1. $$f(x) = x^{(x^2)} \qquad x > 0$$

Use logarithms to find the x coordinate of the stationary point of the curve with equation $y = f(x)$.

(5)

2. The curve C has equation $y = x^{\sin x}$, $\quad x > 0$.

(a) Find the equation of the tangent to C at the point where $x = \dfrac{\pi}{2}$.

(6)

(b) Prove that this tangent touches C at infinitely many points.

(3)

(Total 9 marks)

3.

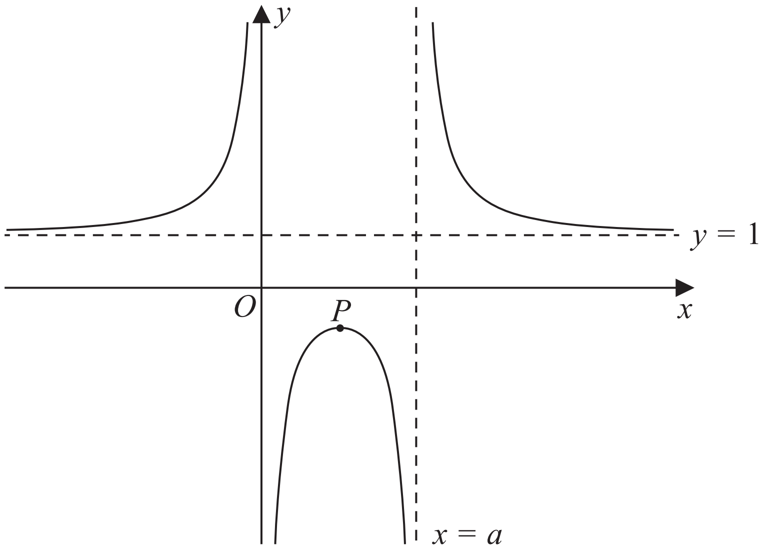

Figure 1

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 1 + \frac{4}{x(x-3)}$$

The curve has a turning point at the point P, and the lines with equations $y = 1$, $x = 0$ and $x = a$ are asymptotes to the curve.

(a) Write down the value of a.

(1)

(b) Find the coordinates of P, justifying your answer.

(4)

(c) Sketch the curve with equation $y = \left|\mathrm{f}\left(x + \frac{3}{2}\right)\right| - 1$

On your sketch, you should show the coordinates of any points of intersection with the coordinate axes, the coordinates of any turning points and the equations of any asymptotes.

(7)

4. The curve C has equation

$$x^2 + y^2 + fxy = g^2,$$

where f and g are constants and $g \neq 0$.

(a) Find an expression in terms of α, β and f for the gradient of C at the point (α, β).

(4)

Given that $f < 2$ and $f \neq -2$ and that the gradient of C at the point (α, β) is 1,

(b) show that $\alpha = -\beta = \dfrac{\pm g}{\sqrt{(2-f)}}$.

(4)

Given that $f = -2$,

(c) sketch C.

(3)

(Total 11 marks)

5.

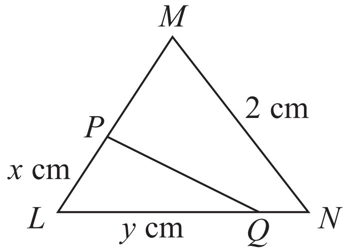

Figure 1

Figure 1 shows the equilateral triangle LMN of side 2 cm. The point P lies on LM such that $LP = x$ cm and the point Q lies on LN such that $LQ = y$ cm. The points P and Q are chosen so that the area of triangle LPQ is half the area of triangle LMN.

(a) Show that $xy = 2$

(2)

(b) Find the shortest possible length of PQ, justifying your answer.

(5)

Mathematicians know that for any closed curve or polygon enclosing a fixed area, the ratio $\dfrac{\text{area enclosed}}{\text{perimeter}}$ is a maximum when the closed curve is a circle.

By considering 6 copies of triangle LMN suitably arranged,

(c) find the length of the shortest line or curve that can be drawn from a point on LM to a point on LN to divide the area of triangle LMN in half. Justify your answer.

(6)

(Total 13 marks)

6. (a) Given that $x^4 + y^4 = 1$, prove that $x^2 + y^2$ is a maximum when $x = \pm y$, and find the maximum and minimum values of $x^2 + y^2$.

