Simple model of atomic decay

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Abstract

Firstly a simplified model of atomic decay is introduced. Then we make some investigation about the free and forced response of a related differential system.

1 Introduction

Let's consider a simplified first order linear model for atomic decay. It is composed of N different kind of atomic elements, X_i (always $i = 1 \dots N$), where X_1 is the top element of the chain (that with maximum mass number) and X_i decays to X_{i+1} with time constant $\tau_i = \frac{1}{r_i} > 0$. Let $x_i(t)$ be the number of type X_i atoms at a given time. In the continous approximation of the model x_i is a continous function of time.

2 Three atoms model

Let's assume $N = 3$ and an initial number $x_i^0 > 0$ of type X_i atoms. The solving system is made up by three first order homogeneous ordinary differential equations

$$\begin{cases} \dot{x}_1 = -r_1 x_1 \\ \dot{x}_2 = r_1 x_1 - r_2 x_2 \\ \dot{x}_3 = r_2 x_2 \end{cases}$$

or, with obvious notation,

$$\dot{\vec{x}}(t) = A\vec{x}(t) \qquad t > 0; \qquad \vec{x}(0) = \vec{x}_0 \tag{2.0.1}$$

with

$$A \equiv \begin{pmatrix} -r_1 & 0 & 0 \\ r_1 & -r_2 & 0 \\ 0 & r_2 & 0 \end{pmatrix} \qquad \vec{x}_0 \equiv \begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{pmatrix}$$

The matrix A is singular. Having all zero elements above the principal diagonal its eigenvalues can be read immediately along the diagonal, $\lambda_1 = -r_1$, $\lambda_2 = -r_2$ and $\lambda_3 = 0$. They are all real and distinct numbers. The general solution is

$$\vec{x}(t) = C_1 e^{-r_1 t}\vec{\eta}^{(1)} + C_2 e^{-r_2 t}\vec{\eta}^{(2)} + C_3\vec{\eta}^{(3)} \tag{2.0.2}$$

We have to determine the eigenvectors $\vec{\eta}^{(i)}$ by solving $(A - \lambda_i I)\vec{\eta}^{(i)} = 0$, and find the constants C_i by imposing the initial conditions.

For eigenvalue $\lambda_1 = -r_1$ we have

$$\begin{pmatrix} 0 & 0 & 0 \\ r_1 & r_1 - r_2 & 0 \\ 0 & r_2 & r_1 \end{pmatrix} \begin{pmatrix} \eta_1^{(1)} \\ \eta_2^{(1)} \\ \eta_3^{(1)} \end{pmatrix} = \vec{0} \qquad \Rightarrow \vec{\eta}^{(1)} = \begin{pmatrix} \frac{r_2}{r_1} - 1 \\ 1 \\ -\frac{r_2}{r_1} \end{pmatrix}$$

For eigenvalue $\lambda_2 = -r_2$

$$\begin{pmatrix} r_2 - r_1 & 0 & 0 \\ r_1 & 0 & 0 \\ 0 & r_2 & r_2 \end{pmatrix} \begin{pmatrix} \eta_1^{(2)} \\ \eta_2^{(2)} \\ \eta_3^{(2)} \end{pmatrix} = \vec{0} \qquad \Rightarrow \vec{\eta}^{(2)} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

For eigenvalue $\lambda_3 = 0$

$$\begin{pmatrix} -r_1 & 0 & 0 \\ r_1 & -r_2 & 0 \\ 0 & r_2 & 0 \end{pmatrix} \begin{pmatrix} \eta_1^{(3)} \\ \eta_2^{(3)} \\ \eta_3^{(3)} \end{pmatrix} = \vec{0} \qquad \Rightarrow \vec{\eta}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

From the initial conditions

$$C_1 \begin{pmatrix} \frac{r_2}{r_1} - 1 \\ 1 \\ -\frac{r_2}{r_1} \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{pmatrix}$$

we find $C_1 = \frac{x_1^0 r_1}{r_2 - r_1}$, $C_2 = \frac{x_1^0 r_1}{r_2 - r_1} - x_2^0$, $C_3 = x_1^0 + x_2^0 + x_3^0$.

Thus, the solution is

$$\begin{cases} x_1(t) = x_1^0 e^{-r_1 t} \\ x_2(t) = \frac{x_1^0 r_1}{r_2 - r_1} e^{-r_1 t} + (x_2^0 - \frac{x_1^0 r_1}{r_2 - r_1}) e^{-r_2 t} \\ x_3(t) = x_1^0 + x_2^0 + x_3^0 - \frac{x_1^0 r_1}{r_2 - r_1} e^{-r_1 t} - (x_2^0 - \frac{x_1^0 r_1}{r_2 - r_1}) e^{-r_2 t} \end{cases}$$

The picture shows how all these quantities vary with time, having $\vec{x}_0 = (x_0, 0, 0); \tau_1 = 300; \tau_2 = 1200$.

