1. $$\frac{9x^2}{(x-1)^2(2x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(2x+1)}$$

Find the values of the constants A, B and C.

(4)

2. A new smartphone was released by a company.

The company monitored the total number of phones sold, n, at time t days after the phone was released.

The company observed that, during this time,

the rate of increase of n was proportional to n

Use this information to write down a suitable equation for n in terms of t.

(*You do not need to evaluate any unknown constants in your equation*.)

(2)

3. (a) Find $\int \frac{9x+6}{x}\,dx$, $x>0$.

(2)

(b) Given that $y=8$ at $x=1$, solve the differential equation

$$\frac{dy}{dx}=\frac{(9x+6)y^{\frac{1}{3}}}{x}$$

giving your answer in the form $y^2=g(x)$.

(6)

4. The rate of decay of the mass of a particular substance is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{5}{2}x, \qquad t \geqslant 0$$

where x is the mass of the substance measured in grams and t is the time measured in days.

Given that $x = 60$ when $t = 0$,

(a) solve the differential equation, giving x in terms of t. You should show all steps in your working and give your answer in its simplest form.

(4)

(b) Find the time taken for the mass of the substance to decay from 60 grams to 20 grams. Give your answer to the nearest minute.

(3)

5.

Given that $y = 2$ at $x = \frac{\pi}{4}$, solve the differential equation

$$\frac{dy}{dx} = \frac{3}{y\cos^2 x}$$

(5)

6. (a) Find $\int (4y+3)^{-\frac{1}{2}} \, dy$

(2)

(b) Given that $y = 1.5$ at $x = -2$, solve the differential equation

$$\frac{dy}{dx} = \frac{\sqrt{(4y+3)}}{x^2}$$

giving your answer in the form $y = f(x)$.

(6)

7. (i) Find

$$\int x\,\mathrm{e}^{4x}\,\mathrm{d}x$$

(3)

(ii) Find

$$\int \frac{8}{(2x-1)^3}\,\mathrm{d}x, \quad x > \frac{1}{2}$$

(2)

(iii) Given that $y = \frac{\pi}{6}$ at $x = 0$, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x \operatorname{cosec} 2y \operatorname{cosec} y$$

(7)

8. (a) Express $\dfrac{5}{(x-1)(3x+2)}$ in partial fractions.

(3)

(b) Hence find $\displaystyle\int \frac{5}{(x-1)(3x+2)}\,\mathrm{d}x$, where $x > 1$.

(3)

(c) Find the particular solution of the differential equation

$$(x-1)(3x+2)\frac{\mathrm{d}y}{\mathrm{d}x} = 5y, \quad x > 1,$$

for which $y = 8$ at $x = 2$. Give your answer in the form $y = \mathrm{f}(x)$.

(6)

9. Water is being heated in a kettle. At time t seconds, the temperature of the water is θ °C.

The rate of increase of the temperature of the water at any time t is modelled by the differential equation

$$\frac{d\theta}{dt} = \lambda(120 - \theta), \qquad \theta \leqslant 100$$

where λ is a positive constant.

Given that $\theta = 20$ when $t = 0$,

(a) solve this differential equation to show that

$$\theta = 120 - 100e^{-\lambda t}$$

(8)

When the temperature of the water reaches 100 °C, the kettle switches off.

(b) Given that $\lambda = 0.01$, find the time, to the nearest second, when the kettle switches off.

(3)

10. (a) Express $\dfrac{2}{P(P-2)}$ in partial fractions.

(3)

A team of biologists is studying a population of a particular species of animal.

The population is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{2}P(P-2)\cos 2t, \quad t \geqslant 0$$

where P is the population in thousands, and t is the time measured in years since the start of the study.

Given that $P = 3$ when $t = 0$,

(b) solve this differential equation to show that

$$P = \frac{6}{3 - \mathrm{e}^{\frac{1}{2}\sin 2t}}$$

(7)

(c) find the time taken for the population to reach 4000 for the first time.
Give your answer in years to 3 significant figures.

(3)

11. A bottle of water is put into a refrigerator. The temperature inside the refrigerator remains constant at 3 °C and t minutes after the bottle is placed in the refrigerator the temperature of the water in the bottle is θ °C.

The rate of change of the temperature of the water in the bottle is modelled by the differential equation,

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{(3 - \theta)}{125}$$

(a) By solving the differential equation, show that,

$$\theta = A\mathrm{e}^{-0.008t} + 3$$

where A is a constant.

(4)

Given that the temperature of the water in the bottle when it was put in the refrigerator was 16 °C,

(b) find the time taken for the temperature of the water in the bottle to fall to 10 °C, giving your answer to the nearest minute.

