1.

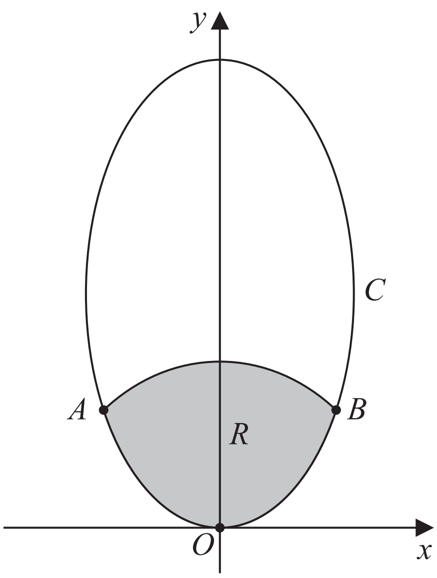

Figure 1

Figure 1 shows the curve C given by the parametric equations

$$x = \frac{5}{\sqrt{3}}\sin t \qquad y = 5(1 - \cos t) \qquad 0 \leqslant t \leqslant 2\pi$$

The circle with centre at the origin O and with radius $\frac{5\sqrt{2}}{2}$ meets the curve C at the points A and B as shown in Figure 1.

(a) Determine the value of t at the point B.

(3)

The region R, shown shaded in Figure 1, is bounded by the curve C and the circle.

(b) Determine the area of the region R.

(7)

2. **Figure 1**

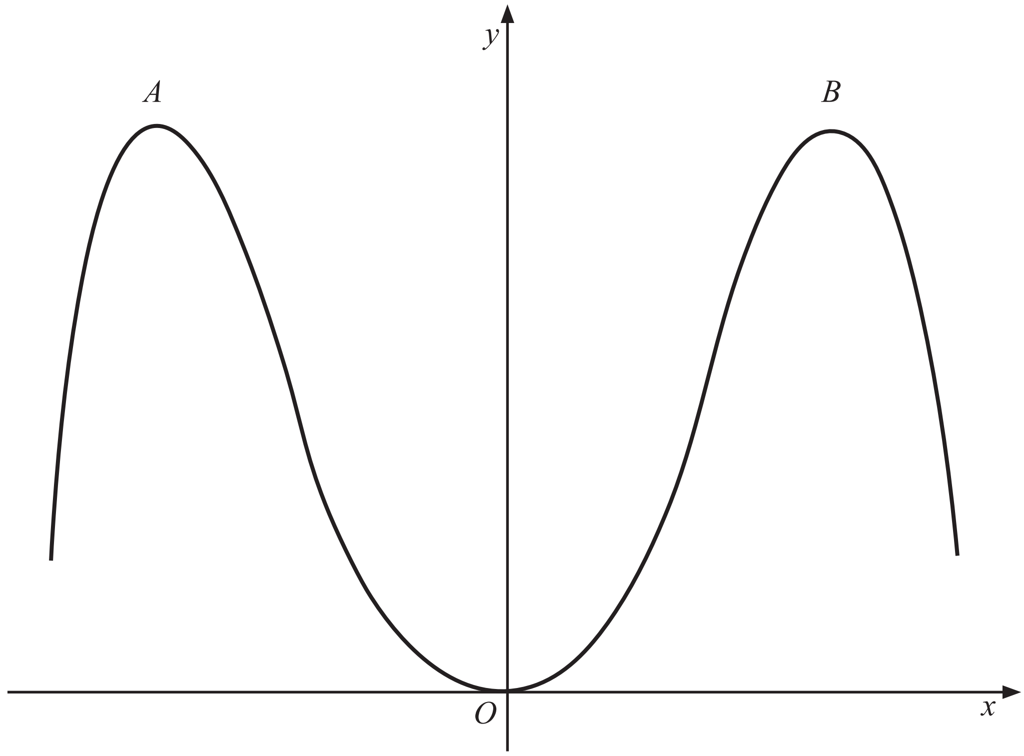

Figure 1 shows a sketch of the curve C with equation

$$y = \cos x \ln(\sec x), \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

The points A and B are maximum points on C.