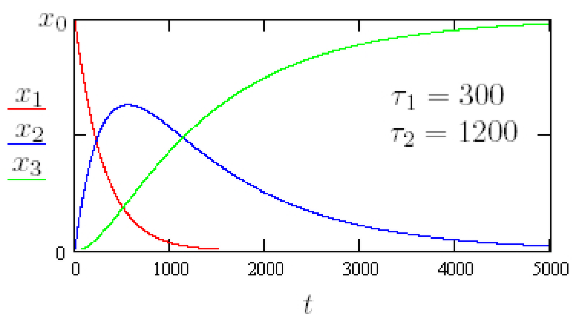

Figure 1: Simplified model of atomic decay - System component's time evolution ($\tau_1 = 300; \tau_2 = 1200$).

3 Digression about closed chains - homogeneous

In the atomic decay model discussed above, the mass flows from x_1 through x_N and stay there indefinitely. Now, with a little modification we make the chain cyclic

$$\begin{cases} \dot{x}_1 = r_3 x_3 - r_1 x_1 \\ \dot{x}_2 = r_1 x_1 - r_2 x_2 \\ \dot{x}_3 = r_2 x_2 - r_3 x_3 \end{cases}$$

i.e. "mass" can go back to x_1 trough the rate r_3. In matrix notation,

$$\dot{\vec{x}}(t) = A\vec{x}(t) \qquad t > 0; \qquad \vec{x}(0) = \vec{x}_0 \tag{3.0.3}$$

$$A \equiv \begin{pmatrix} -r_1 & 0 & r_3 \\ r_1 & -r_2 & 0 \\ 0 & r_2 & -r_3 \end{pmatrix} \qquad \vec{x}_0 \equiv \begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{pmatrix} \qquad \vec{r} \equiv \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

It is an isolated system, no "mass" sources or sinks being involved (i.e. the system is homogeneous).

A graphical representation of the system is sketched below. Each arrow is directed with the "mass" flow and it is labeled with the transfer rate from the originating variable to the next one.

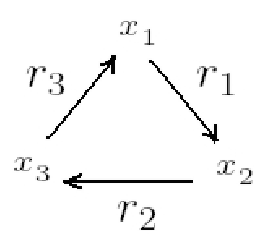

Figure 2: Isolated three-variables unidirectional closed chain.

This model has no obvious physical significance, indeed we investigate such a system just for the pleasure to do it.

Note that

$$\dot{x}_1 + \dot{x}_2 + \dot{x}_3 = 0$$

This holds because the system is an isolated system, and this implies that the scalar quantity

$$\mathcal{E} \equiv x_1 + x_2 + x_3 = x_1^0 + x_2^0 + x_3^0 = const.$$

it is a time invariant. Without any forcing terms into the equations we're sure the system will stay bounded all the time. i.e it is asintotically stable for $t \to +\infty$.

Also, we assume $r_i > 0$. The matrix A is nonsingular. To find the eigenvalues, let's solve the characteristic equation

$$|A - \lambda I| = \begin{vmatrix} -r_1 - \lambda & 0 & r_3 \\ r_1 & -r_2 - \lambda & 0 \\ 0 & r_2 & -r_3 - \lambda \end{vmatrix} = -(\lambda + r_1)(\lambda + r_2)(\lambda + r_3) + r_1 r_2 r_3 = 0$$

This gives

$$\lambda \cdot [\lambda^2 + \lambda(r_1 + r_2 + r_3) + r_1 r_2 + r_2 r_3 + r_3 r_1] = 0 \tag{3.0.4}$$

An eigenvalue is $\lambda_1 = 0$, while λ_2 and λ_3 are the roots of the quadratic factor

$$\lambda_{2,3} = \frac{-S \pm \sqrt{r_1^2 + r_2^2 + r_3^2 - 2P}}{2}$$

with $S \equiv r_1 + r_2 + r_3$ and $P \equiv r_1 r_2 + r_2 r_3 + r_3 r_1$. What about the sign of the quantity under square root? From the Schwartz inequality we know that

$$r_1^2 + r_2^2 + r_3^2 - P > 0$$

Nevertheless $\vec{r} = (r_1, r_2, r_3)$ can be chosen such that

$$r_1^2 + r_2^2 + r_3^2 - 2P \leq 0$$

so any cases could take place. Just let's mention the following cases. When $\vec{r} = (\rho, \rho, \rho)$ the term under the square root is negative and $\lambda_{2,3} = \frac{-3 \pm j\sqrt{3}}{2}\rho$.

If two of the rates are equal to ρ and the third rate is equal to 4ρ, then $\lambda_2 = \lambda_3 = -3\rho$.

Finally, if one of the rates is greater than twice the sum of the others, $\lambda_{2,3}$ are real and distinct - the last statement coming from the inequalities

$$r_1^2 + r_2^2 + r_3^2 \geq 2r_1 r_2 + r_3^2 > 2P \quad \Leftrightarrow \quad r_3^2 > 2P - 2r_1 r_2 = 2r_2 r_3 + 2r_3 r_1 = 2r_3(r_1 + r_2)$$

Note that the assumption $r_i > 0$ is very important for the time evolution of the system. In fact it implies that, whether eigenvalues $\lambda_{2,3}$ will be real or complex, their real part will have to be negative, so that the system not only is stable, but takes a constant value in the long term $t \to +\infty$, i.e.

$$\lim_{t \to +\infty} \vec{x}(t) = \vec{x}_\infty = const. \tag{3.0.5}$$

Now, in order to move toward an explicit solution we have to determine the eigenvectors by solving $(A - \lambda_i I)\vec{\eta}^{(i)} = 0$.

