to John Harsanyi, John Nash and Reinhard Selten for their contributions to the analysis of equilibria in non-cooperative games.

The conflicts which are often chosen to demonstrate the application of game theory -- one thinks immediately of the *prisoners’ dilemma* -- although throwing some light on the complexities of human interaction, sometimes to leave us with a sense of triviality or, at least, of oversimplification. We may conclude that the theory is ultimately of little more than paradigmatic value. This impression is, however, quite wrong, for game theory can be an outstanding tool for *modeling* conflict situations. It is capable of formalizing real, practical problems, it gives us the tools to think about them in a logical and structured way and it can often suggest useful solutions. Applications of game theory range from the analysis of the simple but ubiquitous prisoners’ dilemma to the complex design of auctions for telecommunications frequencies, from the frivolity of writing poker-playing programs to the seriousness of calculating optimal military strategies, from the theory of the market place to the theory of evolution. One recurring theme of this book will be the illustration of the modeling capabilities of game theory, particularly with the class of so-called *inspection games*, applying the algorithmic methods developed in the initial chapters to find solutions quickly.

Many techniques have been proposed for computing solutions to game-theoretical problems and we shall of course be meeting some of these soon. Generally fast methods exist for solving two-person, zero-sum games with finitely many strategies, because such games turn out to be equivalent to linear programs. An efficient algorithm to find at least one solution to a nonzero-sum game was developed by Lemke and Howson (1964). It is of particular interest as it provides an elementary constructive proof of the existence of equilibria in bimatrix games. Unfortunately, determining all equilibria of games with finitely many strategies is NP-hard (Gilboa and Zemel 1989). This means that there most probably exists no computationally efficient algorithm for solving them completely, a fact that we shall have to live with but which will not deter us, in the course of this book, from developing useful *Mathematica* programs.

The three-volume *Handbook on Game Theory* (Aumann and Hart 2002) gives an in-depth overview of the present state of the subject and of the many categories and sub-categories of games and their applications. At the highest level one distinguishes non-cooperative (or strategic) games from cooperative (or coalitional) games. In our excursion with *Mathematica* into the realm of game theory and rational behavior we will in fact restrict ourselves to non-cooperative, two-person games. The former, because non-cooperative games are in some sense more fundamental than their cooperative counterparts. (Harsanyi and Selten (1988) argue that cooperation can and should be modeled non-cooperatively.) The latter, because two-person conflicts are illustrative of the depth and subtlety of non-cooperative games in general, but especially because there exist useful algorithms for solving them. They are thus more amenable both to the powers of *Mathematica* and to the programming addictions of the author.

Keeping for the time being to the subject of quarreling spouses a la James Thurber, we'll begin by introducing some basic concepts informally with the help of a classic two-person game

The Battle of the Sexes

Consider the following standard -- and admittedly stereotyped -- situation on the playing field of wedlock: *She* wants to attend a concert, but there happens to be an important hockey game to which *he* would prefer to go. Fortunately (or, rather, unfortunately in this case) the marriage is intact and neither of the two would like to go out alone. What to do? Let's model this little marital problem as a two-person, non-cooperative game.

Extensive Form: Ladies first

The two alternatives, namely attend the concert or go to the hockey game, define the *strategies* of the wife, whom we shall refer to as player 1, and of her husband, player 2. We'll call these strategies C and H, respectively. Their mutual choice comprises a *strategy profile* which uniquely determines the *outcome* of the conflict. The terms *strategy profile* and *outcome* are in this sense interchangeable. We can denote the possible outcomes as CC, meaning that they attend the concert together, CH, she goes to the concert, he to the hockey match, and so on. The game is non-cooperative in the sense that, although the players may discuss the problem among themselves, binding agreements as to choice of strategy are not allowed. Moreover, each partner is assumed to be *rational* in that he or she chooses a strategy entirely on the basis of personal preference, at the same time taking full account of the strategic alternatives of the other partner. Furthermore, each player is convinced that the other player is similarly rational.

Let $\succeq_i$ denote an ordering relation for the preference of player i. If outcome X is at least as good as outcome Y we write $X \succeq_i Y$. Thus we might reasonably assume

$\mathrm{CC} \succeq_1 \mathrm{HH} \succeq_1 \mathrm{CH} \succeq_1 \mathrm{HC}$

for the wife, and

$\mathrm{HH} \succeq_2 \mathrm{CC} \succeq_2 \mathrm{CH} \succeq_2 \mathrm{HC}$

for the husband. Note that although we are formulating a non-cooperative game, the cooperative .aspect, namely the tendency of the married couple to do things together, is included in our model.

