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On reward sharing in blockchain mining pools ☆

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ABSTRACT

This paper proposes a conceptual framework for the analysis of reward sharing schemes in mining pools, such as those associated with Bitcoin. The framework is centered around the reported shares in a pool instead of agents and introduces two new fairness criteria: absolute and relative redistribution. These criteria impose that the addition of a share to a round affects all previous shares of the round in the same way, either in absolute amount or in relative ratio. We characterize two large classes of reward sharing schemes corresponding to each of these fairness criteria in turn. We further show that the intersection of these classes brings about a generalization of the well-known proportional scheme, which in turn leads to a new characterization of the proportional scheme itself.

1. Introduction

The invention of the first decentralized cryptocurrency, Bitcoin, (Nakamoto (2008)) sparked a huge interest in decentralized networks with distributed trust, both academically and businesswise. The technology behind these decentralized networks is called Blockchain. Loosely speaking, a blockchain is a ledger composed of an immutable chain of transactions organized in *blocks*. Each block is synchronized across the network users (nodes) through a distributed consensus protocol which ensures that all the nodes in the network agree on the latest status of the ledger.[1]

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[1] To agree on the latest status of the ledger, Blockchains utilize various different consensus protocols, see e.g. Nguyen and Kim (2018); Mingxiao et al. (2017).

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In terms of financials, as of May 2021, the global crypto market cap exceeds 2.26 trillion[2] and a single Bitcoin is traded around 56000 USD while its total market cap is comparable to the GDPs of various countries.[3] In terms of other use cases, there are an increasing number decentralized applications on various blockchains, e.g., Ethereum mainnet has about 2700 dApps deployed, providing solutions in banking, finance, law, logistics, and other sectors. Following the global interest in Blockchains, all EU members and European Commission have joined forces to form the European Blockchain Partnership (EBP) while China has already launched trials for its national cryptocurrency, the digital Yuan.

1.1. Consensus protocols and pools

The development of the blockchain technology is still in its infancy. The oldest consensus protocol that has a proven track record is the original Proof-of-Work (PoW) protocol in Nakamoto (2008), while many new blockchains utilize a Proof-of-Stake (PoS) protocol for energy efficiency and increased transaction throughput. The process of achieving consensus is called *mining* in PoW, while this is achieved by *staking* in PoS. To achieve consensus under PoW protocol, some nodes, called *miners*, compete with one another to solve a cryptographic puzzle.[4] Miners search for an integer (a nonce) such that when combined (hashed) together with the list of transactions (the block header), produces another integer known as the hash value. In case, this hash value is less than a predefined number (network target value) set by the network, the so-called puzzle is solved. The solution to this aforementioned puzzle is called a *full solution*, for which the successful miner is given a financial reward. In theory, the probability of finding a full solution in PoW is proportional to the computational power of the miner. However, the computational power of individual miners is negligible in comparison to that of the network.[5] Therefore, mining alone leads to a highly unstable income and incentivizes miners to *pool* their resources and split the resulting rewards. Such cooperative actions result in lower income variation for miners (Romiti et al. (2019); Rosenfeld (2011)). This is why, despite being a decentralized system by design, PoW has been shown to induce centralization both theoretically (Leshno and Strack (2020) and Chen et al. (2019)) and in practice through the emergence and total dominance of centralized mining pools. In fact, as of 2021, almost 90% of the total computational power in Bitcoin blockchain is provided by the top ten mining pools.[6] Consequently, mining pools are probably the most important actors in the blockchain ecosystem.

Typically, a mining pool is maintained and coordinated by a *pool manager*. The success of a pool depends on its computational power, therefore the miners must commit their resources to find a full solution through mining for the pool. To estimate the computing power of each miner, the pool manager sets an easier puzzle to solve and requests the solutions to this puzzle, which are called *partial solutions*.[7] For instance, the full solution to puzzle may require finding a hash value less than, say 100. To estimate how much miners are working, the pool manager may accept any (partial) solution less than, e.g. 200 (which is twice easier than finding the full solution). Miners submit these partial solutions to the manager, each of which forms a *share*. In case one of these shares is a full solution, the pool manager gets the reward and distributes it among the pool members based on their submitted shares and a predefined *reward sharing scheme*. Therefore, the choice of reward sharing scheme is a crucial design element in any mining pool. An ideal scheme must ensure that the pool does not go bankrupt, and incentivize miners to devote their computational resources to mine honestly for the pool. However, it turns out that there are trade-offs involved as we demonstrate in Example 2.

1.2. Our contribution

In this paper, we analyze and design *reward sharing schemes* in mining pools by proposing a comprehensive axiomatic framework. We propose a modelling framework rich enough to allow representation of all known reward sharing schemes which also indicates the wide range of the potential design space. We depart from the existing literature in various ways. First, our axiomatic framework is not on the consensus protocols but on the mining pools in any of these protocols. Second, our model is not restricted to a static single block, since various schemes in practice pay the miners repetitively over time in various blocks. Third, we propose reward sharing schemes and *allocations* not on the miners in a pool but instead on the shares submitted by these miners. This is particularly enriching, because defining allocations on shares (rather than on miners), enables more granular and relevant parameters such as the submission time or order of the shares. Therefore, we can practically formulate any of the existing schemes under our unified axiomatic framework, e.g., the Slush and PPLNS scheme.[8] Moreover, focusing on shares rather than agents is in line with the fact that miners are anonymous and can easily

[2] https://coinmarketcap.com.

[3] For example, Hungary, Kenya, Luxembourg and many others. See the following link for a ranking: https://worldpopulationreview.com/countries/countries-by-gdp. For other financial statistics, see https://bitinfocharts.com/bitcoin/.

[4] For an extensive demonstration of these concepts see Anders Brownworth's blog at https://andersbrownworth.com/blockchain/block.

[5] At the time of writing this paper, the total hashing power of the Bitcoin blockchain is 120^{18} KH/s, while a state-of-the-art CPU has about 11 KH/s. Therefore, the probability of finding a full solution as a solo miner is roughly $\frac{1}{10^{19}}$.

[6] https://btc.com/stats/pool.

[7] This is done by setting an internal pool target value higher (easier) than the network target value for the original puzzle. Note that the set of partial solutions for this easier target value is a superset of the full solutions. Therefore it is a very good approximation of the computational commitment of a miner.

[8] See Rosenfeld (2011) for a comprehensive list of reward sharing schemes, and Section 5 for formal definitions.

create as many aliases as they want. It is therefore questionable whether it is meaningful to base reward sharing schemes on miner identities.

Miners' perception of the fairness of reward sharing schemes is important for the general stability of the pool. It is easy for miners to switch from one pool to another so if miners do not feel appropriately compensated for their work nothing will tie them to a given pool. In recent studies by Tovanich et al. (2021) and Belotti et al. (2018), it is shown that reward sharing schemes and pool fees do indeed influence miners' decisions to join, change, or exit from a mining pool. Therefore, from the perspective of miners as well as the pool manager, the choice of reward sharing scheme is a vital element of a well functioning and stable mining pool.

With a focus on fair allocation, we therefore try to formulate several desirable properties (axioms) of reward sharing schemes. In particular, we propose two axioms with direct fairness implications, i.e., how the awards for the shares should be redistributed[9] when the round is prolonged by an additional share. We show that, together with other axioms related to ensuring economic viability of the pool, each of these fairness axioms, *absolute redistribution* and *relative redistribution*, characterize two relatively rich and distinct classes of reward sharing schemes, the class of absolute fair and relative fair schemes: a common element is the well-known proportional scheme. Thereafter, we characterize the class of generalized proportional reward schemes, which we call k-pseudo proportional schemes, satisfying both our fairness axioms simultaneously. Finally, by imposing an additional *strict positivity* requirement, we single out the proportional reward scheme.

In practice, mining pools use a wide variety of reward sharing schemes and we illustrate how the most commonly used (and debated) schemes relate to our axioms. One main take away from that analysis is that there is a potential trade-off between axioms that relate to distributional fairness (which we show lead to variations of the proportional scheme) and axioms relating to incentive compatibility such as avoiding (short run) pool hopping and delayed reporting of full solutions (which the proportional scheme is known to fail, see e.g. Schrijvers et al. (2016)). Modifying the proportional scheme to be incentive compatible (as in the scheme we dub the IC-scheme) looses the distributional fairness properties.

1.3. Related literature

Our paper relates to the literature on miners' general incentives under the PoW protocol. The seminal paper by Rosenfeld (2011) provides one of the earliest comprehensive mathematical analysis on the reward sharing schemes in mining pools. Schrijvers et al. (2016) discuss strategic behavior of miners under incentive compatibility. Zolotavkin et al. (2017) investigate the interplay between incentive compatibility and the distribution of computational power among miners in the pool. Lewenberg et al. (2015) model mining pools from a cooperative game perspective while Qin et al. (2018) and Chatzigiannis et al. (2019) investigate competition among pools.[10]

In terms of methodology, our paper utilizes tools from the literature on economic design (see e.g., Moulin (1991); Young (1994); Thomson (2018)) and is inspired by welfare economics, in particular the literature on distributional fairness (see e.g., Roemer (1996); Moulin (1987); Young (1994)) and fair allocation in networks (see e.g., Hougaard (2018); Moulin (2018)). As mentioned, we take an axiomatic approach to the analysis of reward sharing schemes: a list of desirable properties of generic schemes (axioms) are identified, and individual schemes are uniquely characterized by different sets of axioms. In turn, this allows a qualitative comparison of various reward sharing schemes based on their axiomatic foundation.

There has been a steadily growing literature, pointing towards utilization of mechanism design in the context of blockchains. Can (2019) proposes the use of economic design on the consensus protocol, while Hougaard et al. (2022) provides a different rationale for PoW by decentralized socially optimal reward schemes. Concerning an axiomatic approach to PoW, Leshno and Strack (2020) and Chen et al. (2019) propose characterizations (on the consensus protocols) with properties such as anonymity, collusion-proofness and sybil-proofness the last two of which are analogous to non-manipulability by merging and splitting.[11]

The remaining part of the paper is organized as follows. In Section 2 we propose our modeling framework and notation. Section 3 defines and comments on various axioms including the two central fairness conditions. Section 4 presents the characterization results. Finally, Section 5 closes with a discussion of well-known schemes under our proposed framework and logical independence of the characterizing axioms.

2. Model and notation

We consider a centralized mining pool. That is, when one of the pool's miners finds a full solution, the resulting block reward is paid out to the pool manager and then redistributed among the miners in accordance with the announced reward sharing scheme of the pool (possibly less a pool fee). The allocation of payment among the miners should ideally depend on

[9] This redistribution could be either in absolute amount or in relative ratio. Fair reward sharing is crucial for pool stability which in turn affects overall system efficiency. Given the symmetric nature of the shares in a pool round, these fairness axioms represent a (natural) stronger versions of the celebrated "population monotonicity" axiom in the literature on fair allocation (e.g. Thomson (2016)).

[10] For a list of further readings regarding miners' behavior, see Eyal and Sirer (2014); Biais et al. (2019); Babaioff et al. (2012); Sapirshtein et al. (2016); Fisch et al. (2017); Carlsten et al. (2016); Kiayias et al. (2016); Cong et al. (2021).

[11] Both papers propose characterizations of the proportional reward scheme which is well studied in the economic design literature and in bankruptcy models (see Banker (1981); Moulin (1987); Chun (1988); de Frutos (1999)).

the amount of "work" that the miners have done for the pool. However, miners can use their computational power to mine for different mining pools at any point in time and the pool manager, therefore, has no knowledge about the amount of resources that the individual miners invest in mining for the pool. As mentioned in Section 1.1, the pool therefore chooses to accept solutions that are "close enough" to a full solution – dubbed partial solutions, or shares – as evidence that the miners have actually tried to mine for the pool. Any scheme that allocates the rewards collected by the pool therefore share these rewards based on the registered partial solutions. Partial solutions come with a time signature (or timestamp) resulting in a natural order. We can therefore picture the process as follows: miners in the pool start to mine for a full solution and will continuously produce partial solutions that are registered in the order in which they arrive. When one of those partial solution turns out to be a full solution the block reward can be paid out based on the history of partial solutions. The partial solutions found between two consecutive full solutions are said to belong to the same "round". Thus, a given miner may have found several partial solutions in a given round and will receive the sum total of payments associated with these solutions.

Formally, let T denote the set of all possible *time signatures*, and let $\mathcal{S}$ denote the set of potential solutions (i.e., admissible hashes that satisfy the difficulty of the pool).

A (partial solution) *share* is then a two-tuple $s = (h, t) \in \mathcal{S} \times T$ consisting of an admissible hash submitted to the pool at time[12] t. We denote the time signature of a share s by a function $\tau(s)$, i.e., for $s = (h, t)$, we have $\tau(s) = t$. The set of all ordered shares submitted in a pool is denoted by $S = \{s_1, s_2, s_3, \dots, s_m\} \subsetneq \mathcal{S} \times T$ where the order is defined by the shares' time signatures. We do not associate shares with individual miners since all schemes that are used in practice are neutral towards the "identity" of the miner.

A *pool round* is an ordered set of shares ending with a share, submitted by the pool, which is a full solution on the blockchain. So everytime a share submitted by the pool is a full solution on the blockchain, a pool round ends and a new round begins. We consider the partitioning of the submitted set of shares by (pool) rounds, and denote it by $\mathcal{P}(S)$: for instance, let $l = |\mathcal{P}(S)|$ and let $P_1, P_2, \dots, P_l$ denote the set of shares submitted in rounds $1, 2, \dots, l$ respectively. Let $H = (S, \mathcal{P}(S))$ denote the *history* of the pool.

Given the partition of a history $\mathcal{P}(S)$ and any share $s \in S$, we denote the set of all shares in the same round as s by $P(s) = X \in \mathcal{P}(S)$ such that $s \in X$. Given a round $P \in \mathcal{P}(S)$, we denote the relative rank of a share s in P by $\rho(s) = |\{s' \in P | \tau(s') < \tau(s)\}| + 1$, i.e., the number of shares that are submitted in round P up to, and including, s. We define the *length* of a round P by the number of shares submitted in that round, i.e. $|P|$.

Example 1. Equation (1) below, illustrates an example of a history: shares in bold are full solutions.

$$\begin{aligned} S &= \{s_1, \dots, \mathbf{s_{10}}, s_{11} \dots, \mathbf{s_{150}}, \dots, s_{564}, \dots, \mathbf{s_{589}}, \dots, \mathbf{s_m}\} \\ \mathcal{P}(S) &= \{\{\underbrace{s_1, s_2, \dots, \mathbf{s_{10}}}_{P_1}\}, \{\underbrace{s_{11} \dots, \mathbf{s_{150}}}_{P_2}\}, \dots, \{\underbrace{s_{564}, \dots, \mathbf{s_{589}}}_{P_r}\}, \dots, \{\underbrace{\dots, \mathbf{s_m}}_{P_l}\}\} \end{aligned} \tag{1}$$

Considering, for instance, the share s_{564} we see that it is the first share of round r, so $P(s_{564}) = P_r = \{s_{564}, \dots, s_{589}\}$ and $\rho(s_{564}) = 1$. The length of round r is $|P_r| = 26$.

Whenever a full solution is found by a miner of the pool, the pool gets a block reward $B > 0$, which we assume is fixed for every block throughout the history. In practice, the payments to miners are made after a certain amount of time, therefore when considering the allocation of rewards, we analyze a round that is terminated and already confirmed on the Blockchain.[13] For each pool round, the pool manager charges a fee to compensate the costs of running the pool. For a history $H = (S, \mathcal{P}(S))$ we denote this fee by the mapping $f : \mathcal{P}(S) \to \mathbb{R}_+$. That is, for every round the pool manager gets a cut of the block reward B. Thus, we denote the actual award to be distributed among the miners in the r^{th} round by $R_r = B - f(P_r)$. To allocate this net reward among the miners the mining pool uses a reward sharing scheme, which is based on individual shares.

Definition 1. Given a history $H = (S, \mathcal{P}(S))$, we define a *reward sharing scheme*, α, by awards

$$\alpha(s, H) \in \mathbb{R}_+$$

to every share $s \in S$.