(a) Find the coordinates of B in terms of e.

(5)

The finite region R lies between C and the line AB.

(b) Show that the area of R is

$$\frac{2}{e}\arccos\left(\frac{1}{e}\right) + 2\ln\left(e + \sqrt{(e^2 - 1)}\right) - \frac{4}{e}\sqrt{(e^2 - 1)}.$$

[arccos$x$ is an alternative notation for $\cos^{-1}x$]

(8)

(Total 13 marks)

3. The curve C has parametric equations

$$x = \cos^2 t$$

$$y = \cos t \sin t$$

where $0 \leqslant t < \pi$

(a) Show that C is a circle and find its centre and its radius.

(5)

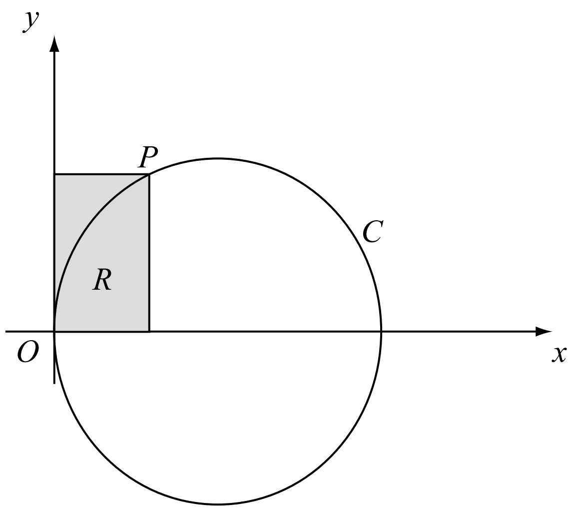

Figure 1

Figure 1 shows a sketch of C. The point P, with coordinates $\left(\cos^2\alpha\ ,\ \cos\alpha\sin\alpha\right)$, $0 < \alpha < \frac{\pi}{2}$, lies on C. The rectangle R has one side on the x-axis, one side on the y-axis and OP as a diagonal, where O is the origin.

(b) Show that the area of R is $\sin\alpha\cos^3\alpha$

(1)

(c) Find the maximum area of R, as α varies.

(7)

(Total 13 marks)

4. Given that $f(x) = e^{x^3-2x}$

(a) find $f'(x)$

(2)

The curves C_1 and C_2 are defined by the functions g and h respectively, where

$$g(x) = 8x^3e^{x^3-2x} \qquad x \in \mathbb{R}, x > 0$$

$$h(x) = (3x^5 + 4x)e^{x^3-2x} \qquad x \in \mathbb{R}, x > 0$$

(b) Find the x coordinates of the points of intersection of C_1 and C_2

(4)

Given that C_1 lies above C_2 between these points of intersection,

(c) find the area of the region bounded by the curves between these two points. Give your answer in the form $A + Be^C$ where A, B, and C are exact real numbers to be found.

(7)

(+S1)

5.

Figure 1

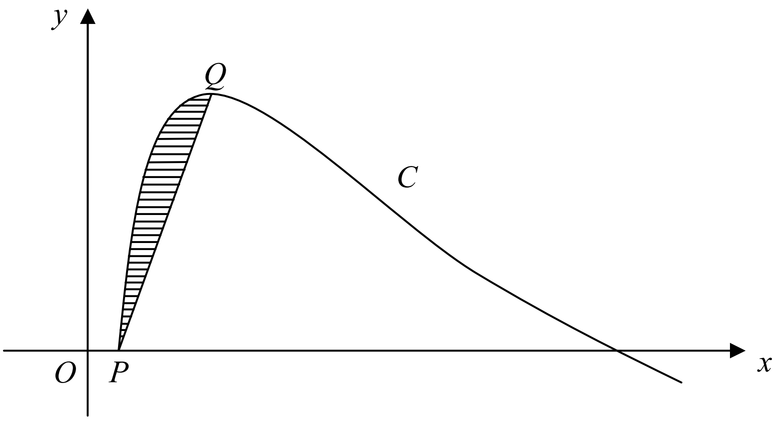

Figure 1 shows a sketch of part of the curve C with equation

$$y = \sin(\ln x), \quad x \geq 1.$$

The point Q, on C, is a maximum.