3.1 Eigenvalue $\lambda_1 = 0$

For the zero eigenvalue we have

$$\begin{pmatrix} -r_1 & 0 & r_3 \\ r_1 & -r_2 & 0 \\ 0 & r_2 & -r_3 \end{pmatrix} \begin{pmatrix} \eta_1^{(1)} \\ \eta_2^{(1)} \\ \eta_3^{(1)} \end{pmatrix} = \vec{0} \qquad \Rightarrow \vec{\eta}^{(1)} = \begin{pmatrix} \frac{r_3}{r_1} \\ \frac{r_3}{r_2} \\ 1 \end{pmatrix}$$

This eigenvector plays an important role because it is associated with the steady state value of the system trajectory, $\vec{x}_\infty$, as we'll see in the next paragraph.

3.2 $\lambda_{2,3}$ real and distinct eigenvalues

For real and distinct eigenvalues $\lambda_{2,3}$ the general solution is

$$\vec{x}(t) = C_1\vec{\eta}^{(1)} + C_2 e^{\lambda_2 t}\vec{\eta}^{(2)} + C_3 e^{\lambda_3 t}\vec{\eta}^{(3)} \tag{3.2.1}$$

We have

$$\begin{pmatrix} -r_1 - \lambda_{2,3} & 0 & r_3 \\ r_1 & -r_2 - \lambda_{2,3} & 0 \\ 0 & r_2 & -r_3 - \lambda_{2,3} \end{pmatrix} \begin{pmatrix} \eta_1^{(2,3)} \\ \eta_2^{(2,3)} \\ \eta_3^{(2,3)} \end{pmatrix} = \vec{0} \qquad \Rightarrow \vec{\eta}^{(2,3)} = \begin{pmatrix} r_3 \\ \frac{r_1 r_3}{r_2 + \lambda_{2,3}} \\ r_1 + \lambda_{2,3} \end{pmatrix}$$

To make things easier, let's consider the special case $\vec{r} = (\rho, \frac{1}{6}\rho, \frac{1}{3}\rho)$, $\vec{x}_0 = (x_0, 0, 0)$, so that $\lambda_2 = -\frac{2}{3}\rho$ and $\lambda_3 = -\frac{5}{6}\rho$. Using this assumptions the eigenvector associated with the zero eigenvalue becomes

$$\vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$$

For eigenvalue $\lambda_2 = -\frac{2}{3}\rho$ we have

$$\vec{\eta}^{(2)} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

For eigenvalue $\lambda_3 = -\frac{5}{6}\rho$

$$\vec{\eta}^{(3)} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

Constants C_i have to be determined by imposing the initial conditions

$$C_1 \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1^0 \\ 0 \\ 0 \end{pmatrix}$$

We get $C_1 = \frac{1}{10}x_1^0$, $C_2 = -\frac{3}{2}x_1^0$, $C_3 = \frac{6}{5}x_1^0$. The solution is

$$\begin{cases} x_1(t) = \frac{x_1^0}{10}(1 - 15e^{-\frac{2\rho t}{3}} + 24e^{-\frac{5\rho t}{6}}) \\ x_2(t) = \frac{x_1^0}{5}(3 + 15e^{-\frac{2\rho t}{3}} - 18e^{-\frac{5\rho t}{6}}) \\ x_3(t) = \frac{x_1^0}{10}(3 - 15e^{-\frac{2\rho t}{3}} + 12e^{-\frac{5\rho t}{6}}) \end{cases}$$

The picture below shows how all the quantities vary in time ($\rho = 1$).

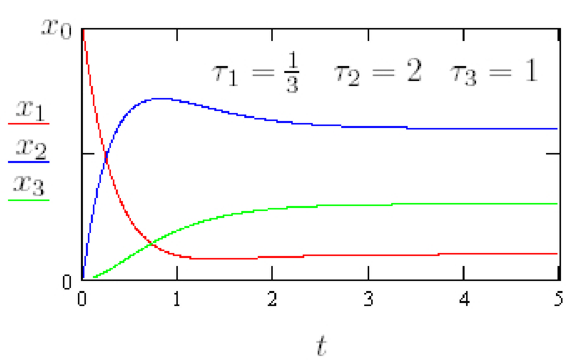

Figure 3: Time course for the three system components with real and distinct eigenvalues ($\tau_1 = \frac{1}{3}; \tau_2 = 2; \tau_3 = 1$).

