课后题答案 1-5 章（部分）

第一章

1.13

$$\begin{cases} U = X + Y & U \in (0, +\infty) \\ V = \dfrac{X}{X+Y} & V \in (0,1) \end{cases}$$

可得

$$X = UV$$
$$Y = U(1-V)$$

计算二维雅可比变换的雅可比因子

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -uv - u + uv = -u$$

$$\begin{aligned} f(u,v) &= f(x,y)\left|J\right| \\ &= \lambda^2 e^{-\lambda(uv+u-uv)} \cdot \left|-u\right| \\ &= \lambda^2 u e^{-\lambda u} \end{aligned}$$

$$f(u,v) = \begin{cases} \lambda^2 u e^{-\lambda u} & u > 0, 0 < v < 1 \\ 0 & \text{其他} \end{cases}$$

$$f(u)=\int_0^1 \lambda^2 u e^{-\lambda u}dv=\lambda^2 u e^{-\lambda u}$$

$$f(u)=\begin{cases}\lambda^2 u e^{-\lambda u} & u>0\\ 0 & \text{其他}\end{cases}$$

$$f(v)=\int_0^{\infty}\lambda^2 u e^{-\lambda u}du=1$$

$$f(v)=\begin{cases}1 & 0<v<1\\ 0 & \text{其他}\end{cases}$$

1.18

$$f(x,y)=f(x)f(y)=\frac{1}{2\sigma^2}\exp\{-\frac{x^2+y^2}{2\sigma^2}\},-\infty<x<\infty,-\infty<y<\infty$$

（1）由 $Z=\sqrt{X^2+Y^2}$，可得 $X=\pm\sqrt{Z^2-Y^2}$

$$\begin{aligned}F_Z(z)&=P\{Z\le z\}=P\{\sqrt{X^2+Y^2}\le z\}\\&=\int_{-z}^{z}\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}}f(x,y)dxdy\\&=4\int_0^z\int_0^{\sqrt{z^2-y^2}}\frac{1}{2\pi\sigma^2}\exp\{-\frac{x^2+y^2}{2\sigma^2}\}dxdy\end{aligned}$$

$$\begin{aligned}f(z)=\frac{dF_Z(z)}{dz}&=4\int_0^z\frac{1}{2\pi\sigma^2}\exp\{-\frac{z^2}{2\sigma^2}\}(\sqrt{z^2-y^2})'dy\\&=\frac{2z}{\pi\sigma^2}\exp\{-\frac{z^2}{2\sigma^2}\}\int_0^z\frac{1}{\sqrt{z^2-y^2}}dy\\&=\frac{2z}{\pi\sigma^2}\exp\{-\frac{z^2}{2\sigma^2}\}(\arcsin\frac{y}{z}\Big|_0^z)\\&=\frac{z}{\sigma^2}\exp\{-\frac{z^2}{2\sigma^2}\}\end{aligned}$$

所以 $f_Z(z)=\begin{cases}\frac{z}{\sigma^2}\exp\{-\frac{z^2}{2\sigma^2}\} & z\ge 0\\ 0 & \text{其他}\end{cases}$

$$
\begin{aligned}
F_W(w) &= P\{\frac{X}{Y} \le w\} \\
&= P\{X \le wY, Y > 0\} + P\{X \ge wY, Y < 0\} \\
&= \int_{y=0}^{y=\infty}\int_{x=-\infty}^{x=wy} f_{XY}(x,y)dxdy + \int_{y=-\infty}^{y=0}\int_{x=wy}^{x=\infty} f_{XY}(x,y)dxdy \\
&= 2\int_{y=0}^{y=\infty}\int_{x=-\infty}^{x=wy} f_{XY}(x,y)dxdy
\end{aligned}
$$

$$
\begin{aligned}
f_W(w) &= \frac{dF_W(w)}{dw} = 2\int_0^{\infty} y f_{XY}(yw, y)dy \\
&= 2\int_0^{\infty} y\frac{1}{2\pi\sigma^2}\exp\{-\frac{(w^2+1)y^2}{2\sigma^2}\}dy \\
&= \frac{1}{\pi(w^2+1)}
\end{aligned}
$$

$$
f_W(w) = \frac{1}{\pi(w^2+1)} \quad (-\infty < w < \infty)
$$

$$
\begin{aligned}
F_\Theta(\theta) &= P\{\Theta \le \theta\} = P\{\frac{X}{Y} \le \tan\theta\} \\
&= P\{X \le Y\tan\theta, Y > 0\} + P\{X \ge Y\tan\theta, Y < 0\} \\
&= 2\int_0^\infty \int_{-\infty}^{y\tan\theta} f_{XY}(x,y)dxdy \\
&=2\int_0^\infty \int_{-\infty}^{y\tan\theta} \frac{1}{2\pi\sigma^2}\exp\{-\frac{x^2+y^2}{2\sigma^2}\}dxdy
\end{aligned}
$$

$$
\begin{aligned}
f_\Theta(\theta) = \frac{dF_\Theta}{d\theta} &= 2\int_0^\infty \frac{y}{\cos^2\theta}\exp\{-\frac{(y\tan\theta)^2+y^2}{2\sigma^2}\}dy \\
&= \frac{2}{\pi}\int_0^\infty \frac{y}{2\sigma^2\cos^2\theta}\exp\{-\frac{y^2}{2\sigma^2\cos^2\theta}\}dy \\
&= \frac{1}{\pi}\int_0^\infty \exp\{-\frac{y^2}{2\sigma^2\cos^2\theta}\}d(\frac{y^2}{2\sigma^2\cos^2\theta}) \\
&= \frac{1}{\pi}\int_0^\infty \exp\{-u\}du \\
&= \frac{1}{\pi}
\end{aligned}
$$

$$
f_\Theta(\theta) = \begin{cases} \dfrac{1}{\pi} & -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2} \\ 0 & \text{其他} \end{cases}
$$

