1. The first and second terms of an arithmetic series are 200 and 197.5 respectively.

The sum to n terms of the series is S_n.

Find the largest positive value of S_n.

(Total 5 marks)

2. The sum of the first p terms of an arithmetic series is q and the sum of the first q terms of the same arithmetic series is p, where p and q are positive integers and $p \neq q$.

Giving simplified answers in terms of p and q, find

(a) the common difference of the terms in this series,

(5)

(b) the first term of the series,

(3)

(c) the sum of the first $(p+q)$ terms of the series.

(3)

(Total 11 marks)

3. The angle K, $0 < K < \frac{\pi}{2}$, satisfies

$$\tan\theta \tan 2\theta = \sum_{r=0}^{\infty} 2\cos^r 2\theta$$

(a) Show that $\tan\theta = 3^p$, where p is a rational number to be found.

(8)

(b) Hence show that $\frac{\pi}{6} < \theta < \frac{\pi}{4}$

(2)

(Total 10 marks)

4. A sequence $\{u_n\}$ is given by

$$\begin{aligned} u_1 &= k \\ u_{2n} &= u_{2n-1} \times p \qquad n \geqslant 1 \\ u_{2n+1} &= u_{2n} \times q \qquad n \geqslant 1 \end{aligned}$$

where k, p and q are positive constants with $pq \neq 1$

(a) Write down the first 6 terms of this sequence.

(3)

(b) Show that $\displaystyle\sum_{r=1}^{2n} u_r = \frac{k(1+p)\left(1-(pq)^n\right)}{1-pq}$

(6)

In part (c) $[x]$ means the integer part of x, so for example $[2.73] = 2$, $[4] = 4$ and $[0] = 0$

(c) Find $\displaystyle\sum_{r=1}^{\infty} 6 \times \left(\frac{4}{3}\right)^{\left[\frac{r}{2}\right]} \times \left(\frac{3}{5}\right)^{\left[\frac{r-1}{2}\right]}$

(4)

(Total 13 marks)

5. Given that

$$(1+x)^n = 1 + \sum_{r=1}^{\infty} \frac{n(n-1)\ldots(n-r+1)}{1\times 2\times\ldots\times r}x^r \qquad (|x|<1,\ x\in\mathbb{R},\ n\in\mathbb{R})$$

(a) show that

$$(1-x)^{-\frac{1}{2}} = \sum_{r=0}^{\infty}\binom{2r}{r}\left(\frac{x}{4}\right)^r$$

(5)

(b) show that $(9-4x^2)^{-\frac{1}{2}}$ can be written in the form $\displaystyle\sum_{r=0}^{\infty}\binom{2r}{r}\frac{x^{2r}}{3^q}$ and give q in terms of r.

(3)

(c) Find $\displaystyle\sum_{r=1}^{\infty}\binom{2r}{r}\times\frac{2r}{9}\times\left(\frac{x}{3}\right)^{2r-1}$

(3)

(d) Hence find the exact value of

$$\sum_{r=1}^{\infty}\binom{2r}{r}\times\frac{2r\sqrt{5}}{9}\times\frac{1}{5^r}$$

giving your answer as a rational number.

(2)

(Total 13 marks)

6. A sequence of positive integers $a_1, a_2, a_3, \ldots$ has rth term given by

$$a_r = 2^r - 1$$

(a) Write down the first 6 terms of this sequence.

(1)

(b) Verify that $a_{r+1} = 2a_r + 1$

(1)

(c) Find $\sum_{r=1}^{n} a_r$

(3)

(d) Show that $\dfrac{1}{a_{r+1}} < \dfrac{1}{2} \times \dfrac{1}{a_r}$

(1)

(e) Hence show that $1 + \dfrac{1}{3} + \dfrac{1}{7} + \dfrac{1}{15} + \dfrac{1}{31} + \ldots < 1 + \dfrac{1}{3} + \left(\dfrac{1}{7} + \dfrac{\frac{1}{2}}{7} + \dfrac{\frac{1}{4}}{7} + \ldots \right)$

(2)

(f) Show that $\dfrac{31}{21} < \sum_{r=1}^{\infty} \dfrac{1}{a_r} < \dfrac{34}{21}$

(5)

(Total 13 marks)

7. (a) Show that

$$\sum_{r=0}^{n} x^{-r} = \frac{x}{x-1} - \frac{x^{-n}}{x-1} \qquad \text{where } x \neq 0 \text{ and } x \neq 1$$

(2)

(b) Hence find an expression in terms of x and n for $\sum_{r=0}^{n} r x^{-(r+1)}$ for $x \neq 0$ and $x \neq 1$

Simplify your answer.