For generic initial conditions we would have had $C_1 = \frac{1}{10}(x_1^0 + x_2^0 + x_3^0)$, $C_2 = \frac{1}{4}(-3x_1^0 - x_2^0 + 3x_3^0)$, $C_3 = \frac{1}{5}(6x_1^0 + x_2^0 - 4x_3^0)$. It's worth to note that the coefficient related with the zero eigenvalue is proportional to $\mathcal{E}$. The same held in the atomic decay model discussed above. In our example the steady state constant is

$$\vec{x}_\infty = C_1 \vec{\eta}^{(1)} = \frac{\mathcal{E}}{10} \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix} \tag{3.2.2}$$

3.3 $\lambda_{2,3}$ complex eigenvalues

For complex eigenvalue $\lambda_2 = \alpha + j\beta$ we have

$$\begin{pmatrix} -r_1 - \alpha - j\beta & 0 & r_3 \\ r_1 & -r_2 - \alpha - j\beta & 0 \\ 0 & r_2 & -r_3 - \alpha - j\beta \end{pmatrix} \begin{pmatrix} \eta_{1R} + j\eta_{1I} \\ \eta_{2R} + j\eta_{2I} \\ \eta_{3R} + j\eta_{3I} \end{pmatrix}$$

This is equivalent to solve the real system

$$\begin{pmatrix} -\alpha_1 & \beta & 0 & 0 & r_3 & 0 \\ \beta & \alpha_1 & 0 & 0 & 0 & -r_3 \\ r_1 & 0 & -\alpha_2 & \beta & 0 & 0 \\ 0 & -r_1 & \beta & \alpha_2 & 0 & 0 \\ 0 & 0 & r_2 & 0 & -\alpha_3 & \beta \\ 0 & 0 & 0 & -r_2 & \beta & \alpha_3 \end{pmatrix} \begin{pmatrix} \eta_{1R} \\ \eta_{1I} \\ \eta_{2R} \\ \eta_{2I} \\ \eta_{3R} \\ \eta_{3I} \end{pmatrix} = \vec{0}$$

where $\alpha_i \equiv r_i + \alpha$.

But let's consider the special case $\vec{r} = (\rho, \rho, \rho)$, so that $\lambda = \frac{-3 \pm j\sqrt{3}}{2}\rho$.

With this assumption the eigenvector coupled with the zero eigenvalue becomes

$$\vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For eigenvalue $\lambda_2 = \frac{-3+j\sqrt{3}}{2}\rho$

$$\begin{pmatrix} \frac{1-j\sqrt{3}}{2}\rho & 0 & \rho \\ \rho & \frac{1-j\sqrt{3}}{2}\rho & 0 \\ 0 & \rho & \frac{1-j\sqrt{3}}{2}\rho \end{pmatrix} \begin{pmatrix} \eta_{1R} + j\eta_{1I} \\ \eta_{2R} + j\eta_{2I} \\ \eta_{3R} + j\eta_{3I} \end{pmatrix} = \vec{0} \qquad \Rightarrow \vec{\eta}^{(2)} = \begin{pmatrix} 1 \\ -\frac{1+j\sqrt{3}}{2} \\ -\frac{1-j\sqrt{3}}{2} \end{pmatrix}$$

We know that $\lambda_3 = \lambda_2^*$ and $\vec{\eta}^{(3)} = (\vec{\eta}^{(2)})^*$. From the initial conditions

$$C_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -\frac{1+j\sqrt{3}}{2} \\ -\frac{1-j\sqrt{3}}{2} \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ -\frac{1-j\sqrt{3}}{2} \\ -\frac{1+j\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} x_1^0 \\ 0 \\ 0 \end{pmatrix}$$

and we obtain $C_1 = C_2 = C_3 = \frac{x_1^0}{3}$. The solution is

$$\vec{x}(t) = \frac{x_1^0}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{x_1^0}{3}\begin{pmatrix} 1 \\ -\frac{1+j\sqrt{3}}{2} \\ -\frac{1-j\sqrt{3}}{2} \end{pmatrix} e^{\frac{-3+j\sqrt{3}}{2}\rho t} + \frac{x_1^0}{3}\begin{pmatrix} 1 \\ -\frac{1-j\sqrt{3}}{2} \\ -\frac{1+j\sqrt{3}}{2} \end{pmatrix} e^{\frac{-3-j\sqrt{3}}{2}\rho t}$$

or

$$\begin{cases} x_1(t) = \frac{x_1^0}{3}(1 + 2e^{-\frac{3\rho t}{2}} \cos \frac{\sqrt{3}\rho t}{2}) \\ x_2(t) = \frac{x_1^0}{3}(1 + \sqrt{3}e^{-\frac{3\rho t}{2}} \sin \frac{\sqrt{3}\rho t}{2} - e^{-\frac{3\rho t}{2}} \cos \frac{\sqrt{3}\rho t}{2}) \\ x_3(t) = \frac{x_1^0}{3}(1 - \sqrt{3}e^{-\frac{3\rho t}{2}} \sin \frac{\sqrt{3}\rho t}{2} - e^{-\frac{3\rho t}{2}} \cos \frac{\sqrt{3}\rho t}{2}) \end{cases}$$

The picture below shows how all the quantities vary in time, with $\tau_1 = \tau_2 = \tau_3 = 1$.

