**1.** (i) A curve with equation $y = \mathrm{f}(x)$ has $\mathrm{f}(x) \geqslant 0$ for $x \geqslant a$ and

$$A = \int_a^b \mathrm{f}(x)\,\mathrm{d}x \quad \text{and} \quad V = \pi\int_a^b [\mathrm{f}(x)]^2\,\mathrm{d}x$$

where $a$ and $b$ are constants with $b > a$.

Use integration by substitution to show that for the positive constants $r$ and $h$

$$\pi\int_{a+h}^{b+h} [r + \mathrm{f}(x-h)]^2\,\mathrm{d}x = \pi r^2\,(b-a) + 2\pi rA + V$$

**(Total 3 marks)**

**2.** Given that

$$\int_0^{\frac{\pi}{2}} \left(1+\tan\left[\tfrac{1}{2}x\right]\right)^2 \, \mathrm{d}x = a + \ln b$$

find the value of $a$ and the value of $b$.

**(Total 7 marks)**

**3.** (a) Show that

$$\sin 3x = 3\sin x - 4\sin^3 x$$

**(3)**

Hence find

(b) $\int \cos x(6\sin x - 2\sin 3x)^{\frac{2}{3}}\ \mathrm{d}x$

**(3)**

(c) $\int (3\sin 2x - 2\sin 3x\cos x)^{\frac{1}{3}}\ \mathrm{d}x$

**(4)**

**(Total 10 marks)**

**4.** (a) Use the substitution $x = \sec\theta$ to show that

$$\int_{\sqrt{2}}^{2} \frac{1}{(x^2 - 1)^{\frac{3}{2}}}\, dx = \frac{\sqrt{6} - 2}{\sqrt{3}}$$

**(5)**

(b) Use integration by parts to show that

$$\int \operatorname{cosec}\theta \cot^2\theta \, d\theta = \frac{1}{2}\left[\ln\left|\operatorname{cosec}\theta + \cot\theta\right| - \operatorname{cosec}\theta \cot\theta\right] + c$$

**(6)**

**(Total 11 marks)**

**5.**

$$I = \int \frac{1}{(x-1)\sqrt{(x^2-1)}}\,\mathrm{d}x, \qquad x>1$$

(a) Use the substitution $x = 1 + u^{-1}$ to show that

$$I = -\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}} + c.$$

**(7)**

(b) Hence show that

$$\int_{\sec\alpha}^{\sec\beta} \frac{1}{(x-1)\sqrt{(x^2-1)}}\,\mathrm{d}x = \cot\left(\frac{\alpha}{2}\right) - \cot\left(\frac{\beta}{2}\right), \qquad 0<\alpha<\beta<\tfrac{\pi}{2}$$

**(5)**

**(Total 12 marks)**

**6.** The points $(x, y)$ on the curve $C$ satisfy

$$(x+1)(x+2)\frac{dy}{dx} = xy.$$

The line with equation $y = 2x + 5$ is the tangent to $C$ at a point $P$.

(a) Find the coordinates of $P$.

**(4)**

(b) Find the equation of $C$, giving your answer in the form $y = f(x)$.

**(8)**

**(Total 12 marks)**

7. (a) Use the trapezium rule with 4 strips to find an approximate value for

$$\int_0^1 16^x \, dx$$

**(2)**

(b) Use the trapezium rule with $n$ strips to write down an expression that would give an approximate value for

$$\int_0^1 16^x \, dx$$

**(2)**

(c) Hence show that

$$\int_0^1 16^x \, dx = \lim_{n \to \infty} \left( \frac{1}{n} \left( 1 + 16^{\frac{1}{n}} + \ldots + 16^{\frac{n-1}{n}} \right) \right)$$

**(3)**

(d) Use integration to determine the exact value of

$$\int_0^1 16^x \, dx$$

**(3)**

Given that the limit exists,

(e) use part (c) and the answer to part (d) to determine the exact value of

$$\lim_{x \to 0} \frac{16^x - 1}{x}$$

**(5)**
**(+S1)**

**8.**

In this question u and v are functions of $x$. Given that $\int \mathrm{u}\ \mathrm{d}x$, $\int \mathrm{v}\ \mathrm{d}x$ and $\int \mathrm{uv}\ \mathrm{d}x$ satisfy

$$\int \mathrm{uv}\ \mathrm{d}x = \left(\int \mathrm{u}\ \mathrm{d}x\right) \times \left(\int \mathrm{v}\ \mathrm{d}x\right) \qquad \mathrm{uv} \neq 0$$

