



CIEM0000 U2. ***Mechanics for Civil Engineering***

Complement to class notes

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Preface

The following notes are based on the classes imparted at TU Delft during the first quarter of the year 2023. CIEM0000 Interdisciplinary mechanics is the first course for all students enrolled in the Civil Engineering MSc. As the name implies, the content covered a wide range of mechanics problems. The course is organized in 14 topics, but I decide to forgo that structure and instead divide the course into 4 chapters.

The classes were imparted by the following professors

Continuum mechanics - Basics: Karel van Dalen & Ad Reniers & Phil Vardon

Linear waves: Robert Jan Labeur & Karel van Dalen

Diffusion- Heat Conduction: Phil Vardon

Amplitude dependant waves: Robert Jan Labeur & Ludovic Leclercq

This document is intended to be a community project from the MSc student at TU Delft, hopefully this document will be used, improved and extended in the following years.

Feel free to either download, share and/or modified this document, the *.tex* files are available in

<https://github.com/GFFollet/CIEM0000-U2.git>

Contributors

Again, if this document helped you, consider reporting or fixing any mistake you may have found, from simple spelling errors (I'm sure there will be lot of them) to more substantial changes.

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Contents

Contents	2
1 Continuum Mechanics: Basics	3
1.1 Kinematics	3
1.2 Action-Deformation	5
1.3 Balance Laws	6
1.4 Simplifications/Reduction of balance laws	7
Simplified differential form	7
Dimensional reduction	8
1.5 Case Study: Darcy's law and potential flow	10
1.6 Case Study: Simplified flow through soil	11
1.7 Case Study: Flow to a well	11
Flow through 1D soil layer	11
Flow through 1D soil, with discontinuity in soil properties	12
Vertical 1D flow through isotropic layer	13
Radial flow	14
1.8 Case study: Static deformation of a bar with distributed elastic foundation	14
2 Linear Waves	16
2.1 Equations of motion	16
2.2 Long Waves	18
Boundary Resistance	18
2.3 Dimensional analysis	19
2.4 Classification/Characterization of long waves	20
Translatory waves	20
Tsunami wave	20
Seiches	20
Tides	21
Flood waves	21
2.5 Wave propagation	21
2.6 General solution: d'Alembert	22
Boundary conditions	23
2.7 Periodic Waves	24
Boundary conditions	24
2.8 Case Study: Dispersive wave in foundation pile	26
Boundary Conditions: Infinite Pile	27
3 Diffusion: Heat Conduction	28
3.1 Homogeneous BC	28
References	30

1 Continuum Mechanics: Basics

Every continuum mechanics problem can be reduced to three basic steps

1. Kinematics
2. Action-Deformation
3. Equilibrium

1.1 Kinematics

Let's consider the following expression

$$\varphi(X) = X + u(X)$$

Where

X : Initial position

$u(X)$: Displacement field

$\varphi(X)$: Final position

Then we can define

Lagrangian Strain Tensor

$$E := \frac{1}{2} (\nabla u + \nabla u^T + \nabla u \nabla u^T)$$

For *small* displacement gradients

Cauchy Strain Tensor

$$E \simeq \varepsilon := \frac{1}{2} (\nabla u + \nabla u^T)$$

In civil engineering, the Cauchy Tensor (Engineering tensor) is almost always used. See [5] for rigorous definition/proof.

In matrix form

$$\varepsilon = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial x} \right) & \frac{1}{2} \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \\ \frac{1}{2} \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right) & \frac{\partial u_y}{\partial y} & \frac{1}{2} \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) \\ \frac{1}{2} \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \right) & \frac{1}{2} \left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} \right) & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

We can also define the shear strain angle as

$$\gamma_{i,j} = \alpha + \beta$$

Where:

$$\tan(\alpha) = \frac{\frac{\partial u_j}{\partial i}}{1 + \frac{\partial i}{\partial u_i}} \quad \tan(\beta) = \frac{\frac{\partial u_i}{\partial j}}{1 + \frac{\partial j}{\partial u_j}}$$

For *small angles* and *small displacement gradients*, the shear strain angle is

$$\gamma_{i,j} = \gamma_{j,i} = \frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i}$$

Then we can express the Cauchy Strain Tensor as

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \epsilon_{yy} & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \epsilon_{zz} \end{bmatrix}$$

Because the Cauchy Strain Tensor is symmetrical we can express it as a vector using Voigt notation

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

In some cases it may be useful to compute the rigid body rotation

$$\boldsymbol{\theta} = \nabla \times \boldsymbol{\epsilon} = \begin{bmatrix} 0 & \frac{1}{2} \left(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial x} \right) & \frac{1}{2} \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right) \\ -\frac{1}{2} \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) & 0 & \frac{1}{2} \left(\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y} \right) \\ -\frac{1}{2} \left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z} \right) & -\frac{1}{2} \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right) & 0 \end{bmatrix}$$

Analogous to the Cauchy Strain Tensor, we define the Strain Rate Tensor

Cauchy Strain Rate Tensor

$$\dot{\boldsymbol{\epsilon}} := \frac{1}{2} (\nabla \dot{\mathbf{u}} + \nabla \dot{\mathbf{u}}^T)$$

Then, the rotation of a fluid element is

$$\boldsymbol{\Omega} = \nabla \times \mathbf{v}$$

From linear algebra we know that we can change basis such that

$$\boldsymbol{\epsilon}' = \mathbf{Q} \boldsymbol{\epsilon} \mathbf{Q}^T$$

If we use a unit vector (\hat{n}) instead of the basis matrix, (\mathbf{Q}) we can obtain the deformation in the direction \hat{n} we obtain the projection in that direction.

$$\epsilon_{axial}(\hat{n}) = \hat{n}^T \boldsymbol{\epsilon} \hat{n}$$

If we want to obtain the principal directions, we need to solve the following minimization problem

$$\begin{aligned} \max/\min \quad & (\hat{N}^T \boldsymbol{\epsilon} \hat{N}) \\ \text{such that} \quad & |\hat{N}| = 1 \end{aligned}$$

Using the Lagrange multipliers method, we get the problem is equivalent to an eigenvalue problem

$$(\epsilon - \lambda I) N = 0 \quad \text{where } |N| = 1$$

The fact that we can always find a basis such that we can diagonalize the strain/stress tensor, means that we can find a frame of reference (or direction) in which there is no shear or only shear. In many cases we can just reduce the problem to principal stresses.