(7)

(b) On the same diagram, sketch the curves C_1 and C_2 with equations $x^4 + y^4 = 1$ and $x^2 + y^2 = 1$ respectively.

(2)

(c) Write down the equation of the circle C_3, centre the origin, which touches the curve C_1 at the points where $x = \pm y$.

(1)

(Total 10 marks)

7. (a) The function $\mathrm{f}(x)$ has $\mathrm{f}'(x) = \dfrac{\mathrm{u}(x)}{\mathrm{v}(x)}$. Given that $\mathrm{f}'(k) = 0$,

show that $\mathrm{f}''(k) = \dfrac{\mathrm{u}'(k)}{\mathrm{v}(k)}$.

(3)

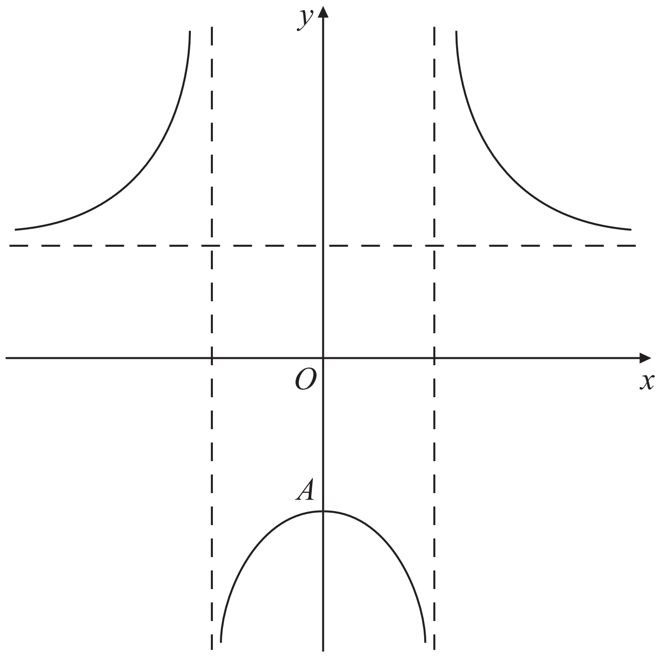

Figure 1

(b) The curve C with equation

$$y = \frac{2x^2 + 3}{x^2 - 1}$$

crosses the y-axis at the point A. Figure 1 shows a sketch of C together with its 3 asymptotes.

(i) Find the coordinates of the point A.

(1)

(ii) Find the equations of the asymptotes of C.

(2)

The point $P(a, b)$, $a > 0$ and $b > 0$, lies on C. The point Q also lies on C with PQ parallel to the x-axis and $AP = AQ$.

(iii) Show that the area of triangle PAQ is given by $\dfrac{5a^3}{a^2 - 1}$.

(2)

(iv) Find, as a varies, the minimum area of triangle PAQ, giving your answer in its simplest form.

(6)

(Total 14 marks)

8.

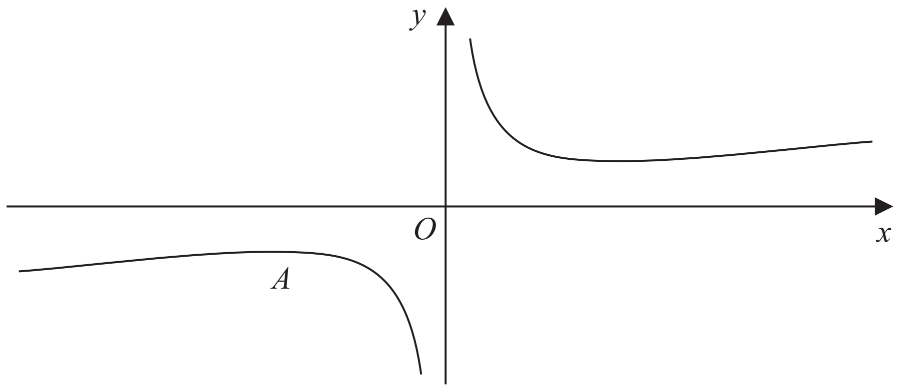

Figure 1

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{x^2 + 16}{3x} \qquad x \neq 0$$

The curve has a maximum at the point A with coordinates (a, b).

(a) Find the value of a and the value of b.

(4)

The function g is defined as

$$\mathrm{g} : x \mapsto \frac{x^2 + 16}{3x} \qquad a \leqslant x < 0$$

where a is the value found in part (a).

(b) Write down the range of g.

