备注：QR 公式（也叫 UR 公式）

求 QR 要点:（详见例子）

1.先用“**许米特**公式”,求**优阵** Q，或半优阵 Q (也叫列优，次优)

2.再用公式 **R=Q^H^A**，求三角阵 R

最后写出分解：A=QR

例 求 $A=\begin{pmatrix}1&-2&0\\1&4&2\\0&0&1\end{pmatrix}$ 的 QR 分解 $A=QR$，已知 $Q=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1&0\\1&1&0\\0&0&\sqrt{2}\end{pmatrix}$.

可知 $R=Q^H A=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1&0\\-1&1&0\\0&0&\sqrt{2}\end{pmatrix}\begin{pmatrix}1&-2&0\\1&4&2\\0&0&1\end{pmatrix}=\begin{pmatrix}\sqrt{2}&\sqrt{2}&\sqrt{2}\\0&3\sqrt{2}&\sqrt{2}\\0&0&1\end{pmatrix}$，得到

$$A=QR=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1&0\\1&1&0\\0&0&\sqrt{2}\end{pmatrix}\begin{pmatrix}\sqrt{2}&\sqrt{2}&\sqrt{2}\\0&3\sqrt{2}&\sqrt{2}\\0&0&1\end{pmatrix}$$

Eg. $A=\begin{pmatrix}1&-2&0\\1&4&2\\0&0&1\end{pmatrix}$, and $A=QR$, 已知 $Q=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1&0\\1&1&0\\0&0&\sqrt{2}\end{pmatrix}$. 求find: $R=?$

答Ans：$R=Q^H A=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1&0\\-1&1&0\\0&0&\sqrt{2}\end{pmatrix}\begin{pmatrix}1&-2&0\\1&4&2\\0&0&1\end{pmatrix}=\begin{pmatrix}\sqrt{2}&\sqrt{2}&\sqrt{2}\\0&3\sqrt{2}&\sqrt{2}\\0&0&1\end{pmatrix}$, ,

we get :

$$A=QR=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1&0\\1&1&0\\0&0&\sqrt{2}\end{pmatrix}\begin{pmatrix}\sqrt{2}&\sqrt{2}&\sqrt{2}\\0&3\sqrt{2}&\sqrt{2}\\0&0&1\end{pmatrix}.$$

$$\text{Eg. } A=\begin{pmatrix}1 & 2i\\ i & 1\\ i & 0\end{pmatrix}=(\alpha_1,\alpha_2),\ \text{求find：} A=QR.$$

答Ans：令Set $\beta_1=\alpha_1=\begin{pmatrix}1\\ i\\ i\end{pmatrix}$,

$$\beta_2 = \alpha_2 - \frac{(\alpha_2,\alpha_1)}{|\alpha_1|^2}\alpha_1 = \alpha_2 - \frac{(\alpha_1{}^H\alpha_2)}{|\alpha_1|^2}\alpha_1 = \frac{1}{3}\begin{pmatrix}5i\\ 4\\ 1\end{pmatrix}$$

单位化Put： $\varepsilon_1=\dfrac{\beta_1}{|\beta_1|}=\dfrac{1}{\sqrt{3}}\begin{pmatrix}1\\ i\\ i\end{pmatrix}$, $\varepsilon_2=\dfrac{\beta_2}{|\beta_2|}=\dfrac{1}{\sqrt{42}}\begin{pmatrix}5i\\ 4\\ 1\end{pmatrix}$

$$\Rightarrow\ Q=(\varepsilon_1,\varepsilon_2)=\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{5i}{\sqrt{42}}\\ \frac{i}{\sqrt{3}} & \frac{4}{\sqrt{42}}\\ \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{42}}\end{pmatrix}\ \text{is C-U (列优或半优)},\quad Q^HQ=I$$

$$\text{令}R=Q^HA=\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{3}} & \frac{-i}{\sqrt{3}}\\ \frac{-5i}{\sqrt{42}} & \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{42}}\end{pmatrix}A=\begin{pmatrix}\sqrt{3} & \frac{i}{\sqrt{3}}\\ 0 & \frac{\sqrt{14}}{\sqrt{3}}\end{pmatrix}$$

$$\Rightarrow \quad A = QR = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{5i}{\sqrt{42}} \\ \frac{i}{\sqrt{3}} & \frac{4}{\sqrt{42}} \\ \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{42}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & \frac{i}{\sqrt{3}} \\ 0 & \frac{\sqrt{14}}{\sqrt{3}} \end{pmatrix}.$$