As the name suggests, the invariants of a tensor are properties of the tensor that do not depend on the basis, the most common ones are

$$\begin{aligned} I_1 &= \text{tr}(\epsilon) \\ I_2 &= \frac{1}{2} (\text{tr}(\epsilon)^2 - \text{tr}(\epsilon^2)) \\ I_3 &= \det(\epsilon) \end{aligned}$$

1.2 Action-Deformation

In linear materials

$$\sigma = A\epsilon$$

For an isotropic linear-elastic material, the expression can be simplified to

Generalized Hook's Law

$$\sigma = C\epsilon = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \epsilon$$

Where C is the Stiffness matrix.

In most civil engineering problems Hook's Law is sufficiently accurate, except maybe in

- (Foliated) Rock masses, the isotropic hypothesis is not accurate, we may need up to 21 elastic constant. Using ultrasonic/seismic methods we can determine the elastic constant in multiple direction[6].
- Soil Mechanics, in particular when the clay content is important it is necessary to use elastoplastic models [3].
- Earthquake Engineering, even without going for the full nonlinear modeling, implementing a hybrid approach using plastic hinges (elastoplasticity) may reduce the complexity but yield accurate enough results, same for the design of metallic dissipation devices[4].

It may be useful to include the effect of temperature and initial stress (prestressed structures)

$$\sigma = C(\epsilon - \epsilon_0) + \sigma_0$$

Likewise, we can also express this relationship in the other way using the Flexibility matrix (S)

$$\epsilon = S\sigma$$

$$\mathbf{S} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & (1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & (1+\nu) \end{bmatrix}$$

The elastic properties of the material can be also expressed with the Lamé constants.

$$\lambda = \frac{E\nu}{(1-\nu)(1-2\nu)}$$

$$\mu = G = \frac{E}{2(1+\nu)}$$

1.3 Balance Laws

Let's consider an intensive property f , then

$$F = \int_B f \rho dV$$

Where F is the equivalent extensive property, then, we can express multiple conservation laws through the following equation

Master Balance Principle

$$\frac{DF}{Dt} = \frac{D}{Dt} \int_B f(x, t) dV = \int_{\partial B} \phi(x, t, \hat{n}) dS + \int_B \Sigma(x, t) dV$$

The Surface(ϕ) and Volume(Σ) density terms represent the external *forces*.

The previous equation is extremely general and not really used in practice, but I think is necessary to at least know the proper formulation, see [5] for full proof.

Then, we can construct multiple conservation laws, choosing different values of f, ϕ, σ .

Conservation of Mass

$$M = \int_V \rho dV$$

$$\frac{DM}{Dt} = \frac{D}{Dt} \int_{\Omega} \rho(x, t) dV = 0$$

Conservation of Linear Momentum

$$\mathbf{p} = \int_V \rho \mathbf{v} dV$$

$$\frac{D\mathbf{p}}{Dt} = \frac{D}{Dt} \int_{\Omega} \rho \mathbf{v} dV = \int_{\partial\Omega} \mathbf{t} dS + \int_{\Omega} \mathbf{b} dV$$

Conservation of Angular Momentum

$$\mathbf{L} = \int_V \mathbf{r} \times \rho \mathbf{v} dV$$

$$\frac{D\mathbf{L}}{Dt} = \frac{D}{Dt} \int_V \mathbf{r} \times \rho \mathbf{v} dV = \int_{\partial\Omega} \mathbf{r} \times \mathbf{t} dS + \int_V \mathbf{r} \times \mathbf{b} dV$$

First Law of Thermodynamics

$$\mathbf{E} = \int_V \rho e dV$$

$$\frac{DE}{Dt} = \frac{D}{Dt} \int_V \left(gz + \frac{v^2}{2} + \hat{u}(T) + \frac{P}{\rho} \right) \rho dV = \int_{\partial\Omega} \phi(x, t, \hat{n}) dS + \int_V \Sigma(x, t) dV$$

In gases, enthalpy is used instead

$$\hat{h}(T) = \hat{u}(T) + \frac{P}{\rho}$$

1.4 Simplifications/Reduction of balance laws

Simplified differential form

As we previously stated, the master conservation principle is not used in practice instead we use simplified equation in differential form.

Simplified Master Balance Principle in Differential form

$$\frac{\partial Q}{\partial t} = -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z}$$

Which essentially translates, "The variation of Q is equal to the fluxes through the boundaries"

Then the conservation laws are

Simplified Mass Conservation

If we choose

$$Q = \rho$$

$$q_i = \rho v_i$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho v_x}{\partial x} - \frac{\partial \rho v_y}{\partial y} - \frac{\partial \rho v_z}{\partial z}$$

Simplified Conservation of Momentum

If we choose

$$Q = \rho v_i$$
$$q_i = \rho v_i v_j$$

$$\begin{aligned}\frac{\partial \rho v_x}{\partial t} &= -\frac{\partial v_x(\rho v_x)}{\partial x} - \frac{\partial v_y(\rho v_x)}{\partial y} - \frac{\partial v_z(\rho v_x)}{\partial z} + \left(\frac{\sigma_{xx}}{\partial x} + \frac{\sigma_{yx}}{\partial y} + \frac{\sigma_{zx}}{\partial z} \right) + f_x \\ \frac{\partial \rho v_y}{\partial t} &= -\frac{\partial v_x(\rho v_y)}{\partial x} - \frac{\partial v_y(\rho v_y)}{\partial y} - \frac{\partial v_z(\rho v_y)}{\partial z} + \left(\frac{\sigma_{xy}}{\partial x} + \frac{\sigma_{yy}}{\partial y} + \frac{\sigma_{zy}}{\partial z} \right) + f_y \\ \frac{\partial \rho v_z}{\partial t} &= -\frac{\partial v_x(\rho v_z)}{\partial x} - \frac{\partial v_y(\rho v_z)}{\partial y} - \frac{\partial v_z(\rho v_z)}{\partial z} + \left(\frac{\sigma_{xz}}{\partial x} + \frac{\sigma_{yz}}{\partial y} + \frac{\sigma_{zz}}{\partial z} \right) + f_z\end{aligned}$$