(a) show that $1 = \dfrac{\int \mathrm{u}\ \mathrm{d}x}{\mathrm{u}} + \dfrac{\int \mathrm{v}\ \mathrm{d}x}{\mathrm{v}}$

**(3)**

Given also that $\dfrac{\int \mathrm{u}\ \mathrm{d}x}{\mathrm{u}} = \sin^2 x$ ,

(b) use part (a) to write down an expression, in terms of $x$, for $\dfrac{\int \mathrm{v}\ \mathrm{d}x}{\mathrm{v}}$ ,

**(1)**

(c) show that

$$\frac{1}{\mathrm{u}}\frac{\mathrm{du}}{\mathrm{d}x} = \frac{1 - 2\sin x \cos x}{\sin^2 x}$$

**(3)**

(d) hence use integration to show that $\mathrm{u} = A\mathrm{e}^{-\cot x}\mathrm{cosec}^2 x$, where $A$ is an arbitrary constant.

**(6)**

(e) By differentiating $\mathrm{e}^{\tan x}$ find a similar expression for v.

**(2)**

**(Total 15 marks)**

**9.** (a) Starting from $\left[\mathrm{f}(x)-\lambda \mathrm{g}(x)\right]^2 \geqslant 0$ show that $\lambda$ satisfies the quadratic inequality

$$\left(\int_a^b \left[\mathrm{g}(x)\right]^2 \mathrm{d}x\right)\lambda^2 - 2\left(\int_a^b \mathrm{f}(x)\mathrm{g}(x)\ \mathrm{d}x\right)\lambda + \int_a^b \left[\mathrm{f}(x)\right]^2 \mathrm{d}x \geqslant 0$$

where $a$ and $b$ are constants and $\lambda$ can take any real value.

**(2)**

(b) Hence prove that

$$\left[\int_a^b \mathrm{f}(x)\mathrm{g}(x)\ \mathrm{d}x\right]^2 \leqslant \left[\int_a^b \left[\mathrm{f}(x)\right]^2 \mathrm{d}x\right] \times \left[\int_a^b \left[\mathrm{g}(x)\right]^2 \mathrm{d}x\right]$$

**(3)**

(c) By letting $\mathrm{f}(x) = 1$ and $\mathrm{g}(x) = \left(1+x^3\right)^{\frac{1}{2}}$ show that

$$\int_{-1}^{2} \left(1+x^3\right)^{\frac{1}{2}}\ \mathrm{d}x \leqslant \frac{9}{2}$$

**(4)**

(d) Show that $\displaystyle\int_{-1}^{2} x^2\left(1+x^3\right)^{\frac{1}{4}}\ \mathrm{d}x = \frac{12\sqrt{3}}{5}$

**(3)**

(e) Hence show that

$$\frac{144}{55} \leqslant \int_{-1}^{2} \left(1+x^3\right)^{\frac{1}{2}}\ \mathrm{d}x$$

**(4)**

**(Total 16 marks)**

**10..** (a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}u}\ln\left(u+\sqrt{u^2-1}\right)=\frac{1}{\sqrt{u^2-1}}$$

**(2)**

(b) Use the result from part (a) and the substitution $x+3=\frac{1}{t}$ to find

$$\int\frac{1}{(x+3)\sqrt{2x+7}}\,\mathrm{d}x$$

**(6)**

(c) Express $\dfrac{1}{2x^2+13x+21}$ in partial fractions.

**(2)**

(d) Find

$$\int_1^9\frac{1}{\left(2x^2+13x+21\right)\sqrt{2x+7}}\,\mathrm{d}x$$

giving your answer in the form $\ln r - s$ where $r$ and $s$ are rational numbers.

**(6)**

**(Total 16 marks)**

**11.**

(a) Given that f is a function such that the integrals exist,

(i) use the substitution $u = a - x$ to show that

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

**(2)**

(ii) Hence use symmetry of $f(\sin x)$ on the interval $[0, \pi]$ to show that

$$\int_0^{\pi} x f(\sin x)\,dx = \pi \int_0^{\frac{\pi}{2}} f(\sin x)\,dx$$

**(4)**

(b) Use the result of (a)(i) to show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx$$

is independent of $n$, and find the value of this integral.

**(4)**

(c) (i) Prove that

$$\frac{\cos x}{1+\cos x} \equiv 1 - \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$$

(ii) Hence use the results from (a) to find

$$\int_0^{\pi} \frac{x\sin x}{1+\sin x}\,dx$$

**(7)**

(d) Find

$$\int_0^{\pi} \frac{x\sin^4 x}{\sin^4 x + \cos^4 x}\,dx$$

**(4)**

**(+S2)**

Question 11 continued

**(Total for Question 11 is 23 marks)**