第二章

2.1

（1）令 $t_i=0$，则 $X(t_i=0)=A$，有

$$f_X(x;0)=f_A(x)\left|\frac{dA}{dX}\right|=\begin{cases}1 & 0\le x\le 1\\ 0 & \text{其他}\end{cases}$$

令 $t_i=\frac{\pi}{4\omega}$，则 $X(t_i=\frac{\pi}{4\omega})=\frac{\sqrt{2}}{2}A$，有

$$f_X(x;\frac{\pi}{4\omega})=f_A(\sqrt{2}x)\left|\frac{dA}{dX}\right|=\begin{cases}\sqrt{2} & 0\le x\le \frac{\sqrt{2}}{2}\\ 0 & \text{其他}\end{cases}$$

令 $t_i=\frac{3\pi}{4\omega}$，则 $X(t_i=\frac{3\pi}{4\omega})=-\frac{\sqrt{2}}{2}A$，有

$$f_X(x;\frac{3\pi}{4\omega})=f_A(-\sqrt{2}x)\left|\frac{dA}{dX}\right|=\begin{cases}\sqrt{2} & -\frac{\sqrt{2}}{2}\le x\le 0\\ 0 & \text{其他}\end{cases}$$

令 $t_i=\frac{\pi}{\omega}$，则 $X(t_i=\frac{\pi}{\omega})=-A$，有

$$f_X(x;\frac{\pi}{\omega})=f_A(-x)\left|\frac{dA}{dX}\right|=\begin{cases}1 & -1\le x\le 0\\ 0 & \text{其他}\end{cases}$$

（2）令 $t_i'=\frac{\pi}{2\omega}$，则 $X(t_i')=0$，有

$f_X(x)=\delta(x)$

一维随机变量的函数的分布函数，公式：$f_X(x)=f_A(a)\left|\frac{dA}{dX}\right|$

$X(t_i)=A\cos\omega t_i$，假设 $0<t_i<\frac{\pi}{2\omega}$

$$f(x_i)=\begin{cases}\frac{1}{\cos\omega t_i} & 0\le x_i\le\cos\omega t_i\\ 0 & \text{其他}\end{cases}$$

$\int_0^{\cos\omega t_i} f(x_i)dx_i=1$

当 $\cos\omega t_i\to 0^+$ 时，$f(x_i)\to\infty$

2.13

$$f_{A\Theta}(a,\theta)=f_A(a)f_\Theta(\theta)=\begin{cases}\dfrac{a}{2\pi\sigma^2}\exp(-\dfrac{a^2}{2\sigma^2}) & a>0,0\le\theta\le 2\pi\\ 0 & \text{其他}\end{cases}$$

$$m_X(t)=E[X(t)]$$

$$=\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}f_A(a)f_\Theta(\theta)a\cos(\omega t+\theta)d\theta]da$$

$$=\int_0^{\infty}\frac{a^2}{\sigma^2}\exp(-\frac{a^2}{\sigma^2})da\int_0^{2\pi}\frac{1}{2\pi}\cos(\omega t+\theta)d\theta$$

$$=0$$

$m_X(t)$ 与时间无关

$$R_X(t_1,t_2)=E[X(t_1)X(t_2)]$$

$$=\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}a^2\cos(\omega t_1+\theta)\cos(\omega t_2+\theta)f_A(a)f_\Theta(\theta)d\theta]d_a$$

$$=\int_0^{\infty}\frac{a^2}{2\sigma^2}\exp(-\frac{a^2}{2\sigma^2})da^2\int_0^{2\pi}\frac{\cos[\omega(t_1+t_2)+2\theta]+\cos[\omega(t_1-t_2)]}{2}\cdot\frac{1}{2\pi}d\theta$$

$$=\sigma^2\cos[\omega(t_1-t_2)]$$

令 $\tau=t_1-t_2$， 则 $R_X(\tau)=\sigma^2\cos\omega\tau$

又因为 $E[X^2(t)]=R_X(0)=\sigma^2<\infty$

故 $X(t)$ 是平稳随机过程

2.27

（1）

$$R_Z(\tau)=E[Z(t+\tau)Z(t)]$$
$$=E[X(t+\tau)Y(t+\tau)X(t)Y(t)]$$
$$=E[X(t+\tau)X(t)]E[Y(t+\tau)Y(t)]$$
$$=R_X(\tau)\cdot R_Y(\tau)$$

$$S_Z(\omega)=\frac{1}{2\pi}S_X(\omega)*S_Y(\omega)$$

（2）

$$S_X(\omega)=\frac{\sin^2(\omega/2)}{(\omega/2)^2}$$

$$R_Y(\tau)=\cos\omega_0\tau\text{，}\quad S_Y(\omega)=\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$$

$$S_Z(\omega)=\frac{1}{4}\{\frac{\sin^2[(\omega-\omega_0)/2]}{[(\omega-\omega_0)/2]^2}+\frac{\sin^2[(\omega+\omega_0)/2]}{[(\omega+\omega_0)/2]^2}\}$$