(a) Show that the point $P(1, 0)$ lies on C.

(1)

(b) Find the coordinates of the point Q.

(5)

(c) Find the area of the shaded region between C and the line PQ.

(9)

6.

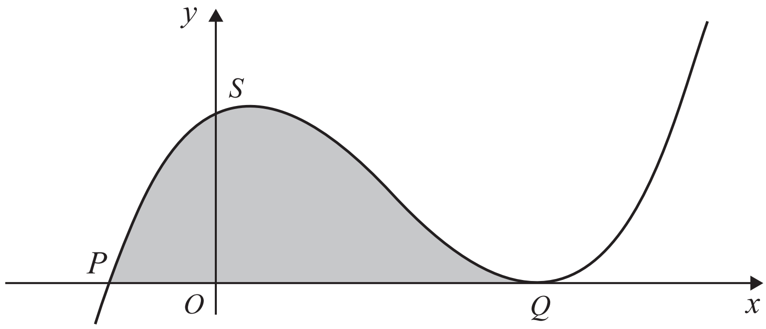

Figure 1

Figure 1 shows a sketch of the curve with equation $y = (x + a)(x - b)^2$, where a and b are positive constants. The curve cuts the x-axis at P and has a maximum point at S and a minimum point at Q.

(a) Write down the coordinates of P and Q in terms of a and b.

(2)

(b) Show that G, the area of the shaded region between the curve PSQ and the x-axis, is given by $G = \dfrac{(a+b)^4}{12}$.

(6)

The rectangle $PQRST$ has RST parallel to QP and both PT and QR are parallel to the y-axis.

(c) Show that $\dfrac{G}{\text{Area of } PQRST} = k$, where k is a constant independent of a and b and find the value of k.

(8)

(Total 16 marks)

7.

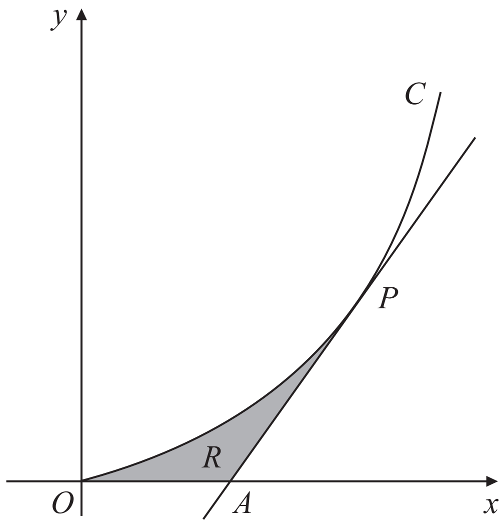

Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = 2\sin t, \qquad y = \ln(\sec t), \qquad 0 \leqslant t < \frac{\pi}{2}.$$

The tangent to C at the point P, where $t = \frac{\pi}{3}$, cuts the x-axis at A.

(a) Show that the x-coordinate of A is $\frac{\sqrt{3}}{3}(3 - \ln 2)$.

(6)

The shaded region R lies between C, the positive x-axis and the tangent AP as shown in Figure 2.

(b) Show that the area of R is $\sqrt{3}(1 + \ln 2) - 2\ln(2 + \sqrt{3}) - \frac{\sqrt{3}}{6}(\ln 2)^2$.