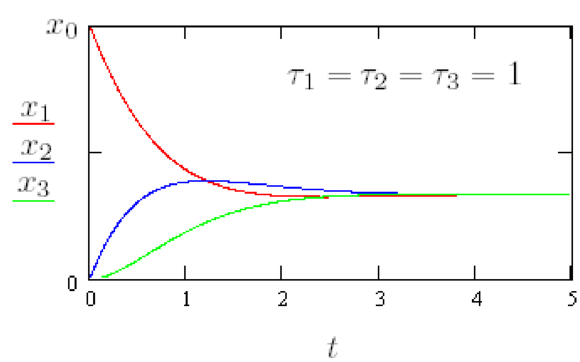

Figure 4: Time course for the three system components with two complex eigenvalues ($\tau_1 = \tau_2 = \tau_3 = 1$).

3.4 $\lambda_{2,3}$ real and equal eigenvalues

Let's consider the special case $\vec{r} = (\rho, \rho, 4\rho)$. In this case $\lambda_2 = \lambda_3 = -3\rho \equiv \lambda$. Now the eigenvector associated with the zero eigenvalue is

$$\vec{\eta}^{(1)} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}$$

From linear algebra we know that the double eigenvalue λ can have one or two independent eigenvectors. In this case it has only one

$$\begin{pmatrix} 2\rho & 0 & 4\rho \\ \rho & 2\rho & 0 \\ 0 & \rho & -\rho \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \vec{0} \qquad \Rightarrow \vec{\eta} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

With these two only independent vectors we cannot fit the solution with the initial conditions. In fact, we need to introduce an auxiliary constant vector $\vec{\theta}$ into the the general solution

$$\vec{x}(t) = C_1\vec{\eta}^{(1)} + C_2 e^{\lambda t}\vec{\eta} + C_3 e^{\lambda t}(t\vec{\eta} + \vec{\theta}) \tag{3.4.1}$$

Replacing the above expression in the equation $\dot{\vec{x}} = A\vec{x}$, with simple matrix algebra we find that $\vec{\theta}$ must satisfy $(A - I\lambda)\vec{\theta} = \vec{\eta}$. After some calculation we arrive at

$$\vec{\theta} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Using the initial conditions, $C_1 = \frac{x_1^0}{9}$, $C_2 = -\frac{x_1^0}{9}$, $C_3 = -\frac{x_1^0}{3}$. The final solution is

$$\begin{cases} x_1(t) = \frac{x_1^0}{9}[4 + e^{-3\rho t}(6t + 5)] \\ \\ x_2(t) = \frac{x_1^0}{9}[4 - e^{-3\rho t}(3t + 4)] \\ \\ x_3(t) = \frac{x_1^0}{9}(1 - 4e^{-3\rho t}) \end{cases}$$

The picture below shows how all the quantities vary in time, with $\tau_1 = \tau_2 = 1$, $\tau_3 = \frac{1}{4}$.

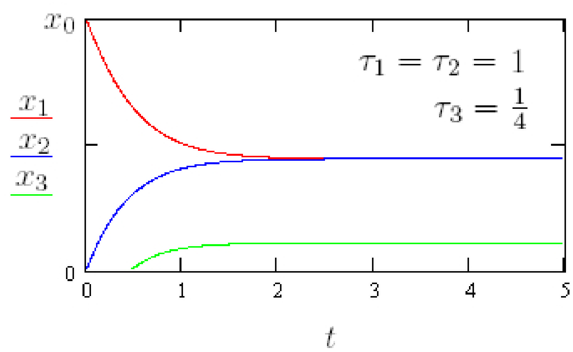

Figure 5: Time course for the three system components with two complex eigenvalues ($\tau_1 = \tau_2 = \tau_3 = 1$).

4 Nonhomogeneous systems

Now we may ask what would happen if forcing functions were introduced into the system. In order to simplify calculations we consider a two variables (i.e. a second order) system. After a quick derivation of the homogeneous solution we will find the Green function kernel, with which the reader is assumed to be familiar, and get the particular solution for a couple of examples.

4.1 Two variables - homogeneous

$$\begin{cases} \dot{x}_1 = r_2 x_2 - r_1 x_1 \\ \dot{x}_2 = r_1 x_1 - r_2 x_2 \end{cases}$$

or, in matrix notation,

$$\dot{\vec{x}}(t) = A\vec{x}(t) \qquad t > 0; \qquad \vec{x}(0) = \vec{x}_0 \tag{4.1.1}$$

$$A \equiv \begin{pmatrix} -r_1 & r_2 \\ r_1 & -r_2 \end{pmatrix} \qquad \vec{x}_0 \equiv \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$$

To find the eigenvalues, let's solve the characteristic equation

$$|A - \lambda I| = (\lambda + r_1)(\lambda + r_2) - r_1 r_2 = 0$$

Eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -(r_1 + r_2) \equiv \lambda$. The general solution is

$$\vec{x}(t) = C_1 \vec{\eta}^{(1)} + C_2 e^{\lambda t} \vec{\eta}^{(2)} \tag{4.1.2}$$