Eg. $A = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = (\alpha_1, \alpha_2)$, find 求： $A = QR.$

Here, $\alpha_1 = (1,\ i)^T$, $\alpha_2 = (i,\ 1)^T$,

Put $\beta_1 = \alpha_1 = (1,\ \ i)^T$, $|\beta_1|^2 = 2,\ \ |\beta_1| = \sqrt{2}$,

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \alpha_1)}{|\alpha_1|^2}\alpha_1 = \alpha_2 - 0\alpha_1 = \alpha_2 = (i,\ 1)^T$$

$$\Rightarrow \beta_1 \perp \beta_2 \ \ (\because\ \alpha_1 \perp \alpha_2)$$

单位化：Set $\varepsilon_1 = \frac{\beta_1}{|\beta_1|} = \frac{1}{\sqrt{2}}\beta_1,\quad \varepsilon_2 = \frac{\beta_2}{|\beta_2|} = \frac{1}{\sqrt{2}}\beta_2$,

$$Q = (\varepsilon_1, \varepsilon_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$ is U(优阵)

Put $R = Q^H A = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$, we get $A = QR = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$

Eg. $A = (\alpha_1, \alpha_2, \alpha_3) = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix}_{4\times 3}$,find求：$A = QR.$

解答 Ans: $\alpha_1 = (1,1,1,1)^T$, $\alpha_2 = (-1,4,4,-1)^T$, $\alpha_3 = (4,-2,2,0)^T$

Let $\beta_1 = \alpha_1 = (1,\ 1,1,1)^T$, $|\beta_1|^2 = 4,\ \ |\beta_1| = 2$.

$$\beta_2=\alpha_2-\frac{(\alpha_2,\alpha_1)}{|\alpha_1|^2}\alpha_1=\left(-\frac{5}{2},\frac{5}{2},\frac{5}{2},-\frac{5}{2}\right)^T=\frac{5}{2}(-1,1,1,-1)^T,\quad |\beta_2|=5$$

$$\beta_3=\alpha_3-\frac{(\alpha_3,\alpha_1)}{|\alpha_1|^2}\alpha_1-\frac{(\alpha_3,\ \beta_2)}{|\beta_2|^2}\beta_2=(2,-2,2,-2)^T,\quad |\beta_3|=4.$$

Put $\varepsilon_1=\frac{\beta_1}{|\beta_1|}=\frac{1}{2}(1,\ 1,1,1)^T$, $\varepsilon_2=\frac{\beta_2}{|\beta_2|}=\frac{1}{2}(-1,1,1,-1)^T$, $\varepsilon_3=\frac{\beta_3}{|\beta_3|}=\frac{1}{2}(1,-1,1,-1)^T$

Put $Q=(\varepsilon_1,\varepsilon_2,\varepsilon_3)=\begin{pmatrix}1/2 & -1/2 & 1/2\\ 1/2 & 1/2 & -1/2\\ 1/2 & 1/2 & 1/2\\ 1/2 & -1/2 & -1/2\end{pmatrix}=1/2\begin{pmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ 1 & 1 & 1\\ 1 & -1 & -1\end{pmatrix}_{4\times 3}$ (it is C-U半优）

$$R=Q^HA=\frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ -1 & 1 & 1 & -1\\ 1 & -1 & 1 & -1\end{pmatrix}A=\begin{pmatrix}2 & 3 & 2\\ 0 & 5 & -2\\ 0 & 0 & 4\end{pmatrix}$$

$$\Rightarrow A=QR=1/2\begin{pmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ 1 & 1 & 1\\ 1 & -1 & -1\end{pmatrix}\begin{pmatrix}2 & 3 & 2\\ 0 & 5 & -2\\ 0 & 0 & 4\end{pmatrix}.$$