(11)

(Total 17 marks)

8. Figure 1 shows a sketch of part of the curve with equation $y = x\sin(\ln x)$, $x \geqslant 1$

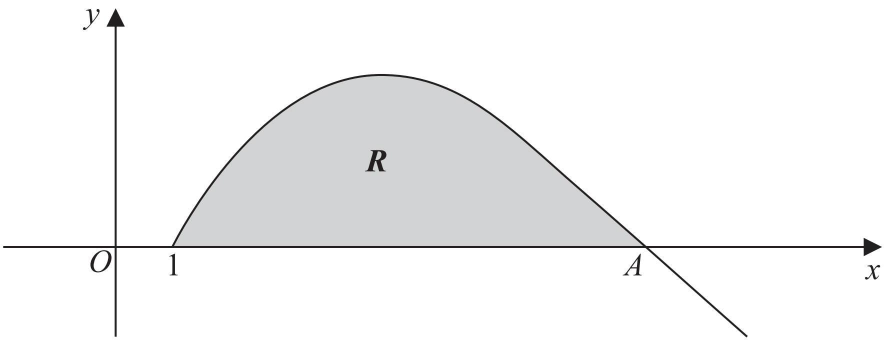

Figure 1

For $x > 1$, the curve first crosses the x-axis at the point A.

(a) Find the x coordinate of A.

(3)

(b) Differentiate $x\sin(\ln x)$ and $x\cos(\ln x)$ with respect to x and hence find

$$\int \sin(\ln x)\,\mathrm{d}x \quad \text{and} \quad \int \cos(\ln x)\,\mathrm{d}x$$

(7)

(c) (i) Find $\int x\sin(\ln x)\,\mathrm{d}x$.

(ii) Hence show that the area of the shaded region $\boldsymbol{R}$, bounded by the curve and the x-axis between the points (1, 0) and A, is

$$\frac{1}{5}\left(e^{2\pi} + 1\right)$$

(9)

9.

$$f(x) = [1 + \cos(x + \tfrac{\pi}{4})][1 + \sin(x + \tfrac{\pi}{4})], \qquad 0 \leqslant x \leqslant 2\pi$$

(a) Show that $f(x)$ may be written in the form

$$f(x) = (\tfrac{1}{\sqrt{2}} + \cos x)^2, \qquad 0 \leqslant x \leqslant 2\pi$$

(5)

(b) Find the range of the function $f(x)$.

(2)

The graph of $y = f(x)$ is shown in Figure 2.

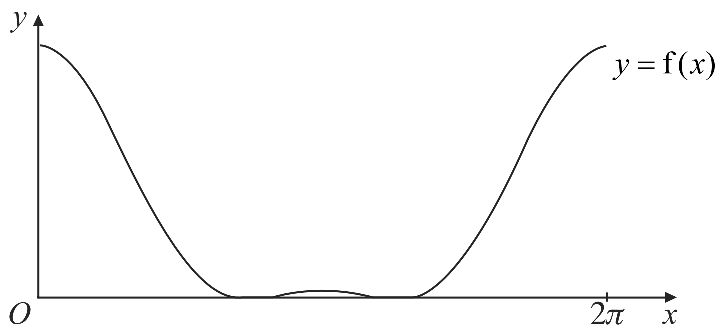

Figure 2

(c) Find the coordinates of all the maximum and minimum points on this curve.

(6)

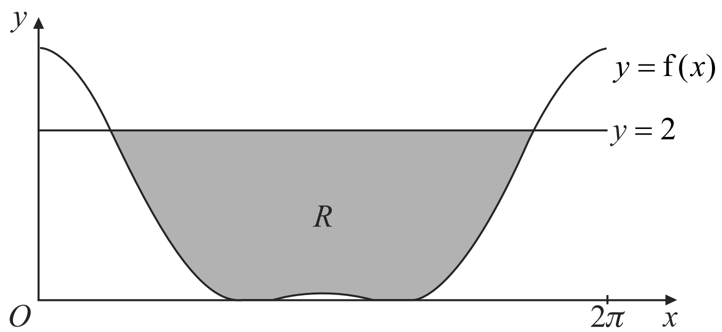

Figure 3

The region R, bounded by $y = 2$ and $y = f(x)$, is shown shaded in Figure 3.

(d) Find the area of R.

(8)

(Total 21 marks)

10.