Linearize Momentum Equation

If we consider that the density is constant, and that the velocity is

$$v_i = \frac{\partial u_i}{\partial t}$$

Also, if we consider that the spacial-temporal cross term are negligible

$$\frac{\partial v_j(\rho v_j)}{\partial x_i} = \rho \frac{\partial}{\partial x_i} \left(\frac{\partial u_i \partial u_j}{\partial t^2} \right) \approx 0$$

We get the linearize momentum equation

$$\begin{aligned}\rho \frac{\partial^2 u_x}{\partial t^2} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x \\ \rho \frac{\partial^2 u_y}{\partial t^2} &= \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + f_y \\ \rho \frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z\end{aligned}$$

Laplace Fluids Law

$$k \nabla^2 h = 0$$

Dimensional reduction

In many cases, we don't really care about the response of the system in all direction. The magnitude of the response in some direction could be negligible or because of symmetries in the system, nevertheless, in many cases we are interested in the 2D system or even in the 1D response.

Plane Stress

Assuming that there are no stress in the out-of-plane direction,

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{zz} \\ \epsilon_{zx} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1 + \nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{zx} \end{bmatrix}$$

$$\epsilon_{yy} = \frac{-\nu}{E}(\sigma_{xx} + \sigma_{zz})$$

Alternatively

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{zx} \end{bmatrix} = \frac{E}{(1 + \nu)(1 - \nu)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1 - \nu \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{zz} \\ \epsilon_{zx} \end{bmatrix}$$

At first glance, everything seems consistent, but if we consider the linearized momentum equation

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + f_y$$

$$\rho \frac{\partial^2 u_y}{\partial t^2} = 0$$

However that implies that there is now acceleration, which is only true in the static case

Plane Strain

Assuming that there are no strain in the out-of-plane direction,

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{zz} \\ \epsilon_{zx} \end{bmatrix} = \frac{1 + \nu}{E} \begin{bmatrix} 1 - \nu & -\nu & 0 \\ -\nu & 1 - \nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{zx} \end{bmatrix}$$

Alternatively

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{zx} \end{bmatrix} = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & \nu & 0 \\ \nu & 1 - \nu & 0 \\ 0 & 0 & 1 - 2\nu \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{zz} \\ \epsilon_{zx} \end{bmatrix}$$

$$\sigma_{yy} = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}(\epsilon_{xx} + \epsilon_{zz})$$

Which is consistent and does not presuppose the static case as plain stress does.

Using plane strain/stress and some assumptions we can model very common cases such as :

Axially loaded bar

Considering bar biaxially loaded, if we integrate over the cross section we get

$$\rho A \frac{\partial^2 \bar{u}_x}{\partial t^2} = EA \frac{\partial^2 \bar{u}_x}{\partial x^2} + q_x$$

Shear layer

Lets consider a layer loaded in pure shear (double plane strain), such that there is only shear deformation. Then replacing in the linearized equation of motion we get

$$\rho \frac{\partial^2 u_x}{\partial t^2} = G \frac{\partial^2 u_x}{\partial z^2} + f_x$$

Diaphragm wall

Lets consider a wall loaded vertically, such that it can be considered to be in plane stress and plane strain, integrating over the thickness of wall (d)

$$\rho d \frac{\partial^2 \bar{u}_x}{\partial t^2} = \frac{Ed}{1 - \nu^2} \frac{\partial^2 \bar{u}_x}{\partial z^2} + p_z$$

1.5 Case Study: Darcy's law and potential flow

Let consider the conservation of mass (volume assuming that the density is constant) for a control volume we get

Continuity equation

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\nabla q \\ \frac{\partial \rho}{\partial t} &= -\rho \nabla v\end{aligned}$$

Considering that there is no storage capacity we get

$$\rho \nabla v = 0$$

Lets remember that the simplest model for flow is the potential flow, also known as Darcy's Law, it says that "*water flow from a higher energy state to a lower one*"

Lets define the potential or head as

$$\Psi = h = z + \frac{p}{\rho g} + \frac{1}{2} \frac{U^2}{2g}$$

In water flow through soil, the velocity term is negligible

$$h \approx z + \frac{p}{\rho g}$$

Then mathematically we can express potential flow as

Darcy's Law

$$q = -k \nabla h$$

k : Hydraulic conductivity

Combining Darcy's Law into the continuity equation we get

Laplace's Fluids Law

$$k \nabla^2 h = 0$$

1.6 Case Study: Simplified flow through soil

Lets consider the case of fully saturated soil, and no chemical of mechanical interaction/degradation of the soil due to the flow/presence of water. We also know that soil fails in term of effective stresses, then it is convenient to define the effective stress tensor as

Effective stress

$$\sigma' = \sigma - I u_w$$

In Soil mechanics it may be useful to consider the invariants of the effective stress tensor, which are analogous to the stress tensor's but are called J_1 J_2 and J_3 respectively, which are used in failure criteria[3]. For the continuity equation we previously assumed that $\phi \neq 0$ and $\Sigma = 0$) but from the first chapter we know that the medium can deform, therefore the control volume can deform and we have to include the change of volume in the continuity equation

In rocks, we have to be careful about porosity vs connected porosity and permeability vs hydraulic conductivity.

Deformable medium - Continuity Equation

For permanent flow

$$\frac{\varepsilon_v}{\partial t} = -k \nabla^2 h$$

With $I_1 = \varepsilon_v$

1.7 Case Study: Flow to a well

Let's consider

- Saturated soils
- Single fluid, i.e. water
- Isotropic soil
- Water is incompressible

Flow through 1D soil layer

The soil has hydraulic conductivity k , and the potential in the boundaries are P_1 and P_2 , therefore it is a Dirichlet problem.