The players' preferences should be consistent with rational behavior. We interpret this as meaning that the preference relation is transitive: if $X \succeq_i$ Y and $Y \succeq_i$ Z then $X \succeq_i Z$, and also that it is complete: for all pairs of outcomes X and Y, either $X \succeq_i$ Y or $Y \succeq_i$ X or both, that is, each player always has a preference or is indifferent.

To begin with, we'll make things easy on ourselves by allowing the wife to choose first and then announce her decision to her husband. She does this, say, by going downtown and buying a ticket, either for the concert or the hockey game, and then showing the *fait accompli* to her husband, who must then decide how to react. This situation can be described as a game of *perfect information*. It is shown in so-called *extensive form* in Figure 1.1.

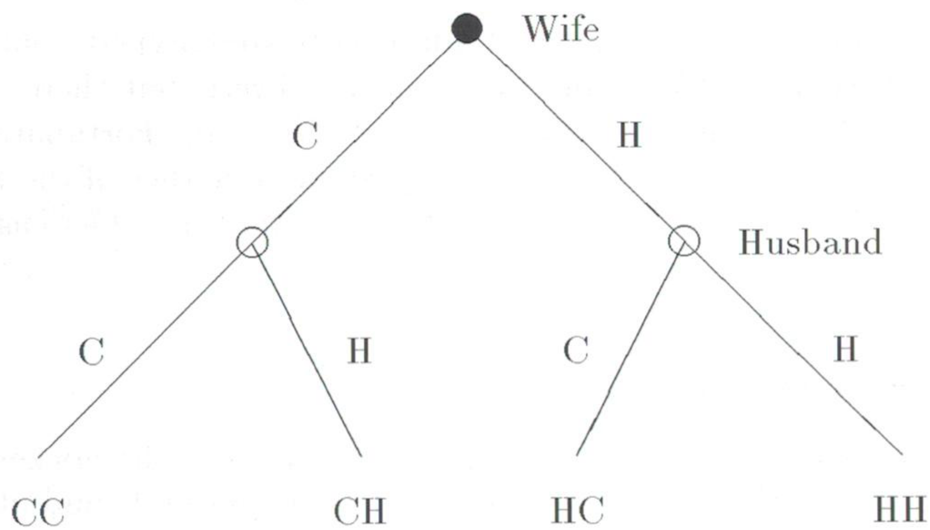


Figure 1.1. The *battle of the sexes* as an extensive form game of perfect informa

The game representation in this figure is almost self-explanatory. The non-terminal nodes of the tree represent decision points for the protagonists, and the terminal nodes correspond to the four possible outcomes. The game is said to be one of perfect information because each player knows the precise situation that has been reached in the game when required to make a decision. The extensive form is a fundamental and complete way of representing a non-cooperative game. Games of perfect information when represented in this way can always be solved, at least in principle, by *backward induction.* Thus the wife knows that if she takes egotistic advantage of her feminine privilege and plays strategy C, her husband will find himself at his left-hand node, whereupon examining the two terminal nodes still available to him, he will also choose strategy C, following his equally egotistic preferences. She can therefore guarantee her optimal outcome CC by playing C and she will no doubt do so.

Normal form: Bimatrix games

But suppose husband and wife order their tickets simultaneously. We can represent this again in extensive form, but now as a game of *imperfect information* as shown in Figure 1.2.

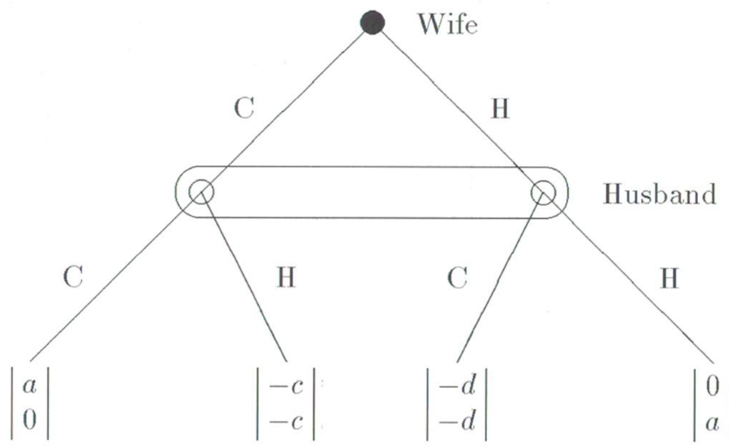


Figure 1.2. The *battle of the sexes* as an extensive form game of imperfect information and with outcomes labeled in terms of utilities (see text). The oval represents the information set of player 2.