To provide an example, we introduce one of the most intuitive schemes, also known as the *Proportional scheme* below.[14]

[12] Since some known reward sharing schemes use time signature as input we denote the shares with (h, t). We shall drop the time signature when it is redundant for the schemes we analyze.

[13] The pool manager should wait for at least 6 block confirmation to make sure the block found by the pool ends up on the longest chain.

[14] While the proportional scheme in effect ensures round wise equal sharing of rewards, this name is used in the blockchain literature since the miners are rewarded in proportion to their relative hashing power in the pool.

Example 2. The Proportional scheme is one of the most straightforward and intuitive reward sharing schemes. For any given round, the pool manager gets a fixed fraction of the reward as fee, leaving R_r to the miners. The proportional scheme then assigns to every share in that round a proportion of the reward relative to the length of the corresponding round. Formally, for any history $H = (S, \mathcal{P}(S))$ and any share $s \in S$:

$$\alpha(s, H) = \frac{R_r}{|P(s)|}.$$

Recall Example 1 above, normalize the block reward to $B = 1$, and fix the pool fee of any round to $f(P) = 0.1$. Then if the pool uses the proportional scheme, we get $\alpha(s_i, H) = \frac{1-0.1}{10} = 0.09$ for $s_i \in P_1$ and $\alpha(s_i, H) = \frac{1-0.1}{140} = 0.006$ for $s_i \in P_2$ etc. Say, miner j has 5 shares in round P_1 and 30 shares in round P_2, then j's total payment for these rounds is $5 \times \frac{0.9}{10} + 30 \times \frac{0.9}{140} = 0.64$. So the pool only pays miners when block rewards are realized. In comparison, say that the pool alternatively pays out a fixed amount for every share it receives: $\alpha(s, H) = 0.02$. Again, using the "data" from Example 1, we see that the pool will run a deficit already after the second round P_2 since $2 - 150 \times 0.02 = -1$.

In Section 5.1 we provide a brief discussion of various reward sharing schemes that are used in practice or proposed to improve existing practices.

For our axiomatic analysis below, we are inspired by the framework in Schrijvers et al. (2016). In particular, we consider situations in which the pool rounds are prolonged by an additional share, i.e., when the pool round ends with the submission of additional share(s) at the end. In the simplest of such cases, we consider histories where a pool round is extended by one additional share at the end of the round. We therefore need some additional definitions.

Formally, let $H = (S, \mathcal{P}(S))$ be the history of a pool. Let $s^* \notin S$ be the so-called additional last share (into the r^{th} round), i.e., $\tau(\underline{s}) < \tau(s^*) < \tau(\overline{s})$ for all $\underline{s} \in P_r$ and for all $\overline{s} \in P_{r+1}$. The history $H' = (S', \mathcal{P}'(S'))$ is an *extension of H at the r^{th} round* whenever $S' = S \cup \{s^*\}$ and:

- $P'_k = P_k$ for all $k \neq r$,
- $P'_r = P_r \cup \{s^*\}$.

Note that one can consider the extension of a round at any position and not only at the end of the round. However, this will be a stronger version and it seems more natural to extend a round by adding one share at the end.

Next, we define the restriction of a history to a single round. Let $H = (S, \mathcal{P}(S))$ be any history. The *restriction of H to the r^{th} round* is denoted as $H|_r = (P_r, \{P_r\})$. That is, the set of shares in the history only consists of those at the r^{th} round and the only elements in the partition of the history is P_r.

Example 3. Recall the situation in Example 1. Below is the example of an extension of the history in Equation (1) at the r^{th} round by the full solution (share) s^*.

$$\begin{aligned} S &= \{s_1, \ldots, \mathbf{s_{10}}, s_{11} \ldots, \mathbf{s_{150}}, \ldots, s_{564}, \ldots, s_{589}, \mathbf{s^*}, \ldots, \mathbf{s_m}\} \\ \mathcal{P}(S) &= \{\{\underbrace{s_1, s_2, \ldots, \mathbf{s_{10}}}_{P_1}\}, \{\underbrace{s_{11} \ldots, \mathbf{s_{150}}}_{P_2}\}, \ldots, \{\underbrace{s_{564}, \ldots, s_{589}, \mathbf{s^*}}_{P_r}\}, \ldots, \{\underbrace{\ldots, \mathbf{s_m}}_{P_l}\}\} \end{aligned}$$

Moreover, an example of restriction of the history in Equation (1) to the r^{th} round is

$$H|_r = \left(\{s_{564}, \ldots, \mathbf{s_{589}}\}, \left\{\{\underbrace{s_{564}, \ldots, \mathbf{s_{589}}}_{P_r}\}\right\}\right)$$

3. Axioms

In the following, we introduce five properties of generic reward sharing schemes, which we find desirable in terms of preventing the pool from going bankrupt and ensuring a fair distribution of rewards among reported shares. In particular, two of the key axioms reflect different aspects of fairness related to how the rewards must be re-distributed in case there is one additional share in the round.[15]

In expectation every round has the same length and effectively the same reward. Thus, it seems natural to require a *fixed total reward* to the miners for any pool round in a history. In turn, this implies that the fee charged by the pool manager is the same for any two rounds in a history. This guarantees that the pool manager can not take advantage (or be harmed) from shorter (longer) rounds, as the pool must distribute the same amount to the miners in any round. Moreover, say we wanted to compensate miners in long rounds by increasing their total reward for such rounds in order to prevent pool

[15] In this sense our fairness axioms can be viewed as relational axioms in case of a variable populations framework, see e.g., Thomson (2016).

hopping. This, however, makes miners prefer longer rounds, which they can strategically ensure by delay reporting the full solution contrary to what the pool manager (network) wants. Therefore our first axiom protects the miners in the short rounds (by distributing the same amount as in the longer rounds) and the pool manger in the long rounds (by ensuring that the pool does not have to subsidize miners). Formally,

- **Fixed Total Reward:** A scheme α satisfies fixed total reward whenever, for any history $H = (S, \mathcal{P}(S))$, and any two rounds $P, P' \in \mathcal{P}(S)$, we have

$$\sum_{s \in P} \alpha(s, H) = \sum_{s \in P'} \alpha(s, H).$$

The next condition, dubbed *budget limit*, requires that the pool manager charges a non-negative fee. This ensures that the pool does not need credit, or eventually go bankrupt, as only actually realized income can be distributed among the miners in every round.

- **Budget Limit:** A scheme α satisfies budget limit whenever, for any history $H = (S, \mathcal{P}(S))$, and any round $P \in \mathcal{P}(S)$, we have:

$$\sum_{s \in P} \alpha(s, H) \leq B.$$

The next condition, dubbed *round based rewards*, requires that the distribution of the reward only depends on the round itself and it is not affected by any other rounds in the history. Formally,

- **Round Based Rewards:** A scheme α satisfies round based rewards whenever, for any history H, and any round P_r, we have for all $s \in P_r$:

$$\alpha(s, H) = \alpha(s, H|_r).$$

Next, we consider situations where the submission time of shares may change. To analyze such situations, we create a ceteris paribus case where the time signature of only a single share in a history changes in a minimal way. Formally, let $H = (S, \mathcal{P}(S))$ be any history, and consider any round $P_r \in \mathcal{P}(S)$, and any share $s_i \in P_r$ in this round. A *time-shift* $H' = (S', \mathcal{P}'(S'))$ of H at s_i is defined as:

1. $P'_j = P_j$ for all $j \neq r$, and
2. $S' = (S \setminus \{s_i\}) \cup \{s'_i\}$ for some $s'_i \in \mathcal{S} \setminus S$ such that $\tau(s_{i-1}) < \tau(s'_i) < \tau(s_{i+1})$.

The next condition, dubbed *delay invariance*, requires that delaying the submission of shares should not affect the reward distribution, so long as the order of those shares is preserved. Without "delay invariance" miners would have direct incentives to speculate in the exact timing of which they report shares, while ideally they should report immediately.

- **Delay Invariance:** A scheme α satisfies delay invariance if, for any $H = (S, \mathcal{P}(S))$, and for any time-shift $H' = (S', \mathcal{P}'(S'))$ of H at any s_i, we have:

$$\alpha(s, H) = \alpha(s, H') \text{ for all } s \in S \setminus \{s_i\}$$

We now turn to our two main fairness conditions. The first is dubbed *absolute redistribution*, and requires that in case a round is extended (prolonged) by one additional share, the award assigned to any existing share in the corresponding round decreases by the same *amount*.[16] Formally,

- **Absolute Redistribution:** A scheme α satisfies absolute redistribution whenever, for any history $H = (S, \mathcal{P}(S))$, any round P_r with $|P_r| > 1$, and any extension $H' = (S', \mathcal{P}'(S'))$ at the r^{th} round we have for any $s_i, s_j \in P_r$:

$$\alpha(s_i, H) - \alpha(s_i, H') = \alpha(s_j, H) - \alpha(s_j, H').$$

[16] This is, in fact, a strong version of Population monotonicity in Thomson (2016) and resembles the spirit of Myerson's fairness axiom Myerson (1977) stating that agents should be affected equally from entering mutual agreements.

In the same spirit, the next condition, dubbed *relative redistribution*, requires that in case a round is extended (prolonged) by one additional share, the reward of each existing share in the corresponding round is decreased by the same *ratio*.[17] Formally,

- **Relative Redistribution:** A scheme α satisfies relative redistribution whenever for any history $H = (S, \mathcal{P}(S))$, any round P_r with $|P_r| > 1$, and any extension $H' = (S', \mathcal{P}'(S'))$ at the r^{th} round, we have for any $s_i, s_j \in P_r$ with $\alpha(s_i, H) \neq 0$ and $\alpha(s_j, H) \neq 0$:

$$\frac{\alpha(s_i, H')}{\alpha(s_i, H)} = \frac{\alpha(s_j, H')}{\alpha(s_j, H)}.$$

Finally, we note that both fairness axioms relate to the situation where a single share is added at the end of a round. Because of the delay invariance axiom the order of shares are relevant and therefore it makes sense to add a share at the end of the round because it depends on the time signature. Without the delay invariance axiom the share can be added anywhere in particular at the end. Therefore all the results presented in the following sections are still going to hold.

We first observe that round based rewards strengthens the fixed total reward condition, such that the latter imposes the fixed total rewards for any rounds in any history.

Lemma 1. *If a scheme α satisfies fixed total reward and round based rewards, then for any two histories $H = (S, \mathcal{P}(S))$ and $H' = (S', \mathcal{P}'(S'))$ and any two rounds $P \in \mathcal{P}(S)$ and $P' \in \mathcal{P}'(S')$ we have*

$$\sum_{s \in P} \alpha(s, H) = \sum_{s \in P'} \alpha(s, H').$$

Proof. See Appendix A. ■

Remark 1. An immediate consequence of Lemma 1 is that, if a scheme satisfies fixed total rewards and round based rewards then the fee must be the same for all histories and all rounds in these histories. Therefore, these two axioms, together with budget limit imply a stronger version of the budget limit[18] in the sense that for a scheme which satisfies fixed total rewards and round based rewards and for any history $H = (S, \mathcal{P}(S))$ and any round $P \in \mathcal{P}(S)$, we have $\sum\limits_{s \in P} \alpha(s, H) = R$, with $R = B - f$ for some $f \in [0, B]$.

Second, we observe that joint with delay invariance, fixed total reward and round based rewards imply that shares with same relative rank in rounds of equal length get identical awards. Formally,

Lemma 2. *If a scheme α satisfies fixed total reward, round based rewards and delay invariance, then for any two histories $H = (S, \mathcal{P}(S))$ and $\bar{H} = (\bar{S}, \bar{\mathcal{P}}(\bar{S}))$ and any two rounds $P \in \mathcal{P}(S)$ and $\bar{P} \in \bar{\mathcal{P}}(\bar{S})$ such that $|P| = |\bar{P}|$, we have for all $s \in P$ and for all $\bar{s} \in \bar{P}$ such that $\rho(s) = \bar{\rho}(\bar{s})$:*

$$\alpha(s, H) = \alpha(\bar{s}, \bar{H}).$$

Proof. See Appendix B. ■

4. Fair reward sharing schemes

4.1. Absolute fairness

In this section we single out a particular class of schemes, called the class of *absolute fair schemes*. We show that this is the only such class that satisfies absolute redistribution together with fixed total reward, budget limit, round based rewards, and delay invariance.

Specifically, a scheme is an absolute fair scheme if there exists $\varepsilon : \mathbb{N} \to [0, 1]$ with $\varepsilon(1) = 1$ and $\varepsilon(\rho(s)) \geq \sum\limits_{i=\rho(s)+1}^{\infty} \frac{\varepsilon(i)}{i-1}$ such that,

[17] The relative redistribution axiom is inspired by the axioms that are used in, for instance, Dietzenbacher and Kondratev (2020) and Hougaard et al. (2017) to characterize geometric schemes. Moreover, the relative redistribution property also has roots in observed practice. For instance, when allocating the prize money in the WCOOP Poker Tournament, the reward for the first place, second place, etc are distributed such that they maintain the same relative difference (see Dietzenbacher and Kondratev (2020)). One may also find many examples in taxation, where the increase in the tax burden is often reflected in a relative manner on the population.

[18] This is conceptually similar to the *strong budget balance* in Chen et al. (2019).

$$\alpha^{**}(s,H) = R\left(\varepsilon(\rho(s)) - \sum_{i=\rho(s)+1}^{|P(s)|} \frac{\varepsilon(i)}{i-1}\right) \tag{2}$$

Note that for the last share $\rho(s) = |P(s)|$, therefore $\sum_{i=|P(s)|+1}^{|P(s)|} \frac{\varepsilon(i)}{i-1}$ is an empty sum and by convention it equals 0.

We can interpret absolute fair schemes in an iterative fashion: if there is only one share s, then $\rho(s) = 1$ so $\varepsilon(1) = 1$ and the share receives the full net reward R. If we add a share s', then $\rho(s') = 2$ and $\varepsilon(2) < 1$ so the first share s gets $R(\varepsilon(1) - \varepsilon(2))$ and the second share s' gets $R\varepsilon(2)$. Now, adding a third share s'' the award to both the first and second share should be reduced by the same amount δ. Thus, $\varepsilon(3) = 2\delta$ and consequently the first share is awarded $R(\varepsilon(1) - \varepsilon(2) - \frac{\varepsilon(3)}{2})$ and the second share is awarded $R(\varepsilon(2) - \frac{\varepsilon(3)}{2})$, and so forth, for any share in the round.

Next, we present our first main result.

Theorem 1. *A reward sharing scheme α satisfies round based rewards, budget limit, fixed total reward, delay invariance and absolute redistribution if and only if it is an absolute fair scheme.*

Proof. See Appendix C. ∎

A prominent member of the class of absolute fair schemes is the proportional scheme as stated below.

Proposition 1. *The proportional scheme is an absolute fair scheme.*

Proof. See Appendix D. ∎

4.2. Relative fairness

In this section we single out a particular class of schemes, called the class of *relative fair schemes*. We show that this is the only such class that satisfies relative redistribution together with fixed total reward, budget limit, round based rewards, and delay invariance.

Specifically, a scheme is a relative fair scheme if there exists $\varepsilon(j) : \mathbb{N} \to [0, 1]$ with $\varepsilon(1) = 1$ such that,

$$\alpha^{*}(s,H) = R\varepsilon(\rho(s)) \prod_{j=\rho(s)+1}^{|P(s)|} \big(1 - \varepsilon(j)\big) \tag{3}$$

Note that for the last share $\rho(s) = |P(s)|$, therefore $\prod_{j=|P(s)|+1}^{|P(s)|} (1 - \varepsilon(j))$ is an empty product and by convention it equals 1.