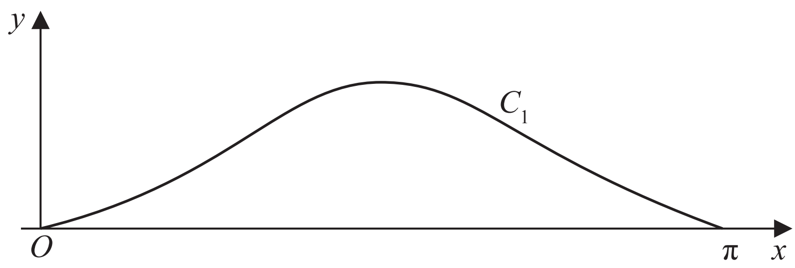

Figure 1

Figure 1 shows a sketch of the curve C_1 with equation

$$y = \cos(\cos x)\sin x \qquad \text{for} \qquad 0 \leqslant x \leqslant \pi$$

(a) Find $\dfrac{dy}{dx}$

(3)

(b) Hence verify that the turning point is at $x = \dfrac{\pi}{2}$ and find the y coordinate of this point.

(2)

(c) Find the area of the region bounded by C_1 and the positive x-axis between $x = 0$ and $x = \pi$

(4)

Figure 2 shows a sketch of the curve C_1 and the curve C_2 with equation

$$y = \sin(\cos x)\sin x \qquad \text{for} \qquad 0 \leqslant x \leqslant \pi$$

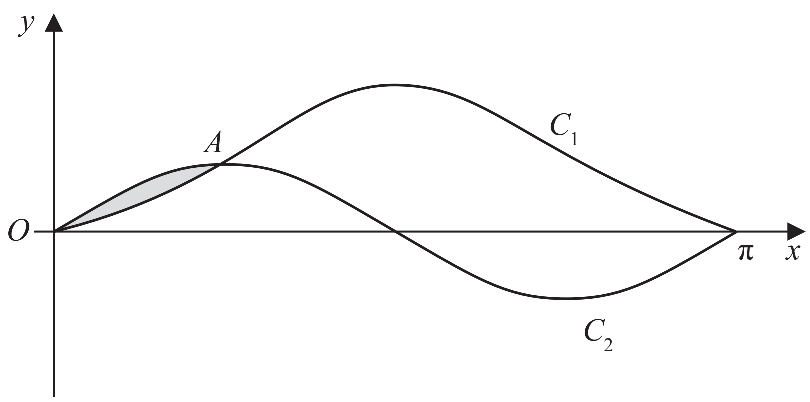

Figure 2

The curves C_1 and C_2 intersect at the origin and the point $A(a, b)$, where $a < \pi$

(d) Find a and b, giving b in a form not involving trigonometric functions.

(5)

(e) Find the area of the shaded region between C_1 and C_2

(8)

(Total 22 marks)

11.

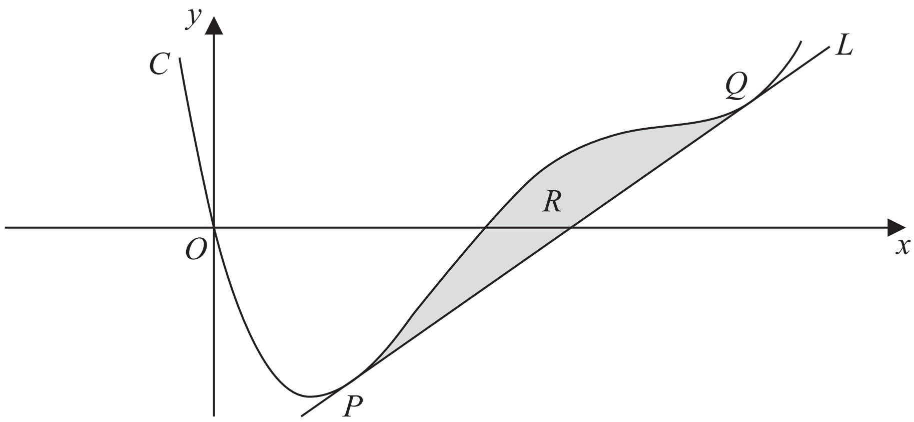

Figure 3

Figure 3 shows part of the curve C with equation $y = x^4 - 10x^3 + 33x^2 - 34x$ and the line L with equation $y = mx + c$.