The husband's *information set*, shown as an oval in the figure, encompasses both of his decision points. This is meant to imply that he doesn't know at which node he is situated when choosing his strategy. Clearly this representation of the game is not unique, since we could just as well have placed the husband's move at the root of the tree. In any case, backward induction no longer works here so we shall have to find a different solution method.

To this end we define *utility functions* h_i which, for player i=1,2, map the game's outcomes onto the real numbers in such a way as to preserve their preference relations. Thus $X \succeq_i Y$ if and only if $h_i(X) \geq h_i(Y)$.

In our *battle of the sexes* we shall assume symmetrically that

$h_1(\text{CC}) = a,\ h_2(\text{CC}) = 0$
$h_1(\text{HH}) = 0,\ h_2(\text{HH}) = a$
$h_1(\text{CH}) = -c,\ h_2(\text{CH}) = -c,$
$h_1(\text{HC}) = -d,\ h_2(\text{HC}) = -d$

where, according to the players' preferences,

$a > 0 > -c > -d.$

The missing letter *b* has simply been set to zero, illustrating a general property of utility functions, namely that any transformation of the form

$h' = \text{mh} + e,\ m > 0,$

can be applied to them without changing the strategic situation. (It is not intended to delve more deeply into utility theory in this text. See Morrow (1994), Chapter 1, for an informal but detailed treatment.) The utilities defined above have been used to label the terminal nodes of the game tree in Figure 1.2.

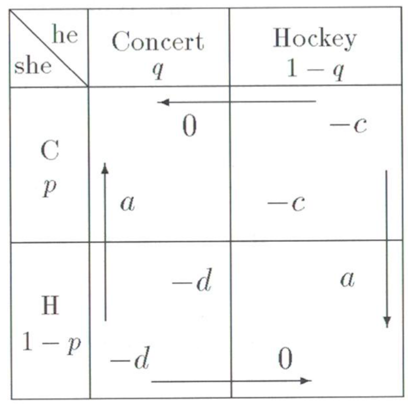


Figure 1.3. The game of Figure 1.2 in normal or bimatrix form. The numbers in the lower left hand corners of each box are the utilities of player 1, also called the *row player*, those in the upper right hand corners are the utilities of player 2, the *column player*. The arrows are the players' preference directions; p and q denote their mixed strategies (see text).

The simultaneous game is shown in its *normal form* in Figure 1.3. For two-person games this is a matrix of utilities or *payoffs* for the outcomes of all strategy combinations of the two players, often referred to as a *bimatrix*. The preference arrows shown in the figure lead us at once to a possible solution of our simultaneous game. The leftmost vertical arrow, for example, indicates what the wife would do if she had chosen strategy H, but was convinced -- on rational grounds -- that her husband was going to opt for C. She would immediately change her decision and go to the concert with him, thereby improving her payoff from -d to a. The other arrows (the horizontal ones refer of course to the husband) have similar interpretations. Thus the strategy combination (C,C) has the special property that *neither player has an incentive to depart from it unilaterally*. It is referred to as a *Nash equilibrium* (or simply *equilibrium*) of the game, and the corresponding strategies are said to constitute an *equilibrium pair*.

Nash equilibrium is the fundamental solution concept for non-cooperative games, since, as we shall try to show shortly, only equilibrium strategies are consistent with the presumed rationality of the players. Moreover, as we shall also see shortly, every bimatrix game has at least one equilibrium. Therefore we can be assured that any conflict modeled in this way will have a solution.

However Figure 1.3 also betrays the greatest weakness of this solution method: A non-cooperative game can have *more* than one Nash equilibrium. The *battle of the sexes* demonstrates the weakness especially clearly, since the second equilibrium indicated by the preference arrows, namely (H,H), is preferred by player 2, while player 1 is happy to stay with (C,C). If they decide simultaneously for their “favorite” equilibria, they will of course land in the upper right hand corner of the bimatrix with negative payoffs for all concerned. The poor game theorist is no better off, for how can he make a recommendation without taking sides? Worse still, the game has yet another equilibrium, as we shall now demonstrate.