Using the Laplace Flow equation and considering that the problem is 1D

$$k \frac{\partial^2 h}{\partial x^2} = 0 \quad \text{and} \quad h|_{L=P_1}^{0=P_2}$$

Integrating

$$\begin{aligned}\int k \frac{\partial^2 h}{\partial x^2} &= 0 \\ k \frac{\partial h}{\partial x^2} + C_1 &= 0 \\ \int k \frac{\partial h}{\partial x^2} + C_1 &= 0 \\ kh + C_1 x + C_2 &= 0\end{aligned}$$

Applying Boundary conditions

$$h(x) = \left(\frac{P_2 - P_1}{L} \right) x + P_1$$

Flow through 1D soil, with discontinuity in soil properties

Because the hydraulic conductivity is discontinuous in the layer, it is easier to consider the problem as 2 problems of 1D isotropic soil, but applying boundary conditions in the interface as to preserve the conservation of mass principle.

If we did not care about the distribution, but only about the global behavior, we could have used the equivalent hydraulic conductivity.

$$\begin{aligned}k_{eq}^h &= \frac{\sum k_i d_i}{\sum d_i} \\ k_{eq}^v &= \frac{\sum d_i}{\sum \frac{d_i}{k_i}}\end{aligned}$$

Then, similarly to the last problem

$$\begin{aligned}k_1 \frac{\partial^2 h_1(x)}{\partial x^2} &= 0 \\ k_2 \frac{\partial^2 h_2(x)}{\partial x^2} &= 0 \\ h_1(0) &= P_1 \\ h_2(L) &= P_2 \\ k_1 \frac{\partial h_1(L/2)}{\partial x} &= k_2 \frac{\partial h_2(L/2)}{\partial x}\end{aligned}$$

Depending on the complexity of the situation, it may be useful to define local coordinates systems to simplify the algebra, for example

$$\begin{aligned}x'_1 &= x \\x'_2 &= x + \frac{L}{2}\end{aligned}$$

However, in the previous exercise because k was uniform in each layer, it is just easier to use the result from question 1, where we know the pressure distribution in terms of the boundary conditions, the problem can be easily reduced to

$$\begin{aligned}h'_1(x_1) &= \frac{P_3 - P_1}{L}x + P_1 \\h'_2(x_2) &= \frac{P_2 - P_3}{L}x + P_3 \\k_1 \frac{\partial h'_1(0)}{\partial x_1} &= k_2 \frac{\partial h'_2(0)}{\partial x_2}\end{aligned}$$

Vertical 1D flow through isotropic layer

Considering the boundary condition of no flux in the bottom and no pressure in the top, and that the energy is

$$h(z) = gz + \frac{v(z)^2}{2} + \frac{P(z)}{\rho(z)} \approx gz + \frac{P(z)}{\rho}$$

It is necessary to pay attention to the unit of energy, dependent on the situation, it may be better to express the energy in terms of height, pressure or just energy.

In addition, the *no-pressure* boundary condition refers to *relative pressure*, because

$$\begin{aligned}P_{top} &= P_{atm} + 0 \\P_{bottom} &= P_{atm} + P_{bottom}\end{aligned}$$

But atmospheric pressure usually affects both boundary condition.

We know from problem 1 that the solution to 1D flow is

$$kh + C_1x + C_2 = 0$$

Then applying Boundary conditions

$$\begin{aligned}h(z_{top}) &= 0 \\ \frac{\partial h(z_{bottom})}{\partial z} &= 0\end{aligned}$$

We get

$$h(z) = \rho g(z_{top} - z)$$

In problems involving atmospheric transport, we need to remember that $g(z)$ and $\rho(z)$. See [1] for examples.

Radial flow

The Laplace Flow equation works in any dimension and coordinates system¹.

$$k \left(\frac{\partial^2 h}{\partial r^2} + \frac{1}{r} \frac{\partial h}{\partial r} + \frac{1}{r^2} \frac{\partial^2 h}{\partial \theta^2} \right) = 0$$

Because the situation is radially symmetric ($\frac{\partial^2 h}{\partial \theta^2} = 0$)

$$k \left(\frac{\partial^2 h}{\partial r^2} + \frac{1}{r} \frac{\partial h}{\partial r} \right) = 0$$
$$h = C_1 \ln r + C_2$$

1.8 Case study: Static deformation of a bar with distributed elastic foundation

Let's consider an elastic pile foundation, with

- Axial stiffness: EA
- Distributed spring stiffness: χ , accounting for soil resistance
- Discrete spring: K , accounting for soil resistance at the tip
- External force: F_0

Kinematics

It is trivial to see that the displacement field of the bar is equal to the displacement field of the later soil

$$u_z^{bar} = u_z^{soil} = u_z$$

And also that the displacement in the tip is equal to the deformation of the soil under the tip

$$\frac{du_z(L)}{dz} = \delta_{tip}$$

Action-Deformation

We have three different action-deformation relationships

- Bar

$$\sigma_{bar} = EA \left(\frac{du_z}{dz} \right)$$

- Skin force (Soil)

$$\sigma_{soil} = \chi u_z$$

Strictly, $\sigma_{soil} = \chi \frac{du'_z}{dz'}$, but the deformation for every *spring* is $\frac{du'_z}{dz'} = u_z - u_z^0$, and in the continuum $u_z^0 = 0$, then the deformation of every *spring* is just the displacement of the bar at that point

¹See [?] for the Laplace operator in any coordinate systems

- *Tip force* (Soil)

$$F_{tip} = K \left(\frac{du_z(L)}{dz} \right)$$

Equilibrium

From the conservation of momentum in a 1D static situation

$$\begin{aligned} \frac{d}{dz} \left(EA \frac{du_z}{dz} \right) + q_z &= 0 \\ \frac{d}{dz} \left(EA \frac{du_z}{dz} \right) - \chi u_z &= 0 \end{aligned}$$

We have the stiffness of the soil and the pile expressed in different forms, it may be useful to solve the problem in terms of the ratio of soil/bar stiffness ($\kappa^2 = \frac{\chi}{EA}$)

$$\frac{d^2 u_z}{dz^2} - \kappa^2 u_z = 0$$

General solution

The solution is

$$u_z(z) = C_1 e^{\kappa z} + C_2 e^{-\kappa z}$$

Boundary Conditions

The boundary conditions are the deformation in the top and bottom

$$\begin{aligned} \left(EA \frac{du_z}{dz} \right) \Big|_{z=0} &= -F_0 \\ \left(EA \frac{du_z}{dz} \right) \Big|_{z=L} &= -K u_z \Big|_{z=L} \end{aligned}$$