(4)

(c) Find $\sum_{r=0}^{n} \left(\frac{3+5r}{2^r} \right)$

Give your answer in the form $a - \frac{b+cn}{2^n}$, where a, b and c are integers.

(7)

(Total 13 marks)

8. [In this question the values of a, x, and n are such that a and x are positive real numbers, with $a > 1$, $x \neq a$, $x \neq 1$ and n is an integer with $n > 1$]

Sam was confused about the rules of logarithms and thought that

$$\log_a x^n = (\log_a x)^n \qquad (1)$$

(a) Given that x satisfies statement (1) find x in terms of a and n.

(3)

Sam also thought that

$$\log_a x + \log_a x^2 + \ldots + \log_a x^n = \log_a x + (\log_a x)^2 + \ldots + (\log_a x)^n \qquad (2)$$

(b) For $n = 3$, x_1 and x_2 $(x_1 > x_2)$ are the two values of x that satisfy statement (2).

(i) Find, in terms of a, an expression for x_1 and an expression for x_2.

(ii) Find the exact value of $\log_a\left(\frac{x_1}{x_2}\right)$.

(5)

(c) Show that if $\log_a x$ satisfies statement (2) then

$$2(\log_a x)^n - n(n+1)\log_a x + (n^2 + n - 2) = 0$$

(6)

(Total 14 marks)

9. (a) The sides of the triangle ABC have lengths $BC = a$, $AC = b$ and $AB = c$, where $a < b < c$. The sizes of the angles A, B and C form an arithmetic sequence.

(i) Show that the area of triangle ABC is $ac\frac{\sqrt{3}}{4}$.

(4)

Given that $a = 2$ and $\sin A = \frac{\sqrt{15}}{5}$, find

(ii) the value of b,

(2)

(iii) the value of c.

(4)

(b) The internal angles of an n-sided polygon form an arithmetic sequence with first term $143°$ and common difference $2°$.

Given that all of the internal angles are less than $180°$, find the value of n.

(5)

(Total 15 marks)

10. A sequence of non-zero real numbers $a_1, a_2, a_3, \ldots$ is defined by

$$a_{n+1} = p + \frac{q}{a_n} \qquad n \in \mathbb{N}$$

where p and q are real numbers with $q \neq 0$

It is known that

- one of the terms of this sequence is a
- the sequence is periodic

(a) Determine an equation for q, in terms of p and a, such that the sequence is constant (of period / order one).

(2)

(b) Determine the value of p that is necessary for the sequence to be of period / order 2.

(3)

(c) Give an example of a sequence that satisfies the condition in part (b), but is **not** of period / order 2.

(2)

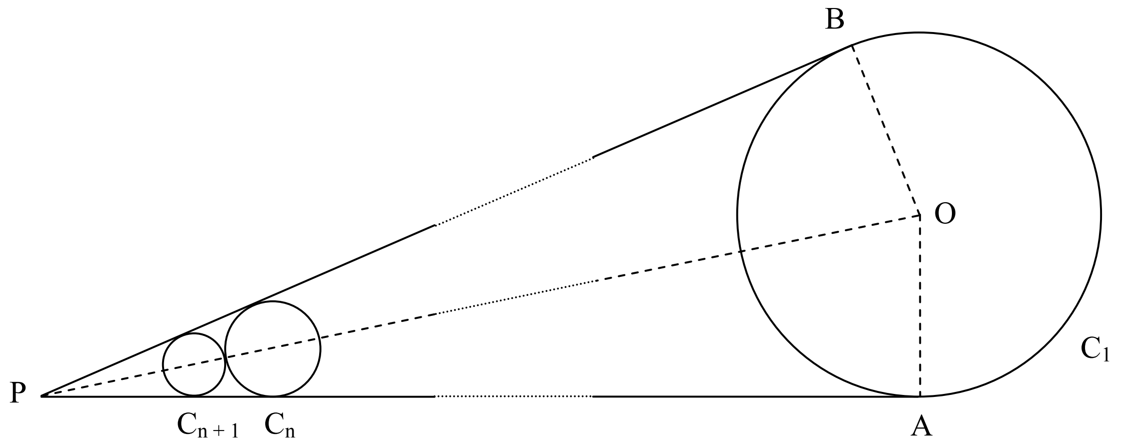
(d) Determine an equation for q, in terms of p only, such that the sequence has period / order 4.