(1)

(c) On the same axes sketch $y = \mathrm{g}(x)$ and $y = \mathrm{g}^{-1}(x)$.

(3)

(d) Find an expression for $\mathrm{g}^{-1}(x)$ and state the domain of g^{-1}

(5)

(e) Solve the equation $\mathrm{g}(x) = \mathrm{g}^{-1}(x)$.

(3)

(Total 16 marks)

9.

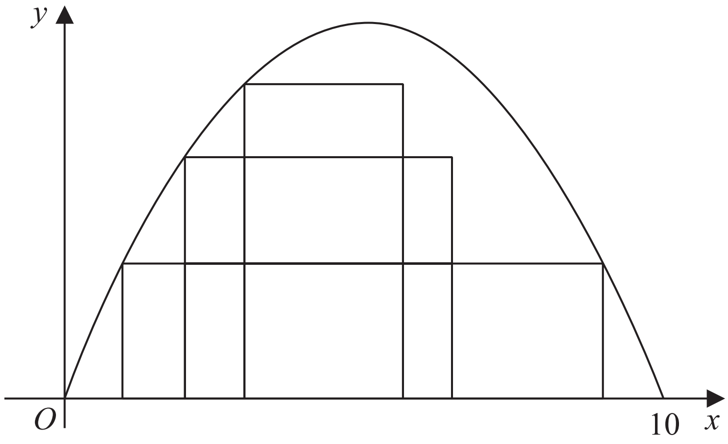

Figure 2

Figure 2 shows a sketch of the parabola with equation $y = \frac{1}{2}x(10 - x)$, $0 \leqslant x \leqslant 10$

This question concerns rectangles that lie under the parabola in the first quadrant. The bottom edge of each rectangle lies along the x-axis and the top left vertex lies on the parabola. Some examples are shown in Figure 2.

Let the x coordinate of the top left vertex be a.

(a) Explain why the width, w, of such a rectangle must satisfy $w \leqslant 10 - 2a$

(2)

(b) Find the value of a that gives the maximum area for such a rectangle.

(5)

Given that the rectangle must be a square,

(c) find the value of a that gives the maximum area for such a square.

(3)

Given that the area of the rectangles is fixed as 36

(d) find the range of possible values for a.

(6)

(+S1)

10.

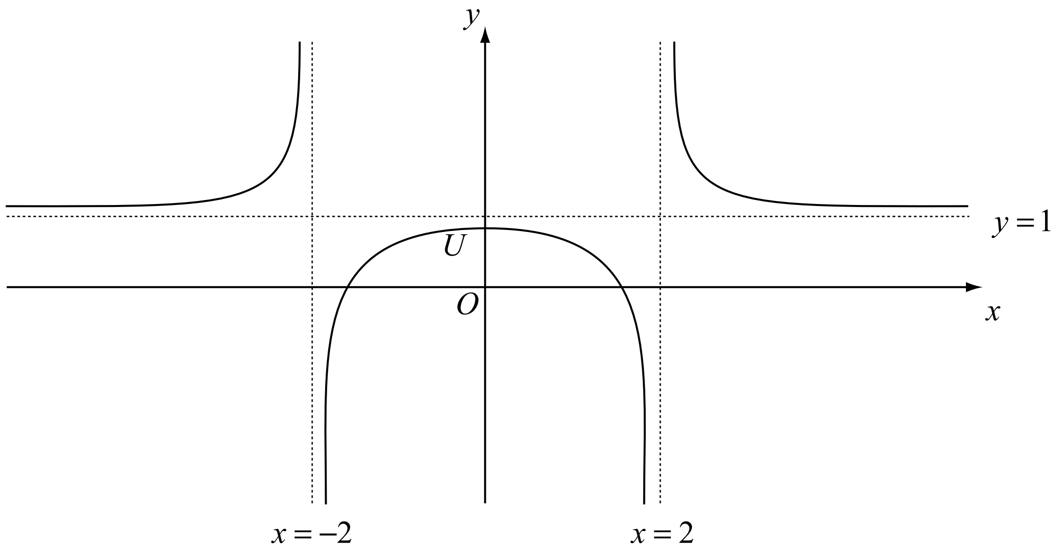

Figure 2

Figure 2 shows a sketch of the curve C with equation $y = \dfrac{x^2 - 2}{x^2 - 4}$ and $x \neq \pm 2$.

The curve cuts the y-axis at U.