(5)

12. (a) Express $\dfrac{1}{P(5-P)}$ in partial fractions.

(3)

A team of conservationists is studying the population of meerkats on a nature reserve. The population is modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{15}P(5 - P), \qquad t \geqslant 0$$

where P, in thousands, is the population of meerkats and t is the time measured in years since the study began.

Given that when $t = 0$, $P = 1$,

(b) solve the differential equation, giving your answer in the form,

$$P = \frac{a}{b + c\mathrm{e}^{-\frac{1}{3}t}}$$

where a, b and c are integers.

(8)

(c) Hence show that the population cannot exceed 5000

(1)

13. (a) Express $\dfrac{3}{(2x-1)(x+1)}$ in partial fractions.

(3)

When chemical A and chemical B are mixed, oxygen is produced.

A scientist mixed these two chemicals and measured the total volume of oxygen produced over a period of time.

The total volume of oxygen produced, $V\,\text{m}^3$, t hours after the chemicals were mixed, is modelled by the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{3V}{(2t-1)(t+1)} \qquad V \geqslant 0 \qquad t \geqslant k$$

where k is a constant.

Given that exactly 2 hours after the chemicals were mixed, a total volume of $3\,\text{m}^3$ of oxygen had been produced,

(b) solve the differential equation to show that

$$V = \frac{3(2t-1)}{(t+1)}$$

(5)

The scientist noticed that

- there was a **time delay** between the chemicals being mixed and oxygen being produced
- there was a **limit** to the total volume of oxygen produced

Deduce from the model

(c) (i) the **time delay** giving your answer in minutes,

(ii) the **limit** giving your answer in m^3

(2)

14. (a) Express $\dfrac{1}{P(11-2P)}$ in partial fractions.

(3)

A population of meerkats is being studied.

The population is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{22}P(11 - 2P), \quad t \geqslant 0, \qquad 0 < P < 5.5$$

where P, in thousands, is the population of meerkats and t is the time measured in years since the study began.

Given that there were 1000 meerkats in the population when the study began,

(b) determine the time taken, in years, for this population of meerkats to double,

(6)

(c) show that

$$P = \frac{A}{B + C\mathrm{e}^{-\frac{1}{2}t}}$$

where A, B and C are integers to be found.

(3)

15.

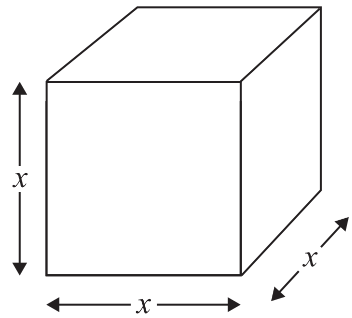

Figure 1

Figure 1 shows a metal cube which is expanding uniformly as it is heated.
At time t seconds, the length of each edge of the cube is x cm,
and the volume of the cube is V cm^3.

(a) Show that $\dfrac{dV}{dx} = 3x^2$

(1)

Given that the volume, V cm^3, increases at a constant rate of 0.048 cm^{3}s^{-1},

(b) find $\dfrac{dx}{dt}$, when $x = 8$

(2)

(c) find the rate of increase of the total surface area of the cube, in cm^{2}s^{-1}, when $x = 8$

(3)

16.

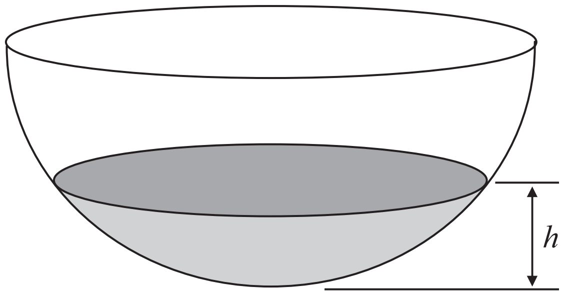

Figure 1

A hollow hemispherical bowl is shown in Figure 1. Water is flowing into the bowl. When the depth of the water is h m, the volume $V\,\text{m}^3$ is given by

$$V = \frac{1}{12}\pi h^2(3-4h), \qquad 0 \leqslant h \leqslant 0.25$$

(a) Find, in terms of π, $\dfrac{\mathrm{d}V}{\mathrm{d}h}$ when $h = 0.1$

(4)

Water flows into the bowl at a rate of $\dfrac{\pi}{800}\,\text{m}^3\text{s}^{-1}$.

(b) Find the rate of change of h, in $\text{m}\,\text{s}^{-1}$, when $h = 0.1$

(2)

17.