2.29

$$S_X(\omega)=\frac{\omega^2+4}{\omega^4+10\omega^2+9}=\frac{3}{8}\cdot\frac{1}{\omega^2+1}+\frac{5}{8}\cdot\frac{1}{\omega^2+9}$$

$$R_X(\tau)=\frac{3}{16}e^{-|\tau|}+\frac{5}{48}e^{-3|\tau|}\text{，}\quad \psi_X^2=R_X(0)=\frac{7}{24}$$

$$S_Y(\omega)=\frac{\omega^2}{\omega^4+3\omega^2+2}=-\frac{1}{\omega^2+1}+\frac{2}{\omega^2+2}$$

$$R_Y(\tau)=-\frac{1}{2}e^{-|\tau|}+\frac{1}{\sqrt{2}}e^{-\sqrt{2}|\tau|}\text{，}\quad \psi_Y^2=R_Y(0)=\frac{\sqrt{2}-1}{2}$$

2.30

由题图可得$Y(t) = X(t) + X(t+T)$

$$E[Y(t)Y(t-\tau)] = E\{[X(t) - X(t-T)][X(t-\tau) - X(t-T-\tau)]\}$$

$$= R_X(\tau) - R_X(\tau+T) - R_X(\tau-T) + R_X(\tau)$$

$$= 2R_X(\tau) - R_X(\tau+T) - R_X(\tau-T)$$

$$S_Y(\omega) = \int_{-\infty}^{\infty}[2R_X(\tau) - R_X(\tau+T) - R_X(\tau-T)]e^{-j\omega\tau}d\tau$$

$$= 2S_X(\omega) + \int_{-\infty}^{\infty}R_X(\tau')e^{-j\omega(\tau'-T)}d\tau' + \int_{-\infty}^{\infty}R_X(\tau'')e^{-j\omega(\tau''+T)}d\tau''$$

$$= 2S_X(\omega) + S_X(\omega)(e^{j\omega T} + e^{-j\omega T})$$

$$= 2S_X(\omega)(1 + \cos\omega T)$$

第三章

3.3

$$R_Y(t_1,t_2)=E[Y(t_1)Y(t_2)]=E[a\frac{dX(t_1)}{dt_1}\cdot a\frac{dX(t_2)}{dt_2}]$$

$$=a^2\frac{\partial^2 R_X(t_1,t_2)}{\partial t_1\partial t_2}\quad(\tau=t_1-t_2)$$

$$=a^2\frac{\partial^2 R_X(\tau)}{\partial\tau^2}\cdot\frac{\partial\tau}{\partial t_1}\cdot\frac{\partial\tau}{\partial t_2}$$

$$=-a^2\frac{\partial^2 R_X(\tau)}{\partial\tau^2}$$

$$\frac{\partial R_X(\tau)}{\partial\tau}=\sigma_X^2\cdot(-2\alpha^2\tau)e^{-\alpha^2\tau^2}$$

$$\frac{\partial^2 R_X(\tau)}{\partial\tau^2}=\sigma_X^2\cdot(-2\alpha^2+4\alpha^4\tau^2)e^{-\alpha^2\tau^2}=-2\sigma_X^2\alpha^2e^{-\alpha^2\tau^2}(1-2\alpha^2\tau^2)$$

$$R_Y(t_1,t_2)=2(1-2\alpha^2\tau^2)a^2\sigma_X^2\alpha^2e^{-\alpha^2\tau^2}$$

3.4

$$R_Y(\tau)=E[Y(t)Y(t-\tau)]$$

$$=E\{[X(t)+\dot X(t)][X(t-\tau)+\dot X(t-\tau)]\}$$

$$=E\{X(t)X(t-\tau)+X(t)\dot X(t-\tau)+\dot X(t)X(t-\tau)+\dot X(t)X(t-\tau)\}$$

$$=R_X(\tau)-R_X'(\tau)+R_X'(\tau)-R_X''(\tau)$$

$$=R_X(\tau)-R_X''(\tau)$$

$$=(3-4\tau^2)e^{-\tau^2}$$

3.5

一阶线性微分方程 $\frac{dy}{dx}+p(x)y=Q(x)$ 的通解：

$$\begin{aligned}y&=e^{-\int p(x)dx}(\int Q(x)e^{\int p(x)dx}dx+C)\\&=Ce^{-\int p(x)dx}+e^{-\int p(x)dx}\int Q(x)e^{\int p(x)dx}dx\end{aligned}$$

求 $E[Y(t)]$：

对微分方程两边取期望

$$\begin{cases}m_{\dot{Y}}(t)+2m_Y(t)=2 \quad t>0\\ m_Y(0)=1\end{cases}$$

$m_Y(t)=Ce^{-2t}+2e^{-2t}\cdot\frac{e^{2t}}{2}=Ce^{-2t}+1$，解得 $C=0$

所以 $m_Y(t)=1$

求 $R_{XY}(t_1,t_2)$：

改写微分方程为

$$\begin{cases}Y'(t_2)+2Y(t_2)=X(t_2)\\ Y(0)=1\end{cases}$$

$$\begin{cases} E[X(t_1)Y'(t_2)] + 2E[X(t_1)Y(t_2)] = E[X(t_1)X(t_2)] \\ E[X(t_1)Y(0)] = E[X(t_1)] \end{cases}$$

可得

$$\begin{cases} \dfrac{\partial R_{XY}(t_1,t_2)}{\partial t_2} + 2R_{XY}(t_1,t_2) = R_X(t_1,t_2) = 4 + 2e^{-|t_1-t_2|} \\ R_{XY}(t_1,0) = 2 \end{cases}$$

分类讨论

当 $t_1 > t_2$ 时，解得 $C = -\frac{2}{3}e^{-t_1}$， $R_{XY}(t_1,t_2) = 2 + \frac{2}{3}e^{-t_1+t_2} - \frac{2}{3}e^{-t_1-2t_2}$

当 $t_1 < t_2$ 时，解得 $C = -2e^{t_1}$， $R_{XY}(t_1,t_2) = 2 + 2(e^{t_1-t_2} + e^{t_1-2t_2})$