An iterative interpretation of relative fair schemes is as follows: if there is only one share s, $\varepsilon(1) = 1$ and the share is awarded the entire net reward R. If we add a second share s', then s' will be awarded $R\varepsilon(2)$ and the first share must get $R(\varepsilon(1)(1 - \varepsilon(2))) = R(1 - \varepsilon(2))$. Now, adding a third share s'' we must now have that both the award to the first share and the second share change by the same ratio δ so $(1 - \varepsilon(2))\delta + \varepsilon(2)\delta + \varepsilon(3) = 1$ implying that $\delta = 1 - \varepsilon(3)$ yielding the award $R((1 - \varepsilon(2))(1 - \varepsilon(3)))$ to the first share and $R(\varepsilon(2)(1 - \varepsilon(3)))$ to the second share, and finally $R\varepsilon(3)$ to the third share, and so forth for any share in the round.

We now present our second main result characterizing the class of relative fair reward sharing schemes.

Theorem 2. *A reward scheme α satisfies round based rewards, budget limit, fixed total reward, delay invariance and relative redistribution if and only if it is a relative fair scheme.*

Proof. See Appendix E. ∎

It turns out that the proportional scheme also takes up a prominent position among the relative fair schemes as recorded below.

Proposition 2. *The proportional scheme is a relative fair scheme.*

Proof. See Appendix F. ∎

4.3. Consensus between absolute and relative redistribution

Next, we search for a consensus between the two fairness concepts that we discussed in the previous sections. Imposing both fairness axioms together with the other conditions characterize a new class of schemes at the intersection of the former two classes. We call this new class of schemes *k-pseudo proportional schemes* and find that this class is also a generalization of the proportional scheme mentioned in Example 2. Formally, given any k and δ, a k-pseudo proportional scheme assigns awards to shares in a round identical to the proportional scheme, $\frac{R}{|P(S)|}$, so long as the round is shorter than k, i.e., when $|P(S)| < k$. In case the round has more shares than k, then the scheme assigns an award of i) δ to the k^{th} share, ii) distributes the rest, $R - \delta$, to the first $k-1$ shares, and iii) awards 0 to any share that is submitted after k. The general structure of the k-pseudo proportional scheme with $k > 1$ is as follows:

$$\alpha^{k,\delta}(s,H) = \begin{cases} \frac{R}{|P(s)|}, & \text{if } |P(s)| < k \\ \frac{R-\delta}{k-1}, & \text{if } |P(s)| \geq k \text{ and } \rho(s) < k \\ \delta, & \text{if } |P(s)| \geq k \text{ and } \rho(s) = k \\ 0, & \text{if } |P(s)| \geq k \text{ and } \rho(s) > k \end{cases} \tag{4}$$

for $0 \leq \delta \leq R$.

The following theorem shows that the class of k-pseudo proportional schemes is the only one at the intersection of relative fair schemes and absolute fair schemes.

Theorem 3. *A reward sharing scheme satisfies round based rewards, budget limit, fixed total reward, delay invariance, absolute redistribution, and relative redistribution if and only if it is a k-pseudo proportional scheme.*

Proof. See Appendix G. ■

Next, we consider a property that ensures strictly positive awards for all shares, for any history. To ensure that all shares get paid can be seen as a fairness requirement since all shares involve "work", but it also relates to miners' incentives. If miners are not paid for their shares they may stop mining for the pool: in particular, if they are not paid after a certain number of shares in a round as in the k-pseudo-proportional schemes.[19] We write this feature as an additional axiom and provide the resulting characterization of the proportional scheme as a corollary to Theorem 3, in effect highlighting that the proportional scheme is the only member of the family (4) that is compatible with miners' incentives to keep mining for the pool.

- **Strict Positivity:** A scheme α satisfies strict positivity whenever, for any history $H = (S, \mathcal{P}(S))$, and any round $P \in \mathcal{P}(S)$, we have:

$$\alpha(s, H) \in \mathbb{R}_{++}.$$

Adding strict positivity to the previous set of axioms provides a new characterization of the proportional rule. However, joint with strict positivity, the axiom delay invariance becomes redundant. Formally,

Corollary 1. *The proportional scheme is the only scheme that satisfies round based rewards, budget limit, fixed total reward, absolute redistribution, relative redistribution, and strict positivity.*

Proof. See Appendix H. ■

Example 2 *(Continued).* In Example 2 the proportional scheme and the fixed payment per share schemes were discussed. Here, we use the same "data" for round P_1 to provide an example of the absolute fair scheme, relative fair scheme and the k-pseudo proportional scheme. Also, recall that $B = 1$ and we fixed the pool fee of any round to $f(P) = 0.1$.

For the absolute fair scheme, let $\varepsilon(1) = 1$ and $\varepsilon(i) = \frac{0.2}{i}$ for any $i > 1$. Then $\alpha^{**}(s_1, H) = (1 - 0.1)\left(\varepsilon(1) - \sum_{i=2}^{10} \frac{\varepsilon(i)}{i-1}\right) = 0.9(1 - \frac{9}{50}) = 0.738$, $\alpha^{**}(s_2, H) = (1-0.1)\left(\varepsilon(2) - \sum_{i=3}^{10} \frac{\varepsilon(i)}{i-1}\right) = 0.9(0.02) = 0.018$. Following the scheme we have $\alpha^{**}(s_i, H) = 0.018$, for $i \in \{3, \ldots, 10\}$.

[19] A weaker version of the axiom would require only strict positivity of the last share in a round. This would likely be enough to incentivize miners to keep mining for the pool, e.g., as in the Pay-Per-Last-N-Shares scheme.

For the relative fair scheme, let $\varepsilon(1)=1$, and $\varepsilon(i)=\frac{1}{2}$ for any $i>1$. Then $\alpha^*(s_1,H)=(1-0.1)\times 1\times\prod_{j=2}^{10}\left(1-\frac{1}{2}\right)=\frac{0.9}{2^9}$, $\alpha^*(s_2,H)=(1-0.1)\times\frac{1}{2}\times\prod_{j=3}^{10}\left(1-\frac{1}{2}\right)=\frac{0.9}{2^9}$. Following the scheme we have $\alpha^*(s_i,H)=\frac{0.9}{2^{11-i}}$, for $i\in\{3,\dots,10\}$.

For the k-pseudo proportional scheme, let $k=3$ and $\delta=0.2$, then $\alpha^{3,0.2}(s_1,H)=\frac{1-0.1-0.2}{2}=0.35$, $\alpha^{3,0.2}(s_2,H)=\frac{1-0.1-0.2}{2}=0.35$, $\alpha^{3,0.2}(s_3,H)=0.2$, and $\alpha^{3,0.2}(s_i,H)=0$ for $i\in\{4,\dots,10\}$.

5. Discussion

This paper provides, for the first time, a rich framework for reward sharing schemes in mining pools through an economic design perspective. To demonstrate the flexibility in the design, we proposed various desirable axioms and put particular emphasis on the fairness concept as an example. We provided three different characterizations of classes within the framework, i.e., absolute fair, relative fair, and k-pseudo proportional schemes.

In what follows, we show that the framework also allows the formalization of various well-known schemes and investigate these schemes axiomatically. The results are summarized in Table 1. We conclude the paper with a discussion of the logical independence of the characterizing axioms and provide a summary in Table 2.

5.1. Known reward sharing schemes

In this section we focus on some of the most popular, and widely applied, reward sharing schemes and examine whether they satisfy the axioms proposed in Section 3. In addition, we comment on several potentially interesting schemes suggested in Rosenfeld (2011) and Schrijvers et al. (2016) respectively.

- **Pay Per Share (PPS):** The strict egalitarian Pay-Per-Share scheme fails to ensure against potential bankruptcy of the mining pool (as noted in Rosenfeld (2011)) but has been applied, for instance by F2Pool[20] and Poolin,[21] probably due to its immediate simplicity and transparency. In pay per share, every submitted share receives a fixed reward regardless of when it is submitted, and the round it is submitted in. Formally, $\alpha(s,H)=c$ for some constant c. The payments to the shares are usually adjusted by the network difficulty and the length of a round. Trivially, PPS fails fixed total reward and budget limit, but satisfies relative redistribution, absolute redistribution, delay invariance and round based rewards.
- **Pay-Per-Last-N-Shares:** The pay-per-last-N-shares (PPLNS for short) has also been popular and is applied, for instance in GHash.IO[22] and P2Pool.[23] In PPLNS, the pool manager gets a fixed fee, say f, and the net-reward, $R=B-f$, is distributed equally among the N last shares (including the full the solution), regardless of the round boundaries. Therefore, the reward of a share at the time it is submitted depends on the number of full solutions among the next $N-1$ shares. That is, if no full solution is found among the next $N-1$ shares, the share receives no reward, if one full solution is found it is rewarded once with $\frac{1}{N}R$, if two full solutions are found it is rewarded twice with $\frac{1}{N}R$, and so forth.

To formally define the PPLNS, consider a history $H=(S,\mathcal{P}(S))$ and let $\Omega(s)$ denote the index of the pool round that share s belongs to i.e., $\Omega(s)=\{x\le|\mathcal{P}(S)|\mid s\in P_x\}$. Then the PPLNS scheme is defined as

$$\alpha(s_i,H)=\frac{\Omega(s_{N+i})-\Omega(s_i)}{N}R.$$

It is trivial that PPLNS satisfies delay invariance. However, Example 4 (in Appendix I.1) shows that PPLNS fails to satisfy budget limit, fixed total rewards, round based rewards, absolute redistribution and relative redistribution.

Besides the two popular schemes analyzed above, the academic literature has suggested additional and more sophisticated schemes designed to provide miners with improved incentives. For instance, Rosenfeld (2011) suggests to use a so-called geometric scheme.

- **Geometric**: In this scheme, the rewards are distributed among the shares in a round using a geometric series based on the order of their submission. Unlike other schemes the fees are variable in this model and they depend on the size of the round. Formally, let $r>1$. The fee for a round is defined as $f(P(s))=\frac{1}{r^{|P(s)|}}$ and the reward to each share is

$$\alpha(s,H)=\frac{(r-1)}{r^{|P(s)|-\rho(s)+1}}B.$$

[20] https://www.f2pool.com/.
[21] https://www.poolin.com/.
[22] https://ghash.io/.
[23] http://p2pool.in/.

It is straight forward to see that the geometric scheme satisfies relative redistribution, round based rewards and delay invariance. It also satisfies budget limit (see Proposition 3 in Appendix I.2). However, the geometric scheme fails to satisfy the fixed total reward axiom. Hence, we propose the following modification of this scheme to fix this failure.

- **Constrained Geometric:** This scheme is defined as $\alpha(s,H) = \dfrac{(r-1)}{r^{|P(s)|-\rho(s)+1}} \times \dfrac{r^{|P(s)|}}{r^{|P(s)|}-1}$. The constrained geometric scheme is included in the class of relative fairness reward sharing schemes (with $\epsilon_1 = 1$ and $\epsilon_j = \frac{r^{j-1}-1}{r^j-1}$). Therefore, it satisfies budget limit, total fix rewards, delay invariance and relative redistribution. To show that it fails absolute re-distribution consider the history H and its extension H' as presented in Example 5 (in Appendix I.2). It can be verified that $\alpha(s_1,H) = \frac{r-1}{r^2} \times \frac{r^2}{r^2-1}B$, $\alpha(s_2,H) = \frac{r-1}{r} \times \frac{r^2}{r^2-1}B$, and $\alpha(s_1,H') = \frac{r-1}{r^3} \times \frac{r^3}{r^3-1}B$, $\alpha(s_2,H') = \frac{r-1}{r^2} \times \frac{r^3}{r^3-1}B$, $\alpha(s^*,H') = \frac{r-1}{r} \times \frac{r^3}{r^3-1}B$. Since $r>1$ and $\alpha(s_1,H) - \alpha(s_1,H') \neq \alpha(s_2,H) - \alpha(s_2,H')$ then the constrained geometric scheme fails to satisfy the absolute redistribution.

Next, we formulate another scheme which approaches the reward sharing problem from the "incentive compatibility" perspective, i.e., it provides miners with the incentive to report shares immediately.

- **IC scheme:** The IC scheme is proposed by Schrijvers et al. (2016) as an *incentive compatible* reward sharing scheme. In words, let $1/D$ denote the probability of a share to be a full solution. Then, if the round length is at least of the same size as D, the scheme distributes the reward proportionally according to the length of the round; if the round length is shorter, every share receives $1/D$ and the residual budget is given to the last (full) share of the round. Formally,

$$\alpha(s,H) = \begin{cases} \frac{R}{D} & \textit{if} \ \ |P(s)| \le D \text{ and } \rho(s) < |P(s)| \\ \frac{R}{D} + (1 - \frac{|P(s)|}{D})R, & \textit{if} \ \ |P(s)| \le D \text{ and } \rho(s) = |P(s)| \\ \frac{R}{|P(s)|}, & \textit{if} \ \ |P(s)| \ge D \end{cases}$$

This scheme obviously satisfies fixed total rewards, delay invariance, and round based rewards, while it fails both absolute and relative redistribution.

We conclude this section by formulating one more reward sharing scheme, which is interesting. The Slush scheme, named after the Slush mining pool,[24] is the only reward sharing scheme that uses time signatures as a parameter for distributing the rewards. Our framework is rich enough to capture this feature. Formally:

- **Slush:** Consider a history $H = (S, \mathcal{P}(S))$ and let $\Omega(s)$ denote the index of the pool round that share s belongs to i.e., $\Omega(s) = \{x \le |\mathcal{P}(S)| \mid s \in P_x\}$. Let $\bar{s}_j$ denote the last share in the j^{th} round, i.e., $\bar{s}_j = \{s \in P_j \mid \tau(s) \ge \tau(s'), \ \forall\, s' \in P_j\}$. Let

$$score(s,j) = \frac{e^{\frac{\tau(s)-\tau(\bar{s}_j)}{\lambda}}}{\sum\limits_{\tau(s') \le \tau(\bar{s}_j)} e^{\frac{\tau(s')-\tau(\bar{s}_j)}{\lambda}}}$$ for any $\Omega(s) \le j \le l$. Then,

$$\alpha(s,H) = R \sum_{i=\Omega(s)}^{l} score(s,i).$$

The parameter λ is set to 1200 in the Slush pool. In what follows, we therefore assume $\lambda = 1200$. Example 6 (in Appendix I.3) demonstrates that the slush scheme does not satisfy fixed total reward, delay invariance, budget limit, round based reward, absolute redistribution and relative redistribution.

The results for the well-known schemes are summarized in Table 1 below.

5.2. Logical independence

We define six reward sharing schemes in order to demonstrate logical independence of the axioms in Section 3. The results are summarized in Table 2 below. Defining schemes 1–6, let $R = B - f$ for some fixed $f \in [0, B]$.

- **Scheme 1:**

$$\alpha(s,H) = \begin{cases} \frac{R}{|P(s)|}, & \text{if } |P(s)| \text{ is odd} \\ \frac{R}{2|P(s)|}, & \text{if } |P(s)| \text{ is even} \end{cases}$$

This scheme fails *fixed total rewards*, but meets all the other axioms.

[24] https://slushpool.com/.

Table 1
Summary of the well-known schemes.

Scheme	Fixed total reward	Relative redistribution	Absolute redistribution	Round based rewards	Budget limit	Delay invariance
PPS	-	+	+	+	-	+
PPLNS	-	-	-	-	-	+
Geometric	-	+	-	+	+	+
Modified Geometric	+	+	-	+	+	+
IC	+	-	-	+	+	+
Slush	-	-	-	-	-	-

Table 2
Logical independence of the axioms.

Scheme	Fixed total reward	Relative redistribution	Absolute redistribution	Round based rewards	Budget limit	Delay invariance
Scheme 1	-	+	+	+	+	+
Scheme 2	+	-	+	+	+	+
Scheme 3	+	+	-	+	+	+
Scheme 4	+	+	+	-	+	+
Scheme 5	+	+	+	+	-	+
Scheme 6	+	+	+	+	+	-

- **Scheme 2:**

$$\alpha(s,H)=\begin{cases} R, & \text{if } |P(s)|=1 \\ (R-\lambda)+\frac{\lambda}{|P(s)|}, & \text{if } |P(s)|>1 \text{ and } \rho(s)=1 \\ \frac{\lambda}{|P(s)|}, & \text{if } |P(s)|>1 \text{ and } \rho(s)\neq 1 \end{cases}$$

where $0<\lambda<R$ is a constant number.
This scheme fails *relative redistribution*, but meets all the other axioms.