Mixed Strategies

Before proceeding further, it is about time that we invoked *Mathematica*, if only for a few mundane tasks. First of all let's load the Wolfram language script `bimatrix.m`, the explanation and use of which will occupy a good part of this book.

```
<< src/bimatrix.m
```

General::compat : Combinatorica Graph and Permutations functionality has been superseded by preloaded functionality . The package now being loaded may conflict with this . Please see the Compatibility Guide for details .

One of the functions exported by this package is `BimatrixForm[A,B,Options]`, which constructs a simple bimatrix from the players' individual payoff matrices:

```
A = {{a, -c}, {-d, 0}};
B = {{0, -c}, {-d, a}};
BimatrixForm [A, B]
```

	S_1	S_2
R_1	a \ 0	-c \ -c
R_2	-d \ -d	0 \ a

The strategies of player 1 (row player) are automatically labeled $R_1 \ldots R_m$, those of player 2 (column player) are labeled $S_1 \ldots S_n$ after the German word *Spalte* = column. Optionally the user can provide his own labels.

Rather than deciding on a single pure strategy as the best plan of action, a player can choose at random from some appropriate set of his or her pure strategies. We say that the player is using a *mixed strategy*, which is a probability distribution over the actual strategic alternatives. The latter shall from now on be referred to as *pure strategies* in order to make the distinction clear.

Although playing a mixed strategy can never improve a player's chances against any given pure strategy of the opponent (there is always a pure strategy which is just as good), it can deter the opponent from using strategies which are harmful. Thus the purpose of a mixed strategy is essentially defensive. If the husband knows that his spouse may behave with some degree of randomness, he may have to rethink his own actions and perhaps respond with a mixed strategy himself.

The married couple's mixed strategies can be represented as column vectors in the form

$P = (p, 1 - p)^T$ and $Q = (q, 1 - q)^T$

for wife and husband respectively, where T denotes transposition and where p and q are the respective probabilities of playing the pure strategy C. Since the mixed strategies P and Q are chosen independently, the resulting *expected payoff* to player 1 (with payoff matrix A) is given by

$$H_1(P, Q) = p \cdot [\, q\, (A)_{11} + (1 - q)\, (A)_{12}\,] + (1 - p) \cdot [\, q\, (A)_{21} + (1 - q)\, (A)_{22}\,],$$

where $(A)_{ij}$ is th ijth element of A. This can be written more simply in matrix notation as

$$H_1(P, Q) = P^T A Q.$$

Mathematica, by the way, doesn't differentiate between row and column vectors, rather it represents both by a simple list structure. We won't take this as an excuse to do likewise, however we should keep in mind that a quadratic form like $P^T AQ$ is coded in *Mathematica* as P.A.Q:

```
P = {p, 1 - p};
Q = {q, 1 - q};
P.A.Q
```

```
-c p (1 - q) + (-d (1 - p) + a p) q
```

If both players can choose mixed strategies -- call them P^* and Q^* -- such as to make the respective opponent *completely indifferent* as to his or her strategy choice, then we have by definition another equilibrium. Neither player has any incentive to depart unilaterally from (P^*, Q^*). The *battle of the sexes* indeed possesses such an equilibrium, its third:

```
{p*} = p /. Solve[Transpose[B][[1]].P == Transpose[B][[2]].P, p];
{q*} = q /. Solve[A[[1]].Q == A[[2]].Q, q];
H1 = P.A.Q /. {p → p*, q → q*};
H2 = P.B.Q /. {p → p*, q → q*};
Simplify[{{p*, 1 - p*}, H1, {q*, 1 - q*}, H2}]
```

$$\left\{\left\{\frac{a+d}{a+c+d}, \frac{c}{a+c+d}\right\}, -\frac{c\,d}{a+c+d}, \left\{\frac{c}{a+c+d}, \frac{a+d}{a+c+d}\right\}, -\frac{c\,d}{a+c+d}\right\}$$

The calculation method used here is very simple. For instance the second line in the input cell above,

```
{q*} = q /. Solve[A[[1]].Q == A[[2]].Q, q];
```

determines the mixed strategy for player 2 which will make player 1 indifferent as to her choice of either of her two pure strategies, and hence to any mixture of them as well. That is, the matrix equation

$(1, 0)\, A\, Q = (0, 1)\, A\, Q$ or equivalently $(A)_{1\cdot}\, Q = (A)_{2\cdot}\, Q$

is solved for q, with the result shown in the output cell, namely

$$q^* = \frac{c}{a+c+d}.$$

We are using the convention that $(A)_{i\,\cdot}$ and $(A)_{\cdot j}$ refer respectively to the ith row and jth column of the matrix A. In *Mathematica* these are coded as `A[[i]]` and `Transpose[A][[j]]`.