求 $R_Y(t_1,t_2)$：

将原方程改写为

$$\begin{cases} \dfrac{\partial R_Y(t_1,t_2)}{\partial t_1} + 2R_Y(t_1,t_2) = R_{XY}(t_1,t_2) \\ R_Y(0,t_2) = 1 \end{cases}$$

当 $t_1 > t_2$ 时，解得 $R_Y(t_1,t_2) = 1 + \frac{2}{3}(e^{-t_1+t_2} - e^{-t_1-2t_2} + e^{-2t_1-2t_2} - e^{-2t_1+t_2})$

当 $t_1 < t_2$ 时，解得 $R_Y(t_1,t_2) = 1 + \frac{2}{3}(e^{t_1-t_2} - e^{t_1-2t_2} + e^{-2t_1-2t_2} - e^{-2t_1-t_2})$

3.10

$$RC\frac{dy(t)}{dt}+y(t)=x(t)$$

对上式两边同时取傅里叶变换，利用微分性质可得

$$RC\cdot j\omega Y(j\omega)+Y(j\omega)=X(j\omega)$$

则传递函数为

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{1}{1+j\omega RC}=\frac{a}{a+j\omega}\quad (a=\frac{1}{RC})$$

输出过程的物理功率谱密度为

$$F_Y(\omega)=\left|H(j\omega)\right|^2 F_X(\omega)=\frac{a^2N_0}{a^2+\omega^2}\quad (0<\omega<\infty)$$

输出过程的自相关函数为

$$R_Y(\tau)=\mathcal{F}^{-1}[\frac{F_Y(\omega)}{2}]=\frac{N_0}{4}ae^{-a|\tau|}\quad (a=\frac{1}{RC})$$

由于$t_3>t_2>t_1$，则有

$$\frac{R_Y(t_3-t_2)R_Y(t_2-t_1)}{R_Y(0)}=\frac{\frac{N_0a}{4}e^{-(t_3-t_2)}\cdot\frac{N_0a}{4}e^{-(t_2-t_1)}}{\frac{N_0a}{4}}$$

$$=\frac{N_0a}{4}e^{-(t_3-t_1)}=R_Y(t_3-t_1)$$

故

$$R_Y(t_3-t_1)=\frac{R_Y(t_3-t_2)R_Y(t_2-t_1)}{R_Y(0)}$$

3.13

输出过程的自相关函数为

$$R_Y(t_1,t_2)=E[\int_0^{t_1}X(u)du\int_0^{t_2}X(v)dv]$$
$$=\int_0^{t_1}\int_0^{t_2}E[X(u)X(v)]dudv$$

当 $t_1<t_2$ 时，令 $\tau=u-v, d\tau=-dv$ ，则有

$$R_Y(t_1,t_2)=\int_0^{t_1}[\int_u^{u-t_2}\sigma^2\delta(\tau)d(-\tau)]du$$
$$=\int_0^{t_1}[\int_{u-t_2}^{u}\sigma^2\delta(\tau)d(\tau)]du$$
$$=\int_0^{t_1}\sigma^2du$$
$$=\sigma^2t_1$$

当 $t_1>t_2$ 时，令 $\tau=u-v, d\tau=du$ ，则有

$$R_Y(t_1,t_2)=\int_0^{t_2}[\int_{-v}^{t_1-v}\sigma^2\delta(\tau)d\tau]dv$$
$$=\int_0^{t_2}[\int_{-v}^{t_1-v}\sigma^2\delta(\tau)d(\tau)]dv$$
$$=\int_0^{t_2}\sigma^2dv$$
$$=\sigma^2t_2$$

综上可得

$$R_Y(t_1,t_2)=\sigma^2\min(t_1,t_2)$$

$$\psi^2=E[Y^2(t)]=R_Y(t,t)=\sigma^2t$$

3.16

$$R_W(t_1,t_2)=\delta(t_1-t_2)$$

$$R_X(t_1,t_2)=\delta(t_1-t_2)\cdot[U(t_1)-U(t_1-T)]\cdot[U(t_2)-U(t_2-T)]$$

$$R_X(t_1,t_2)=\begin{cases}\delta(t_1-t_2) & 0\le t_1\le T,0\le t_2\le T\\ 0 & \text{其他}\end{cases}$$

$$h(t)=e^{-\alpha t}U(t),\alpha>0$$

输出 $Y(t)$ 自相关函数为

$$\begin{aligned}R_Y(t_1,t_2)&=E[Y(t_1)Y(t_2)]=E[\int_{-\infty}^{\infty}X(u)h(t_1-u)du\int_{-\infty}^{\infty}X(v)h(t_2-v)dv]\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E[X(u)X(v)]h(t_1-u)h(t_2-v)dudv\\&=e^{-\alpha(t_1+t_2)}\int_0^{t_2}\int_0^{t_1}\delta(u-v)e^{\alpha(u+v)}dudv\end{aligned}$$

令 $t=\min(t_1,t_2)$，则有

（I）当 $t<0$ 时，$R_Y(t_1,t_2)=0$

（II）当 $0\le t\le T$ 时，

假设 $t_1>t_2$，$\int_0^{t_2}\int_0^{t_1}\delta(u-v)e^{\alpha(u+v)}dudv=\int_0^{t_2}e^{2\alpha v}dv=\dfrac{e^{2\alpha t_2}-1}{2\alpha}$

假设 $t_1<t_2$，$\int_0^{t_2}\int_0^{t_1}\delta(u-v)e^{\alpha(u+v)}dudv=\int_0^{t_1}e^{2\alpha u}du=\dfrac{e^{2\alpha t_1}-1}{2\alpha}$