- **Scheme 3:**

$$\alpha(s,H)=\begin{cases} \frac{R}{2^{|P(s)|-1}}, & \text{if } \rho(s)=1 \\ \frac{2^{\rho(s)-2}R}{2^{|P(s)|-1}}, & \text{if } \rho(s)\neq 1 \end{cases}$$

This scheme fails *absolute redistribution*, but meets all the other axioms.

- **Scheme 4:**

$$\alpha(s,H)=\begin{cases} \frac{R}{|P(s)|}, & \text{if the number of shares in the first round of the history is odd} \\ \frac{R}{2|P(s)|}, & \text{if the number of shares in the first round of the history is even} \end{cases}$$

This scheme fails *round based rewards*, but meets all the other axioms.

- **Scheme 5:**

$$\alpha(s,H)=\frac{2R}{|P(s)|}$$

This scheme fails *budget limit*, but meets all the other axioms.

- **Scheme 6:**

$$\alpha(s,H)=\begin{cases} R, & \textit{if } \ |P(s)|=1 \\ \alpha^{2,\frac{R}{2}}(s,H), & \textit{if } \ \tau(\rho(s)=1)-\tau(\rho(s)=2)<T \\ \alpha^{2,\frac{R}{3}}(s,H), & \textit{if } \ \tau(\rho(s)=1)-\tau(\rho(s)=2)\geq T \end{cases}$$

where T is a time threshold.
This scheme fails *delay invariance*, but meets all the other axioms.

To show the logical independence of the axioms in Corollary 1, note that all the Schemes 1–5 presented above satisfy the strict positivity. For a reward sharing scheme that satisfies all the axioms (in Corollary 1) except strict positivity, one can simply consider any k-pseudo proportional scheme with any finite k.

Declaration of competing interest

On behalf of all co-authors I hereby declare that none of the authors have any relevant, or material financial interests in the research described in our paper.

Appendix A. Proof of Lemma 1

Lemma 1. *If a scheme α satisfies fixed total reward and round based rewards, then for any two histories $H = (S, \mathcal{P}(S))$ and $H' = (S', \mathcal{P}'(S'))$ and any two rounds $P \in \mathcal{P}(S)$ and $P' \in \mathcal{P}'(S')$ we have*

$$\sum_{s \in P} \alpha(s, H) = \sum_{s \in P'} \alpha(s, H').$$

Proof. Take any two histories H, H' and P, P' as in the lemma. Let r (and r') denote the round number of P (and P') in history H (and H'). As α satisfies fixed total reward, there exist fixed rewards for both histories, say K and K':

$$\sum_{s \in P} \alpha(s, H) = K \text{ and } \sum_{s \in P'} \alpha(s, H') = K' \tag{5}$$

Consider the restriction of H to r^{th} round (and of H' to r'^{th} round). As α satisfies round based rewards, for all $s \in P$ (and for all $s' \in P'$) we have: $\alpha(s, H) = \alpha(s, H|_r)$ and $\alpha(s', H') = \alpha(s', H'|_{r'})$. This implies:

$$\sum_{s \in P} \alpha(s, H|_r) = K \text{ and } \sum_{s \in P'} \alpha(s, H'|_{r'}) = K' \tag{6}$$

Next we consider two cases. We say two rounds in different histories *overlap* when there are some shares in the these two rounds which have overlapping time signatures. Formally, we say P and P' are overlapping whenever: $\min_{s \in P'} \le \min_{s \in P} \tau(s) \le \max_{s \in P'} \tau(s)$ or $\min_{s \in P'} \le \max_{s \in P} \tau(s) \le \max_{s \in P'} \tau(s)$. Based on whether these rounds P and P' in histories H and H' overlap or not, we shall separately prove the equality of rewards, $K = K'$, in Equation (5).

Case 1. No overlap. Without loss of generality, assume all the shares in P have an earlier time signature than those in P'. In this case, consider a history with two rounds $H'' = (P \cup P', \{P, P'\})$. Note that H'' is a history consisting only of two rounds, P as the first and P' as the second round. Note also that $\max_{s \in P} \tau(s) < \min_{s \in P'} \tau(s)$.

As α satisfies fixed total reward, there exists a fixed reward for this history, say K'':

$$\sum_{s \in P} \alpha(s, H'') = \sum_{s \in P'} \alpha(s, H'') = K'' \tag{7}$$

Consider the restriction of H'' to the 1^{st} and the 2^{nd} rounds. As α satisfies round based rewards for all $s \in P$ we have $\alpha(s, H'') = \alpha(s, H''|_1)$, and for all $s \in P'$ we have $\alpha(s, H'') = \alpha(s, H''|_2)$. This implies:

$$\sum_{s \in P} \alpha(s, H''|_1) = K'' \text{ and } \sum_{s \in P'} \alpha(s, H''|_2) = K''. \tag{8}$$

Note that the restriction of H'' to the 1^{st} round is equivalent to the restriction of H to the r^{th} round. Similarly the restriction of H'' to the 2^{nd} round is equivalent to the restriction of H' to the r'^{th} round. That is, $H''|_1 = (P, \{P\}) = H|_r$ and $H''|_2 = (P', \{P'\}) = H'|_{r'}$, therefore we have:

$$\sum_{s \in P} \alpha(s, H''|_1) = \sum_{s \in P} \alpha(s, H|_r) \text{ and } \sum_{s \in P'} \alpha(s, H''|_2) = \sum_{s \in P'} \alpha(s, H'|_{r'})$$

The former equality implies $K'' = K$ and the latter implies $K'' = K'$. This completes the proof for Case 1.

Case 2. Overlap. In this case we consider two additional histories with two rounds each: $\overline{H} = (P \cup \overline{P}, \{P, \overline{P}\})$ and $\overline{H}' = (P' \cup \overline{P}', \{P', \overline{P}'\})$ such that

$$\max_{s \in P \cup P'} \tau(s) < \min_{s \in \overline{P}} \tau(s) < \max_{s \in \overline{P}} \tau(s) < \min_{s \in \overline{P}'} \tau(s).$$

Note that $\overline{H}$ is a history consisting only of two rounds, P as the first and $\overline{P}$ as the second round. Similarly, $\overline{H}'$ is a history consisting only of two rounds, P' as the first and $\overline{P}'$ as the second round.

Since $\max_{s \in P \cup P'} \tau(s) < \min_{s \in \overline{P}} \tau(s)$ there is no overlap between P and $\overline{P}$, therefore applying Case 1 on H and $\overline{H}$ yields:

$$i) \quad \sum_{s \in P} \alpha(s, H) = \sum_{s \in \overline{P}} \alpha(s, \overline{H}) \tag{9}$$

Since $\max_{s \in P \cup P'} \tau(s) < \min_{s \in \overline{P}'} \tau(s)$ there is no overlap between P' and $\overline{P}'$, therefore applying Case 1 on H' and $\overline{H}'$ yields:

$$ii) \quad \sum_{s \in P'} \alpha(s, H') = \sum_{s \in \overline{P}'} \alpha(s, \overline{H}'). \tag{10}$$

Since $\max_{s\in\overline{P}} \tau(s) < \min_{s\in\overline{P}'} \tau(s)$ there is no overlap between $\overline{P}$ and $\overline{P}'$, therefore applying Case 1 on $\overline{H}$ and $\overline{H}'$ yields:

$$\sum_{s\in\overline{P}} \alpha(s,\overline{H}) = \sum_{s\in\overline{P}'} \alpha(s,\overline{H}'),$$

which – combined with Equations (9) and (10) – completes the proof. ■

Appendix B. Proof of Lemma 2

Lemma 2. *If a scheme α satisfies fixed total reward, round based rewards and delay invariance, then for any two histories $H = (S,\mathcal{P}(S))$ and $\bar{H} = (\bar{S},\bar{\mathcal{P}}(\bar{S}))$ and any two rounds $P \in \mathcal{P}(S)$ and $\bar{P} \in \bar{\mathcal{P}}(\bar{S})$ such that $|P| = |\bar{P}|$, we have for all $s \in P$ and for all $\bar{s} \in \bar{P}$ such that $\rho(s) = \bar{\rho}(\bar{s})$:*

$$\alpha(s,H) = \alpha(\bar{s},\bar{H}).$$

Proof. Take any scheme α that satisfies these two axioms. Let $H = (S,\mathcal{P}(S))$ and $\bar{H} = (\bar{S},\bar{\mathcal{P}}(\bar{S}))$ be two histories and let $P \in \mathcal{P}(S)$ and $\bar{P} \in \bar{\mathcal{P}}(\bar{S})$ be any two rounds with $|P| = |\bar{P}| = k$ for some k. Let $P = P_r$ and $\bar{P} = \bar{P}_{\bar{r}}$, i.e., P is the r^{th} round in H and $\bar{P}$ is the $\bar{r}^{th}$ round in $\bar{H}$. As α satisfies round based rewards we have i) $\alpha(s,H) = \alpha(s,H|_r)$ for all $s \in S$ and ii) $\alpha(\bar{s},\bar{H}) = \alpha(\bar{s},\bar{H}|_{\bar{r}})$ for all $\bar{s} \in \bar{S}$. Let $H|_r = (\{s_1,\ldots,s_k\},\{\{s_1,\ldots,s_k\}\})$ and $\bar{H}|_{\bar{r}} = (\{\bar{s}_1,\ldots,\bar{s}_k\},\{\{\bar{s}_1,\ldots,\bar{s}_k\}\})$ denote these histories. In what follows we show for all $i \le k$,

$$\alpha(s_i,H) = \alpha(\bar{s}_i,\bar{H})$$

Consider a time-shift $\hat{H}^1$ of H at s_1 (first share) with $\hat{S}^1 = (S\setminus\{s_1\})\cup\{\hat{s}_1\})$ such that $\tau(\hat{s}_1) = \min\{\tau(s_1),\tau(\bar{s}_1)\}$. Similarly consider a time-shift $\hat{\bar{H}}^1$ of $\bar{H}$ at $\bar{s}_1$ (first share) with $\hat{\bar{S}}^1 = (\bar{S}\setminus\{\bar{s}_1\})\cup\{\hat{\bar{s}}\}$ such that $\tau(\hat{\bar{s}}_1) = \min\{\tau(s_1),\tau(\bar{s}_1)\}$. Then by delay invariance of α, we have for all $s \in S\setminus\{s_1\}$ and for all $\bar{s} \in \bar{S}\setminus\{\bar{s}_1\}$:

$$\alpha(s,H|_r) = \alpha(\hat{s},\hat{H}^1)$$
$$\alpha(\bar{s},\bar{H}|_{\bar{r}}) = \alpha(\hat{\bar{s}},\hat{\bar{H}}^1)$$

Note that by Remark 1, the total reward is fixed both in P and in $\bar{P}$, therefore we conclude that for all $s \in S$ and for all $\bar{s} \in \bar{S}$:

$$\alpha(s,H|_r) = \alpha(\hat{s},\hat{H}^1)$$
$$\alpha(\bar{s},\bar{H}|_{\bar{r}}) = \alpha(\hat{\bar{s}},\hat{\bar{H}}^1)$$

Continuing iteratively and letting $\hat{H}^i$ be the time-shift of H at the i^{th} share and $\hat{\bar{H}}^i$ as the time-shift of $\bar{H}$ at the i^{th} share, the same argument above yields for all $i \le k$:

$$\alpha(s,H|_r) = \alpha(\hat{s},\hat{H}^1) = \alpha(\hat{s},\hat{H}^2) = \cdots = \alpha(\hat{s},\hat{H}^k)$$
$$\alpha(\bar{s},\bar{H}|_{\bar{r}}) = \alpha(\hat{\bar{s}},\hat{\bar{H}}^1) = \alpha(\hat{\bar{s}},\hat{\bar{H}}^2) = \cdots = \alpha(\hat{\bar{s}},\hat{\bar{H}}^k)$$

Note that by construction, the time signature of all shares at $\hat{H}^k$ are the same with those at $\hat{\bar{H}}^k$, i.e., $\hat{H}^k = \hat{\bar{H}}^k$. Therefore, $\alpha(\hat{s},\hat{H}^k) = \alpha(\hat{\bar{s}},\hat{\bar{H}}^k)$. This – together with the two equations above – implies $\alpha(s,H|_r) = \alpha(\bar{s},\bar{H}|_{\bar{r}})$. ■

Appendix C. Proof of Theorem 1

Theorem 1. *A reward sharing scheme α satisfies round based rewards, budget limit, fixed total reward, delay invariance and absolute redistribution if and only if it is an absolute fair scheme.*

Proof. If part.
Round based rewards. Let $H = (S,\mathcal{P}(S))$ be a history, and let r be any round. Note that, by restricting a round to the r^{th} round the relative rank of a share s in the round, as well as its time signature and value are the same at both H and $H|_r$. Therefore, $\alpha^{**}(s,H) = \alpha^{**}(s,H|_r)$.
Budget limit. Let $H = (S,\mathcal{P}(S))$ be any history. Take any round r. Let $|P_r| = k$. As the relative fair scheme is based on $\rho(s)$, without loss of generality, we assume $P_r = \{1,2,\ldots,k\}$ so that $\rho(s) = s$. Note that $\sum_{s=1}^{k} \alpha^{**}(s,H) = R\sum_{s=1}^{k}\left(\varepsilon(s) - \sum_{i=s+1}^{k} \frac{\varepsilon(i)}{i-1}\right)$.
Therefore,

$$
\begin{aligned}
\frac{1}{R}\sum_{s=1}^{k}\alpha^{**}(s,H) &= \sum_{s=1}^{k}\left(\varepsilon(s)-\sum_{i=s+1}^{k}\frac{\varepsilon(i)}{i-1}\right)\\
&= \sum_{s=1}^{k-1}\left(\varepsilon(s)-\sum_{i=s+1}^{k}\frac{\varepsilon(i)}{i-1}\right)+\varepsilon(k)\\
&= \sum_{s=1}^{k-1}\varepsilon(s)-\sum_{s=1}^{k-1}\Big(\sum_{i=s+1}^{k}\frac{\varepsilon(i)}{i-1}\Big)+\varepsilon(k)\\
&= \sum_{s=1}^{k-1}\varepsilon(s)-\sum_{s=1}^{k-1}\Big(\sum_{i=s+1}^{k-1}\frac{\varepsilon(i)}{i-1}+\frac{\varepsilon(k)}{k-1}\Big)+\varepsilon(k)\\
&= \sum_{s=1}^{k-1}\varepsilon(s)-\sum_{s=1}^{k-1}\sum_{i=s+1}^{k-1}\frac{\varepsilon(i)}{i-1}-\sum_{s=1}^{k-1}\frac{\varepsilon(k)}{k-1}+\varepsilon(k)\\
&= \sum_{s=1}^{k-1}\varepsilon(s)-\sum_{s=1}^{k-1}\sum_{i=s+1}^{k-1}\frac{\varepsilon(i)}{i-1}-(k-1)\frac{\varepsilon(k)}{k-1}+\varepsilon(k)\\
&= \sum_{s=1}^{k-1}\varepsilon(s)-\sum_{s=1}^{k-1}\sum_{i=s+1}^{k-1}\frac{\varepsilon(i)}{i-1}\\
&\cdots\cdots\cdots\\
&= \sum_{s=1}^{2}\varepsilon(s)-\sum_{s=1}^{2}\sum_{i=s+1}^{2}\frac{\varepsilon(i)}{i-1}\\
&= (\varepsilon(1)+\varepsilon(2))-(\varepsilon(2))=\varepsilon(1)
\end{aligned}
$$

By definition of the ε we have $0\le\varepsilon(1)\le 1$, hence $\sum_{s=1}^{k}\alpha^{**}(s,H)\le R$ the absolute redistribution scheme satisfies the budget limit.

Fixed total rewards. It follows from a similar proof of the above.