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补例：求下列 A 的 QR（UR）分解，其中 R=Q^{H}A

$$A=\begin{pmatrix}0 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}$$

解：记 $\alpha_1=(0,1,1)^T,\alpha_2=(1,1,0)^T,\alpha_3=(1,0,1)^T$。由 Schmidt 正交方法，

令 $\beta_1=(0,1,1)^T,\|\beta\|=\sqrt{2}$, 单位化令$\varepsilon_1=\frac{1}{\sqrt{2}}(0,1,1)^T$，计算可知：

$$(\alpha_2,\varepsilon)=\frac{1}{\sqrt{2}},\beta_2=\alpha_2-(\alpha_2,\varepsilon_1)\varepsilon_1=\left(1,\frac{1}{2},-\frac{1}{2}\right)^T,\|\beta_2\|=\frac{\sqrt{6}}{2},\text{单位化}\varepsilon_2=\frac{1}{\sqrt{6}}(2,1-1)^T$$

且可知

$(\alpha_3,\varepsilon_1)=\frac{1}{\sqrt{2}},(\alpha_3,\varepsilon_2)=\frac{1}{\sqrt{6}},\beta_3=\alpha_3-(\alpha_3,\varepsilon_1)\varepsilon_1-(\alpha_3,\varepsilon_2)\varepsilon_2=\frac{2}{3}(1,-1,1)^T$,

$\|\beta_3\|=\frac{2}{\sqrt{3}}$,单位化$\varepsilon_3=\frac{1}{\sqrt{3}}(1,-11)^T$

令优阵：$Q=(\varepsilon_1,\varepsilon_2,\varepsilon_3)=\begin{pmatrix}0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{pmatrix}$

可知 $R=Q^HA=\begin{pmatrix}\|\beta_1\| & (\alpha_2,\varepsilon_1) & (\alpha_3,\varepsilon_1)\\ 0 & \|\beta_2\| & (\alpha_3,\varepsilon_1)\\ 0 & 0 & \|\beta_2\|\end{pmatrix}=\begin{pmatrix}\sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ 0 & \frac{\sqrt{6}}{2} & \frac{1}{\sqrt{6}}\\ 0 & 0 & \frac{2}{\sqrt{3}}\end{pmatrix}$

可得 $A=QR$

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补充例子：

1.已知列向量$\alpha=\begin{pmatrix}1\\1\end{pmatrix},\beta=\begin{pmatrix}|\alpha|\\0\end{pmatrix}=\begin{pmatrix}\sqrt{2}\\0\end{pmatrix}$,令$X=\alpha-\beta=\begin{pmatrix}1-\sqrt{2}\\1\end{pmatrix}$

求 $P=I-\frac{2XX^H}{|X|^2}=$？ 验证 $P\alpha=\beta$, $P\beta=\alpha$

答：$P=\frac{\sqrt{2}}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

2. 设 $A=\begin{pmatrix}1 & 2i\\ i & 1\end{pmatrix},(i^2=-1)$ **求 QR 分解** $A=QR$

答：A=QR=$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}\sqrt{2} & \frac{i}{\sqrt{2}}\\ 0 & 3/\sqrt{2}\end{pmatrix}$

若用镜面阵方法，也有其它答案：A=$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ i & -1\end{pmatrix}\begin{pmatrix}\sqrt{2} & \frac{i}{\sqrt{2}}\\ 0 & -3/\sqrt{2}\end{pmatrix}$

3.已知 $\alpha=\begin{pmatrix}1\\2\\2\end{pmatrix},\beta=\begin{pmatrix}|\alpha|\\0\\0\end{pmatrix}=\begin{pmatrix}3\\0\\0\end{pmatrix}$，令 $X=\alpha-\beta=\begin{pmatrix}-2\\2\\2\end{pmatrix}$