整理可得 $R_Y(t_1,t_2)=\dfrac{1}{2\alpha}(e^{2\alpha t}-1)e^{-\alpha(t_1+t_2)}$

（III）当$t \geq T$时，$R_Y(t_1,t_2)=\frac{1}{2\alpha}(e^{2\alpha T}-1)e^{-\alpha(t_1+t_2)}$

故

$$R_Y(t_1,t_2)=\begin{cases} 0 & t \leq 0 \\ \frac{1}{2\alpha}(e^{2\alpha t}-1)e^{-\alpha(t_1+t_2)} & 0<t\leq T \\ \frac{1}{2\alpha}(e^{2\alpha T}-1)e^{-\alpha(t_1+t_2)} & t>T \end{cases}$$

式中$t=\min(t_1,t_2)$

3.19

（1）由系统框图可得

$$Y(t)=X(t)-X(t-T)=X(t)*[\delta(t)-\delta(t-T)]$$

$$Z(t)=\int_{-\infty}^{\infty}Y(\lambda)u(t-\lambda)d\lambda=Y(t)*u(t)$$

故有

$$Z(t)=X(t)*[\delta(t)-\delta(t-T)]*u(t)=X(t)*[u(t)-u(t-T)]$$

$$h(t)=u(t)-u(t-T)$$

$$H(j\omega)=T\mathrm{Sa}(\frac{\omega T}{2})e^{-j\frac{\omega T}{2}}$$

（2） $S_Z(\omega)=\left|H(j\omega)\right|^2 S_X(\omega)=S_0T^2\mathrm{Sa}^2(\frac{\omega T}{2})$

$$\psi_Z^2=R_Z(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_0T^2\mathrm{Sa}^2(\frac{\omega T}{2})d\omega$$

$$=\frac{2}{\pi}S_0T\int_0^{\infty}\frac{\sin^2 x}{x^2}dx \quad (x=\frac{\omega T}{2})$$

$$=S_0T$$

3.28

（1）求 $m_Y(t)$：

$m_X(t)=E[X(t)]=E[W(t)*h_1(t)]=E[W(t)]*h_1(t)=0$

$m_Y(t)=E[Y(t)]=E[X(t)-X(t-T)]=0$

（2）求 σ_Y^2

对于系统 1，由 $h_1(t)=e^{-\alpha t}U(t),\alpha>0$ 可得 $H_1(j\omega)=\dfrac{1}{\alpha+j\omega}$

对于系统 2，由 $Y(t)=X(t)-X(t-T)=X(t)*[\delta(t)-\delta(t-T)]$

$h_2(t)=\delta(t)-\delta(t-T)$ 可得 $H_2(j\omega)=1-e^{-j\omega T}$

故

$$\begin{aligned}S_Y(\omega)&=\left|H_1(j\omega)\right|^2\left|H_2(j\omega)\right|^2S_X(\omega)\\&=\frac{N_0}{2}\cdot\frac{2-2\cos\omega T}{\alpha^2+\omega^2}\end{aligned}$$

$$\sigma_Y^2=R_Y(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_Y(\omega)d\omega=\frac{N_0}{2\alpha}(1-e^{-\alpha T})$$

故

$$P_Y(y)=\frac{1}{\sigma_Y\sqrt{2\pi}}\exp\{-\frac{y^2}{2\sigma_Y^2}\}$$

式中 $\sigma_Y^2=\dfrac{N_0}{2\alpha}(1-e^{-\alpha T}),\sigma_Y>0$

第四章

4.13

$$
\begin{aligned}
R_{sc}(\tau) &= E\left\{\left[\hat{X}(t)\cos\omega_0 t - X(t)\sin\omega_0 t\right]\times\left[X(t-\tau)\cos\omega_0(t-\tau)+\hat{X}(t-\tau)\sin\omega_0(t-\tau)\right]\right\} \\
&= E\left[\hat{X}(t)X(t-\tau)\right]\cos\omega_0 t\cos\omega_0(t-\tau) \\
&\quad + E\left[\hat{X}(t)\hat{X}(t-\tau)\right]\cos\omega_0 t\sin\omega_0(t-\tau) \\
&\quad - E\left[X(t)X(t-\tau)\right]\sin\omega_0 t\cos\omega_0(t-\tau) \\
&\quad - E\left[X(t)\hat{X}(t-\tau)\right]\sin\omega_0 t\sin\omega_0(t-\tau) \\
&= R_{\hat{X}X}(\tau)\left[\cos\omega_0 t\cos\omega_0(t-\tau)-\sin\omega_0 t\sin\omega_0(t-\tau)\right]-R_X(\tau)\left[\sin\omega_0 t\cos\omega_0(t-\tau)-\cos\omega_0 t\sin\omega_0(t-\tau)\right] \\
&= R_{\hat{X}}(\tau)\cos\omega_0\tau - R_X(\tau)\sin\omega_0\tau
\end{aligned}
$$

4.19

（1）

$n(t)$ 和 $\hat{n}(t)$：

$$n(t)=n_c(t)\cos\omega_0 t-n_s(t)\sin\omega_0 t$$

$$\hat{n}(t)=n_c(t)\sin\omega_0 t+n_s(t)\cos\omega_0 t$$

两正交分量 $n_c(t)$ 和 $n_s(t)$：

$$n_c(t)=n(t)\cos\omega_0 t+\hat{n}(t)\sin\omega_0 t$$

$$n_s(t)=\hat{n}(t)\cos\omega_0 t-n(t)\sin\omega_0 t$$

两正交分量的自相关函数 $R_{n_c}(\tau)$ 和 $R_{n_s}(\tau)$：

$$R_{n_c}(\tau)=R_{n_s}(\tau)=R_n(\tau)\cos\omega_0\tau+\hat{R}_n(\tau)\sin\omega_0\tau$$