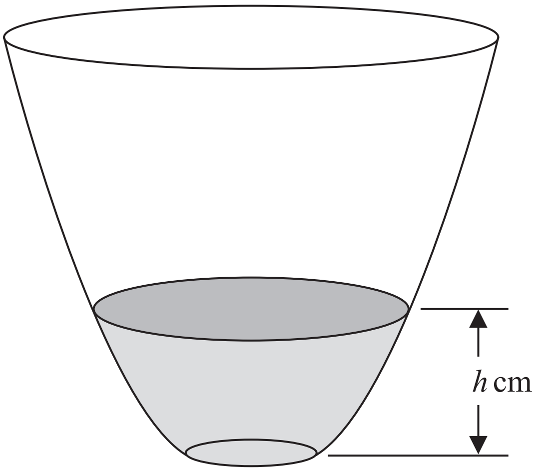

Figure 2

A vase with a circular cross-section is shown in Figure 2. Water is flowing into the vase.

When the depth of the water is h cm, the volume of water V cm^3 is given by

$$V = 4\pi h(h + 4), \qquad 0 \leqslant h \leqslant 25$$

Water flows into the vase at a constant rate of 80π cm^{3}s^{-1}

Find the rate of change of the depth of the water, in cm s^{-1}, when $h = 6$

(5)

18. The area A of a circle is increasing at a constant rate of 1.5 $cm^2\ s^{-1}$. Find, to 3 significant figures, the rate at which the radius r of the circle is increasing when the area of the circle is $2cm^2$.

(5)

19.

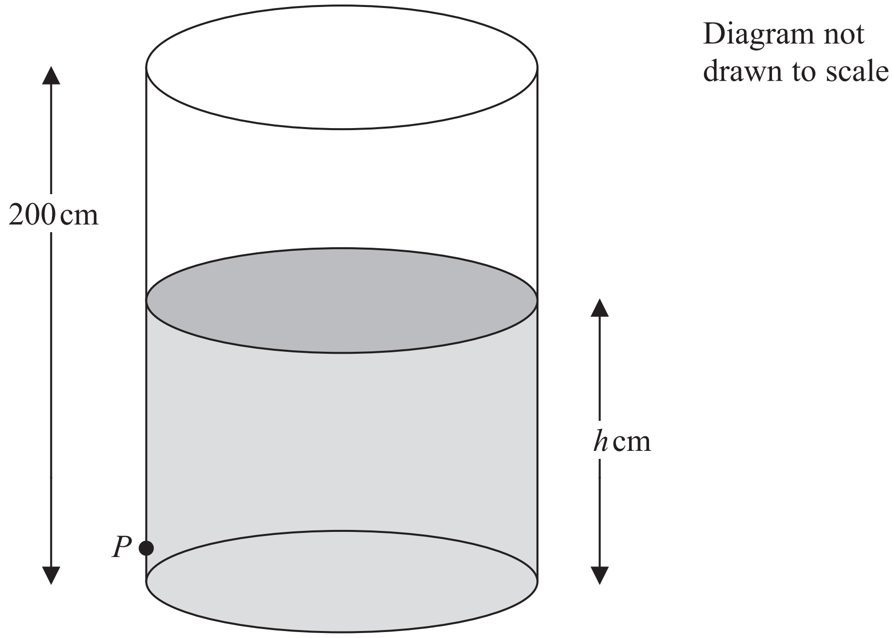

Figure 3

Figure 3 shows a vertical cylindrical tank of height 200 cm containing water.
Water is leaking from a hole P on the side of the tank.

At time t minutes after the leaking starts, the height of water in the tank is h cm.

The height h cm of the water in the tank satisfies the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = k(h-9)^{\frac{1}{2}}, \qquad 9 < h \leqslant 200$$

where k is a constant.

Given that, when $h = 130$, the height of the water is falling at a rate of 1.1 cm per minute,

(a) find the value of k.

(2)

Given that the tank was full of water when the leaking started,

(b) solve the differential equation with your value of k, to find the value of t when $h = 50$

(6)

20.

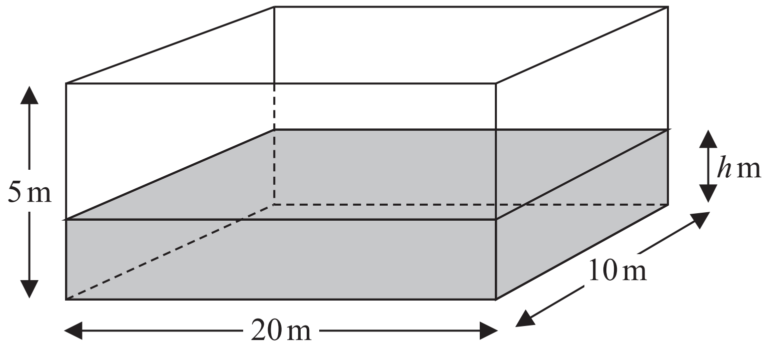

Figure 1

A tank in the shape of a cuboid is being filled with water.