Now, in order to move toward an explicit solution we have to determine the eigenvectors by solving $(A - \lambda_i I)\vec{\eta}^{(i)} = 0$.

$$\begin{pmatrix} -r_1 & r_2 \\ r_1 & -r_2 \end{pmatrix} \begin{pmatrix} \eta_1^{(1)} \\ \eta_2^{(1)} \end{pmatrix} = \vec{0} \qquad \Rightarrow \vec{\eta}^{(1)} = \begin{pmatrix} \frac{r_2}{r_1} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} r_2 & r_2 \\ r_1 & r_1 \end{pmatrix} \begin{pmatrix} \eta_1^{(2)} \\ \eta_2^{(2)} \end{pmatrix} = \vec{0} \qquad \Rightarrow \vec{\eta}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Constants C_i have to be determined by imposing the initial conditions

$$C_1 \begin{pmatrix} \frac{r_2}{r_1} \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$$

We get $C_1 = \frac{r_1(x_1^0 + x_2^0)}{r_1 + r_2}$, $C_2 = \frac{r_1 x_1^0 - r_2 x_2^0}{r_1 + r_2}$. The solution is

$$\begin{cases} x_1(t) = \frac{r_2(x_1^0 + x_2^0)}{r_1 + r_2} + \frac{r_1 x_1^0 - r_2 x_2^0}{r_1 + r_2} e^{\lambda t} \\ \\ x_2(t) = \frac{r_1(x_1^0 + x_2^0)}{r_1 + r_2} - \frac{r_1 x_1^0 - r_2 x_2^0}{r_1 + r_2} e^{\lambda t} \end{cases}$$

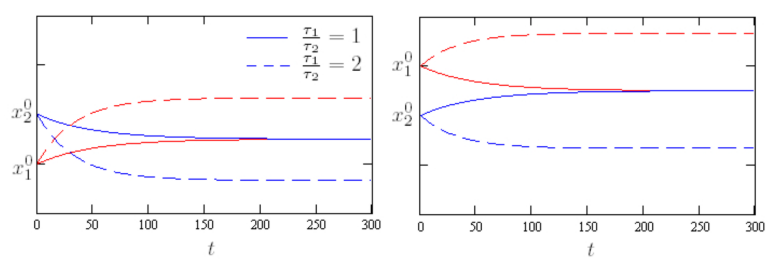

Figure 6: Time course for the two system's components.

4.2 Two variables - nonhomogeneous

Now the system looks like

$$\begin{cases} \dot{x}_1 = r_2 x_2 - r_1 x_1 + f_1 \\ \dot{x}_2 = r_1 x_1 - r_2 x_2 + f_2 \end{cases}$$

where $f_i(t) \in C^1[0, +\infty)$ are the forcing functions. Taking the derivative of the first equation and using $\dot{x}_1 + \dot{x}_2 = f_1 + f_2$ we obtain the equivalent second order nonhomogeneous ODE with nonhomogeneous (i.e. non zero) BCs

$$\begin{cases} \ddot{x}_1 - \lambda \dot{x}_1 = g(t) \qquad 0 \le t < +\infty \\ x_1(0) = x_1^0 \\ \dot{x}_1(0) = r_2 x_2^0 - r_1 x_1^0 + f_1(0) \equiv K \end{cases}$$

where $\lambda \equiv -r_1 - r_2$ and $g \equiv r_2(f_1 + f_2) + \dot{f}_1$. This is called a Cauchy problem, the solution to which is assumed to have the form

$$x_1(t) = x_1^{(o)} + x_1^{(p)} \tag{4.2.1}$$

where $x_1^{(o)}$ is the solution to the homogeneous problem ($g(t) \equiv 0$) with nonhomogeneous BCs, and $x_1^{(p)}$ is a particular solution to the nonhomogeneous problem with homogeneous (i.e. zero) BCs. From the previous paragraph we know that

$$x_1^{(o)} = x_1^0 + \frac{K}{\lambda}(e^{\lambda t} - 1) \tag{4.2.2}$$

4.2.1 Green Function method

The Green function method serves to find $x_1^{(p)}$. The Green function $G(t, \xi)$ is an integral kernel and must satisfy the homogeneous equation for $t \neq \xi$, thus

$$G(t, \xi) = \begin{cases} A_1(\xi) e^{\lambda t} + B_1(\xi) & per\ t < \xi \\ A_2(\xi) e^{\lambda t} + B_2(\xi) & per\ t > \xi \end{cases}$$

