Brachistochrone Problem

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1 Introduction

The **Brachistochrone Problem** (**BP**) is: given two points in a vertical plane not aligned vertically nor horizontally, say P_1 and P_2, find the curve that a point mass m, moving without friction must follow such that, starting from rest in P_1, reaches P_2 in the shortest time under its own gravity. The term derives from the Greek (brachistos) "the shortest" and (chronos) "time".

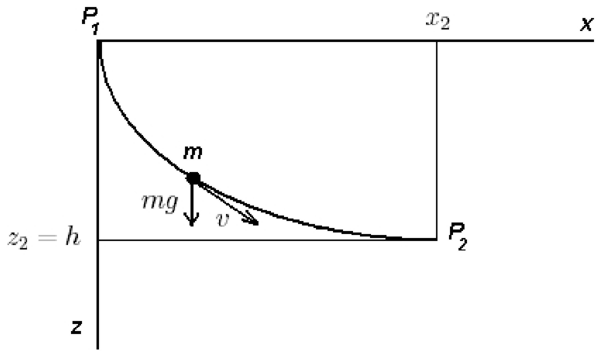

Figure 1:

2 Historical Background

The brachistochrone problem was one of the earliest problems posed in the calculus of variations. Galileo in 1638 had studied the problem in his famous work *Discourse on two new sciences*. He correctly showed the straight line was not the solution:

> "*If one considers motions with the same initial and terminal points then the shortest distance between them being a straight line, one might think that the motion along it needs least time. It turns out that this is not so.*".

But he made an error when he next argued that the path of quickest descent would be an arc of a circle.

Newton was challenged to solve the problem by Johann Bernoulli and Leibniz in 1696. According to Newton's biographer Conduitt, he solved the problem in an evening after returning home from the Royal Mint. Newton wrote:

> "*... in the midst of the hurry of the great recoinage, did not come home till four (in the afternoon) from the Tower very much tired, but did not sleep till he had solved it, which was by four in the morning.*".

In fact, *the solution, which is a segment of a cycloid*, was found by Leibniz, L'Hospital, Newton, and the two Bernoullis. Johann Bernoulli solved the problem using the analogous one of considering the path of light refracted by transparent layers of varying density (Mach 1893, Gardner 1984, Courant and Robbins 1996). Actually, Johann Bernoulli had originally found an incorrect proof that the curve is a cycloid, and challenged his brother Jakob to find the required curve. When Jakob correctly did so, Johann tried to substitute the proof for his own (Boyer and Merzbach 1991, p. 417).

The method which the brothers developed to solve the BP was put in a general setting by Euler in *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes sive solutio problematis isoperimetrici latissimo sensu accepti* published in 1744. In this work, he found what has now come to be known as the Euler-Lagrange differential equation for maximising or minimising function and its derivative.

3 Exact Solution to the classical BP

The time to travel from a point P_1 to another point P_2 is given by the integral

$$t_{12} = \int_{P_1}^{P_2} \frac{ds}{v} \tag{3.0.1}$$

where s is the arc length and v is the speed. The speed at any point is given by a simple application of conservation of energy equating kinetic energy variation ΔT to gravitational potential energy variation $-\Delta U$,

$$T(P) - T(P_1) = U(P_1) - U(P) \tag{3.0.2}$$

Here $P = P(x(t), z(t))$ is the position of the mass at any time instant t. With the force of gravity directed like the z axis, $P_1 \equiv (0,0)$, $P_2 \equiv (x_2, z_2)$ and $v_1 = 0$,

$$\frac{1}{2}mv^2 = mgz \tag{3.0.3}$$

giving

$$v = \sqrt{2gz} \tag{3.0.4}$$

Plugging this into 3.0.1 together with the identity

$$ds = \sqrt{dx^2 + dz^2} = \sqrt{1 + z_x^2}dx \tag{3.0.5}$$

then gives

$$t_{12} = \int_0^{x_2} \sqrt{\frac{1+z_x^2}{2gz}} dx \tag{3.0.6}$$

A fundamental equation of calculus of variations states that if J is defined by an integral of the form

$$J = \int f(x, z, z_x) dx \tag{3.0.7}$$

then J has a stationary value ($J_x = 0$) if the **Euler-Lagrange differential equation**

$$\frac{\partial f}{\partial z} - \frac{d}{dx}(\frac{\partial f}{\partial z_x}) = 0 \tag{3.0.8}$$

is satisfied. Problems in the calculus of variations often can be solved by solution of the appropriate Euler-Lagrange equation. We may verify that manipulation of the Euler-Lagrange differential equation brings to

$$\frac{d}{dx}\big(f - z_x \frac{\partial f}{\partial z_x}\big) - \frac{\partial f}{\partial x} = 0 \tag{3.0.9}$$

