I.S. *iff* V.C. *iff* Clique & Proving NP-Complete

Dianne Foreback

Topics

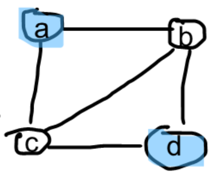
- Relating 3 Problems
- Proving IS iff VC iff IS
- Proving Clique is in NP-Complete
  - More easily done due to our proof IS iff VC iff IS

Relating 3 Decision Problems: Graph G=(V,E)

independent set (IS) of a graph G is a subset S of vertices such that there is no edge between vertices in S; i.e. no 2 vertices are adjacent.

G

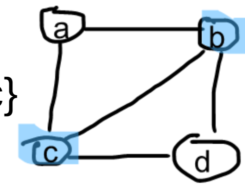
V' = {a,d}



vertex cover (VC) of a graph G is a subset of graph G s.t. every edge is incident on at least 1 vertex in the VC

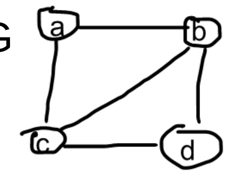
G

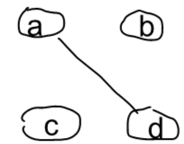
V-V' = {b,c}



clique of a graph G is a subset of vertices s.t. every pair of vertices in the set are adjacent; i.e. all pairs of vertices in the set share an incident edge.

G





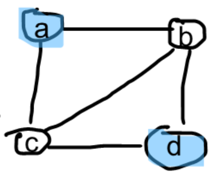
$\overline{G}$, V' = {a,d}

Relating 3 Decision Problems: Graph G=(V,E)

independent set (IS) of a graph G is a subset S of vertices such that there is no edge between vertices in S; i.e. no 2 vertices are adjacent.

G

V' = {a,d}

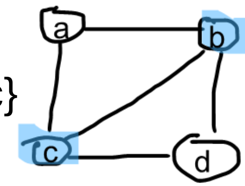


IS (decision) Problem: Given graph G & int k, does G have IS of at least k? Aside: we'll let V' represent IS.

vertex cover (VC) of a graph G is a subset of graph G s.t. every edge is incident on at least 1 vertex in the VC

G

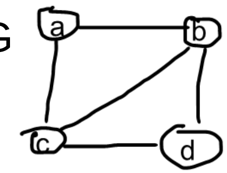
V-V' = {b,c}

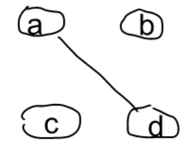


VC (decision) Problem: Given graph G & int r = |V| - k, does G have a VC of r or less? Aside: we'll let V-V' represent VC.

clique of a graph G is a subset of vertices s.t. every pair of vertices in the set are adjacent; i.e. all pairs of vertices in the set share an incident edge.

G





$\overline{G}$, V' = {a,d}

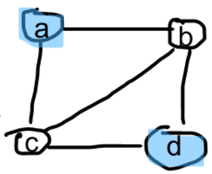
Clique (decision) Problem: Given graph $\overline{G}$ & int k, does $\overline{G}$ have clique of size k? Aside: we'll let $\overline{G}$ represent our graph and V' represent clique.

Equivalent Statements

independent set (IS) of a graph G is a subset S of vertices such that there is no edge between vertices in S; i.e. no 2 vertices are adjacent.

G

V' = {a,d}

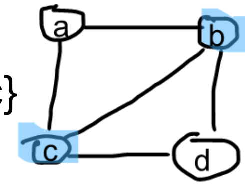


IS (decision) Problem: Given graph G & int k, does G have IS of at least k? Aside: we'll let V' represent IS.

vertex cover (VC) of a graph G is a subset of graph G s.t. every edge is incident on at least 1 vertex in the VC

G

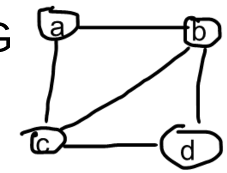
V-V' = {b,c}

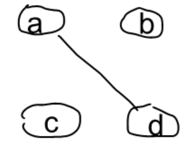


VC (decision) Problem: Given graph G & int r = |V| - k, does G have a VC of r or less? Aside: we'll let V-V' represent VC.

clique of a graph G is a subset of vertices s.t. every pair of vertices in the set are adjacent; i.e. all pairs of vertices in the set share an incident edge.