Then we get that

$$\begin{aligned} C_1 &= -\frac{F_0}{EA\kappa} \frac{Re^{-2\kappa L}}{1 + e^{-2\kappa L}} \\ C_2 &= \frac{F_0}{EA\kappa} \frac{1}{1 + e^{-2\kappa L}} \end{aligned}$$

with

$$R = \frac{K - EA\kappa}{K + EA\kappa}$$

2 Linear Waves

2.1 Equations of motion

Let's assume an hydrostatic vertical pressure distribution and constant density, From the conservation of momentum equation in 1D we get

Unitary discharge

$$q(s, t) = \int_{z_b}^{h(t)} v_s dz$$

If we extend it to the whole cross-section

Cross section discharge

$$Q(s, t) = \int_{A(t)} v_s dA$$

Using the conservation of mass (volume if we consider incompressible flow) we get

$$\frac{\partial A(s, t)}{\partial t} + \frac{\partial Q(s, t)}{\partial s} = 0$$

In channel flow, our main concern is the piezometric height (h), therefore is useful to use the following expression

$$\frac{\partial A(s, t)}{\partial t} = \frac{\partial A}{\partial h} \frac{\partial h}{\partial t} = B(s, t) \frac{\partial h}{\partial t}$$

If we consider small flow velocities, and/or small, semi-enclosed areas

Small-basin approximation

$$\frac{\partial h}{\partial s} \approx 0 \longrightarrow Q(t) \approx A_b \frac{dh_b}{dt}$$

As we saw in the continuum mechanics chapter, the integral version of the conservation principle is more general and powerful than the differential one, but in most cases, assuming a number of symmetries, the differential conservation is more useful. Nevertheless, we will briefly consider the integral case. Let's start with the continuity equation for permanent flow in 1D, and integrate it in the vertical direction

$$\int_{z_b}^{h(t)} \left(\frac{\partial \rho}{\partial t} + \rho \nabla v \right) dz = 0$$

$$\int_{z_b}^{h(t)} \frac{\partial \rho}{\partial t} dz + \int_{z_b}^{h(t)} \frac{\partial \rho v_x}{\partial x} dz + \int_{z_b}^{h(t)} \frac{\partial \rho v_z}{\partial z} dz = 0$$

Using the Leibniz integration rule

$$\left[\frac{\partial}{\partial t} \int_{z_b}^{h(t)} \rho dz - \left(\rho \frac{\partial h}{\partial t} \right)_{z=h} + \left(\rho \frac{\partial z}{\partial t} \right)_{z=z_b} \right] + \left[\frac{\partial}{\partial x} \int_{z_b}^{h(t)} \rho v_x dz - \left(\rho v_x \frac{\partial h}{\partial x} \right)_{z=h} + \left(\rho v_x \frac{\partial z}{\partial x} \right)_{z=z_b} \right] + \left[(\rho v_z)_{z=h} + (\rho v_z)_{z=z_b} \right] = 0$$

The colored terms are the kinematic condition in the boundaries, in particular, the **cyan** on the surface and the **orange** terms in the channel bed. Re-ordering we get

$$\frac{\partial}{\partial t} \int_{z_b}^{h(t)} \rho dz + \frac{\partial}{\partial x} \int_{z_b}^{h(t)} \rho v_x dz = \left[\rho \left(\frac{\partial h}{\partial t} + v_x \frac{\partial h}{\partial x} + v_z \right) \right]_{z=h} - \left[\rho \left(\frac{\partial z}{\partial t} + v_x \frac{\partial z_b}{\partial x} + v_z \right) \right]_{z=z_b}$$

$$\frac{\partial}{\partial t} \int_{z_b}^{h(t)} \rho dz + \frac{\partial}{\partial x} \int_{z_b}^{h(t)} \rho v_x dz = 0$$

The kinematics conditions terms are zero in both boundaries.

The kinematic conditions are zero if we consider them to be *streamlines*, intuitively we can see that the kinematic condition terms that involve temporal derivatives are zero because we assumed that the velocity normal to the flow is zero for the surface and seabed, the other terms relate to the fact that we assume that a fluid particle approaching the seabed will move alongside it (cannot go into the seabed) and a fluid particle moving towards the surface will move along the surface (not splash-out).

The density assumption is usually adequate for canals, but may not be appropriated in the presence of temperature/salinity gradients.

Considering that the density is uniform throughout the cross-section

Mass Balance equation

$$\frac{\partial}{\partial t} \int_{z_b}^{h(t)} \rho dz + \frac{\partial}{\partial x} \int_{z_b}^{h(t)} \rho v_x dz = 0$$

$$\frac{\partial(h(t) - z_b)}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$$\frac{\partial d(t)}{\partial t} + \frac{\partial q}{\partial x} = 0$$

Similarly to mechanics of solids, using the mass balance and the conservation of momentum we get

Euler equation

For incompressible, uniform density, negligible viscous stresses, and only gravity as an external force we get

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} = g_i$$

So far, the equations of motion are formulated in a Cartesian coordinate system, but for geometries different than a straight canal, it can become algebraically tedious to work on $(\hat{x}, \hat{y}, \hat{z})$, therefore we introduce the natural coordinate system.

- R = Radius of curvature
- \hat{s} = Flow direction

- \hat{n} = Normal to the flow direction, towards the inside of the radius of curvature, defines the normal plane
- \hat{b} = Bi-normal direction to the flow, defines osculation plane

Euler equation in natural coordinates

$$\begin{aligned}\frac{\partial v_s}{\partial t} + v_s \frac{\partial v_s}{\partial s} + g \frac{\partial h}{\partial s} &= 0 \\ \frac{\partial v_n}{\partial t} + \frac{v_s^2}{R} + g \frac{\partial h}{\partial n} &= 0 \\ \frac{\partial v_b}{\partial t} + g \frac{\partial h}{\partial b} &= 0\end{aligned}$$

2.2 Long Waves

Let's consider the case of a 1D *long waves* with a constant cross section piezometric level, then we can assume that

- Hydrostatic pressure distribution $\longrightarrow \frac{\partial h}{\partial b} \approx 0$
- Negligible surface slope $\longrightarrow \frac{\partial h}{\partial n} \approx 0$
- Acceleration only in \hat{s}
- $\frac{\partial h}{\partial s}$ is constant within the cross-section
- Considering cross-section averaged velocity $U = \int_A v \, dA / \int dA$

Boundary Resistance

So far, we have only considered conservative forces, we can easily identify at least two types of non-conservative forces, exactly as we did with the master conservation principle, we have the volume force(Σ) and the surface force(ϕ). If we ignore for now the volumetric forces(turbulence,etc), and ignore the air-water interaction we get the frictional force of the canal walls/bed (wet perimeter).

$$F_r = \tau_b P = \rho c_f |U| U P$$

All the other forces have been expressed as force per unit mass, then

$$\frac{F_r}{\rho A_c} = \frac{\rho c_f |U| U P}{\rho A_c} = c_f \frac{|U| U}{R}$$

This assumes that c_f is constant in the wet perimeter, which is not always the case. See Orton's and Lotter's method for compound cross-sections.