In many physical problems, f_x (the partial derivative of f with respect to x) turns out to be 0, in which case 3.0.9 reduces to the greatly simplified and partially integrated form known as the **Beltrami identity**,

$$f - z_x \frac{\partial f}{\partial z_x} = C \tag{3.0.10}$$

For the BP this is the case, because

$$f = \sqrt{\frac{1+z_x^2}{2gz}} \tag{3.0.11}$$

doesn't depend on x. Physically this means that if we translate the whole curve $z(x)$ in the x-direction, without changing its shape, the time of descent is unchanged. Computing and simplifying the Beltrami identity then gives

$$\frac{1}{\sqrt{2gz(1+z_x^2)}} = C \tag{3.0.12}$$

Squaring both sides and rearranging slightly results in

$$\big[1 + (\frac{dz}{dx})^2\big] z = \frac{1}{2gC^2} \equiv a^2 \tag{3.0.13}$$

where the right side has been expressed in terms of a new (positive) constant a^2. This equation is solved by the parametric equations

$$x(\varphi) = \frac{a^2}{2}[\varphi - sin(\varphi)] \tag{3.0.14}$$

$$z(\varphi) = \frac{a^2}{2}[1 - cos(\varphi)] \tag{3.0.15}$$

with $P(\varphi_1) = P(0) = (0,0)$ and $P(\varphi_2) = (x_2, h)$. This curve is a cycloid, the curve described by a point on the circumference of a circle as the circle is rolled along a line without slipping.

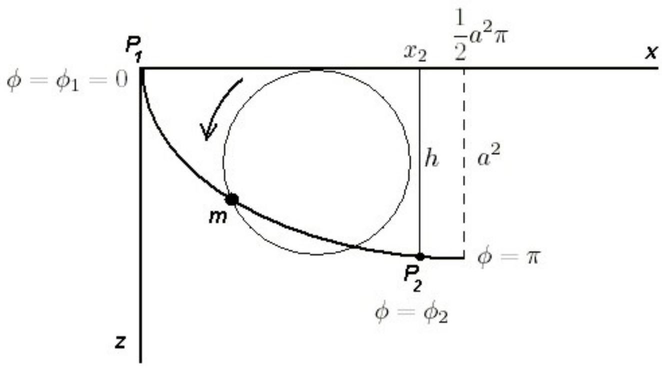

Figure 2:

Now we want to find t_{min}, solving 3.0.1. The parametric equations of the solution give

$$z_x(\varphi) = \frac{dz}{dx} = \frac{\frac{dz}{d\varphi}}{\frac{dx}{d\varphi}} = \frac{\sin\varphi}{1-\cos\varphi} \tag{3.0.16}$$

$$dx = \frac{a^2}{2}(1-\cos\varphi)d\varphi \tag{3.0.17}$$

Substituting into 3.0.1 yield

$$t_{min} = \int_0^{\varphi_2} \sqrt{\frac{a^2}{2g}}d\varphi = \varphi_2\sqrt{\frac{a^2}{2g}} \tag{3.0.18}$$

But, at the arrival point parametric equations rewrite

$$x_2 = \frac{a^2}{2}[\varphi_2 - \sin\varphi_2] \tag{3.0.19}$$

$$h = \frac{a^2}{2}[1 - \cos\varphi_2] \tag{3.0.20}$$

Eliminating a^2 in the equations above, and substituting into 3.0.18 we obtain a couple of equations with φ_2 as the only unknown

$$\varphi_2 - \sin\varphi_2 - \frac{x_2}{h}(1 - \cos\varphi_2) = 0 \tag{3.0.21}$$

$$t_{min} = \varphi_2\sqrt{\frac{h}{2g}(1 - \cos\varphi_2)} \tag{3.0.22}$$

$$\tag{3.0.23}$$

where the first is a trascendental equation to be solved for φ_2. A simple case is given by the parameter set $x_2 = \pi$, $h = 2$, $\varphi_2 = \pi$, so that $t_{min} = \frac{\pi}{\sqrt{g}}$.