G





$\overline{G}$, V' = {a,d}

Clique (decision) Problem: Given graph $\overline{G}$ & int k, does $\overline{G}$ have clique of size k? Aside: we'll let $\overline{G}$ represent our graph and V' represent clique.

Given a Graph G and vertices V' the following 3 statements are equivalent.

(i) V' is an IS of G **iff** (ii) V-V' is a VC of G **iff** (iii) V' is a clique of $\overline{G}$

Prove: (i) → (ii) → (iii) → (i)

Prove (i) → (ii)

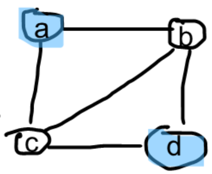
independent set (IS) of a graph G is a subset S of vertices such that there is no edge between vertices in S; i.e. no 2 vertices are adjacent.

vertex cover (VC) of a graph G is a subset of graph G s.t. every edge is incident on at least 1 vertex in the VC

clique of a graph G is a subset of vertices s.t. every pair of vertices in the set are adjacent; i.e. all pairs of vertices in the set share an incident edge.

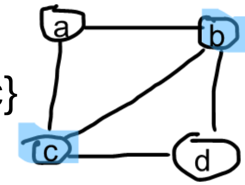
G

V' = {a,d}

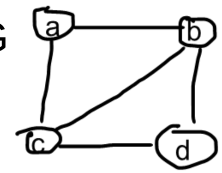


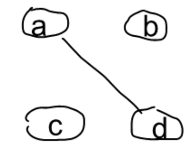
G

V-V' = {b,c}



G





$\overline{G}$, V' = {a,d}

IS (decision) Problem: Given graph G & int k, does G have IS of at least k? Aside: we'll let V' represent IS.

VC (decision) Problem: Given graph G & int r = |V| - k, does G have a VC of r or less? Aside: we'll let V-V' represent VC.

Clique (decision) Problem: Given graph $\overline{G}$ & int k, does $\overline{G}$ have clique of size k? Aside: we'll let $\overline{G}$ represent our graph and V' represent clique.

Theorem IS, VC, Clique Similarity

Given a Graph G and vertices V' the following 3 statements are equivalent.
(i) V' is an IS of G **iff** (ii) V-V' is a VC of G **iff** (iii) V' is a clique of $\overline{G}$

Claim: (i) → (ii) : If V' is an IS of G then (V-V') is a VC of G
Proof: (Goal: want to explain how the IS V' of G can be manipulated to to V-V' that will be a VC of G

Since V' is an IS of G, there is no edge (u,v) such that u & v are in V'; i.e. at least one of the endpoints do not belong to V'.

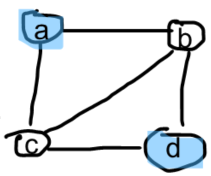
Thus, for any edge in the graph, at least one of the 2 endpoints DOES belong to V-V'. (Notice, this is the definition of VC). □

Prove (ii) → (iii)

independent set (IS) of a graph G is a subset S of vertices such that there is no edge between vertices in S; i.e. no 2 vertices are adjacent.