Following the standard procedure, we apply homogeneous BCs as well as continuity and jump conditions to $G(t, \xi)$ in order to get the unknown functions A_1, B_1, A_2 and B_2

$$G(0, \xi) = 0 \quad \Rightarrow A_1(\xi) + B_1(\xi) = 0 \tag{4.2.3}$$

$$\frac{\partial G}{\partial t}|_{t=0} = 0 \quad \Rightarrow A_1(\xi) = 0 \quad \Rightarrow B_1(\xi) = 0 \tag{4.2.4}$$

$$G(\xi^+, \xi) - G(\xi^-, \xi) = 0 \quad \Rightarrow A_2(\xi) e^{\lambda \xi} + B_2(\xi) = 0 \tag{4.2.5}$$

$$\frac{\partial G}{\partial t}|_{t=\xi^-}^{t=\xi^+} = 1 \quad \Rightarrow A_2(\xi) = \frac{1}{\lambda}e^{-\lambda\xi} \quad \Rightarrow B_2(\xi) = -\frac{1}{\lambda} \tag{4.2.6}$$

So

$$G(t,\xi) = \begin{cases} 0 & for\ t < \xi \\ \frac{1}{\lambda}[e^{\lambda(t-\xi)} - 1] & for\ t \geq \xi \end{cases}$$

Then, the particular solution for the nonhomogeneous system is

$$x_1^{(p)}(t) = \int_0^{+\infty} G(t,\xi)g(\xi)d\xi = \tfrac{1}{\lambda}\int_0^t [e^{\lambda(t-\xi)} - 1]g(\xi)d\xi \tag{4.2.7}$$

The solution to the original Cauchy problem is

$$x_1(t) = x_1^0 + \frac{K}{\lambda}(e^{\lambda t} - 1) + \tfrac{1}{\lambda}\int_0^t [e^{\lambda(t-\xi)} - 1]g(\xi)d\xi \tag{4.2.8}$$

One can repeat exactly the same for the component $x_2(t)$ (the only change affects the integration constants in the homogeneous solution and the forcing function in the particular solution). Putting all together the final result is

$$\begin{cases} x_1(t) = \frac{r_2(x_1^0+x_2^0)}{r_1+r_2} + \frac{r_1x_1^0-r_2x_2^0}{r_1+r_2}e^{\lambda t} + \frac{1}{\lambda}\int_0^t [e^{\lambda(t-\xi)} - 1][r_2(f_1(\xi) + f_2(\xi)) + \dot{f}_1(\xi)]d\xi \\ \\ x_2(t) = \frac{r_1(x_1^0+x_2^0)}{r_1+r_2} - \frac{r_1x_1^0-r_2x_2^0}{r_1+r_2}e^{\lambda t} + \frac{1}{\lambda}\int_0^t [e^{\lambda(t-\xi)} - 1][r_1(f_1(\xi) + f_2(\xi)) + \dot{f}_2(\xi)]d\xi \end{cases}$$

4.2.2 Variation of Parameters method

An alternative method to find a particular solution to the second order nonhomogeneous equations is that of variation of parameters. Basically it consists in applying the formula

$$y^{(p)}(t) = -y_1(t)\int \frac{y_2(\xi)g(\xi)}{W(y_1,y_2)}d\xi + y_2(t)\int \frac{y_1(\xi)g(\xi)}{W(y_1,y_2)}d\xi \tag{4.2.9}$$

where y_1 and y_2 represents a fundamental set of solutions for the homogeneous problem, and W their Wronskian. For our problem we can put $y_1 = e^{\lambda t}$ and $y_2 = 1$ (Do not worry about which of our two solutions is y_1 and which one is y_2. It doesn't matter).

$$W(y_1,y_2) \equiv \begin{vmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{vmatrix} = \begin{vmatrix} e^{\lambda t} & 1 \\ \lambda e^{\lambda t} & 0 \end{vmatrix} = -\lambda e^{\lambda t}$$

Substitution in 4.2.9 and integration gives ($y = x_1$)

$$x_1^{(p)}(t) = \frac{1}{\lambda} \int_0^t [e^{\lambda(t-\xi)} - 1] g(\xi) d\xi \tag{4.2.10}$$

This coincides with the previously stated result. We now look at a series of examples.

4.3 Example I

Let's take the example

$$\begin{cases} f_1(t) = Du(t - t_0) \\ f_2(t) = -Du(t - t_0) \end{cases}$$

with $t_0 > 0$ and $t \geq 0$. Here $u(t)$ denotes the unit (or Heaviside) step function. We have

$$g(t) \equiv r_2(f_1 + f_2) + \dot{f}_1 = D\delta(t - t_0)$$

$$x_1^{(p)}(t) = \tfrac{1}{\lambda} \int_0^t [e^{\lambda(t-\xi)} - 1] D\delta(\xi - t_0) d\xi \tag{4.3.1}$$

$$= \tfrac{D}{\lambda} [e^{\lambda(t-t_0)} - 1] u(t - t_0) \tag{4.3.2}$$

$$= -\tfrac{D}{r_1 + r_2} [e^{-(r_1+r_2)(t-t_0)} - 1] u(t - t_0) \tag{4.3.3}$$

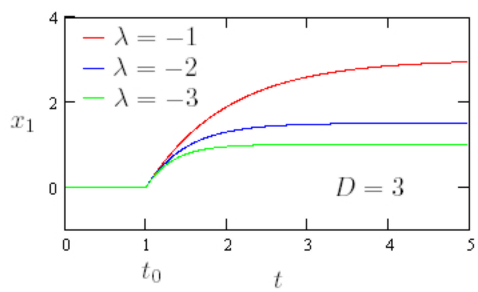

Figure 7: Example I. The particular solution ($\lambda = -1, -2, -3; D = 3$)