Also, see Manning and Chézy equation for other ways to estimate boundary resistance. Detail in [1]

Then, the equation on the whole cross-section

Saint-Venant Equation

$$B \frac{\partial h}{\partial t} + \frac{\partial Q}{\partial s} = 0$$
$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial s} \left(\frac{Q^2}{A_c} \right) + g A_c \frac{\partial h}{\partial s} + c_f \frac{|Q|Q}{A_c R} = 0$$

2.3 Dimensional analysis

We can characterize a wave by four parameter

- Wave length (λ or L_0)
- Wave Period (T_0)
- Flow velocity (U_0)
- Water depth (h or D_0)

If we look at the Saint-Venant equation, by inspection is evident that the storage ($B \frac{\partial h}{\partial t}$) and volume transport ($\frac{\partial Q}{\partial s}$) terms cannot be neglected.

Lets consider the *importance* of the local inertia and use it to normalize the effect of other terms in the momentum equation

$$\text{Normalized local inertia} \propto \left(\frac{U}{T_0} \right) / \left(\frac{U}{T_0} \right) = 1$$
$$\text{Normalized Advective Inertia} \propto \left(U \frac{U}{L} \right) / \left(\frac{U}{T_0} \right) = \frac{U T_0}{L} \approx Fr$$
$$\text{Normalized Gravity forcing} \propto \left(g \frac{H_0}{L_0} \right) / \left(\frac{U}{T_0} \right) = \frac{g H_0 T_0}{L_0 U_0}$$
$$\text{Normalized Resistance} \propto \left(c_f \frac{U_0^2}{D_0} \right) / \left(\frac{U}{T_0} \right) = c_f \frac{U_0 T_0}{D_0}$$

In general the adjective inertia and/or resistance can be neglected, is then useful to define the ratio between them

$$\sigma = c_f \frac{U_0}{D_0} \frac{T_0}{2\pi} = c_f \frac{U_0}{\omega D_0}$$

Let's define the velocity of a wave relative to the flow as

Celerity

For shallow water the propagation speed (phase velocity) can be estimated as

$$c_p \approx \sqrt{gh}$$

For deep water, $\frac{D_0}{L_0} > 0.5$

$$c_p \approx \frac{g}{2\pi} T$$

2.4 Classification/Characterization of long waves

In practice we consider a wave as *long* if $\frac{L_0}{D_0} > 20$.

Translatory waves

These waves are formed when control structures, such as gates, valves, etc are manipulated.

Tsunami wave

Impulsive wave due to tsunami. Can be produced either by a mega-rupture tsunami or due to an earthquake-produced landslide.

- $\lambda \sim 10^{2-3}$ km
- $h \sim 10^3$ km
- $T \sim 10 - 20$ mins
- Then, $c \sim 10^2 m/s$
- Then, $\sigma \sim 10^{-4}$

Seiches

Natural oscillation in lakes and basins, due to geometric, gravitational and atmospheric variations.

- $\lambda \sim 20$ km
- $h \sim 10$ m
- $T \sim 20$ mins
- Then, $c \sim 10 m/s$
- Then, $\sigma \sim 10^{-2}$

Tides

Produce to gravitational effects of the moon and sun. In the ocean

- $\lambda \sim 8 - 9 \times 10^3$ km
- $h \sim 3 - 5 \times 10^3$ km
- $T \sim 745$ mins
- Then, $c \sim 200$ m/s
- Then, $\sigma \sim 10^{-3}$

The resistance factor of tides in coastal areas, channels, tide flats could be in the order of $\sigma \sim [10^{-1}, 10^1]$

Flood waves

Cause by excessive precipitation or sudden melting of snow.

- $\lambda \sim 10^{2-3}$ km
- $h \sim 1 - 2 \times 10$ m
- $T \sim 3 - 5 \times 10^3$ mins
- Then, $c \sim 7$ m/s
- The effect of bed friction is usually considerable in the order of $\sigma \sim 5 \times 10^1$

2.5 Wave propagation

Let's consider a canal with a gate that separates two piezometric levels. If we open the gate, a translatory wave will propagate, the water will flow from the *higher* to the *lower* level, creating a positive amplitude wave in the *lower* side and a negative amplitude in the *higher* side.

We will define the amplitude of the wave (respect to the piezometric level at t_0) as ζ

Simplified wave propagation

Lets consider a control volume such that

$$h(s_1, t) = h_0 + \zeta \quad || \quad h(s_2, t) = h_0$$

Using the mass conservation principle (volume, if we consider ρ to be constant)

$$\begin{aligned} B \frac{\partial h}{\partial t} + \frac{\partial Q}{\partial s} &\approx B \frac{\zeta}{\Delta t} + \frac{Q}{\Delta s} \\ &\approx B \zeta \Delta s = Q \Delta t \longrightarrow Q = Bc\zeta \end{aligned}$$

Then, for the momentum equation ignoring the resistance and advective acceleration terms

$$\begin{aligned} \frac{\partial Q}{\partial t} + gA_c \frac{\partial h}{\partial s} &\approx \frac{Q}{\Delta t} + gA_c \frac{\Delta h}{\Delta s} \\ &\approx \frac{Q}{\Delta t} + gA_c \frac{\zeta}{\Delta s} \longrightarrow gA_c \zeta = cQ \end{aligned}$$

Considering a rectangular cross section we obtained the expression for the celerity that we mentioned before