G

V' = {a,d}

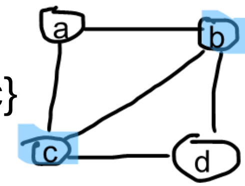


IS (decision) Problem: Given graph G & int k, does G have IS of at least k? Aside: we'll let V' represent IS.

vertex cover (VC) of a graph G is a subset of graph G s.t. every edge is incident on at least 1 vertex in the VC

G

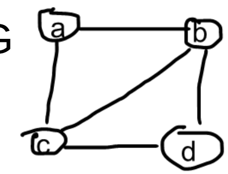
V-V' = {b,c}

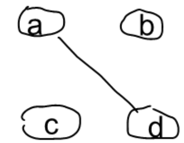


VC (decision) Problem: Given graph G & int r = |V| - k, does G have a VC of r or less? Aside: we'll let V-V' represent VC.

clique of a graph G is a subset of vertices s.t. every pair of vertices in the set are adjacent; i.e. all pairs of vertices in the set share an incident edge.

G





$\overline{G}$, V' = {a,d}

Clique (decision) Problem: Given graph $\overline{G}$ & int k, does $\overline{G}$ have clique of size k? Aside: we'll let $\overline{G}$ represent our graph and V' represent clique.

Theorem IS, VC, Clique Similarity

Given a Graph G and vertices V' the following 3 statements are equivalent.
(i) V' is an IS of G **iff** (ii) V-V' is a VC of G **iff** (iii) V' is a clique of $\overline{G}$

Claim: (ii) → (iii) : If V-V' is VC of G, then V' is a clique of G
Proof: (Goal: explain V' is a clique of $\overline{G}$)

If V-V' is a VC of G, then for any pair of vertices u,v in V' the vertices are not in VC of G. (This is what taking V' out of V means.)

Thus, edge (u, v) is not an edge in G (by definition of VC since VC would have to have at least one of these vertices in it if the edge existed to "cover" the edge). Since the edge DNE in G, the edge exists in $\overline{G}$.

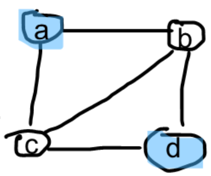
Thus, for any pair of vertices u, v in V', edge (e,v) is in G and this is a clique in $\overline{G}$ as every pair of vertices share an incident edge. □

Prove (iii) → (i)

independent set (IS) of a graph G is a subset S of vertices such that there is no edge between vertices in S; i.e. no 2 vertices are adjacent.

G

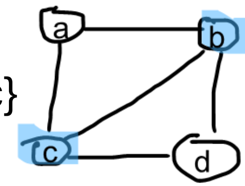
V' = {a,d}



vertex cover (VC) of a graph G is a subset of graph G s.t. every edge is incident on at least 1 vertex in the VC

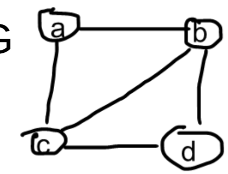
G

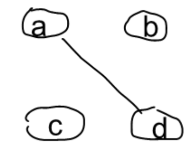
V-V' = {b,c}



clique of a graph G is a subset of vertices s.t. every pair of vertices in the set are adjacent; i.e. all pairs of vertices in the set share an incident edge.

G





$\overline{G}$, V' = {a,d}

IS (decision) Problem: Given graph G & int k, does G have IS of at least k? Aside: we'll let V' represent IS.

VC (decision) Problem: Given graph G & int r = |V| - k, does G have a VC of r or less? Aside: we'll let V-V' represent VC.

Clique (decision) Problem: Given graph $\overline{G}$ & int k, does $\overline{G}$ have clique of size k? Aside: we'll let $\overline{G}$ represent our graph and V' represent clique.

Theorem IS, VC, Clique Similarity

Given a Graph G and vertices V' the following 3 statements are equivalent.
(i) V' is an IS of G **iff** (ii) V-V' is a VC of G **iff** (iii) V' is a clique of $\overline{G}$

Claim: (iii) → (i) : If V' is clique of $\overline{G}$, then V' is an IS of G
Proof: (Goal: explain how to derive an IS using V' and $\overline{G}$)

V' is a clique of $\overline{G}$. Since $\overline{G}$ is a compliment of G, this means for any vertices u,v in V', there is not an edge in G.