两正交分量的功率谱密度 $G_{n_c}(\omega)$ 和 $G_{n_s}(\omega)$：

$$G_{n_c}(\omega)=G_{n_s}(\omega)=\begin{cases}G_n(\omega-\omega_0)+G_n(\omega+\omega_0) & |\omega|<\omega_c\\ 0 & \text{其他}\end{cases}$$

（2）

两正交分量的互相关函数 $R_{cs}(\tau)$ 和 $R_{sc}(\tau)$：

$$R_{cs}(\tau)=R_n(\tau)\sin\omega_0\tau-\hat{R}_n(\tau)\cos\omega_0\tau$$

$$R_{sc}(\tau)=\hat{R}_n(\tau)\cos\omega_0\tau-R_n(\tau)\sin\omega_0\tau$$

$$R_{cs}(\tau)=-R_{sc}(\tau)$$

两正交分量的互功率谱密度 $G_{cs}(\omega)$ 和 $G_{sc}(\omega)$：

$$G_{cs}(\omega)=-G_{sc}(\omega)=\begin{cases}-j\left[G_n(\omega-\omega_0)-G_n(\omega+\omega_0)\right] & |\omega|<\omega_c\\ 0 & \text{其他}\end{cases}$$

4.22

（1）

$$R_{Y_1Y_2}(\tau)=R_X(\tau)*h_1(\tau)*h_2(-\tau)$$

$$S_{Y_1Y_2}(\omega)=S_X(\omega)H_1(j\omega)H_2^*(j\omega)=\frac{N_0}{2}H_1(j\omega)H_2^*(j\omega)$$

$R_{Y_1Y_2}(\tau)$ 为偶函数等价于 $S_{Y_1Y_2}(\omega)$ 为偶函数

所以当 $H_1(j\omega)H_2^*(j\omega)$ 为实对称函数时，互相关函数 $R_{Y_1Y_2}(\tau)$ 为偶函数

（2）

$Y_1(t)$， $Y_2(t)$ 统计独立等价于 $Y_1(t)$， $Y_2(t)$ 不相关，因此有 $R_{Y_1Y_2}(\tau)=0$

因此 $h_1(t)$ 和 $h_2(t)$ 应满足 $h_1(t)*h_2(-t)=0$

在频域里 $H_1(j\omega)H_2^*(j\omega)=0$

即在频域里要求两个系统的通带不混叠

第五章

5.5

由于U和V是统计独立的高斯随机变量，容易知道$X(t)$是高斯随机过程，因此欲求其一维和二维概率密度函数，只需求其前两阶矩即可。

先求一维概率密度函数。因为

$$m_X(t)=E\left\{X(t)\right\}=E(U)\cos\omega t+E(V)\sin\omega t=0$$

和

$$\sigma_X^2(t)=E\left\{X^2(t)\right\}=E(U^2)\cos^2\omega t+E(V^2)\sin^2\omega t+2E(UV)\sin\omega t\cos\omega t=\sigma^2$$

故有

$$p_X(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{x^2}{2\sigma^2}\}$$

现在求二维概率密度函数。因为

$$\boldsymbol{a}=\begin{bmatrix}m_X(t_1)\\m_X(t_2)\end{bmatrix}=\boldsymbol{0}$$

$$\boldsymbol{C}=\begin{bmatrix}E\left\{X^2(t_1)\right\} & E\left\{X(t_1)X(t_2)\right\}\\E\left\{X(t_2)X(t_1)\right\} & E\left\{X^2(t_2)\right\}\end{bmatrix}$$

$$p_X(x_1,x_2;\tau)=\frac{1}{2\pi|\boldsymbol{C}|^{1/2}}\exp\left\{-\frac{1}{2}[x_1,x_2]\boldsymbol{C}\begin{bmatrix}x_1\\x_2\end{bmatrix}\right\}$$

5.6

对于任意时刻t，$Y(t)$为一个高斯随机变量。这样，$Y(t)$的均值和方差分别为

$$m_Y(t)=E\{Y(t)\}=\frac{1}{\varepsilon}\left[E\{X(t+\varepsilon)\}-E\{X(t)\}\right]=0$$

$$\begin{aligned}\sigma_Y^2(t)&=E\{Y^2(t)\}\\&=\frac{1}{\varepsilon^2}\left[E\{X^2(t+\varepsilon)\}+E\{X^2(t)\}-2E\{X(t+\varepsilon)X(t)\}\right]\\&=\frac{2}{\varepsilon^2}[R_X(0)-R_X(\varepsilon)]\end{aligned}$$

故有

$$p_Y(y)=\frac{1}{\sqrt{4\pi\left[R_X(0)-R_X(\varepsilon)\right]/\varepsilon^2}}\exp\left\{-\frac{\varepsilon^2y^2}{4\left[R_X(0)-R_X(\varepsilon)\right]}\right\}$$

注意到$R_X(\tau)$为偶函数，故$R_X'(0)=0$。故有

$$\begin{aligned}\lim_{\varepsilon\to0}\frac{2\left[R_X(0)-R_X(\varepsilon)\right]}{\varepsilon^2}&=\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\cdot\frac{1}{\varepsilon}\left\{\left[R_X(0)-R_X(\varepsilon)\right]+\left[R_X(0)-R_X(\varepsilon)\right]\right\}\\&=\lim_{\varepsilon\to0}\frac{-\left[R_X'(0)+R_X'(\varepsilon)\right]}{\varepsilon}\\&=\lim_{\varepsilon\to0}\frac{-R_X'(\varepsilon)}{\varepsilon}\\&=-R_X''(0)\end{aligned}$$