4.4 Example II

Consider another example

$$\begin{cases} f_1(t) = 0 \\ f_2(t) = D\sin(\omega t) \end{cases}$$

We have

$$g(t) = r_2 D \sin(\omega t)$$

$$x_1(t) = \tfrac{D}{\lambda}\int_0^t [e^{\lambda(t-\xi)} - 1] r_2 \sin(\omega\xi) d\xi \tag{4.4.1}$$

$$= \tfrac{r_2 D}{\lambda} e^{\lambda t} [I_1]_0^t - \tfrac{r_2 D}{\lambda}\int_0^t \sin(\omega\xi) d\xi \tag{4.4.2}$$

where

$$I_1 = \int e^{-\lambda\xi}\sin(\omega\xi)d\xi = -\frac{\lambda\sin(\omega\xi) + \omega\cos(\omega\xi)}{\lambda^2+\omega^2}e^{-\lambda\xi}$$

By substitution we find

$$x_1^{(p)}(t) = \tfrac{r_2 D}{\lambda(\lambda^2+\omega^2)}[\lambda\sin(\omega t) + \omega\cos(\omega t) - \omega e^{\lambda t}] - \tfrac{r_2 D}{\lambda\omega}[1-\cos(\omega t)] \tag{4.4.3}$$

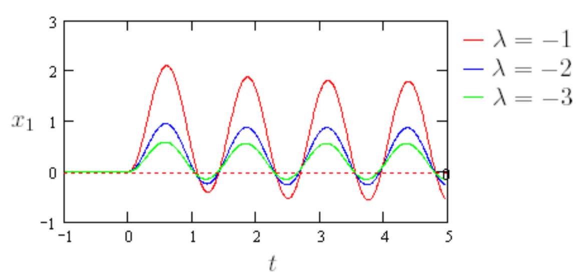

Figure 8: Example II. The particular solution $(\lambda = -1, -2, -3; D = 3)$

5 Table of atomic decay chains

Elemento	Z	A	tipo di disintegrazione	tempo di dimezzamento
La catena radioattiva naturale $4n$				
Th	90	232	alfa	$1{,}39 \cdot 10^{10}$ anni
Ra	88	228	beta	6,7 anni
Ac	89	228	beta	6,13 ore
Th	90	228	alfa	1,90 anni
Ra	88	224	alfa	3,64 ore
Em	86	220	alfa	54,5 secondi
Po	84	216	alfa, beta	0,16 secondi
Pb	82	212	beta	10,6 ore
At	85	216	alfa	$3 \cdot 10^{-4}$ secondi
Bi	83	212	alfa, beta	47 minuti
Po	84	212	alfa	$3{,}0 \cdot 10^{-7}$ secondi
Tl	81	208	beta	2,1 minuti
Pb	82	208	stabile	
La catena radioattiva naturale $4n+2$				
U	92	238	alfa	$4{,}50 \cdot 10^{9}$ anni
Th	90	234	beta	24,1 giorni
Pa	91	234	beta	1,18 minuti
Pa	91	234	beta	6,7 ore
U	92	234	alfa	$2{,}50 \cdot 10^{5}$ anni
Th	90	230	alfa	$8{,}0 \cdot 10^{4}$ anni
Ra	88	226	alfa	1.620 anni
Em	86	222	alfa	3,82 giorni
Po	84	218	alfa, beta	3,05 minuti
Pb	82	214	beta	26,8 minuti
At	85	218	alfa	2 secondi
Bi	83	214	alfa, beta	19,7 minuti
Po	84	214	alfa	$1{,}64 \cdot 10^{-4}$ secondi
Tl	81	210	beta	1,32 minuti
Pb	82	210	beta	22 anni
Bi	83	210	beta	5,0 giorni
Po	84	210	alfa	138,3 giorni
Tl	81	206	beta	4,2 minuti
Pb	82	206	stabile	
La catena radioattiva naturale $4n+3$				
U	92	235	alfa	$7{,}10 \cdot 10^{8}$ anni
Th	90	231	beta	24,6 ore
Pa	91	231	alfa	$3{,}43 \cdot 10^{4}$ anni
Ac	89	227	alfa, beta	22,0 anni
Th	90	227	alfa	18,6 giorni
Fr	87	223	beta	21 minuti
Ra	88	223	alfa	11,2 giorni
Em	86	219	alfa	3,92 secondi
Po	84	215	alfa, beta	$1{,}83 \cdot 10^{-3}$ secondi
Pb	82	211	beta	36,1 minuti
At	85	215	alfa	10^{-4} secondi
Bi	83	211	alfa, beta	2,16 minuti
Po	84	211	alfa	0,52 secondi
Tl	81	207	beta	4,79 minuti
Pb	82	207	stabile	

Figure 9: Table of atomic decay chains.