Lets re-examine the Saint-Venant equation

$$B \frac{\partial h}{\partial t} + \frac{\partial Q}{\partial s} = 0$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial s} \left(\frac{Q^2}{A_c} \right) + g A_c \frac{\partial h}{\partial s} + c_f \frac{|Q|Q}{A_c R} = 0$$

For low and long waves we can neglected advective acceleration and boundary resistance term

$$\begin{aligned} \frac{\partial}{\partial s} \left(\frac{Q^2}{A_c} \right) &\approx 0 \\ c_f \frac{|Q|Q}{A_c R} &\approx 0 \end{aligned}$$

Assuming a constant rectangular cross-section and combining the equation we get the

Elementary wave equation

$$\begin{aligned} \frac{\partial^2 h}{\partial t^2} - \frac{g A_c}{B} \frac{\partial^2 h}{\partial s^2} &= 0 \\ \frac{\partial^2 h}{\partial t^2} - c^2 \frac{\partial^2 h}{\partial s^2} &= 0 \end{aligned}$$

2.6 General solution: d'Alembert

The general solution to the elementary wave equation is

d'Alembert's solution

$$\begin{aligned} h(s, t) &= h^+(s - ct) + h^-(s + ct) \\ h(s, t) &= h^+(S^+) + h^-(S^-) \end{aligned}$$

Sometimes, for situation with non-homogeneous boundary conditions, its better to use the following formulation

$$h(s, t) = h^+(t - s/c) + h^-(t + s/c)$$

The solution is the superposition of two waves traveling at a constant speed in opposite directions. Because the wave has no distortion, it *simply* propagates with fixed velocity, then the value of h is constant in S , or alternative $s = ct + \text{constant}$ ²

Then it may be useful to define to total derivative as

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + V \frac{\partial h}{\partial s}$$

If consider the case of our propagating wave we get that the total derivative of h is zero

$$\frac{dh^+}{dt} = \frac{\partial h^+}{\partial t} + c \frac{\partial h^+}{\partial s}$$

²See, method of characteristic for more detail [put here hyper-reference to this section](#)

This relationship is very useful because it lets us relate the temporal and spatial derivative of h

$$\frac{\partial h^+}{\partial x} = -\frac{1}{c} \frac{\partial h^+}{\partial t}$$

Boundary conditions

We will consider common boundary conditions that happen canals, however they can be easily related to boundary condition of propagating wave in a bar as it was shown in Assignment 2 ($h \sim \varepsilon_x$ and $Q \sim N$)

Canal with close end

As the wave approaches the end of the canal, locally

$$Q = \delta Q^+ + \delta Q^- = Bc(\delta h^+ - \delta h^-) = 0 \longrightarrow \delta h^+ = \delta h^-$$

We know that the solution is the superposition of the waves traveling in opposite directions then we get *fully positive reflection* Which is equivalent to the case of a semi-infinite bar restricted at the end

$$h = \delta h^+ + \delta h^- = 2\delta h^+$$

Canal connected to reservoir

As the wave approaches the reservoir, locally

$$\delta h = \delta h^+ + \delta h^- = 0 \longrightarrow \delta h^+ = -\delta h^-$$

We know that the solution is the superposition of the waves traveling in opposite directions then we get *fully negative reflection*.

Which is equivalent to the case of a semi-infinite bar restricted at the end

Canal with rapidly varying cross-section

The transition is characterized by a discontinuous cross-section.

Qualitatively we can expect *partial reflection*, and that is exactly what we get.

Defining compatibility conditions in the transition

$$\text{Continuous water level at the transition} \longrightarrow \delta h_t = \delta h_i + \delta h_r$$

$$\text{Continuous discharge at the transition} \longrightarrow \delta Q_t = \delta Q_i + \delta Q_r$$

We define the following adimensional parameters

$$r_t = \frac{\delta h_t}{\delta h_i}$$

$$r_r = \frac{\delta h_r}{\delta h_i}$$

$$\gamma = \frac{B_2 c_2}{B_1 c_1} = \sqrt{\frac{A_{c,2} B_2}{A_{c,1} B_1}}$$

Then, the compatibility conditions are expressed as

$$r_t = 1 + r_r$$

$$\gamma r_t = 1 - r_r$$

Then, the reflected and transmitted wave ratios are

$$r_r = \frac{1 - \gamma}{1 + \gamma}$$

$$r_t = \frac{2}{1 + \gamma} = 1 + r_r$$

Multiple Channels

If the abrupt transition consist of multiple channel, we can extend the previous principle, we define an *equivalent channel* such that

$$B^* c^* = \sum B_i c_i$$

Then applying conservation of mass and momentum in the transmitted channel, we determine the height/discharge in each channel

We are assuming that the wave is *insensitive to changes in propagation direction*. Also, see example of how having a side close channel the affect the reflection int the transmitted and original channel, in [2] p.50.

Gradually varying cross-section

This boundary conditions escapes the scope of the course, see chapter 4.3.2 of [2] for basic explanation.

2.7 Periodic Waves

In the previous section we considered the case of a single wave, let it be a translatory wave or a single wave, lets extend the same concepts to a periodic wave.

Periodic wave

$$\zeta^+(s, t) = \hat{\zeta} \cos\left(\frac{2\pi}{L}(s - ct)\right)$$

$$\zeta^+(s, t) = \hat{\zeta} \cos(ks - \omega t)$$

With:

$$\text{Wave number}(k) := \frac{2\pi}{L}$$

$$\text{Angular frequency}(\omega) := \frac{2\pi}{T}$$

Boundary conditions

Canal with closed end

The boundary condition is

$$Q = \delta Q^+ + \delta Q^-$$

Using the d'Alembert method, considering the boundary conditions

$$\zeta = \zeta^+ + \zeta^-$$

$$\zeta = 2\hat{\zeta} \cos(ks) \cos(\omega t)$$

Closed basin

The boundary condition is

$$Q = 0$$

Then the antinodes of the wave are on the boundaries, then

$$k_n = \frac{n\pi}{\ell}$$

Because we are dealing with un-forced oscillations, the natural frequencies are

$$\frac{\omega_n}{k_n} = \sqrt{\frac{g A_c}{B}}$$
$$\omega_n = k_n \sqrt{\frac{g A_c}{B}}$$