The equilibrium payoffs $H_1{}^*$ and $H_2{}^*$ turn out to be -- one is inclined to say `thank heaven' -- negative:

$$H_1(P^*, Q^*) = H_2(P^*, Q^*) = \frac{-c\,d}{a+c+d},$$

and hence less attractive for both players. The mixed equilibrium (P^*, Q^*) is said to be *payoff dominated*, and would probably neither be recommended by an outsider nor be chosen spontaneously by the players. As we'll see in the next section, mixed strategy equilibria cannot in general be dismissed so easily. At any rate, neither of the other two equilibria are dominated in this sense and non-cooperative game theory is simply not able to offer a recommendation to our married couple. Put positively, the theory at least shows clearly why, for two egotistical protagonists in such a situation, there cannot be a satisfactory resolution.

One further remark: The perfect information game of Figure 1.1 can be also be represented in normal form:

```
A = {{a, a, -c, -c}, {-d, 0, -d, 0}};
B = {{0, 0, -c, -c}, {-d, a, -d, a}};
BimatrixForm [A, B]
```

	S_1	S_2	S_3	S_4
R_1	a, 0	a, 0	−c, −c	−c, −c
R_2	−d, −d	0, a	−d, −d	0, a

Here, the husband's strategies are

S_1 = choose C regardless of what ticket my wife shows me,

S_2 = go along with whatever she chooses,

S_3 = perversely buy a ticket for the opposite event, and

S_4 = just go to the damned hockey game.

The wife's strategies are R_1 = C and R_2 = H, as before. We see that, according to our definition, there are two pure strategy Nash equilibria (R_1, S_1) and (R_1, S_2) whose outcomes correspond to the extensive form game's unique backward induction solution, but also an equilibrium (R_2, S_4) with outcome HH which does not. There is no real problem here, since for games of perfect information there is usually no need to resort to the normal form at all. However this is not the case for imperfect information games, and the problem of "unreasonable" equilibria can become a serious one. We will have more to say about this in Chapter 7.

Rationality

Unlike the *battle of the sexes*, the following (2×2)-dimensional bimatrix game, taken from Van Damme (1991),

```
A = {{2, 4}, {4, 3}};
B = {{2, 1}, {1, 3}};
BimatrixForm [A, B]
```

	S_1	S_2
R_1	2 / 2	4 / 1
R_2	4 / 1	3 / 3

obviously possesses no equilibria in pure strategies. The (imaginary) preference arrows are cyclic. However we can easily find a mixed
strategy equilibrium using the same trick as in the preceding section:

```
{p*} = p /. Solve[Transpose[B][[1]].P == Transpose[B][[2]].P, p];
{q*} = q /. Solve[A[[1]].Q == A[[2]].Q, q];
H1 = P.A.Q /. {p → p*, q → q*};
H2 = P.B.Q /. {p → p*, q → q*};
{{p*, 1 - p*}, H1, {q*, 1 - q*}, H2}
```

$$\left\{\left\{\frac{2}{3}, \frac{1}{3}\right\}, \frac{10}{3}, \left\{\frac{1}{3}, \frac{2}{3}\right\}, \frac{5}{3}\right\}$$

Thus player 1 plays mixed strategy $P^* = (2/3, 1/3)^T$ at equilibrium and player 2 counters with $Q^* = (1/3, 2/3)^T$. They receive expected payoffs $H_1{}^* = 10/3$ and $H_2{}^* = 5/3$ respectively. Since this is in fact the unique Nash equilibrium of the game, and since rational players should only play Nash equilibrium strategies, we are done.

Or are we?

Jein, as Germans love to say when things get pleasantly ambivalent. We've jumped into deep water without knowing it, so we'd better learn how to swim.