Thus, V' is an IS of G. (see definition of independent set) □

Prove Decision Problem is NP-Complete

- To Prove a problem U is in NP-Complete must:
  1. Prove it is in NP
     - state U as a verification problem and provide an algorithm that verifies it in polynomial time)
  2. Prove it is in NP-Hard
     - given a problem H that is assumed to be in NP-complete, provide a reduction from H to U in polynomial time. Written $H \leq_P U$
       - ❑ the reduction is an "if and only if" (must prove in both directions)
       - ❑ A problem H is polynomial-time reducible to problem U, if any algorithm that can solve U can be used to solve H such that the increase in time is no more than polynomial

This will be much easier now that we have our *Theorem IS, VC, Clique Similarity .* $IS \leq_P Clique$

Prove Clique (Decision) Problem is NP-Complete Part 1: Clique is in NP

1. Prove Clique is in NP

Clique verification problem: Given a graph G, an integer k, and a vertex set V', if V' a clique of size k?

Prove by explaining an algorithm that runs in polynomial time to verify that V' is a clique of G of size k.

Leaving for the students to do.

Prove Clique (Decision) Problem is NP-Complete Part 2: Clique is in NP-Hard and Give Transform Algo proving its correctness

2. Prove Clique is in NP $\quad\quad IS \leq_P Clique$

Proof by Contradiction: Assume Clique (decision) problem is decidable in polynomial time. Then there exists a polynomial time algorithm A, that given a graph G and integer k returns *true* if G contains a clique of size k, or returns *false* otherwise.

Algo: Transform IS to Clique (G, k)
Input: Graph G, int k > 0
Output: True if G contains a clique of size k
Create the graph $\overline{G}$ complement of Graph G. Invoke algorithm A passing in G and k. (We assumed algorithm A exists). Return true if algorithm A returns true and false otherwise.

Proof of Correctness for Algo Transform IS to Clique

- Algorithm Transform IS to Clique runs in polynomial time since we can compute $\overline{G}$ from G in polynomial time (must explain this in detail and discuss its runtime. We did in prior lecture and will leave for student to write in).
- By assumption, Algorithm A runs in polynomial time.

Prove Clique (Decision) Problem is NP-Complete Part 2: Clique is in NP-Hard and prove reduction $IS \leq_P Clique$

$$IS \leq_P Clique$$

(→) If $\overline{G}$ has a clique of size k, then G has an IS of size k as proven by our Theorem *IS, VC, Clique Similarity* . Recall: G has an IS of size k iff G has a clique of size k. (Works if algo A returns true but what if it returns false. Prove in the other direction.)

(←) If G has an IS of size k, then $\overline{G}$ has a clique of size k by our Theorem IS, VC, Clique Similarity since G has IS of size k iff $\overline{G}$ has clique of size k.

Prove Clique (Decision) Problem is NP-Complete : Finish our proof by contradiction

(→) If $\overline{G}$ has a clique of size k, then G has an IS of size k as proven by our Theorem *IS, VC, Clique Similarity* . Recall: G has an IS of size k iff G has a clique of size k. (Works if algo A returns true but what if it returns false. Prove in the other direction.)

(←) If G has an IS of size k, then $\overline{G}$ has a clique of size k by our Theorem IS, VC, Clique Similarity since G has IS of size k iff G has clique of size k.

Proof by Contradiction: Assume Clique (decision) problem is decidable in polynomial time. Then there exists a polynomial time algorithm A, that given a graph G and integer k returns *true* if G contains a clique of size k, or returns *false* otherwise.

We prove that $IS \leq_P Clique$. But algorithm A does not exist since IS is in known to be in NP-Complete. Thus, Clique (Decision) problem is in NP-Hard.

Aside: We showed Clique is in NP and in NP-Hard, so it is in NP-Complete. ▯

Thank You !

Questions ?