于是

$$p_Y(y)\xrightarrow[\varepsilon\to0]{}\frac{1}{\sqrt{2\pi\left[-R_X''(0)\right]}}\exp\left\{\frac{y^2}{2R_X''(0)}\right\}$$

问题：

$$\begin{aligned}\lim_{\varepsilon\to0}\frac{2\left[R_X(0)-R_X(\varepsilon)\right]}{\varepsilon^2}&=\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\cdot\frac{1}{\frac{\varepsilon}{2}}\left\{\left[R_X(0)-R_X(\frac{\varepsilon}{2})\right]+\left[R_X(\frac{\varepsilon}{2})-R_X(\varepsilon)\right]\right\}\\&=\lim_{\varepsilon\to0}\frac{-\left[R_X'(0)+R_X'(\frac{\varepsilon}{2})\right]}{\varepsilon}\\&=\lim_{\varepsilon\to0}\frac{-R_X'(\frac{\varepsilon}{2})}{\varepsilon}\\&=-\frac{1}{2}R_X''(0)\end{aligned}$$

5.7

$$E(Y)=E\left\{\int_0^1 X(t)dt\right\}=\int_0^1 E\{X(t)\}dt=0$$

$$\begin{aligned}E(Y^2)&=E\left\{\int_0^1 X(t)dt\int_0^1 X(\tau)d\tau\right\}\\&=\int_0^1\int_0^1 E\{X(t)X(\tau)\}dtd\tau\\&=\int_0^1\int_0^1 e^{-|t-\tau|}dtd\tau\\&=\int_0^1\left[\int_0^t e^{-(t-\tau)}d\tau+\int_t^1 e^{t-\tau}d\tau\right]dt\\&=2e^{-1}\end{aligned}$$

于是可得

$$p_Y(y)=\frac{1}{\sqrt{4\pi e^{-1}}}\exp\left\{-\frac{y^2}{4e^{-1}}\right\}$$

5.8

（1）

Z 是高斯随机变量，可得 $E(Z)=0$， $E(Z^2)=\sigma^2$

$$p_Z(z)=\frac{1}{\sqrt{2\pi\sigma}}\exp\{-\frac{z^2}{2\sigma^2}\}$$

（2）

先计算随机变量 R 的分布函数

$$\begin{aligned}F_R(r)&=P(R\le r)=P\left(\sqrt{X^2+Y^2}\le r\right)\\&=\iint_{\sqrt{x^2+y^2}\le r}p(x,y)dxdy\\&=\int_0^{2\pi}\int_0^r r\cdot\frac{1}{2\pi\sigma^2}\exp\{-\frac{r^2}{2\sigma^2}\}drd\theta\\&=\int_0^r\frac{r}{\sigma^2}\exp\{-\frac{r^2}{2\sigma^2}\}dr\end{aligned}$$

相应的概率密度函数为

$$p_R(r)=F_R'(r)=\begin{cases}\frac{r}{\sigma^2}\exp\{-\frac{r^2}{2\sigma^2}\} & r\ge 0\\ 0 & 其他\end{cases}$$

（3）

根据题意，该样本是

$$Z(0)=X,\quad Z(\frac{1}{4})=Y,\quad Z(\frac{1}{2})=-X$$

故 $Z(0)$ 和 $Z(\frac{1}{4})$ 统计独立，而 $Z(\frac{1}{2})$ 是 $Z(0)$ 的线性函数，由 $p_Z(z)=\frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{z^2}{2\sigma^2}\}$ 可得

$$p_{z_1z_2}(z_1,z_2)=\frac{1}{2\pi\sigma^2}\exp\left\{-\frac{z_1^2+z_2^2}{2\sigma^2}\right\}$$

$$p(z_3\mid z_1z_2)=p(z_3\mid z_1)=\delta(z_3+z_1)$$

故

$$p(z_1,z_2,z_3)=\frac{1}{2\pi\sigma^2}\exp\{-\frac{z_1^2+z_2^2}{2\sigma^2}\}\delta(z_1+z_3)$$

5.13

设线性变换为

$$\begin{aligned}
Y_1&=L_{11}X_1+L_{12}X_2+\cdots+L_{1n}X_n\\
Y_2&=L_{21}X_1+L_{22}X_2+\cdots+L_{2n}X_n\\
&\vdots\\
Y_n&=L_{n1}X_1+L_{n2}X_2+\cdots+L_{nn}X_n
\end{aligned}$$

线性变换矩阵形式为

$$\boldsymbol{Y}=\boldsymbol{L}\boldsymbol{X}$$

式中

$$\boldsymbol{L}=\begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1n}\\ L_{21} & L_{22} & \cdots & L_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ L_{n1} & L_{n2} & \cdots & L_{nn}\end{bmatrix}$$

$$\boldsymbol{Y}=\left[Y_1,Y_2,\cdots,Y_n\right]^{\mathrm{T}},\quad \boldsymbol{X}=\left[X_1,X_2,\cdots,X_n\right]^{\mathrm{T}}$$

$$f_X(x)=\frac{1}{(2\pi)^{\frac{n}{2}}|\boldsymbol{C}|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{a})^{\mathrm{T}}\boldsymbol{C}^{-1}(\boldsymbol{x}-\boldsymbol{a})\right\}$$