Absolute redistribution. Let $H=(S,\mathcal{P}(S))$ be any history and $P_r=\{s_1,s_2,\dots,s_k\}$ be any round. Let $H'=(S',\mathcal{P}'(S'))$ be any extension of H at the r^{th} round. Take any $s_a\in P_r$. Note that $|P'(s_a)|=|P(s_a)|+1$.

$$
\begin{aligned}
\alpha^{**}(s_a,H)-\alpha^{**}(s_a,H') &= R\left(\varepsilon(\rho(s_a))-\sum_{i=\rho(s_a)+1}^{|P(s_a)|}\frac{\varepsilon(i)}{i-1}\right)-R\left(\varepsilon(\rho(s_a))-\sum_{i=\rho(s_a)+1}^{|P'(s_a)|}\frac{\varepsilon(i)}{i-1}\right)\\
&= -R\Big(\sum_{i=\rho(s_a)+1}^{|P(s_a)|}\frac{\varepsilon(i)}{i-1}\Big)+R\Big(\sum_{i=\rho(s_a)+1}^{|P(s_a)|+1}\frac{\varepsilon(i)}{i-1}\Big)\\
&= R\frac{\varepsilon(|P(s_a)|+1)}{|P(s_a)|}=R\frac{\varepsilon(k+1)}{k}
\end{aligned}
$$

Only if part. Take any reward sharing scheme α that satisfies the axioms. Take any history $H=(S,\mathcal{P}(S))$ and any $s\in S$. Let $s\in P_r$ for some $1\le r\le l=|\mathcal{P}(S)|$. As α satisfies round based rewards $\alpha(s,H)=\alpha(s,H|_r)$. Similarly for α^{**}, we have $\alpha^{**}(s,H)=\alpha^{**}(s,H|_r)$. Hence, it suffices to prove $\alpha(s,H|_r)=\alpha^{**}(s,H|_r)$ at the restricted histories, i.e., histories with only a single round.

Note that by Lemma 2, if the rounds in any two restricted histories are of the same size, then the shares are awarded based on their ranks. Hence, without loss of generality, we can denote these rounds $P_r=\{s_1,s_2,\dots,s_n\}$ and the restricted histories as $H^n=H|_r$. Therefore, it suffices to prove $\alpha(s_i,H^n)=\alpha^{**}(s_i,H^n)$ for all n and for all $i\le n$.

By absolute redistribution, for any $j>1$, we can let δ_j denote the absolute decrease in the rewards of all shares whilst moving from history H^{j-1} to H^j, i.e., $\alpha(s,H^{j-1})-\alpha(s,H^j)$. In addition let $\delta_1=\alpha(s_1,H^1)$. As α is well-defined, for all $n\ge 1$, $\alpha(s_1,H^n)\ge 0$. Therefore, $\delta_1\ge\sum_{i=2}^{\infty}\delta_i$ (otherwise at some history, s_1 would get a negative reward). Similarly, as α is well-defined, for all $n>1$, and for all $p\le n$, $\alpha(s_p,H^n)\ge 0$. Note that by construction and Remark 1, $\alpha(s_p,H^p)=(p-1)\delta_p$. Therefore, $(p-1)\delta_p\ge\sum_{i=p+1}^{\infty}\delta_i$ (otherwise at some history, s_p would get a negative reward). Now, consider a function ε such that $\varepsilon(1)=1$ and for all $j>1$,

$$\varepsilon(j) = \frac{(j-1)\delta_j}{R} \tag{11}$$

Note that we have the following two properties for ε:

1. for all $j \geq 1$, $\varepsilon(j) \to [0, 1]$. By definition δ_j denotes the absolute decrease in the rewards of all shares whilst moving from history H^{j-1} to H^j. Therefore, the last share at H^j gets $(j-1)\delta_j$ (as there are $j-1$ that each payouts δ_j). As the scheme satisfies total fixed rewards and round based rewards then by Remark 1 the total reward at round H^j is R. Therefore, it must be the case that $(j-1)\delta_j \leq R$ so $\delta_j \leq \frac{R}{j-1}$. Therefore, $\varepsilon(j) = \frac{(j-1)\delta_j}{R} \leq \frac{j-1}{R} \times \frac{R}{j-1}$. As $\delta_j \geq 0$ then $\varepsilon(j) \geq 0$. All in all, $\varepsilon(j) \to [0, 1]$ for all $j \geq 1$.
2. for all $j \geq 1$, $\varepsilon(j) \geq \sum_{i=j+1}^{\infty} \frac{\varepsilon(i)}{i-1}$.

   For $j = 1$ we have $\varepsilon(j) = 1$ we have $1 \geq \sum_{i=2}^{\infty} \frac{\varepsilon(i)}{i-1}$. Replacing from Equation (11) we have $1 \geq \sum_{i=2}^{\infty} \frac{1}{i-1} \frac{(i-1)\delta_i}{R}$ which implies $R \geq \sum_{i=2}^{\infty} \delta_i \ldots$.

   For $\varepsilon(j)$ we have $\varepsilon(j) = \frac{(j-1)\delta_j}{R} \geq \sum_{i=j+1}^{\infty} \frac{\varepsilon(i)}{i-1}$. Replacing from Equation (11) we have $(j-1)\delta_j \geq \sum_{i=j+1}^{\infty} \delta_i$.

In what follows we shall show that $\alpha(s, H^n) = \alpha^{**}(s, H^n)$ for the aforementioned ε function. We do it first for a single share rounds and then for multi-share rounds.

Single-share round: Let H^1 be any history with a single round with a single share s. By Remark 1, $\alpha(s, H^1) = R$. Setting $\varepsilon(1) = 1$ we have:

$$\alpha(s, H^1) = \alpha^{**}(s, H^1) = R\varepsilon(1) \tag{12}$$

Multi-share round: Let H^n be any history with a single round with multiple shares $s_1, \ldots, s_n$. In what follows, we show that $\alpha(s, H^n) = \alpha^{**}(s, H^n)$ for any $s \in P_r$, by induction on n, i.e., the size of the round P_r.

Induction Basis: Let $n = 2$. Let $H^2 = (\{s_1, s_2\}, \{\{s_1, s_2\}\})$. We will show that $\alpha(s_1, H^2) = \alpha^{**}(s_1, H^2)$ and $\alpha(s_2, H^2) = \alpha^{**}(s_2, H^2)$. By Remark 1 for two histories H^1 and H^2, and by Equation (12), we have $\alpha(s_1, H^2) + \alpha(s_2, H^2) = R$. So, $\alpha(s_1, H^2) = R - \delta_2$ and $\alpha(s_2, H^2) = \delta_2$. Setting $\varepsilon(2) = \frac{\delta_2}{R}$, yields $\alpha(s_1, H^2) = R\varepsilon(1) - R\varepsilon(2) = \alpha^{**}(s_1, H^2)$ and $\alpha(s_2, H^2) = R\varepsilon(2) = \alpha^{**}(s_2, H^2)$.

Induction Hypothesis: Let $n = k$ with $k > 1$. Suppose we have $\alpha(s_i, H^k) = \alpha^{**}(s_i, H^k)$ for all $i \leq k$.

To prove for $n = k+1$, consider any $H^{k+1} = (\{s_1, \ldots, s_k, s_{k+1}\}, \{\{s_1, \ldots, s_k, s_{k+1}\}\})$. We will show that $\alpha(s_i, H^{k+1}) = \alpha^{**}(s_i, H^{k+1})$ for all $i \leq k+1$. Let $H^k = (\{s_1, \ldots, s_k\}, \{\{s_1, \ldots, s_k\}\})$.

By construction, $\alpha(s_i, H^{k+1}) = \alpha(s_i, H^k) - \delta_{k+1}$ for all $i \leq k$. By the induction hypothesis $\alpha(s_i, H^k) = \alpha^{**}(s_i, H^k)$ for all $i \leq k$ which implies $\alpha(s_i, H^{k+1}) = \alpha^{**}(s_i, H^k) - \delta_{k+1}$. Therefore,

$$\alpha(s_i, H^{k+1}) = R\left(\varepsilon(i) - \sum_{j=i+1}^{k} \frac{\varepsilon(j)}{j-1}\right) - \delta_{k+1} \text{ for all } i \leq k \tag{13}$$

As $\delta_{k+1} = \frac{\varepsilon(k+1)R}{k}$, the above equation is simplified as:

$$\alpha(s_i, H^{k+1}) = R\left(\varepsilon(i) - \sum_{j=i+1}^{k+1} \frac{\varepsilon(j)}{j-1}\right) \text{ for all } i \leq k \tag{14}$$

Note that $\alpha(s_i, H^{k+1}) = \alpha(s_i, H^k) - \delta_{k+1}$, and by Lemma 1 (on H^k and on H^1), we have

$$\sum_{i=1}^{k} \alpha(s_i, H^{k+1}) = \sum_{i=1}^{k} \alpha(s_i, H^k) - k\delta_{k+1} = R - k\delta_{k+1} \tag{15}$$

Note also that $\sum_{i=1}^{k+1} \alpha(s_i, H^{k+1}) = \sum_{i=1}^{k} \alpha(s_i, H^{k+1}) + \alpha(s_{k+1}, H^{k+1})$. Plugging Equation (15) into this yields $\sum_{i=1}^{k+1} \alpha(s_i, H^{k+1}) = R - k\delta_{k+1} + \alpha(s_{k+1}, H^{k+1})$. By Remark 1, $\sum_{i=1}^{k+1} \alpha(s_i, H^{k+1}) = R$, therefore we have:

$$\alpha(s_{k+1}, H^{k+1}) = k\delta_{k+1} = R\varepsilon(k+1). \tag{16}$$

Equations (14) and (16) together prove $\alpha(s_i, H^{k+1}) = \alpha^{*}(s_i, H^{k+1})$ for $i \in \{1, \ldots, k\}$ and for $i = k+1$, respectively. ■

Appendix D. Proof of Proposition 1

Proposition 1. *The proportional scheme is an absolute fair scheme.*

Proof. Let $H = (S, \mathcal{P}(S))$ be any history. It is easy to see the proportional scheme satisfies the round-baseness condition, hence we only consider a history with a single round, i.e., $H = (\{s_1, \ldots, s_k\}, \{\{s_1, \ldots, s_k\}\})$. Let $\varepsilon(\rho(s_i)) = \frac{1}{\rho(s_i)}$ for all $i \in \{1, \ldots, k\}$. Note that $\rho(s_i) = i$. Therefore, for any share s_j we have:

$$\alpha^{**}(s_j, H) = R\left(\varepsilon(\rho(s_j)) - \sum_{i=\rho(s_j)+1}^{|P(s_j)|} \frac{\varepsilon(i)}{i-1}\right) = R\left(\frac{1}{j} - \sum_{i=j+1}^{k} \frac{1}{i \times (i-1)}\right)$$

$$= R\left(\frac{1}{j} - \sum_{i=j+1}^{k} \left(\frac{1}{i-1} - \frac{1}{i}\right)\right) = R\left(\frac{1}{j} - \left(\frac{1}{j} - \frac{1}{k}\right)\right) = \frac{R}{k}$$

Note that for the last share $\sum_{i=k+1}^{k} \varepsilon(i)$ is an empty sum and by convention it equals 0. ■

Appendix E. Proof of Theorem 2

Theorem 2. *A reward scheme α satisfies round based rewards, budget limit, fixed total reward, delay invariance and relative redistribution if and only if it is a relative fair scheme.*

Proof. If part.
Round based rewards. To show that the relative fair scheme satisfies $\alpha^*(s, H) = \alpha^*(s, H|_r)$, let $H = (S, \mathcal{P}(S))$ be a history, and let r be any round. Note that, by restricting a history to the r^{th} round, the round and the shares remain intact, hence the relative rank of a share s is the same at both H and $H|_r$. Therefore, the relative fair scheme satisfies round based rewards.
Budget limit. Let $H = (S, \mathcal{P}(S))$ be any history. Take any round r. Let $|P_r| = k$. As the relative fair scheme is based on $\rho(s)$, without loss of generality, we assume $P_r = \{1, 2, \ldots, k\}$ so that $\rho(s) = s$. Therefore to show $\sum_{s \in P_r} \alpha^{**}(s, H) \leq R$, we have

$$\begin{aligned}
\sum_{s \in P_r} \alpha(s, H) &= \sum_{s=1}^{k} \left(R \times \varepsilon(\rho(s)) \times \prod_{i=\rho(s)+1}^{k} (1 - \varepsilon(i))\right) \\
&= R\left(\varepsilon(1) \times \prod_{i=2}^{k} (1 - \varepsilon(i))\right) + R \sum_{s=2}^{k} \left(\varepsilon(\rho(s)) \times \prod_{i=\rho(s)+1}^{k} (1 - \varepsilon(i))\right) \\
&= R\Big((1 - \varepsilon(2))(\prod_{i=3}^{k} 1 - \varepsilon(i))\Big) + R \sum_{s=2}^{k} \left(\varepsilon(s) \times \prod_{i=s+1}^{k} (1 - \varepsilon(i))\right) \\
&= R\Big(\prod_{i=3}^{k} \Big(1 - \varepsilon(i)\Big) - \varepsilon(2) \prod_{i=3}^{k} \Big(1 - \varepsilon(i)\Big)\Big) + R \sum_{s=2}^{k} \left(\varepsilon(s) \times \prod_{i=s+1}^{k} (1 - \varepsilon(i))\right) \\
&= R \prod_{i=3}^{k} \Big(1 - \varepsilon(i)\Big) - R\varepsilon(2) \prod_{i=3}^{k} \Big(1 - \varepsilon(i)\Big) + R\varepsilon(2) \prod_{i=3}^{k} (1 - \varepsilon(i)) + R \sum_{s=3}^{k} \left(\varepsilon(s) \times \prod_{i=s+1}^{k} (1 - \varepsilon(i))\right) \\
&= R \prod_{i=3}^{k} \Big(1 - \varepsilon(i)\Big) + R \sum_{s=3}^{k} \left(\varepsilon(s) \times \prod_{i=s+1}^{k} (1 - \varepsilon(i))\right) \\
&= \cdots \\
&= R \times \Big((1 - \varepsilon(k))\Big) + R \sum_{s=k}^{k} \left(\varepsilon(s) \times \prod_{i=s+1}^{k} (1 - \varepsilon(i))\right) \\
&= R \times \Big((1 - \varepsilon(k))\Big) + R\varepsilon(k) = R
\end{aligned}$$

Fixed total rewards. It follows from a similar proof of the above.

Relative redistribution. To show relative fairness let $H = (S, \mathcal{P}(S))$ be any history and $P_r = \{s_1, s_2, \dots, s_k\}$ be any round. Let $H' = (S', \mathcal{P}'(S'))$ be any extension of H at the r^{th} round. Then for any $s_a, s_b \in P_r$ $\alpha^*(s_a, H) \neq 0$ and $\alpha^*(s_b, H) \neq 0$ we have:

$$\frac{\alpha^*(s_a, H')}{\alpha^*(s_a, H)} = \frac{R \times \varepsilon(\rho(s_a)) \times \prod_{i=\rho(s_a)+1}^{|P(s_a)|+1} \big(1 - \varepsilon(i)\big)}{R \times \varepsilon(\rho(s_a)) \times \prod_{i=\rho(s_a)+1}^{|P(s_a)|} \big(1 - \varepsilon(i)\big)} = 1 - \varepsilon\big(|P(s_a)| + 1\big) = 1 - \varepsilon\big(k+1\big)$$

$$\frac{\alpha^*(s_b, H')}{\alpha^*(s_b, H)} = \frac{R \times \varepsilon(\rho(s_b)) \times \prod_{i=\rho(s_b)+1}^{|P(s_b)|+1} \big(1 - \varepsilon(i)\big)}{R \times \varepsilon(\rho(s_b)) \times \prod_{i=\rho(s_b)+1}^{|P(s_b)|} \big(1 - \varepsilon(i)\big)} = 1 - \varepsilon\big(|P(s_b)| + 1\big) = 1 - \varepsilon\big(k+1\big)$$

The above two equations show that $\frac{\alpha^*(s_a,H')}{\alpha^*(s_a,H)} = \frac{\alpha^*(s_b,H')}{\alpha^*(s_b,H)}$.