The line L touches C at the points P and Q with x coordinates p and q respectively.

(a) Explain why

$$x^4 - 10x^3 + 33x^2 - (34 + m)x - c = (x - p)^2(x - q)^2$$

(2)

The finite region R, shown shaded in Figure 3, is bounded by C and L.

(b) Use integration by parts to show that the area of R is $\dfrac{(q-p)^5}{30}$

(6)

(c) Show that

$$(x - p)^2(x - q)^2 = x^4 - 2(p + q)x^3 + Sx^2 - Tx + U$$

where S, T and U are expressions to be found in terms of p and q.

(5)

(d) Using part (a) and part (c) find the value of p, the value of q and the equation of L.

(8)

(Total 21 marks)

12.

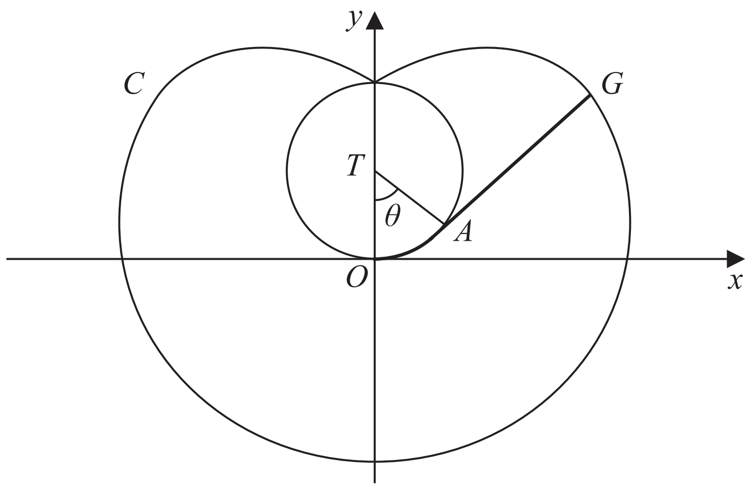

Figure 2

A circular tower stands in a large horizontal field of grass. A goat is attached to one end of a string and the other end of the string is attached to the fixed point O at the base of the tower. Taking the point O as the origin $(0, 0)$, the centre of the base of the tower is at the point $T(0, 1)$. The radius of the base of the tower is 1. The string has length π and you may ignore the size of the goat. The curve C represents the edge of the region that the goat can reach as shown in Figure 2.

(a) Write down the equation of C for $y < 0$.

(1)

When the goat is at the point $G(x, y)$, with $x > 0$ and $y > 0$, as shown in Figure 2, the string lies along OAG where OA is an arc of the circle with angle $OTA = \theta$ radians and AG is a tangent to the circle at A.

(b) With the aid of a suitable diagram show that

$$x = \sin\theta + (\pi - \theta)\cos\theta$$

$$y = 1 - \cos\theta + (\pi - \theta)\sin\theta$$

(5)

(c) By considering $\int y \frac{\mathrm{d}x}{\mathrm{d}\theta}\,\mathrm{d}\theta$, show that the area between C, the positive x-axis and the positive y-axis can be expressed in the form

$$\int_0^\pi u \sin u \,\mathrm{d}u + \int_0^\pi u^2 \sin^2 u \,\mathrm{d}u + \int_0^\pi u \sin u \cos u \,\mathrm{d}u$$

(5)

(d) Show that $\int_0^\pi u^2 \sin^2 u \,\mathrm{d}u = \frac{\pi^3}{6} + \int_0^\pi u \sin u \cos u \,\mathrm{d}u$

(4)

(e) Hence find the area of grass that can be reached by the goat.

(8)

(Total 23 marks)