(7)

(+S1)

11.

Figure 2



The circle C_1 has centre O and radius R. The tangents AP and BP to C_1 meet at the point P and angle APB = 2α, $0 < \alpha < \frac{\pi}{2}$. A sequence of circles $C_1, C_2, \ldots, C_n, \ldots$ is drawn so that each new circle C_{n+1} touches each of C_n, AP and BP for n = 1, 2, 3, ... as shown in Figure 2. The centre of each circle lies on the line OP.

(a) Show that the radii of the circles form a geometric sequence with common ratio

$$\frac{1-\sin\alpha}{1+\sin\alpha}.$$

(5)

(b) Find, in terms of R and α, the total area enclosed by all the circles, simplifying your answer.

(3)

The area inside the quadrilateral PAOB, not enclosed by part of C_1 or any of the other circles, is S.

(c) Show that

$$S = R^2\left(\alpha + \cot\alpha - \frac{\pi}{4}\operatorname{cosec}\alpha - \frac{\pi}{4}\sin\alpha\right).$$

(5)

(d) Show that, as α varies,

$$\frac{dS}{d\alpha} = R^2\cot^2\alpha\left(\frac{\pi}{4}\cos\alpha - 1\right).$$

(4)

(e) Find, in terms of R, the least value of S for $\frac{\pi}{6} \le \alpha \le \frac{\pi}{4}$.

(3)

12.

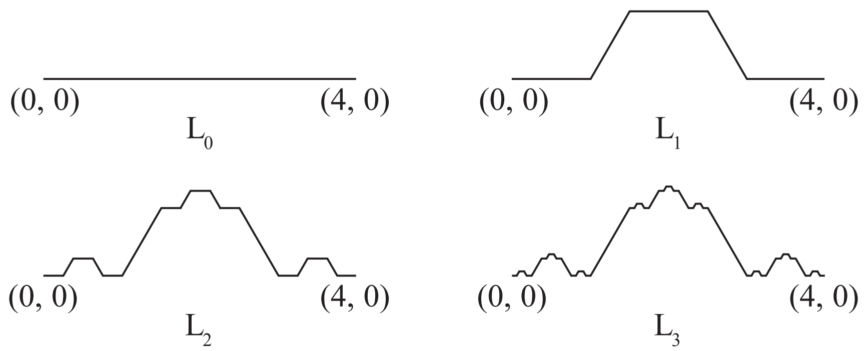


Figure 2

Figure 2 shows the first few iterations in the construction of a curve, L.

Starting with a straight line L_0 of length 4, the middle half of this line is replaced by three sides of a trapezium above L_0 as shown, such that the length of each of these sides is $\frac{1}{4}$ of the length of L_0

After the first iteration each line segment has length one.

In subsequent iterations, each line segment parallel to L_0 similarly has its middle half replaced by three sides of a trapezium above that line segment, with each side $\frac{1}{4}$ the length of that line segment.

Line segments in L_n are either parallel to L_0 or are sloped.

(a) Show that the length of L_2 is $\frac{23}{4}$

(2)

(b) Write down the number of

(i) line segments in L_n that are parallel to L_0

(ii) sloped line segments in L_2 that are not in L_1

(iii) **new** sloped line segments that are created by the (n + 1)th iteration.

(3)

(c) Hence find the length of L_n as $n \to \infty$

(6)

The area enclosed between L_0 and L_n is A_n

(d) Find the value of A_1

(2)

(e) Find, in terms of n, an expression for $A_{n+1} - A_n$

(3)

(f) Hence find the value of A_n as $n \to \infty$

(3)

Question 12 continued

The same construction as described above is applied externally to the three sides of an equilateral triangle of side length a.

Given that the limit of the area of the resulting shape is $26\sqrt{3}$

(g) find the value of a.

(3)

(+S2)