为了方便推导，假定其均值矢量为零 $\boldsymbol{a}=\mathbf{0}$

令 $\boldsymbol{\Gamma}=\boldsymbol{L}^{-1}$，则有

$$\boldsymbol{X}=\boldsymbol{\Gamma}\boldsymbol{Y}$$

$$p_Y(\boldsymbol{y}) = p(\boldsymbol{\Gamma y})|\boldsymbol{J}|$$

$$|\boldsymbol{J}| = \left|\frac{\partial \boldsymbol{\Gamma Y}}{\partial \boldsymbol{Y}}\right| = |\boldsymbol{\Gamma}| = 1/|\boldsymbol{L}|$$

于是有

$$f_Y(y) = \frac{1}{(2\pi)^{\frac{n}{2}}\left(|\boldsymbol{L}|^2|\boldsymbol{C}|\right)^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}(\boldsymbol{\Gamma y})^{\mathrm{T}}\boldsymbol{C}^{-1}(\boldsymbol{\Gamma y})\right\}$$
$$= \frac{1}{(2\pi)^{\frac{n}{2}}\left(|\boldsymbol{L}|^2|\boldsymbol{C}|\right)^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\boldsymbol{y}^{\mathrm{T}}\left(\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{C}^{-1}\boldsymbol{\Gamma}\right)\boldsymbol{y}\right\}$$

令 $\boldsymbol{F}^{-1} = \boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{C}^{-1}\boldsymbol{\Gamma} = \left(\boldsymbol{L}^{\mathrm{T}}\right)^{-1}\boldsymbol{C}^{-1}\boldsymbol{L}^{-1} = \left(\boldsymbol{LCL}^{\mathrm{T}}\right)^{-1}$， $|\boldsymbol{F}| = |\boldsymbol{L}||\boldsymbol{C}|\left|\boldsymbol{L}^{\mathrm{T}}\right| = |\boldsymbol{L}|^2\boldsymbol{C}$

所以有

$$f_Y(y) = \frac{1}{(2\pi)^{\frac{n}{2}}|\boldsymbol{F}|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{b})^{\mathrm{T}}\boldsymbol{F}^{-1}(\boldsymbol{y}-\boldsymbol{b})\right\}$$

5.15

$$E[A(t)] = \int_{-\infty}^{\infty} Ap(A)dA = \int_0^{\infty} A\frac{A}{\sigma_X^2}e^{-\frac{A^2}{2\sigma_X^2}}dA = \int_0^{\infty} -Ade^{-\frac{A^2}{2\sigma_X^2}},(A>0)$$
$$= -Ae^{-\frac{A^2}{2\sigma_X^2}}\Big|_0^{\infty} + \int_0^{\infty} e^{-\frac{A^2}{2\sigma_X^2}}dA = 0 + \sqrt{2\pi}\int_0^{\infty}\frac{1}{\sqrt{2\pi}\sigma_X}e^{-\frac{A^2}{2\sigma_X^2}}dA = \sqrt{2\pi}\sigma_X \times \frac{1}{2} = \sqrt{\frac{\pi}{2}}\sigma_X$$

$$E\left[A^2(t)\right] = \int_0^{\infty} A^2\frac{A}{\sigma_X^2}e^{-\frac{A^2}{2\sigma_X^2}}dA = 2\sigma_X^2\int_0^{\infty}\frac{A^2}{2\sigma_X^2}e^{-\frac{A^2}{2\sigma_X^2}}d\frac{A^2}{2\sigma_X^2} = 2\sigma_X^2\int_0^{\infty}xde^{-x},(x>0)$$
$$= 2\sigma_X^2$$

$$D[A(t)] = E\left[A^2(t)\right] - E^2[A(t)] = \left(2-\frac{\pi}{2}\right)\sigma_X^2$$

5.16

$$R_Y(\tau)=E\left[Y(t)Y(t-\tau)\right]=E\left[X^2(t)X^2(t-\tau)\right]$$

$$E\left[X_1^n X_2^k\right]=(-j)^{n+k}\left.\frac{\partial^{n+k}\phi_X(v_1,v_2)}{\partial v_1^n \partial v_2^k}\right|_{v_1=v_2=0}$$

随机变量的各阶矩可以通过对特征函数求导得到

$$E\left[X^2(t)X^2(t-\tau)\right]=(-j)^4\left.\frac{\partial^4\phi_X(v_1,v_2;t,t-\tau)}{\partial v_1^2 \partial v_2^2}\right|_{v_1=v_2=0}$$

已知 $X(t)$ 为均值为 0 的高斯平稳随机过程，特征函数为

$$\phi_X(v_1,v_2;\tau)=\exp\left\{j\boldsymbol{a}^{\mathrm{T}}\boldsymbol{v}-\frac{1}{2}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{v}\right\}$$

式中

$$\boldsymbol{a}=\begin{bmatrix}a_1\\a_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\quad \boldsymbol{v}=\begin{bmatrix}v_1\\v_2\end{bmatrix},\quad \boldsymbol{C}=\begin{bmatrix}R_X(0) & R_X(\tau)\\R_X(\tau) & R_X(0)\end{bmatrix}$$

所以

$$\phi_X(v_1,v_2;\tau)=\exp\left\{-\frac{1}{2}\left[R_X(0)v_1^2+2R_X(\tau)v_1v_2+R_X(0)v_2^2\right]\right\}$$

故

$$R_Y(\tau)=(-j)^4\left.\frac{\partial^4\exp\left\{-\frac{1}{2}\left[R_X(0)v_1^2+2R_X(\tau)v_1v_2+R_X(0)v_2^2\right]\right\}}{\partial v_1^2\partial v_2^2}\right|_{v_1=v_2=0}$$

$$=\left[R_X(0)\right]^2+2\left[R_X(\tau)\right]^2$$