1. (18 points) Prove that the following language is in **P**.

   Horn-satisfiability:

   $CNF_H = \{< \phi > | \phi \text{ is a Horn formula}\}$

   A Horn clause is a clause with at most one positive literal and any number of negative literals. A Horn formula is a propositional formula formed by conjunction of Horn clauses.

2. (14 points) Suppose $L_1, L_2 \in \mathbf{NP}$. Then is $L_1 \cup L_2 \in \mathbf{NP}$? What about $L_1 \cap L_2$?

3. (12 points) Prove that $L = \{w | w \text{ is a binary representation of a prime number}\}$ belongs to $\mathcal{NP}$.

   Tip: A natural number n is a prime number if and only if for every prime factor q of $(n-1)$, there exists an $a \in \{2, \cdots, n-1\}$ such that $a^{n-1} = 1 \mod n$ and $a^{(n-1)/q} \neq 1 \mod n$.

4. (14 points) Let HALT be the Halting language. Show that HALT is **NP**-hard. Is it **NP**-complete?

5. (14 points) Show that, if $\mathbf{P} = \mathbf{NP}$, then every language $A \in \mathbf{P}$, except $A = \emptyset$ and $A = \Sigma^*$, is **NP**-complete.

6. (28 points) Let ϕ be a 3CNF. An $\neq$-assignment to the variables of ϕ is one where each clause contains two literals with unequal truth values. In other words, an $\neq$-assignment satisfies ϕ without assigning three true literals in any clause.

   - Show that the negation of any $\neq$-assignment to ϕ is also an $\neq$-assignment.
   - Let $\neq$SAT be the collection of 3CNFs that have an $\neq$-assignment. Show that we obtain a polynomial time reduction from 3SAT to $\neq$SAT by replacing each clause

     $$c_i = (y_1 \vee y_2 \vee y_3)$$

     with the two clauses

     $$(y_1 \vee y_2 \vee z_i) \text{ and } (\bar{z}_i \vee y_3 \vee b),$$

     where z_i is a new variable for each clause c_i and b is a single additional new variable.
   - Conclude that $\neq$SAT is **NP**-complete.