(a) Write down the coordinates of the point U.

(1)

The point P with x-coordinate a $(a \neq 0)$ lies on C.

(b) Show that the normal to C at P cuts the y-axis at the point

$$\left(0 \;,\; \left[\frac{a^2 - 2}{a^2 - 4} - \frac{\left(a^2 - 4\right)^2}{4}\right]\right)$$

(6)

The circle E, with centre on the y-axis, touches all three branches of C.

(c) (i) Show that

$$\left[\frac{a^2}{2(a^2 - 4)} - \frac{(a^2 - 4)^2}{4}\right]^2 = a^2 + \frac{(a^2 - 4)^4}{16}$$

(ii) Hence, show that

$$\left(a^2 - 4\right)^2 = 1$$

(iii) Find the centre and radius of E.

(10)

(Total 17 marks)

11.

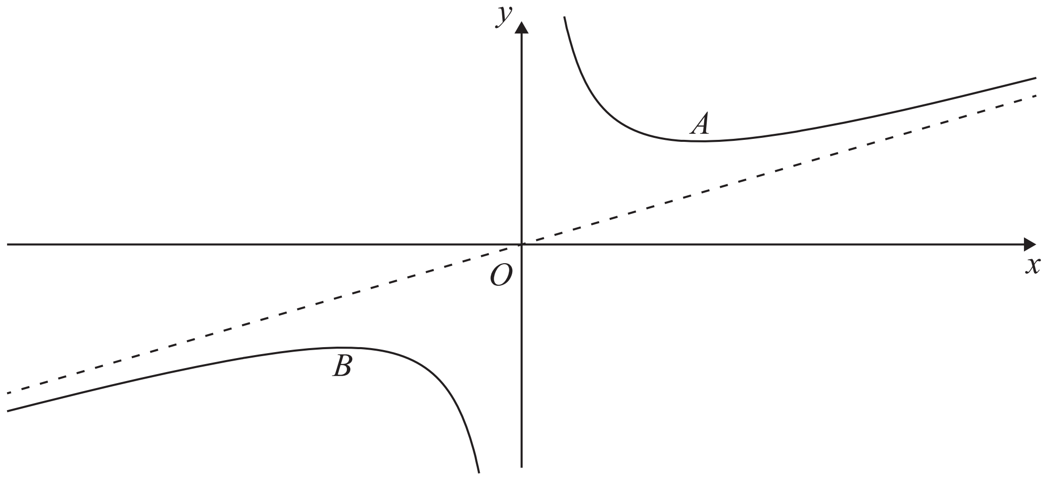

Figure 1

Figure 1 shows a sketch of the curve C_1 with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{x}{3} + \frac{12}{x} \qquad x \neq 0$$

The lines $x = 0$ and $y = \frac{x}{3}$ are asymptotes to C_1. The point A on C_1 is a minimum and the point B on C_1 is a maximum.

(a) Find the coordinates of A and B.

(4)

There is a normal to C_1, which does not intersect C_1 a second time, that has equation $x = k$, where $k > 0$.

(b) Write down the value of k.

(1)

The point P (α, β), $\alpha > 0$ and $\alpha \neq k$, lies on C_1. The normal to C_1 at P does not intersect C_1 a second time.

(c) Find the value of α, leaving your answer in simplified surd form.

(5)

(d) Find the equation of this normal.

(3)

The curve C_2 has equation $y = |\mathrm{f}(x)|$

(e) Sketch C_2 stating the coordinates of any turning points and the equations of any asymptotes.

(4)

The line with equation $y = mx + 1$ does not touch or intersect C_2.

(f) Find the set of possible values for m.

(5)

(Total 22 marks)

12.

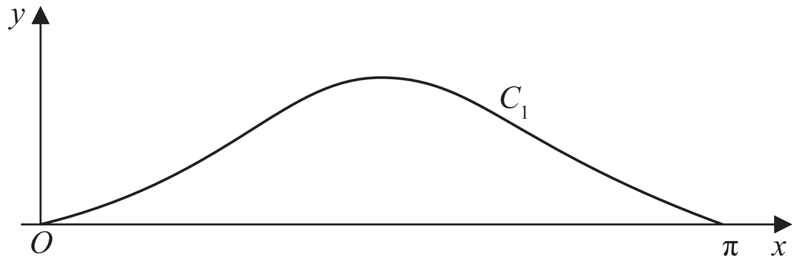

Figure 1

Figure 1 shows a sketch of the curve C_1 with equation

$$y = \cos(\cos x)\sin x \qquad \text{for} \qquad 0 \leqslant x \leqslant \pi$$

(a) Find $\dfrac{dy}{dx}$

(3)

(b) Hence verify that the turning point is at $x = \dfrac{\pi}{2}$ and find the y coordinate of this point.