4 Exact Solution to the BP into a central gravity potential

In the central gravitational field generated by a mass M placed at the origin O of the cartesian coordinate system, the mass m moves from P_1 to P_2 following the brachistochrone path on the plane to which P_1, P_2 and O belong. We prefer to use a polar system (r, θ), being $P = P(r(t), \theta(t))$ the position of the mass at any time instant t. We can fix $P_1 \equiv (r_1, \theta_1)$, $P_2 \equiv (r_2, \theta_2)$, with $r_1 > r_2$, $\theta_1 = 0$ and $v_1 = 0$.

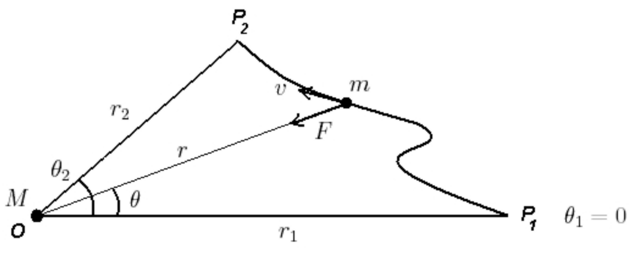

Figure 3:

In such a field the central potential is $U(P) = U(r) = -\frac{km}{r}$, $\overrightarrow{F} = -U_r\frac{\overrightarrow{r}}{r} = -\frac{km\overrightarrow{r}}{r^3}$. The principle of conservation of mechanical energy 3.0.2 yields

$$\frac{1}{2}mv^2 = \frac{km}{r} - \frac{km}{r_1} \tag{4.0.24}$$

giving

$$v = \sqrt{2k(\frac{1}{r} - \frac{1}{r_1})} \tag{4.0.25}$$

Plugging this into 3.0.1 together with the identity

$$ds = \sqrt{r^2 d\theta^2 + dr^2} = \sqrt{r^2 + r_\theta^2}d\theta \tag{4.0.26}$$

then gives

$$t_{12} = \sqrt{\frac{1}{2k}} \int_0^{\theta_2} \sqrt{\frac{r^2 + r_\theta^2}{\frac{1}{r} - \frac{1}{r_1}}} d\theta \tag{4.0.27}$$

Computing, rearranging and simplifying the Beltrami identity

$$f - r_\theta \frac{\partial f}{\partial r_\theta} = C \tag{4.0.28}$$

gives

$$b^2 r^4 = (\frac{1}{r} - \frac{1}{r_1})(r^2 + r_\theta^2) \tag{4.0.29}$$

where $b^2 \equiv \frac{1}{2kC^2} = \frac{1}{2GMC^2}$. In the SI system, $G = 6.673 \times 10^{-11} m^3 kg^{-1} s^{-2}$

$$\frac{dr}{d\theta} = -r\sqrt{\frac{b^2 r^2}{\frac{1}{r} - \frac{1}{r_1}} - 1} \tag{4.0.30}$$

$$\theta = \int_r^{r_1} \frac{du}{u\sqrt{\frac{b^2 r_1 u^3}{r_1 - u} - 1}} \tag{4.0.31}$$

5 Classical BP in the presence of friction

In the case there is also a frictional force resisting the motion, which we assume to be proportional to the speed of the mass with a coefficient of friction k, using the Newton's second law of motion, we can write the equation of motion in the z direction as

$$m\frac{dv}{dt} = mg - kv \tag{5.0.32}$$

This is important to get an expression for $v(z)$ to put in 3.0.1. Thus, let's write

$$mdv = (mg - kv)dt = (mg - kv)\frac{dz}{v} \tag{5.0.33}$$

$$mvdv = (mg - kv)dz \tag{5.0.34}$$

Separating variables we get

$$\int_0^v \frac{udu}{g - \frac{k}{m}u} = \int_0^z d\zeta \tag{5.0.35}$$

Then, Carrying out integration, we obtain

$$-\frac{m}{k}v - \frac{m^2 g}{k^2}\ln(|\frac{kv}{mg} - 1|) + C = z \tag{5.0.36}$$

References

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[2] J. Gibbons, M2A2 Dynamics - Course. March 25, 2003

[3] G. Buttazzo, M. Mintchev Curve Brachistocrone in campi di gravità. Università di Pisa, 2004