Only if part. Take any reward sharing scheme α that satisfies the axioms. Take any history $H = (S, \mathcal{P}(S))$ and any $s \in S$. Let $s \in P_r$ for some $1 \le r \le l = |\mathcal{P}(S)|$. As α satisfies round based rewards $\alpha(s, H) = \alpha(s, H|_r)$. Similarly for α^{**}, we have $\alpha^{**}(s, H) = \alpha^{**}(s, H|_r)$. Hence, it suffices to prove $\alpha(s, H|_r) = \alpha^{**}(s, H|_r)$ at the restricted histories, i.e., histories with only a single round.

Note that by Lemma 2, if the rounds in any two restricted histories are of the same size, then the shares are awarded based on their ranks. Hence, without loss of generality, we can denote these rounds $P_r = \{s_1, s_2, \dots, s_n\}$ and the restricted histories as $H^n = H|_r$. Therefore, it suffices to prove $\alpha(s_i, H^n) = \alpha^{**}(s_i, H^n)$ for all n and for all $i \le n$.

Single-share round: Let H^1 be any history with a single round with a single share s. As α satisfies fixed total rewards and round based rewards then by Remark 1, $\alpha(s, H^1) = R$. Setting $\varepsilon(1) = 1$ we have:

$$\alpha(s, H^1) = \alpha^*(s, H^1) = R. \tag{17}$$

Multi-share round: Let H^n be any history with a single round with multiple shares $s_1, \dots, s_n$. In what follows, we show that $\alpha(s, H^n) = \alpha^*(s, H^n)$ any $s \in P_r$, by induction on n, i.e., the size of the round P_r.

Induction Basis: Let $n = 2$. Let $H^2 = (\{s_1, s_2\}, \big\{\{s_1, s_2\}\big\})$. We will show that $\alpha(s_1, H^2) = \alpha^*(s_1, H^2)$ and $\alpha(s_2, H^2) = \alpha^*(s_2, H^2)$. By Lemma 1 for two histories H^1 and H^2, and by Equation (17), we have $\alpha(s_1, H^2) + \alpha(s_2, H^2) = R$. So, $\alpha(s_1, H^2) = R(1 - \delta_2)$ and $\alpha(s_2, H^2) = R\delta_2$ for some $\delta_2 \in [0, 1]$. Setting $\varepsilon(2) = \delta_2$, yields $\alpha(s_1, H^2) = R(1 - \varepsilon(2))$ and $\alpha(s_2, H^2) = R\varepsilon(2)$. Therefore, $\alpha(s_1, H^2) = R\varepsilon(1)(1 - \varepsilon(2)) = \alpha^*(s_1, H^2)$ and $\alpha(s_2, H^2) = R\varepsilon(2) = \alpha^*(s_2, H^2)$. Note that, $\varepsilon(1) \in [0, 1]$, and as $\delta_2 \in [0, 1]$ then $\varepsilon(2) \in [0, 1]$.

Induction Hypothesis: Let $n = k$ with $k > 1$. Suppose we have $\alpha(s_i, H^k) = \alpha^*(s_i, H^k)$ for all $i \le k$.

To prove for $n = k + 1$, consider any $H^{k+1} = (\big\{s_1, \dots, s_k, s_{k+1}\big\}, \big\{\{s_1, \dots, s_k, s_{k+1}\}\big\})$. We will show that $\alpha(s_i, H^{k+1}) = \alpha^*(s_i, H^{k+1})$ for all $i \le k + 1$. Let $H^k = (\big\{s_1, \dots, s_k\big\}, \big\{\{s_1, \dots, s_k\}\big\})$.

By relative redistribution for all $i, j \in \{1, \dots, k\}$ we have

$$\frac{\alpha(s_i, H^{k+1})}{\alpha(s_i, H^k)} = \frac{\alpha(s_j, H^{k+1})}{\alpha(s_j, H^k)} \tag{18}$$

Let us denote this ratio above by $1 - \delta_{k+1}$. Therefore, $\alpha(s_i, H^{k+1}) = (1 - \delta_{k+1})\alpha(s_i, H^k)$ for all $i \le k$. By the induction hypothesis $\alpha(s_i, H^k) = \alpha^*(s_i, H^k)$ for all $i \le k$ which implies $\alpha(s_i, H^{k+1}) = (1 - \delta_{k+1})\alpha^*(s_i, H^k)$. Therefore,

$$\alpha(s_i, H^{k+1}) = (1 - \delta_{k+1})R\varepsilon(i) \prod_{j=i+1}^{k} (1 - \varepsilon(j)) \text{ for all } i \le k \tag{19}$$

By Equation (18), $\alpha(s_i, H^{k+1}) = (1 - \delta_{k+1})\alpha(s_i, H^k)$, and by Lemma 1 (on H^k and on H^1), we have

$$\sum_{i=1}^{k} \alpha(s_i, H^{k+1}) = (1 - \delta_{k+1}) \sum_{i=1}^{k} \alpha(s_i, H^k) = (1 - \delta_{k+1})R \tag{20}$$

Note that $\sum_{i=1}^{k+1} \alpha(s_i, H^{k+1}) = \sum_{i=1}^{k} \alpha(s_i, H^{k+1}) + \alpha(s_{k+1}, H^{k+1})$. Plugging Equation (20) into this, yields $\sum_{i=1}^{k+1} \alpha(s_i, H^{k+1}) = (1 - \delta_{k+1})R + \alpha(s_{k+1}, H^{k+1})$. By Remark 1, $\sum_{i=1}^{k+1} \alpha(s_i, H^{k+1}) = R$, therefore we have:

$$\alpha(s_{k+1}, H^{k+1}) = \delta_{k+1}R. \tag{21}$$

Setting $\varepsilon(k+1)=\delta_{k+1}$ in Equations (19) and (21), proves $\alpha(s_i,H^{k+1})=\alpha^*(s_i,H^{k+1})$ for all $i\in\{1,\dots,k\}$ and for $i=k+1$, respectively.

Note that, $\varepsilon(k+1)\in[0,1]$ for all $k>0$. Otherwise, if $\varepsilon(k+1)<0$ then $\delta_{k+1}<0$. However, by Equation (21), this implies $\alpha(s_{k+1},H^{k+1})<0$ which contradicts Definition 1. If $\varepsilon(k+1)>1$ then $\delta_{k+1}>1$ which implies $1-\delta_{k+1}<0$. However, by Equation (18), $\alpha(s_i,H^{k+1})=(1-\delta_{k+1})\alpha(s_i,H^k)$ for all $i\le k$, this implies $\alpha(s_i,H^{k+1})<0$ for all $i\in\{1,\dots,k\}$ which contradicts Definition 1. ■

Appendix F. Proof of Proposition 2

Proposition 2. *The proportional scheme is a relative fair scheme.*

Proof. Let $H=(S,\mathcal{P}(S))$ be any history. It is easy to see the proportional scheme satisfies the round-baseness condition, hence we only consider a history with a single round, i.e., $H=(\{s_1,\dots,s_k\},\{\{s_1,\dots,s_k\}\})$. Let $\varepsilon(\rho(s_i))=\frac{1}{\rho(s_i)}$ for all $i\in\{1,\dots,k\}$. Note that $\rho(s_i)=i$. Therefore, for any share s_j we have:

$$
\begin{aligned}
\alpha^*(s_j,H) &= R\times\varepsilon(\rho(s_j))\times\prod_{i=\rho(s_j)+1}^{k}(1-\varepsilon(i))\\
&= R\times(\frac{1}{j})\times(1-\frac{1}{j+1})\times(1-\frac{1}{j+2})\times\cdots\times(1-\frac{1}{k})\\
&= R\times(\frac{1}{j})\times(\frac{j}{j+1})\times(\frac{j+1}{j+2})\times\cdots\times(\frac{k-2}{k-1})\times(\frac{k-1}{k})=\frac{R}{k}
\end{aligned}
$$

Note that for the last share $\prod_{i=k+1}^{k}(1-\varepsilon(i))$ is *empty product* and by convention equals 1. ■

Appendix G. Proof of Theorem 3

Theorem 3. *A reward sharing scheme satisfies round based rewards, budget limit, fixed total reward, delay invariance, absolute redistribution, and relative redistribution if and only if it is a k-pseudo proportional scheme.*

Proof. Take any reward sharing scheme α that satisfies the axioms. Take any history $H=(S,\mathcal{P}(S))$ and any $s\in S$. Let $s\in P_r$ for some $1\le r\le l=|\mathcal{P}(S)|$. As α satisfies round based rewards $\alpha(s,H)=\alpha(s,H|_r)$. Similarly for $\alpha^{k,\delta}$, we have $\alpha^{k,\delta}(s,H)=\alpha^{k,\delta}(s,H|_r)$. Hence, it suffices to prove $\alpha(s,H|_r)=\alpha^{k,\delta}(s,H|_r)$ at the restricted histories, i.e., histories with only a single round.

Note that by Lemma 2, if the rounds in any two restricted histories are of the same size, then the shares are awarded based on their ranks. Hence, without loss of generality, we can denote these rounds $P_r=\{s_1,s_2,\dots,s_n\}$ and the restricted histories as $H^n=H|_r$. Therefore, it suffices to prove that there exist some k and some δ such that $\alpha(s_i,H^n)=\alpha^{k,\delta}(s_i,H^n)$ for all n and for all $i\le n$. First of all, note that for all single share histories, H^1, Remark 1 implies that α coincides with $\alpha^{k,\delta}$ regardless of the choice of k and δ. So let $n\ge 2$.

In what follows, we find – by iterating on n – that there exist k and δ such that $\alpha(s_i,H^n)=\alpha^{k,\delta}(s_i,H^n)$ for all $n\ge 2$ and for all $i\le n$. At each step, we ask if α distributes the awards proportionally, or not. The proof structure is as follows:

1. At step h, if α distributes the awards proportionally, then we move to step $h+1$.
2. At step h, if α distributes the awards disproportionately (while it was proportional at step $h-1$), then we set $k=h$, and $\delta=\alpha(s_h,H^h)$, i.e., the award of the last share in the round. Thereafter we show that the scheme α coincides with $\alpha^{k,\delta}$ for all possible rounds and all shares in these rounds.

Note a round of size $n=2$ is a special case, since a round of size 1 cannot be decided to be proportional or disproportionate. Therefore we first treat such histories.

STEP 2: Let $n=2$, and $H^2=(\{s_1,s_2\},\{\{s_1,s_2\}\})$. By Remark 1, we have $\alpha(s_1,H^2)+\alpha(s_2,H^2)=R$. So $(\alpha(s_1,H^2),\alpha(s_2,H^2))=(R-\gamma_2,\gamma_2)$ for some $\gamma_2\in[0,R]$. Note also that $\alpha^{k,\delta}(s_1,H^2)+\alpha^{k,\delta}(s_2,H^2)=R$ for any k and for any δ. Now, there are two cases, either the awards are proportional, or not.

Case 1: If $R-\gamma_2=\gamma_2$, then $k>2$, therefore continue to the next step, (Step 3, i.e., $n=3$).

Case 2: If $R-\gamma_2\ne\gamma_2$, then we set $k=2$ and $\delta=\gamma_2$. Note that $\alpha^{2,\gamma_2}(s_1,H^2)=\frac{R-\gamma_2}{2-1}$ and $\alpha^{2,\gamma_2}(s_2,H^2)=\gamma_2$. Hence, for $n=2$, we have $\alpha(s_i,H^n)=\alpha^{2,\gamma_2}(s_i,H^n)$ for all $i\le n$. Next we also show, for any $n>2$, we have $\alpha(s_i,H^n)=\alpha^{2,\gamma_2}(s_i,H^n)$ for all $i\le n$.

Case 2a. If $\gamma_2=R$, then $\Big(\alpha(s_1,H^2),\alpha(s_2,H^2)\Big)=(0,R)$. Consider any extension H^3 of H^2. As α does not assign negative awards, Remark 1 and absolute redistribution implies

$$\Big(\alpha(s_1, H^3), \alpha(s_2, H^3), \alpha(s_3, H^3)\Big) = (0, R, 0).$$

A similar argument can be extended to H^4 and further, e.g., $(0, R, 0, 0, \ldots, 0)$ which shows $\alpha(s_i, H^n) = \alpha^{2,\gamma_2}(s_i, H^n)$.

Case 2b. If $\gamma_2 = 0$, then $\Big(\alpha(s_1, H^2), \alpha(s_2, H^2)\Big) = (R, 0)$. Consider any extension H^3 of H^2. As α does not assign negative awards, Remark 1 and absolute redistribution implies

$$\Big(\alpha(s_1, H^3), \alpha(s_2, H^3), \alpha(s_3, H^3)\Big) = (R, 0, 0).$$

A similar argument can be extended to H^4 and further, e.g., $(R, 0, 0, 0, \ldots, 0)$ which shows $\alpha(s_i, H^n) = \alpha^{2,\gamma_2}(s_i, H^n)$.

Case 2c. If $\gamma_2 \in (0, R)$, then $\Big(\alpha(s_1, H^2), \alpha(s_2, H^2)\Big) = (R - \gamma_2, \gamma_2)$. Consider any extension H^3 of H^2. By absolute redistribution we have $\alpha(s_1, H^2) - \alpha(s_1, H^3) = \alpha(s_2, H^2) - \alpha(s_2, H^3)$. By relative redistribution we have

$$\frac{\alpha(s_1, H^3)}{\alpha(s_1, H^2)} = \frac{\alpha(s_2, H^3)}{\alpha(s_2, H^2)} = \theta$$

Combining these equations, we have $\alpha(s_1, H^2) - \alpha(s_1, H^2)\theta = \alpha(s_2, H^2) - \alpha(s_2, H^2)\theta$ which implies $\alpha(s_1, H^2)(1 - \theta) = \alpha(s_2, H^2)(1 - \theta)$. As $R - \gamma_2 \neq \gamma_2$, the previous equation only holds if $\theta = 1$. This results in $\alpha(s_1, H^3) = \alpha(s_1, H^2)$ and $\alpha(s_2, H^3) = \alpha(s_2, H^2)$. Finally, as α does not assign negative awards, Remark 1 implies

$$\Big(\alpha(s_1, H^3), \alpha(s_2, H^3), \alpha(s_3, H^3)\Big) = (R - \gamma_2, \gamma_2, 0).$$

A similar argument can be extended to H^4 and further, e.g., $(R - \gamma_2, \gamma_2, 0, 0, \ldots, 0)$ which shows $\alpha(s_i, H^n) = \alpha^{2,\gamma_2}(s_i, H^n)$.

STEP h: Let $n = h$, and $H^h = (\{s_1, \ldots, s_h\}, \big\{\{s_1, \ldots, s_h\}\big\})$. Reaching to step h implies the awards to shares at $H^{h-1} = (\{s_1, \ldots, s_{h-1}\}, \big\{\{s_1, \ldots, s_{h-1}\}\big\})$ were distributed proportionally and hence all are equal. By relative redistribution for all $i, j \leq h - 1$ we have, $\frac{\alpha(s_i, H^h)}{\alpha(s_i, H^{h-1})} = \frac{\alpha(s_j, H^h)}{\alpha(s_j, H^{h-1})}$. This implies $\alpha(s_i, H^h) = \alpha(s_j, H^h)$ for all $i, j \leq h - 1$. By Remark 1, $\sum_{i=1}^{h} \alpha(s_i, H^h) = R$, which implies $(h - 1)\alpha(s_i, H^h) + \alpha(s_h, H^h) = R$ for any $i \leq h - 1$. Therefore, $\alpha(s_i, H^h) = \frac{R - \alpha(s_h, H^h)}{h-1}$. Let us denote $\alpha(s_h, H^h) = \gamma_h$ for some $\gamma_h \in [0, R]$, so $\Big(\alpha(s_1, H^h), \alpha(s_2, H^h), \ldots, \alpha(s_{h-1}, H^h), \alpha(s_h, H^h)\Big) = (\frac{R-\gamma_h}{h-1}, \frac{R-\gamma_h}{h-1}, \ldots, \frac{R-\gamma_h}{h-1}, \gamma_h)$. Now, there are two cases, either the awards are proportional, or not.

Case 1: If $\frac{R-\gamma_h}{h-1} = \gamma_h$, then $k > h$, therefore continue to the next step, (Step $h + 1$, i.e., $n = h + 1$).