(2)

(c) Find the area of the region bounded by C_1 and the positive x-axis between $x = 0$ and $x = \pi$

(4)

Figure 2 shows a sketch of the curve C_1 and the curve C_2 with equation

$$y = \sin(\cos x)\sin x \qquad \text{for} \qquad 0 \leqslant x \leqslant \pi$$

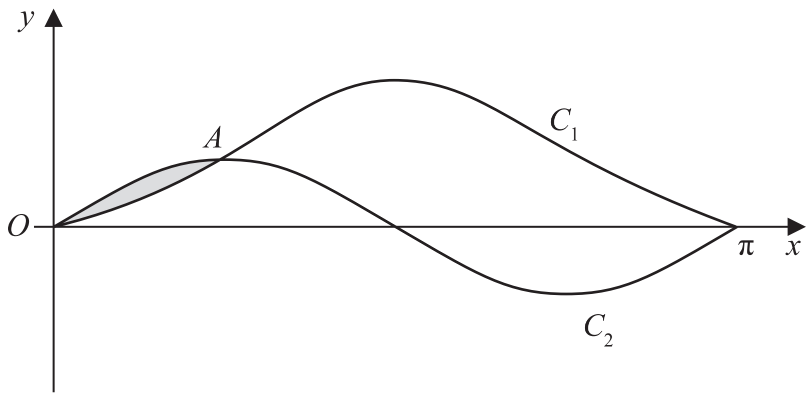

Figure 2

The curves C_1 and C_2 intersect at the origin and the point $A(a, b)$, where $a < \pi$

(d) Find a and b, giving b in a form not involving trigonometric functions.

(5)

(e) Find the area of the shaded region between C_1 and C_2

(8)

(Total 22 marks)

13. *In this question you may assume the following formulae for the volume and curved surface area of a cone of base radius r and height h and of a sphere of radius r.*

***Cone**: volume* $V = \frac{1}{3}\pi r^2 h$ *and curved surface area* $S = \pi r\sqrt{h^2 + r^2}$

***Sphere**: volume* $V = \frac{4^{\sim}}{3}r^3$ *and curved surface area* $S = 4\pi r^2$

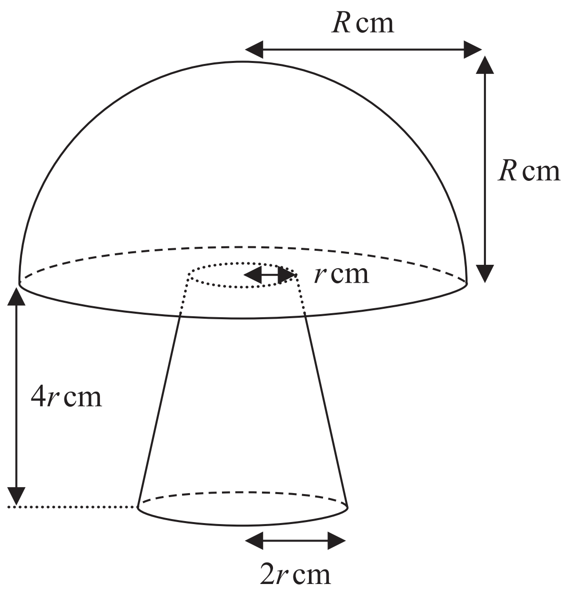

Figure 3

Figure 3 shows the design for a garden ornament.

The ornament is made of a hemisphere on top of a truncated cone.

The truncated cone has base radius $2r$ cm, top radius r cm and height $4r$ cm.

The hemisphere has radius R cm.

Given that the volume of the ornament is $2100\pi\,\text{cm}^3$

(a) show that

$$R^3 = 3150 - 14r^3$$

(5)

(b) Find an expression involving $\frac{dR}{dr}$ in terms of r and/or R.

(2)

The base of the truncated cone of the ornament is fixed to the ground.