Let's examine the situation at equilibrium from the point of view of player 2. He plays strategy Q^*, but his opponent's strategy P^* actually makes him thoroughly indifferent as to what he could play. He might just as well play $(1, 0)^T$, $(1/2, 1/2)^T$ or, indeed, anything at all, for his payoff of 5/3 is assured. He has no incentive to depart from Q^*, but he apparently also has no reason *not* to do so. But if he does depart from equilibrium, *player 1 will not receive her expected payoff*. If for instance player 2 were to choose $(1, 0)^T$, player 1 would get a more modest 8/3 instead of 10/3 as reward for her loyalty to the equilibrium principle. Of course the argument goes the other way round, so that both players might well question the wisdom of the game theoretical recommendation to play only equilibrium strategies.

In fact there exists a rather attractive alternative. Player 1 can play a so-called *maxmin* strategy which guarantees her a certain payoff no matter what her opponent decides to do. She simply chooses that strategy $\hat{P}$ which gives her the same payoff against both pure strategies of player 2, and hence against any mixture of the two. In essence, player 1 is being deliberately paranoid, assuming that player 2 has lost interest in his own utilities and is trying to hurt her as much as possible. The best she can then do is to make him indifferent to whichever pure strategy he "hits her with". (Note that *his* indifference is now in terms of *her* utilities!) Player 2 can reason similarly, and we obtain the following strategies and payoffs:

```
{p̂} = p /. Solve[P.A[[1]] == P.A[[2]], p];
{q̂} = q /. Solve[Transpose[B][[1]].Q == Transpose[B][[2]].Q, q];
Ĥ1 = P.A.Q /. {p → p̂, q → q̂};
Ĥ2 = P.B.Q /. {p → p̂, q → q̂};
{{p̂, 1 - p̂}, Ĥ1, {q̂, 1 - q̂}, Ĥ2}
```

$$\left\{\left\{\frac{1}{3}, \frac{2}{3}\right\}, \frac{10}{3}, \left\{\frac{2}{3}, \frac{1}{3}\right\}, \frac{5}{3}\right\}$$

whereupon we see how perfidious this little game is. The maxmin strategies

$\hat{P} = (1/3, 2/3)$ and $\hat{Q} = (2/3, 1/3)$

actually guarantee both players their Nash equilibrium payoffs. Then why in heaven's name should they prefer to play the unsure equilibrium pair (P^*, Q^*)? Now we have to start swimming or, together with non-cooperative game theory, drown. What is going through the mind of our friend, player 2, when he decides to play his maxmin strategy?

I am, in this game at any rate, a rational, egotistic person and, as such, I am exclusively concerned with maximizing my utility, of course taking full account of the strategic alternatives of my opponent. Therefore I shall play the strategy $\hat{Q}$, since no matter what she does, I am guaranteed the expected payoff 5/3. It goes without saying that my opponent, who I am convinced is as rational and egotistic as I, will play her maxmin strategy $\hat{P}$ as well. Let her, it's her business.

Objection, your Honor! Player 2 might be egotistic , but he is **not** rational. For if he really believes everything he has just thought through, then he must switch to the strategy $Q = (0, 1)^T$ and obtain the expected payoff

```
P.B.Q /. p → p̂ /. q → 0
```

$$\frac{7}{3}$$

His opponent, who is apparently also not stupid, will surely realize this and counter with ... Oh dear!

The maxmin strategies ($\hat{P}$,$\hat{Q}$) are simply not equilibrium strategies. (In Chapter 3 we'll see that, for zero-sum games, the distinction between maxmin and equilibrium strategies disappears.) They cannot be played by players who claim to be behaving rationally, and the same goes for *all* strategy combinations other than (P^*, Q^*). Myerson (1991) puts it best:

[It is] not directly argued that intelligent rational players must use equilibrium strategies in a game. When asked why players in a game should behave as in some Nash equilibrium, my favorite response is to ask "Why not?" and to let the challenger specify what he thinks the players should do. If this specification is not an equilibrium, then we can show that it would destroy its own validity if the players believed it to be an accurate description of each other's behavior.

So we seem to be doomed for better or for worse to cling to Nash equilibrium as our solution concept, despite its weaknesses. Were we now to determine that some games don't possess any equilibria at all, we would surely throw in the towel and turn to a less pathological discipline than game theory. The remainder of the chapter will fortunately spare us this humiliation.

Continue with this Chapter in resolvingconflicts1a.nb