Case 2: If $\frac{R-\gamma_h}{h-1} \neq \gamma_h$, then we set $k = h$ and $\delta = \gamma_h$. Note that $\alpha^{h,\gamma_h}(s_i, H^h) = \frac{R-\gamma_h}{h-1}$ for all $i < h$ and $\alpha^{h,\gamma_h}(s_h, H^h) = \gamma_h$. Hence, for $n = h$, we have $\alpha(s_i, H^n) = \alpha^{h,\gamma_h}(s_i, H^n)$ for all $i \leq n$. Next we also show, for any $n > h$, we have $\alpha(s_i, H^n) = \alpha^{h,\gamma_h}(s_i, H^n)$ for all $i \leq n$.

Case 2a. If $\gamma_h = R$, then $\Big(\alpha(s_1, H^h), \alpha(s_2, H^h), \ldots, \alpha(s_{h-1}, H^h), \alpha(s_h, H^h)\Big) = (0, \ldots, 0, R)$. Consider any extension H^{h+1} of H^h. As α does not assign negative awards, Remark 1 and absolute redistribution implies

$$\Big(\alpha(s_1, H^{h+1}), \ldots, \alpha(s_h, H^{h+1}), \alpha(s_{h+1}, H^{h+1})\Big) = (0, \ldots, 0, R, 0)$$

A similar argument can be extended to H^{h+2} and further, e.g., $(0, \ldots, 0, R, 0, \ldots, 0)$ which shows $\alpha(s_i, H^n) = \alpha^{h,\gamma_h}(s_i, H^n)$.

Case 2b. If $\gamma_h = 0$, then $\Big(\alpha(s_1, H^h), \alpha(s_2, H^h), \ldots, \alpha(s_{h-1}, H^h), \alpha(s_h, H^h)\Big) = (\frac{R}{h-1}, \ldots, \frac{R}{h-1}, 0)$. Consider any extension H^{h+1} of H^h. As α does not assign negative awards, Remark 1 and absolute redistribution implies

$$\Big(\alpha(s_1, H^{h+1}), \ldots, \alpha(s_h, H^{h+1}), \alpha(s_{h+1}, H^{h+1})\Big) = (\frac{R}{h-1}, \ldots, \frac{R}{h-1}, 0, 0)$$

A similar argument can be extended to H^{h+2} and further, e.g., $(\frac{R}{h-1}, \ldots, \frac{R}{h-1}, 0, \ldots, 0)$ which shows $\alpha(s_i, H^n) = \alpha^{h,\gamma_h}(s_i, H^n)$.

Case 2c. If $\gamma_h \in (0, R)$, then $\Big(\alpha(s_1, H^h), \alpha(s_2, H^h), \ldots, \alpha(s_{h-1}, H^h), \alpha(s_h, H^h)\Big) = (\frac{R-\gamma_h}{h-1}, \frac{R-\gamma_h}{h-1}, \ldots, \frac{R-\gamma_h}{h-1}, \gamma_h)$. Consider any extension H^{h+1} of H^h. By absolute redistribution for all $i, j \leq h$ we have $\alpha(s_i, H^h) - \alpha(s_i, H^{h+1}) = \alpha(s_j, H^h) - \alpha(s_j, H^{h+1})$. By relative redistribution for all $i, j \leq h$ we have

$$\frac{\alpha(s_i, H^{h+1})}{\alpha(s_i, H^h)} = \frac{\alpha(s_j, H^{h+1})}{\alpha(s_j, H^h)} = \theta$$

Combining these equations, we have $\alpha(s_i, H^h) - \alpha(s_i, H^h)\theta = \alpha(s_j, H^h) - \alpha(s_j, H^h)\theta$ which implies $\alpha(s_i, H^h)(1 - \theta) = \alpha(s_j, H^h)(1 - \theta)$. As $\alpha(s_i, H^h) \neq \alpha(s_j, H^h)$ for all $i, j \leq h$, the previous equation only holds if $\theta = 1$. This results in $\alpha(s_i, H^{h+1}) = \alpha(s_i, H^h)$ for all $i \leq h$. Finally, as α does not assign negative awards, Remark 1 implies

$$\Big(\alpha(s_1, H^{h+1}), \ldots, \alpha(s_{h-1}, H^{h+1}), \alpha(s_h, H^{h+1}), \alpha(s_{h+1}, H^{h+1})\Big) = (\frac{R-\gamma_h}{h-1}, \ldots, \frac{R-\gamma_h}{h-1}, \gamma_h, 0).$$

A similar argument can be extended to H^4 and further, e.g., $(\frac{R-\gamma_h}{h-1}, \ldots, \frac{R-\gamma_h}{h-1}, \gamma_h, 0, \ldots, 0)$ which shows $\alpha(s_i, H^n) = \alpha^{2,\gamma_2}(s_i, H^n)$.

Note that in case α never distributes the awards "disproportionately", then this implies that h goes to infinity and therefore, we set $k = \infty$ and it is clear to see that $\alpha = \alpha^{\infty,\delta}$ for any δ, i.e., α is the proportional scheme, which is an element of k-pseudo proportional class. All in all, this completes the proof. ■

Appendix H. Proof of Corollary 1

Corollary 1. *The proportional scheme is the only scheme that satisfies round based rewards, budget limit, fixed total reward, absolute redistribution, relative redistribution, and strict positivity.*

Proof. Take any reward sharing scheme α that satisfies the axioms. Take any history $H = (S, \mathcal{P}(S))$ and any $s \in S$. Let $s \in P_r$ for some $1 \le r \le l = |\mathcal{P}(S)|$. As α satisfies round based rewards $\alpha(s, H) = \alpha(s, H|_r)$. Similarly for α^{prop}, we have $\alpha^{prop}(s, H) = \alpha^{prop}(s, H|_r)$. Hence, it suffices to prove $\alpha(s, H|_r) = \alpha^{prop}(s, H|_r)$ at the restricted histories, i.e., histories with only a single round. Let $P_r = \{s_1, s_2, \ldots, s_n\}$ and $H^n = H|_r$ i.e., a history that consists of n shares. Therefore, it suffices to prove $\alpha(s_i, H^n) = \alpha^{prop}(s_i, H^n)$ for all n and for all $i \le n$.

Single-share round: Let H^1 be any history with a single round with a single share s. As α satisfies fixed total rewards and round based rewards then by Remark 1, $\alpha(s, H^1) = R$.

Multi-share round: Let H^n be any history with a single round with multiple shares $s_1, \ldots, s_n$. In what follows, we show that $\alpha(s, H^n) = \alpha^{prop}(s, H^n)$ any $s \in P_r$, by induction on n, i.e., the size of the round P_r.

Induction Basis. Let $n = 2$. Let $H^2 = (\{s_1, s_2\}, \big\{\{s_1, s_2\}\big\})$. By Lemma 1, we have $\alpha(s_1, H^2) + \alpha(s_2, H^2) = \alpha(s, H^1) = R$. So, $\alpha(s_1, H^2) = \lambda R$ and $\alpha(s_2, H^2) = (1-\lambda)R$ for some $\lambda \in (0, 1)$. Note that $\lambda = 0$ results in $\alpha(s_1, H^2) = 0$ which violates strict positivity and $\lambda = 1$ results in $\alpha(s_2, H^2) = 0$ which violates strict positivity. In case $\lambda = \frac{1}{2}$, then the shares receive proportionally. So on the contrary assume that $\lambda \neq \frac{1}{2}$. Consider an extension of H^2 denoted by H^3 by a share s'. Then by relative redistribution we have $\frac{\alpha(s_1,H^3)}{\alpha(s_1,H^2)} = \frac{\alpha(s_2,H^3)}{\alpha(s_2,H^2)} = \frac{\alpha(s_1,H^3)}{\lambda R} = \frac{\alpha(s_2,H^3)}{(1-\lambda)R}$. Therefore,

$$\alpha(s_1, H^3) = \frac{\lambda}{1-\lambda}\alpha(s_2, H^3) \tag{22}$$

By absolute redistribution we have, $\alpha(s_1, H^2) - \alpha(s_1, H^3) = \alpha(s_2, H^2) - \alpha(s_2, H^3)$. So, $\lambda R - \alpha(s_1, H^3) = (1-\lambda)R - \alpha(s_2, H^3)$, which implies $2\lambda R - \alpha(s_1, H^3) = R - \alpha(s_2, H^3)$. Now plugging in Equation (22) we have $2\lambda R - \frac{\lambda}{1-\lambda}\alpha(s_2, H^3) = R - \alpha(s_2, H^3)$. Since $\lambda \neq \frac{1}{2}$, we have $\frac{2\lambda-1}{1-\lambda}\alpha(s_2, H^3) = R(2\lambda - 1)$. This implies that $\alpha(s_2, H^3) = (1-\lambda)R$. This together with Equation (22) yields $\alpha(s_1, H^3) = \lambda R$. Using Lemma 1, $R = \alpha(s_1, H^2) + \alpha(s_2, H^2) = \alpha(s_1, H^3) + \alpha(s_2, H^3) + \alpha(s_3, H^3)$, which implies $\alpha(s_3, H^3) = 0$. Note that this contradicts strict positivity, hence λ must equal $\frac{1}{2}$.

Induction Hypothesis: Let $n = k$ with $k > 1$. Suppose we have $\alpha(s_i, H^k) = \alpha^{prop}(s_i, H^k)$ for all $i \le k$.

To prove for $n = k + 1$, consider any $H^{k+1} = (\big\{s_1, \ldots, s_k, s_{k+1}\big\}, \big\{\{s_1, \ldots, s_k, s_{k+1}\}\big\})$. We show that $\alpha(s_i, H^{k+1}) = \alpha^{prop}(s_i, H^{k+1})$ for all $i \le k+1$. Let $H^k = (\big\{s_1, \ldots, s_k\big\}, \big\{\{s_1, \ldots, s_k\}\big\})$. By induction hypothesis $\alpha(s_i, H^k) = \alpha^{prop}(s_i, H^k) = \frac{1}{k}$ for all $i \in \{1, \ldots, k\}$. Take $\lambda_i \in (0, 1)$ for $i \in \{1, \ldots, k\}$, and let $\alpha(s_i, H^{k+1}) = \lambda_i R$ and $\alpha(s_{k+1}, H^{k+1}) = (1 - \sum_{i=1}^{k}\lambda_i)R$. By relative redistribution for all $i, j \in \{1, \ldots, k\}$ we have $\frac{\alpha(s_i,H^{k+1})}{\alpha(s_i,H^k)} = \frac{\alpha(s_j,H^{k+1})}{\alpha(s_j,H^k)}$ and by induction hypothesis $\alpha(s_i, H^k) = \alpha(s_j, H^k) = \frac{R}{k}$. Putting these two together implies $\lambda_i = \lambda_j = \lambda$ for all $i, j \in \{1, \ldots, k\}$ and some $\lambda \in (0, 1)$. In case $\lambda = \frac{1}{k+1}$, then the shares receive proportionally. So on the contrary assume $\lambda \neq \frac{1}{k+1}$. Consider an extension of H^{k+1} denoted by H^{k+2} by a share s'. Then by relative redistribution we have $\frac{\alpha(s_i,H^{k+2})}{\alpha(s_i,H^{k+1})} = \frac{\alpha(s_j,H^{k+2})}{\alpha(s_j,H^{k+1})} = \frac{\alpha(s_i,H^{k+2})}{\lambda R} = \frac{\alpha(s_j,H^{k+2})}{(1-\lambda)R}$ for all $i, j \in \{1, \ldots, k\}$ which implies

$$\alpha(s_i, H^{k+2}) = \alpha(s_j, H^{k+2}) \quad \text{for all } i \in \{1, \ldots, k\}. \tag{23}$$

Also, for the $k+1$ share by relative redistribution we have $\frac{\alpha(s_i,H^{k+2})}{\alpha(s_i,H^{k+1})} = \frac{\alpha(s_{k+1},H^{k+2})}{\alpha(s_{k+1},H^{k+1})} = \frac{\alpha(s_i,H^{k+2})}{\lambda R} = \frac{\alpha(s_{k+1},H^{k+2})}{(1-k\lambda)R}$ for any $i \in \{1, \ldots, k\}$. The latter equation yields

$$\alpha(s_{k+1}, H^{k+2}) = \frac{1-k\lambda}{\lambda}\alpha(s_i, H^{k+2}). \tag{24}$$

By absolute redistribution, for any $i \in \{1, \ldots, k\}$ we have $\alpha(s_i, H^{k+1}) - \alpha(s_i, H^{k+2}) = \alpha(s_{k+1}, H^{k+1}) - \alpha(s_{k+1}, H^{k+2})$. Hence, $\lambda R - \alpha(s_i, H^{k+2}) = (1-k\lambda)R - \alpha(s_{k+1}, H^{k+2})$. Plugging Equation (24) into this implies

$$\lambda R - \alpha(s_i, H^{k+2}) = (1-k\lambda)R - \alpha(s_{k+1}, H^{k+2})$$
$$\lambda R - \alpha(s_i, H^{k+2}) = (1-k\lambda)R - \frac{1-k\lambda}{\lambda}\alpha(s_i, H^{k+2})$$
$$(\lambda - 1 + k\lambda)R = (\frac{\lambda-1+k\lambda}{\lambda})\alpha(s_i, H^{k+2})$$
$$\lambda R = \alpha(s_i, H^{k+2})$$

Where in the third equation $\lambda - 1 + k\lambda$ cancels out since $\lambda \neq \frac{1}{k+1}$ Also, by Equation (24) we have $\alpha(s_{k+1}, H^{k+2}) = (1 - k\lambda)R$. All in all, $\alpha(s_i, H^{k+1}) = \alpha(s_i, H^{k+2})$ for all $i \in \{1, \dots, k+1\}$. Using Lemma 1, $R = \sum_{i=1}^{k+2} \alpha(s_i, H^{k+2}) = \sum_{i=1}^{k+1} \alpha(s_i, H^{k+2}) + \alpha(s_{k+2}, H^{k+2}) = k\lambda R + (1-k\lambda)R + \alpha(s_{k+2}, H^{k+2})$, which implies $\alpha(s_{k+2}, H^{k+2}) = 0$, however this contradicts the strict positivity axiom. ■

Appendix I. Discussion

I.1. PPLNS

Example 4. Consider the PPLNS reward sharing scheme with $N = 3$. Let $H = (S, \mathcal{P}(S))$ be a history as follows,

$$S = \{s_1, s_2, \dots, s_{1000}\}$$
$$\mathcal{P}(S) = \{\underbrace{\{s_1, s_2, s_3, s_4, \mathbf{s_5}\}}_{P_1}, \underbrace{\{\ \mathbf{s_6}\ \}}_{P_2}, \underbrace{\{\ \mathbf{s_7}\ \}}_{P_3}, \underbrace{\{\ \mathbf{s_8}\ \}}_{P_4}, \underbrace{\{s_9, \dots, \mathbf{s_{20}}\}}_{P_5}, \dots, \underbrace{\{\dots, \mathbf{s_{1000}}\}}_{P_{132}}\}$$

To show that PPLNS fails to satisfy the budget limit and fixed total rewards, note that the rewards of the shares in the first and second rounds of the aforementioned history are

$$\alpha(s_1, H) = 0 \qquad \alpha(s_2, H) = 0 \qquad \alpha(s_3, H) = \frac{1}{3}R$$
$$\alpha(s_4, H) = \frac{2}{3}R \qquad \alpha(s_5, H) = R \qquad \alpha(s_6, H) = \frac{3}{3}R$$

Therefore, as $\sum_{s\in P_1} \alpha(s, H) = \frac{6}{3}R$, the PPLNS fails to satisfy the budget limit. Also, as $\sum_{s\in P_1} \alpha(s, H) \neq \sum_{s\in P_2} \alpha(s, H)$ the PPLNS fails to satisfy the fixed total rewards.