(c) Show that the visible surface area of the ornament, A cm^2, is given by

$$A = \left(3\sqrt{17} - 1\right)\pi r^2 + 3\pi R^2$$

(5)

Question 13 continued

(d) Hence show that

$$\frac{\mathrm{d}A}{\mathrm{d}r} = \gamma \pi r - \frac{\sim\circ\ r^2}{R}$$

where γ and δ are real numbers to be determined.

(2)

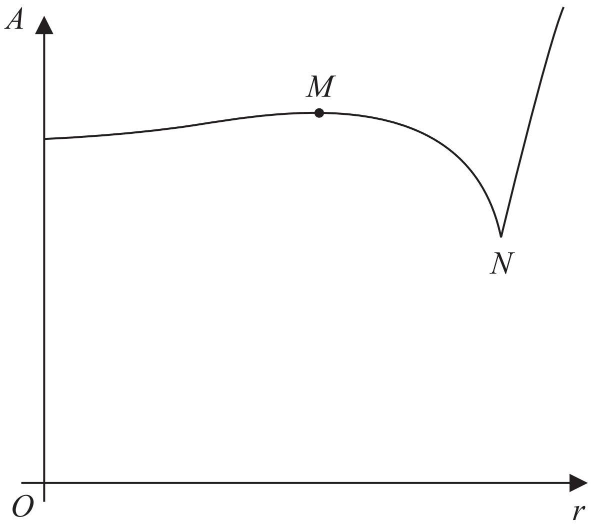

Figure 4

Figure 4 shows a sketch of A against r, for $r \geqslant 0$

There is a local minimum at $r = 0$ and a local maximum at the point M. The overall minimum point is at the point N, where the gradient of the curve is undefined.

(e) (i) Determine the r coordinate of the point N.

(ii) Explain why, for the ornament, r must be less than this value.

(2)

(f) Show that the r coordinate of the point M is

$$\sqrt[3]{\frac{p\left(3\sqrt{17}-1\right)^3}{3q^2+\left(3\sqrt{17}-1\right)^3}}$$

where p and q are integers to be determined.

(5)
(+S2)

14.

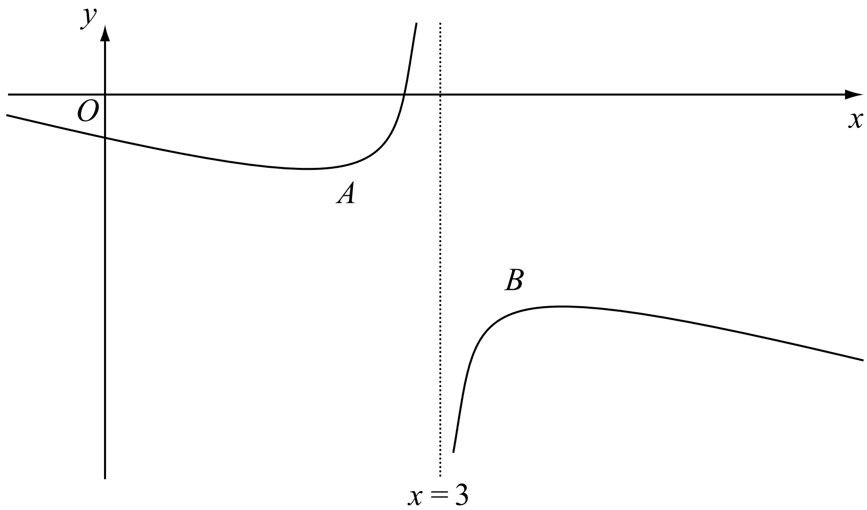

Figure 4

(a) Figure 4 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{x^2 - 5}{3 - x}, \quad x \in \mathbb{R},\ x \neq 3$$

The curve has a minimum at the point A, with x-coordinate α, and a maximum at the point B, with x-coordinate β.

Find the value of α, the value of β and the y-coordinates of the points A and B.

(5)

(b) The functions g and h are defined as follows

$$\mathrm{g} : x \rightarrow x + p \qquad x \in \mathbb{R}$$

$$\mathrm{h} : x \rightarrow |x| \qquad x \in \mathbb{R}$$

where p is a constant.

