Ex1. $X = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ 求 $A = I - \dfrac{2XX^H}{\|X\|^2} = ?$ $AX = ?$ 用干扰法求 $\lambda(A)$, A 是否为优阵

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{2}{3} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$AX = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$\lambda(A-I) = \lambda\left(-\frac{2XX^H}{\|X\|^2}\right) = \left\{ tr\left(-\frac{2XX^H}{\|X\|^2}\right), 0, 0 \right\} = \{-2, 0, 0\}$$

根据干扰法则 $\lambda(A) = \{-1, 1, 1\}$

$$A^H A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I, \quad 故 A 为优阵$$

Ex2. 对以下矩阵计算 QR 分解

① $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ 正交 $Q_0 = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$ 单位化 $Q = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$

$$R = Q^H A = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \frac{3}{2}\sqrt{2} \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$$

于是 $A = QR = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{3}{2}\sqrt{2} \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$ 是 A 的 QR 分解

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Ex2 (续) 对以下矩阵计算 QR 分解

① $A = \begin{pmatrix} 1 & 2 \\ i & i \end{pmatrix}$ 正交化 $Q_0 = \begin{pmatrix} 1 & \frac{1}{2} \\ i & -\frac{1}{2}i \end{pmatrix}$ 单位化 $Q = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2}i & -\frac{\sqrt{2}}{2}i \end{pmatrix}$

$R = Q^H A = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}i \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}i \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ i & i \end{pmatrix} = \begin{pmatrix} \sqrt{2} \cdot & \frac{3}{2}\sqrt{2} \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$

故 $A = QR = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2}i & -\frac{\sqrt{2}}{2}i \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & \frac{3}{2}\sqrt{2} \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$ 是 A 的 QR 分解

③ $A = \begin{pmatrix} i & i \\ -1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$ 正交化 $Q_0 = \begin{pmatrix} i & \frac{1}{2}i \\ -1 & \frac{3}{2} \\ 1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}$ 单位化 $Q = \begin{pmatrix} \frac{1}{2}i & \frac{i}{2\sqrt{3}} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & -\frac{1}{2\sqrt{3}} \\ \frac{1}{2} & -\frac{1}{2\sqrt{3}} \end{pmatrix}$

$R = Q^H A = \begin{pmatrix} -\frac{1}{2}i & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{i}{2\sqrt{3}} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} \end{pmatrix} \cdot \begin{pmatrix} i & i \\ -1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{pmatrix}$

故 $A = QR = \begin{pmatrix} \frac{1}{2}i & \frac{1}{2\sqrt{3}}i \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{pmatrix}$ 是 A 的 QR 分解

满秩分解：习题

$$A=\begin{pmatrix} 2 & 1 & 6 & 1 & 0 \\ 3 & 2 & 10 & 1 & 0 \\ 2 & 3 & 10 & -1 & 3 \\ 4 & 4 & 16 & 0 & -1 \end{pmatrix} \xrightarrow{r_2-r_1} \begin{pmatrix} 2 & 1 & 6 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 \\ 2 & 3 & 10 & -1 & 3 \\ 4 & 4 & 16 & 0 & -1 \end{pmatrix} \xrightarrow[\substack{r_3-2r_2 \\ r_4-4r_2}]{r_1-2r_2} \begin{pmatrix} 0 & -1 & -2 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\xrightarrow{r_1\leftrightarrow r_2} \begin{pmatrix} 1 & 1 & 4 & 0 & 0. \\ 0 & -1 & -2 & 1 & 0. \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \xrightarrow{r_3+r_2} \begin{pmatrix} 1 & 1 & 4 & 0 & 0. \\ 0 & -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \xrightarrow[\substack{r_3\leftrightarrow r_4}]{r_3+3r_4} \begin{pmatrix} 1 & 1 & 4 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[\substack{-r_2 \\ -r_3}]{r_1+r_2} \begin{pmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0. \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ 这说明 rank(A)=3 ，取第 1,2,5列为代表列

$$A=\underset{B}{\underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 2 & 3 & 3 \\ 4 & 4 & -1 \end{pmatrix}}} \times \underset{C}{\underbrace{\begin{pmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}} = \begin{pmatrix} 2 & 1 & 6 & 1 & 0 \\ 3 & 2 & 10 & 1 & 0 \\ 2 & 3 & 10 & -1 & 3 \\ 4 & 4 & 16 & 0 & -1 \end{pmatrix}$$

提 A=BC 为A的满秩分解