To show that PPLNS fails to satisfy the round based rewards, consider the restriction of H to the first round, i.e., $H|_1 = \left(\{s_1, \dots, \mathbf{s_5}\}, \left\{\underbrace{\{s_1, \dots, \mathbf{s_5}\}}_{P_1}\right\}\right)$. The rewards of each share would be $\alpha(s_1, H|_1) = \alpha(s_2, H|_1) = 0$, and $\alpha(s_3, H|_1) = \alpha(s_4, H|_1) = \alpha(s_5, H|_1) = \frac{R}{3}$. Comparing these with the reward of each share at the history H shows that the PPLNS fails to satisfy the round based rewards.

To show that PPLNS fails to satisfy absolute redistribution and relative redistribution, consider $H' = (S', \mathcal{P}'(S'))$ as an extension of H at the first round as follows:

$$S' = \{s_1, s_2, \dots, s^*, \dots, s_{1000}\}$$
$$\mathcal{P}'(S') = \{\underbrace{\{s_1, s_2, s_3, s_4, s_5, \mathbf{s^*}\}}_{P_1}, \underbrace{\{\ \mathbf{s_6}\ \}}_{P_2}, \underbrace{\{\ \mathbf{s_7}\ \}}_{P_3}, \underbrace{\{\ \mathbf{s_8}\ \}}_{P_4}, \underbrace{\{s_9, \dots, \mathbf{s_{20}}\}}_{P_5}, \dots, \underbrace{\{\dots, \mathbf{s_{1000}}\}}_{P_{132}}\}$$

It is easy to verify that $\alpha(s_1, H') = \alpha(s_2, H') = \alpha(s_3, H') = 0$, $\alpha(s_4, H') = \frac{1}{3}R$, $\alpha(s_5, H') = \frac{2}{3}R$, and $\alpha(s^*, H') = R$.

Since, $\alpha(s_1, H) - \alpha(s_1, H') \neq \alpha(s_3, H) - \alpha(s_3, H')$ then the PPLNS fails to satisfy the absolute redistribution. Also as $\frac{\alpha(s_4, H')}{\alpha(s_4, H)} \neq \frac{\alpha(s_5, H')}{\alpha(s_5, H)}$ the PPLNS fails to satisfy the relative redistribution.

I.2. Geometric

Proposition 3. *The geometric scheme satisfies budget limit.*

Proof. Let $H = (S, \mathcal{P}(S))$ be any history. Take any round r. Let $|P_r| = k$. Without loss of generality, assume $P_r = \{1, 2, \dots, k\}$ so that $\rho(s) = s$. Then

$$\sum_{s=1}^{k}\alpha(s,H)=\sum_{s=1}^{k}\frac{(r-1)}{r^{k-s+1}}B=B(r-1)\sum_{s=1}^{k}\frac{1}{r^{s}}$$
$$=B(r-1)\frac{1}{r}\left(\frac{1-\frac{1}{r^{n}}}{1-\frac{1}{r}}\right)=B(r-1)\left(\frac{1-\frac{1}{r^{n}}}{r-1}\right)=B(1-\frac{1}{r^{n}}) \tag{25}$$

As $r>1$ then Equation (25) is always less than B. ■

The following example shows the geometric scheme fails to satisfy the fixed total reward and absolute redistribution.

Example 5. Consider a history $H=(S,\mathcal{P}(S))$ as follows,

$$S=\{s_1,s_2,\ldots,s_{1000}\}$$
$$\mathcal{P}(S)=\{\underbrace{\{s_1,\mathbf{s_2}\}}_{P_1},\underbrace{\{\ \mathbf{s_3}\ \}}_{P_2},\underbrace{\{s_4,\ldots,\mathbf{s_{20}}\}}_{P_3},\ldots,\underbrace{\{\ldots,\mathbf{s_{1000}}\}}_{P_{132}}\}$$

It is easy to verify that $\alpha(s_1,H)=\frac{r-1}{r^2}B$, $\alpha(s_2,H)=\frac{r-1}{r}B$, and $\alpha(s_3,H)=\frac{r-1}{r}B$.

As $\sum_{s\in P_1}\alpha(s,H)\neq\sum_{s\in P_2}\alpha(s,H)$ then geometric scheme fails to satisfy the fixed total rewards.

To show that geometric scheme fails to satisfy absolute redistribution consider $H'=(S',\mathcal{P}'(S'))$ as an extension of H at the first round as follows:

$$S'=\{s_1,s_2,\ldots,s^*,\ldots,s_{1000}\}$$
$$\mathcal{P}'(S')=\{\underbrace{\{s_1,s_2,\mathbf{s^*}\}}_{P_1},\underbrace{\{\ \mathbf{s_3}\ \}}_{P_2},\underbrace{\{s_4,\ldots,\mathbf{s_{20}}\}}_{P_3},\ldots,\underbrace{\{\ldots,\mathbf{s_{1000}}\}}_{P_{132}}\}$$

It is easy to verify that $\alpha(s_1,H')=\frac{r-1}{r^3}B$, $\alpha(s_2,H')=\frac{r-1}{r^2}B$, $\alpha(s^*,H')=\frac{r-1}{r}B$. Since $r>1$ and $\alpha(s_1,H)-\alpha(s_1,H')\neq\alpha(s_2,H)-\alpha(s_2,H')$ then the geometric scheme fails to satisfy the absolute redistribution.

I.3. Slush

Example 6. Consider a history $H=(S,\mathcal{P}(S))$ as follows,

$$S=\{s_1,s_2,s_3\}$$
$$\mathcal{P}(S)=\{\underbrace{\{s_1,\mathbf{s_2}\}}_{P_1},\underbrace{\{\ \mathbf{s_3}\ \}}_{P_2}\}$$

Let $\tau(s_1)=1$, $\tau(s_2)=2$ and $\tau(s_1)=3$. Therefore, we have:

$$score(s_1,1)=\frac{e^{\frac{1-2}{1200}}}{e^{\frac{1-2}{1200}}+e^{\frac{2-2}{1200}}}\approx 0.49$$
$$score(s_1,2)=\frac{e^{\frac{1-3}{1200}}}{e^{\frac{1-3}{1200}}+e^{\frac{2-3}{1200}}+e^{\frac{3-3}{1200}}}\approx 0.33$$
$$score(s_2,1)=\frac{e^{\frac{2-2}{1200}}}{e^{\frac{1-2}{1200}}+e^{\frac{2-2}{1200}}}\approx 0.5$$
$$score(s_2,2)=\frac{e^{\frac{2-3}{1200}}}{e^{\frac{1-3}{1200}}+e^{\frac{2-3}{1200}}+e^{\frac{3-3}{1200}}}\approx 0.33$$
$$score(s_3,2)=\frac{e^{\frac{3-3}{1200}}}{e^{\frac{1-3}{1200}}+e^{\frac{2-3}{1200}}+e^{\frac{3-3}{1200}}}\approx 0.33$$

Therefore, $\alpha(s_1,H)=0.82R$, $\alpha(s_2,H)=0.83R$ and $\alpha(s_3,H)=0.33R$.

As $\sum_{s\in P_1}\alpha(s,H)=1.65R$, the Slush scheme violates the budget limit. As $\sum_{s\in P_1}\alpha(s,H)\neq\sum_{s\in P_2}\alpha(s,H)$, it fails to satisfy the fixed total rewards. It is easy to see that the Slush scheme also fails the round based reward axiom, e.g., for the second round in this example. In addition, one can verify that changing the time signature of any of the shares will have an effect on the award of shares, therefore the Slush scheme also violates delay invariance.

Now consider $H' = (S', \mathcal{P}'(S'))$ as an extension of H at the first round as follows:

$$S' = \{s_1, s_2, s^*, s_3\}$$
$$\mathcal{P}'(S') = \{\underbrace{\{s_1, s_2, \mathbf{s}^*\}}_{P_1}, \underbrace{\{\mathbf{s_3}\}}_{P_2}\}$$

such that $\tau(s^*) = 2.5$ Then the award of each share would be:

$$score(s_1, 1) = \frac{e^{\frac{1-2.5}{1200}}}{e^{\frac{1-2.5}{1200}} + e^{\frac{2-2.5}{1200}} + e^{\frac{2.5-2.5}{1200}}} \approx 0.33$$
$$score(s_1, 2) = \frac{e^{\frac{1-3}{1200}}}{e^{\frac{1-3}{1200}} + e^{\frac{2-3}{1200}} + e^{\frac{2.5-3}{1200}} + e^{\frac{3-3}{1200}}} \approx 0.25$$
$$score(s_2, 1) = \frac{e^{\frac{2-2.5}{1200}}}{e^{\frac{1-2.5}{1200}} + e^{\frac{2-2.5}{1200}} + e^{\frac{2.5-2.5}{1200}}} \approx 0.33$$
$$score(s_2, 2) = \frac{e^{\frac{2-3}{1200}}}{e^{\frac{1-3}{1200}} + e^{\frac{2-3}{1200}} + e^{\frac{2.5-3}{1200}} + e^{\frac{3-3}{1200}}} \approx 0.25$$
$$score(s^*, 1) = \frac{e^{\frac{2.5-2.5}{1200}}}{e^{\frac{1-2.5}{1200}} + e^{\frac{2-2.5}{1200}} + e^{\frac{2.5-2.5}{1200}}} \approx 0.33$$
$$score(s^*, 2) = \frac{e^{\frac{2.5-3}{1200}}}{e^{\frac{1-3}{1200}} + e^{\frac{2-3}{1200}} + e^{\frac{2.5-3}{1200}} + e^{\frac{3-3}{1200}}} \approx 0.25$$
$$score(s_3, 2) = \frac{e^{\frac{3-3}{1200}}}{e^{\frac{1-3}{1200}} + e^{\frac{2-3}{1200}} + e^{\frac{2.5-3}{1200}} + e^{\frac{3-3}{1200}}} \approx 0.25$$

Therefore, $\alpha(s_1, H') = 0.58R$ and $\alpha(s_2, H') = .58R$. Comparing these to $\alpha(s_1, H) = 0.82R$ and $\alpha(s_2, H) = .83R$, shows that the Slush scheme fails both absolute and fair redistribution axioms.

References

Babaioff, M., Dobzinski, S., Oren, S., Zohar, A., 2012. On bitcoin and red balloons. In: Proceedings of the 13th ACM Conference on Electronic Commerce, pp. 56–73.

Banker, R., 1981. Equity Considerations in Traditional Full Cost Allocation Practices: An Axiomatic Perspective. University of Oklahoma, Norman.

Belotti, M., Kirati, S., Secci, S., 2018. Bitcoin pool-hopping detection. In: 2018 IEEE 4th International Forum on Research and Technology for Society and Industry. RTSI. IEEE, pp. 1–6.

Biais, B., Bisiere, C., Bouvard, M., Casamatta, C., 2019. The blockchain folk theorem. Rev. Financ. Stud. 32 (5), 1662–1715.

Can, B., 2019. Economic design of things. In: The Future of Economic Design. Springer, pp. 487–493.

Carlsten, M., Kalodner, H., Weinberg, S.M., Narayanan, A., 2016. On the instability of bitcoin without the block reward. In: Proceedings of the 2016 ACM SIGSAC Conference on Computer and Communications Security, pp. 154–167.

Chatzigiannis, P., Baldimtsi, F., Griva, I., Li, J., 2019. Diversification across mining pools: optimal mining strategies under PoW. arXiv preprint. arXiv:1905.04624.

Chen, X., Papadimitriou, C., Roughgarden, T., 2019. An axiomatic approach to block rewards. In: Proceedings of the 1st ACM Conference on Advances in Financial Technologies, pp. 124–131.

Chun, Y., 1988. The proportional solution for rights problems. Math. Soc. Sci. 15 (3), 231–246.

Cong, L.W., He, Z., Li, J., 2021. Decentralized mining in centralized pools. Rev. Financ. Stud. 34 (3), 1191–1235.

de Frutos, M.A., 1999. Coalitional manipulations in a bankruptcy problem. Rev. Econ. Des. 4 (3), 255–272.

Dietzenbacher, B.J., Kondratev, A.Y., 2020. Fair and consistent prize allocation in competitions. arXiv preprint. arXiv:2008.08985.

Eyal, I., Sirer, E.G., 2014. Majority is not enough: bitcoin mining is vulnerable. In: International Conference on Financial Cryptography and Data Security. Springer, pp. 436–454.

Fisch, B., Pass, R., Shelat, A., 2017. Socially optimal mining pools. In: International Conference on Web and Internet Economics. Springer, pp. 205–218.

Hougaard, J.L., 2018. Allocation in Networks. MIT Press.

Hougaard, J.L., Moreno-Ternero, J.D., Østerdal, L.P., 2022. Optimal management of evolving hierarchies. Manag. Sci. 68 (8), 6024–6038.

Hougaard, J.L., Moreno-Ternero, J.D., Tvede, M., Østerdal, L.P., 2017. Sharing the proceeds from a hierarchical venture. Games Econ. Behav. 102, 98–110.

Kiayias, A., Koutsoupias, E., Kyropoulou, M., Tselekounis, Y., 2016. Blockchain mining games. In: Proceedings of the 2016 ACM Conference on Economics and Computation, pp. 365–382.

Leshno, J.D., Strack, P., 2020. Bitcoin: an axiomatic approach and an impossibility theorem. Am. Econ. Rev. 2 (3), 269–286.

Lewenberg, Y., Bachrach, Y., Sompolinsky, Y., Zohar, A., Rosenschein, J.S., 2015. Bitcoin mining pools: a cooperative game theoretic analysis. In: Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems. Citeseer, pp. 919–927.

Mingxiao, D., Xiaofeng, M., Zhe, Z., Xiangwei, W., Qijun, C., 2017. A review on consensus algorithm of blockchain. In: 2017 IEEE International Conference on Systems, Man, and Cybernetics. SMC. IEEE, pp. 2567–2572.

Moulin, H., 1987. Equal or proportional division of a surplus, and other methods. Int. J. Game Theory 16 (3), 161–186.

Moulin, H., 1991. Axioms of Cooperative Decision Making, vol. 15. Cambridge University Press.

Moulin, H., 2018. Fair division in the age of internet. Annu. Rev. Econ.

Myerson, R.B., 1977. Graphs and cooperation in games. Math. Oper. Res. 2 (3), 225–229.
Nakamoto, S., 2008. Bitcoin: a peer-to-peer electronic cash system. Technical report, Manubot.
Nguyen, G.-T., Kim, K., 2018. A survey about consensus algorithms used in blockchain. J. Inf. Process. Syst. 14 (1).
Qin, R., Yuan, Y., Wang, F.-Y., 2018. Research on the selection strategies of blockchain mining pools. IEEE Trans. Comput. Soc. Syst. 5 (3), 748–757.
Roemer, J., 1996. Theories of Distributive Justice. Harvard University Press.
Romiti, M., Judmayer, A., Zamyatin, A., Haslhofer, B., 2019. A deep dive into bitcoin mining pools: an empirical analysis of mining shares. arXiv preprint. arXiv:1905.05999.
Rosenfeld, M., 2011. Analysis of bitcoin pooled mining reward systems. arXiv preprint. arXiv:1112.4980.
Sapirshtein, A., Sompolinsky, Y., Zohar, A., 2016. Optimal selfish mining strategies in bitcoin. In: International Conference on Financial Cryptography and Data Security. Springer, pp. 515–532.
Schrijvers, O., Bonneau, J., Boneh, D., Roughgarden, T., 2016. Incentive compatibility of bitcoin mining pool reward functions. In: International Conference on Financial Cryptography and Data Security. Springer, pp. 477–498.
Thomson, W., 2016. Fair allocation. Ch. 7. In: Handbook of Well-being and Public Policy. Oxford University Press, New York.
Thomson, W., 2018. How to Divide When There Isn't Enough, vol. 62. Cambridge University Press.
Tovanich, N., Soulié, N., Heulot, N., Isenberg, P., 2021. An empirical analysis of pool hopping behavior in the bitcoin blockchain. In: 2021 IEEE International Conference on Blockchain and Cryptocurrency.
Young, H., 1994. Equity: In Theory and Practice. Princeton University Press.
Zolotavkin, Y., García, J., Rudolph, C., 2017. Incentive compatibility of pay per last n shares in bitcoin mining pools. In: International Conference on Decision and Game Theory for Security. Springer, pp. 21–39.