1. (i) Given that p and q are integers such that

$$pq \text{ is even}$$

use algebra to prove by contradiction that at least one of p or q is even.

(3)

(ii) Given that x and y are integers such that

- $x < 0$
- $(x + y)^2 < 9x^2 + y^2$

show that $y > 4x$

(2)

2. Prove, using algebra, that

$$(n + 1)^3 - n^3$$

is odd for all $n \in \mathbb{N}$

(4)

3

Prove, using algebra, that

$$n(n^2 + 5)$$

is even for all $n \in \mathbb{N}$.

(4)

4. A student attempts to answer the following question:

> Given that x is an obtuse angle, use algebra to prove by contradiction that
>
> $$\sin x - \cos x \geqslant 1$$

The student starts the proof with:

> Assume that $\sin x - \cos x < 1$ when x is an obtuse angle
>
> $$\Rightarrow (\sin x - \cos x)^2 < 1$$
>
> $$\Rightarrow \ldots$$

The start of the student's proof is reprinted below.

Complete the proof.

(3)

Assume that $\sin x - \cos x < 1$ when x is an obtuse angle

$$\Rightarrow (\sin x - \cos x)^2 < 1$$

5. (i) Use proof by exhaustion to show that for $n \in \mathbb{N}, n \leqslant 4$

$$(n+1)^3 > 3^n$$

(2)

(ii) Given that $m^3 + 5$ is odd, use proof by contradiction to show, using algebra, that m is even.

(4)

6. (i) Prove that for all $n \in \mathbb{N}$, $n^2 + 2$ is not divisible by 4

(4)

(ii) "Given $x \in \mathbb{R}$, the value of $|3x - 28|$ is greater than or equal to the value of $(x - 9)$."
State, giving a reason, if the above statement is always true, sometimes true or never true.

(2)

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A geometric series has common ratio r and first term a.

Given $r \neq 1$ and $a \neq 0$

(a) prove that

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

(4)

Given also that S_{10} is four times S_5

(b) find the exact value of r.

(4)

8. Use algebra to prove that the square of any natural number is **either** a multiple of 3 **or** one more than a multiple of 3

(4)

9. Prove by contradiction that there are no positive integers p and q such that

$$4p^2 - q^2 = 25$$

(4)

10. A student is attempting to prove that there are infinitely many prime numbers.

The student's attempt to prove this is in the box below.

Assume there are only finitely many prime numbers, then there is a biggest prime number, p. Let $n = 2p + 1$. Then n is bigger than p and since $2p + 1$ is not divisible by p, n is a prime number. Hence n is a prime number bigger than p, contradicting the initial assumption. So we conclude there are infinitely many prime numbers.

(a) Use $p = 7$ to show that the following claim made in the student's proof is **not** true:

since $2p + 1$ is not divisible by p, n is a prime number.

(1)

The student changes their proof to use $n = 6p + 1$ instead of $n = 2p + 1$

(b) Show, by counter example, that this does not correct the student's proof.

(2)

(c) Write out a correct proof by contradiction to show that there are infinitely many prime numbers.

(5)
(+S1)