求 $P=I-\dfrac{2XX^H}{|X|^2}=?$ 验证 $P\alpha=\beta$

答：$P=\dfrac{1}{3}\begin{pmatrix}1&2&2\\2&1&-2\\2&-2&1\end{pmatrix}$

2. 设 $A=\begin{pmatrix}1&2&1\\2&4&1\\2&4&1\end{pmatrix}$，**求 QR 分解** $A=QR$

答：令镜面阵 $P=\dfrac{1}{3}\begin{pmatrix}1&2&2\\2&1&-2\\2&-2&1\end{pmatrix}$，可知 $P\alpha_1=\begin{pmatrix}3\\0\\0\end{pmatrix}$，$P\alpha_2=\begin{pmatrix}6\\0\\0\end{pmatrix}$，$P\alpha_3=\dfrac{1}{3}\begin{pmatrix}5\\1\\1\end{pmatrix}$

可知 $PA=\begin{pmatrix}3&6&5/3\\0&0&1/3\\0&0&1/3\end{pmatrix}$=R（为上三角），又 $P^{-1}=P^H=P$

可得QR分解：$A=PR=P\begin{pmatrix}3&6&5/3\\0&0&1/3\\0&0&1/3\end{pmatrix}$

例：$A=\begin{pmatrix}1&2i\\i&1\\i&0\end{pmatrix}=(\alpha_1,\alpha_2)$，求QR分解

令 $\beta_1=\alpha_1=\begin{pmatrix}1\\i\\i\end{pmatrix}$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2,\alpha_1)}{|\alpha_1|^2}\alpha_1 = \alpha_2 - \frac{(\alpha_1{}^H\alpha_2)}{|\alpha_1|^2}\alpha_1 = \frac{1}{3}\begin{pmatrix}5i\\4\\1\end{pmatrix}$$

单位化：$\varepsilon_1 = \dfrac{\beta_1}{|\beta_1|} = \dfrac{1}{\sqrt{3}}\begin{pmatrix}1\\i\\i\end{pmatrix}$，$\varepsilon_2 = \dfrac{\beta_2}{|\beta_2|} = \dfrac{1}{\sqrt{42}}\begin{pmatrix}5i\\4\\1\end{pmatrix}$

令$Q=(\varepsilon_1,\varepsilon_2)=\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{5i}{\sqrt{42}}\\ \frac{i}{\sqrt{3}} & \frac{4}{\sqrt{42}}\\ \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{42}}\end{pmatrix}$ （列 U 阵或半优阵）

令$R=Q^HA=\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{3}} & \frac{-i}{\sqrt{3}}\\ \frac{-5i}{\sqrt{42}} & \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{42}}\end{pmatrix}A=\begin{pmatrix}\sqrt{3} & \frac{i}{\sqrt{3}}\\ 0 & \frac{\sqrt{14}}{\sqrt{3}}\end{pmatrix}$

可得：$A=QR=\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{5i}{\sqrt{42}}\\ \frac{i}{\sqrt{3}} & \frac{4}{\sqrt{42}}\\ \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{42}}\end{pmatrix}\begin{pmatrix}\sqrt{3} & \frac{i}{\sqrt{3}}\\ 0 & \frac{\sqrt{14}}{\sqrt{3}}\end{pmatrix}$

习题 Ex. find求: $A=QR$. （1） $A=\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}$，（2） $A=\begin{pmatrix}1 & i\\ 1 & 1\\ 1 & -1\\ i & 0\end{pmatrix}$

(3) $A=\begin{pmatrix}0 & 3 & 1\\ 0 & 4 & -2\\ 1 & 1 & 2\end{pmatrix}$, (4) $A=\begin{pmatrix}1 & 1\\ 1 & -1\\ 1 & 1\\ 1 & 0\end{pmatrix}_{4\times 2}$