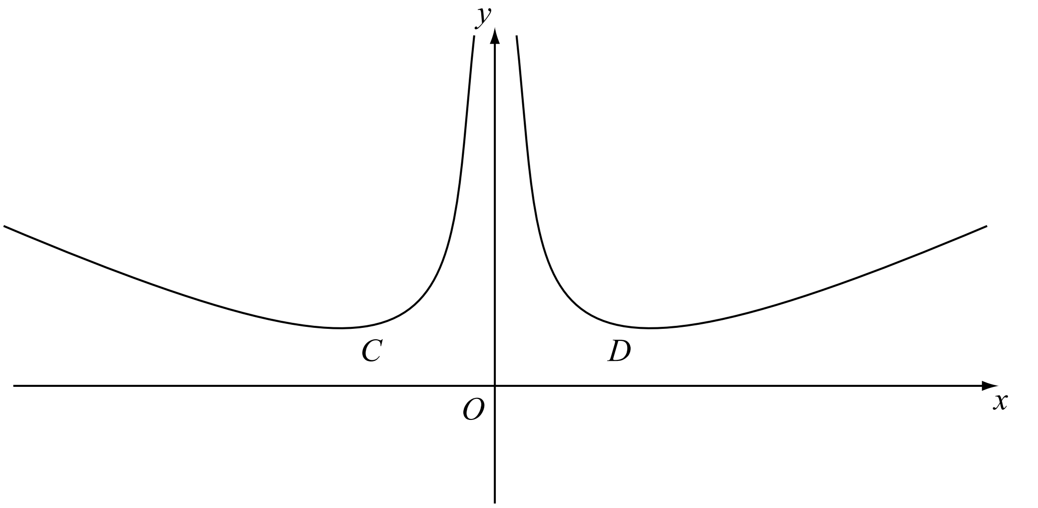

Figure 5

Figure 5 shows a sketch of the curve with equation $y = \mathrm{h}(\mathrm{fg}(x) + q),\ x \in \mathbb{R},\ x \neq 0,$ where q is a constant. The curve is symmetric about the y-axis and has minimum points at C and D.

(i) Find the value of p and the value of q.

(ii) Write down the coordinates of D.

(5)

(c) The function m is given by

$$\mathrm{m}(x) = \frac{x^2 - 5}{3 - x}, \quad x \in \mathbb{R}, x \leqslant \alpha$$

where α is the x-coordinate of A as found in part (a).

(i) Find m^{-1}

(ii) Write down the domain of m^{-1}

(iii) Find the value of t such that $\mathrm{m}(t) = \mathrm{m}^{-1}(t)$

(10)

(Total 20 marks)

15. [$\arccos x$ and $\arctan x$ are alternative notation for $\cos^{-1} x$ and $\tan^{-1} x$ respectively]

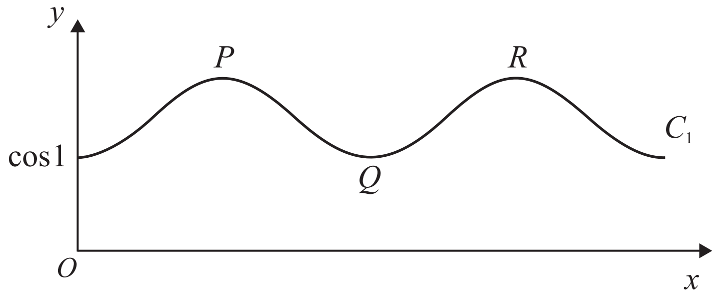

Figure 2

Figure 2 shows a sketch of the curve C_1 with equation $y = \cos(\cos x)$, $0 \leqslant x < 2\pi$.

The curve has turning points at (0, cos1), P, Q and R as shown in Figure 2.

(a) Find the coordinates of the points P, Q and R.

(4)

The curve C_2 has equation $y = \sin(\cos x)$, $0 \leqslant x < 2\pi$. The curves C_1 and C_2 intersect at the points S and T.

(b) Copy Figure 2 and on this diagram sketch C_2 stating the coordinates of the minimum point on C_2 and the points where C_2 meets or crosses the coordinate axes.

(5)

The coordinates of S are (α, d) where 0 < ĩ < ƀ .

(c) Show that $\alpha = \arccos\left(\frac{\pi}{4}\right)$.

(2)

(d) Find the value of d in surd form and write down the coordinates of T.

(3)

The tangent to C_1 at the point S has gradient $\tan\beta$.

(e) Show that $\beta = \arctan\sqrt{\left(\frac{16 - \pi^2}{32}\right)}$.

(5)

(f) Find, in terms of , the obtuse angle between the tangent to C_1 at S and the tangent to C_2 at S.

(5)

(Total 24 marks)