The base of the tank measures 20 m by 10 m and the height of the tank is 5 m, as shown in Figure 1.

At time t minutes after water started flowing into the tank the height of the water was h m and the volume of water in the tank was $V\text{ m}^3$

In a model of this situation

- the sides of the tank have negligible thickness
- the rate of change of V is inversely proportional to the square root of h

(a) Show that

$$\frac{dh}{dt} = \frac{\lambda}{\sqrt{h}}$$

where λ is a constant.

(3)

Given that

- initially the height of the water in the tank was 1.44 m
- exactly 8 minutes after water started flowing into the tank the height of the water was 3.24 m

(b) use the model to find an equation linking h with t, giving your answer in the form

$$h^{\frac{3}{2}} = At + B$$

where A and B are constants to be found.

(5)

(c) Hence find the time taken, from when water started flowing into the tank, for the tank to be completely full.

(2)

21.

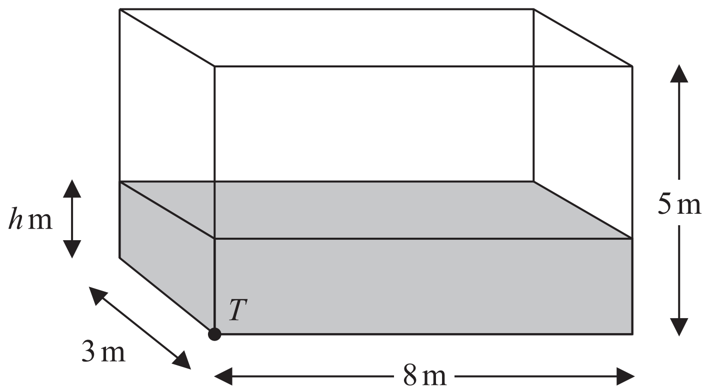

Figure 5

Water flows at a constant rate into a large tank.

The tank is a cuboid, with all sides of negligible thickness.

The base of the tank measures 8 m by 3 m and the height of the tank is 5 m.

There is a tap at a point T at the bottom of the tank, as shown in Figure 5.

At time t minutes after the tap has been opened

- the depth of water in the tank is h metres
- water is flowing into the tank at a constant rate of $0.48\,\text{m}^3$ per minute
- water is modelled as leaving the tank through the tap at a rate of $0.1h\,\text{m}^3$ per minute

(a) Show that, according to the model,

$$1200\frac{\mathrm{d}h}{\mathrm{d}t} = 24 - 5h$$

(4)

Given that when the tap was opened, the depth of water in the tank was 2 m,

(b) show that, according to the model,

$$h = A + B\mathrm{e}^{-kt}$$

where A, B and k are constants to be found.

(6)

Given that the tap remains open,

(c) determine, according to the model, whether the tank will ever become full, giving a reason for your answer.

(2)

22.

A large spherical balloon is deflating.

At time t seconds the balloon has radius r cm and volume V cm^3

The volume of the balloon is modelled as decreasing at a constant rate.

(a) Using this model, show that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{k}{r^2}$$

where k is a positive constant.

(3)

Given that

- the initial radius of the balloon is 40 cm
- after 5 seconds the radius of the balloon is 20 cm
- the volume of the balloon continues to decrease at a constant rate until the balloon is empty

(b) solve the differential equation to find a complete equation linking r and t.

(5)

(c) Find the limitation on the values of t for which the equation in part (b) is valid.

(2)

23. (a) Use the substitution $u = 4 - \sqrt{h}$ to show that

$$\int \frac{\mathrm{d}h}{4 - \sqrt{h}} = -8\ln\left|4 - \sqrt{h}\right| - 2\sqrt{h} + k$$

where k is a constant

(6)

A team of scientists is studying a species of slow growing tree.

The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{0.25}\left(4 - \sqrt{h}\right)}{20}$$

where h is the height in metres and t is the time, measured in years, after the tree is planted.

(b) Find, according to the model, the range in heights of trees in this species.

(2)

One of these trees is one metre high when it is first planted.

According to the model,

(c) calculate the time this tree would take to reach a height of 12 metres, giving your answer to 3 significant figures.

(7)