Semi-closed basin connected to reservoir

At the close end

$$Q = 0$$

and at the reservoir

$$\zeta = 0$$

Then, to fulfill this condition, at all times there is a antinode in the closed end, and a node in the reservoir end.

$$k_n = \frac{1}{2}\pi + n\pi$$

And the natural frequencies are

$$\omega_n = \left(\frac{1}{2}\pi + n\pi\right) \frac{1}{\ell} \sqrt{\frac{g A_c}{B}}$$

Semi-closed basin connected to tidal see

At the close end

$$Q = 0$$

and at the sea-basin interface

$$\zeta(\ell, t) = \zeta_{sea}$$

The oscillation in the basin are forced by the oscillation of the tide, then

$$\zeta(0, t) = \frac{\zeta(\ell, t)}{|\cos k\ell|}$$

Then we get resonance when

$$\cos k\ell \rightarrow 0 \implies \zeta(0, t) \rightarrow \infty$$

For small basins,

$$(\ell \ll L \quad \text{or} \quad k\ell \ll 2\pi) \implies \cos(k\ell) \approx 1$$

Then "the water level responds almost in unison to the tidal forcing at the mouth, rising and falling with the tide but being virtually horizontal at all times"([2]), this phenomenon is called **Pumping or Helmholtz mode**

2.8 Case Study: Dispersive wave in foundation pile

We previously derived the equation of motion for the bar

"Telegraph equation"

$$\frac{1}{c_0^2} \frac{\partial^2 u_z}{\partial t^2} - \frac{\partial u_z}{\partial z^2} + \kappa_d^2 u_z - q_z = 0$$

$$c_0^2 = \frac{E}{\rho}$$

$$\kappa_d^2 = \frac{\xi}{EA} = \frac{\xi}{c_0^2 \rho A}$$

This wave equation is dispersive, therefore we cannot use the d'Alembert method to solve it. Lets consider a SDOF system

SDOF: Undamped, Harmonic Excitation

We know that the response of a undamped system, due to an harmonic excitation is simply the modulated signal shifted in time, or simply

$$u(t) = \text{Re}[\tilde{u}(\omega) \exp(i\omega t)]$$

with

$$\tilde{u} = \hat{u} e^{i\alpha}$$

$$\hat{u} = \left| \frac{1}{k - m\omega^2} \right| \hat{F}_0$$

$$\alpha = \begin{cases} 0 & \omega \leq \sqrt{\frac{k}{m}} \\ \pi & \sqrt{\frac{k}{m}} \leq \omega \end{cases}$$

Replacing in the original equation we get

$$\frac{d^2 \tilde{u}}{dz^2} + \left(\frac{\omega^2}{c_0^2} - \kappa_d \right) \tilde{u} = 0$$

The solution of the ODE is known to be

$$u(z, \omega) = C_1 e^{-i\gamma z} + C_2 e^{i\gamma z}$$

$$\gamma = \frac{1}{c} \sqrt{\omega^2 - (c\kappa)^2} = \frac{1}{c} \sqrt{\omega^2 - \omega_c^2}$$

The response depends on the value of γ , let's suppose that

$$\gamma = \begin{cases} k & \omega_c \leq \omega \\ -i\mu & \omega \leq \omega_c \end{cases}$$

Then we can characterize the spatial response in term of γ , if

- $\gamma = k$

The solution is a propagating harmonic wave

$$u_z(s, t) \sim \tilde{u}_z(s, t)e^{i\omega t} = C_1 e^{i(t-kz)} + C_2 e^{i(\omega t+kz)}$$

- $\gamma = -i\mu$

The solutions is a evanescent wave

$$u_z(s, t) \sim \tilde{u}_z(s, t)e^{i\omega t} = C_1 e^{-\mu z} e^{i\omega t} + C_2 e^{-\mu z} e^{i\omega t}$$

This is analogous to the damped/critically damped/over-damped behavior that we are familiar in Structural Dynamics. Lets consider to different boundary conditions

Boundary Conditions: Infinite Pile

We can easily deduce two boundary conditions. First, the Normal force is the "top face" is opposite to the driving force by simple summation of force. Second, because the pie is infinite the deformation at the end is null (radiation condition), mathematically

3 Diffusion: Heat Conduction

Using the master conservation equation for the case of energy, if we only consider internal energy (in this case temperature)

$$\frac{\partial Q}{\partial t} = -\nabla \cdot q$$

Considering that the "Volumetric Storage Capacity"(Q) is

$$Q = c\rho T$$

c : Specific heat capacity

ρ : Density

And considering that the flow of heat is proportional to the temperature gradient (conduction)

Fourier's Law

$$q = \alpha \nabla T$$

λ : Thermal conductivity

If we ignore advection, radiation and convection mechanisms, we can define the heat diffusion equation

Simplified Heat Transfer or Heat Diffusion

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

Solving the equation is more difficult than the continuity equation, but we can still find an analytic solution

3.1 Homogeneous BC

Lets consider homogeneous boundary conditions, and know initial value, and lets assume that

$$T(x, t) = v(x)w(t)$$

Then

$$v(0)w(t) = 0 \text{ only if } v(0) = 0$$

$$v(L)w(L) = 0 \text{ only if } w(L) = 0$$

If we replace the new expression for T in the differential equation and separate the spacial from the temporal variables

$$\begin{aligned} \frac{\partial T}{\partial t} &= \alpha \nabla^2 T \\ \frac{1}{\alpha w(t)} \frac{\partial w(t)}{\partial t} &= \frac{\nabla^2 v(x)}{v(x)} \end{aligned}$$

Because the equation is linear and we separate variable we can say that the temporal variables equals to a constant, and the spacial variable to the same constant(P), where P does not depend neither on t or x

$$\frac{1}{\alpha w(t)} \frac{\partial w(t)}{\partial t} = -P$$

$$\frac{\nabla^2 v(x)}{v(x)} = -P$$

Using separation of variable, for the temporal part we get that

$$w(t) = e^{-\alpha P t} e_1^C$$

and for the spatial part

$$v(x) = C_2 \cos(\sqrt{P}x) + C_3 \sin(\sqrt{P}x)$$

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