期末考试：14:15 ~ 16:15

一、简答题

1. 两随机过程：

独立：概率分布可乘 $f_{XY}(x_1,y_1;t_1,t_2)=f_X(x_1;t_1)\cdot f_Y(y_1;t_2)$.

不相关：相关系数为零，$r(t_1,t_2)=\dfrac{Cov(t_1,t_2)}{\sqrt{Cov(t_1,t_1)}\sqrt{Cov(t_2,t_2)}}$

正交：乘积的期望为零 即 $E\{X(t)Y(t)\}=0$.

关系：独立随机过程一定互不相关，互不相关的随机过程不一定独立

互不相关的高斯过程相互独立

2. 设 $X(t)$, $Y(t)$ 为实随机过程 则 $Z(t)=X(t)+jY(t)$ 为复随机过程

均值 $E[Z(t)]=E[X(t)]+j\cdot E[Y(t)]$

方差：$D[Z(t)]=E\{|Z(t)-E[Z(t)]|^2\}=E\{Z(t)Z^*(t)\}-(E^2[X(t)]+E^2[Y(t)])$

复平稳指：$X(t)$ 与 $Y(t)$ 均平稳且 $R_Z(t_1,t_2)$ 只与 t_1-t_2 有关，要求 $Z(t)$ 方差有限

自相关函数 $R_Z(t_1,t_2)=E\{Z(t_1)Z^*(t_2)\}$

3. 设 C 为 N维高斯变量的协方差矩阵 则

$$f_X(x_1,\dots,x_n)=\frac{1}{(\sqrt{2\pi})^n|C|^{\frac{1}{2}}}\cdot e^{-\frac{1}{2}\{(x-a)^T C^{-1}(x-a)\}}$$

当 C 为对角阵 时高斯变量互不相关

$$=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\cdot\sigma_i}\cdot e^{-\frac{1}{2}\left\{\frac{(x_i-a)^2}{\sigma_i^2}\right\}}$$

即满足概率密度的可乘性.

得不相关高斯过程独立，又因为独立一定不相关，故高斯过程独立与不相关等价。

4. 泊松过程是满足 (1)零初值 (2)单位跳跃 (3)独立增量 (4)平稳增量的随机过程. 是概率质量为 $P\{N(t)=k\}=\dfrac{e^{-\lambda t}(\lambda t)^k}{k!}$

$E[N(t)]=\lambda t$，$D[N(t)]=\lambda t$，$R_N(t_1,t_2)$ 设 $t_1<t_2$ $=E\{N(t_1)\cdot\{N(t_1)+S(t_1,t_2)\}\}=\lambda\min(t_1,t_2)+\lambda^2\min(t_1,t_2)\cdot|t_2-t_1|$

二. $Z(t) = X\sin\omega_0 t + Y\cos\omega_0 t$.

1.

$E[Z(t)] = (\sin\omega_0 t + \cos\omega_0 t)\cdot E[X]$　　$E[X] = \frac{2}{3}\times(-1) + \frac{1}{3}\times 2 = 0$

$= 0$

$D[Z(t)] = E\{Z^2(t)\} - \cancel{E^2[Z(t)]}^{0} = E\{X^2\sin^2\omega_0 t + Y^2\cos^2\omega_0 t + \cancel{2XY\sin\omega_0 t\cos\omega_0 t}^{0}\}$

$= \sin^2\omega_0 t\cdot E[X^2] + \cos^2\omega_0 t\, E[Y^2]$.　其中 $E[X^2] = \frac{2}{3}\times(-1)^2 + \frac{1}{3}\times(2)^2 = 2$.

$= 2$.

$R_Z(t_1,t_2) = E\{Z(t_1)Z(t_2)\} = E\{\{X\sin\omega_0 t_1 + Y\cos\omega_0 t_1\}\{X\sin\omega_0 t_2 + Y\cos\omega_0 t_2\}\}$

$= E\{X^2\sin\omega_0 t_1\sin\omega_0 t_2 + \cancel{XY}^{0}(\sin\omega_0 t_1\cos\omega_0 t_2 + \cos\omega_0 t_1\sin\omega_0 t_2) + Y^2\cos\omega_0 t_1\cdot\cos\omega_0 t_2\}$

$= (\sin\omega_0 t_1\sin\omega_0 t_2 + \cos\omega_0 t_1\cos\omega_0 t_2)\cdot E[X^2] = 2\cdot\cos\omega_0(t_1 - t_2)$,

2. ~~由于 E[X]~~

由于 $E[Z^2(t)] = D(Z(t)) + E^2(Z(t)) = 2 < \infty$ 故为二阶矩过程

$E[Z(t)] = 0$ 与时间无关. $R_Z(t_1,t_2) = 2\cos\omega_0(t_1 - t_2)$ 只与时间差有关 故广义平稳.

不是狭义平稳过程. 狭义平稳要求 $\forall t$ $f_Z(z,t)$ 均不变.

但 $f_Z(z,0) = \begin{cases} \frac{2}{3}, & z = -1 \\ \frac{1}{3}, & z = 2. \end{cases}$　　$f_Z(z,\frac{\pi}{4}) = \begin{cases} \frac{4}{9}, & z = -2\sqrt{2}. \\ \frac{4}{9} & z = \sqrt{2}. \\ \frac{1}{9}, & z = 4\sqrt{2}. \end{cases}$

不符合. 狭义平稳的要求.

3. $\overline{Z(t)} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} Z(t)\,d(t) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[X\sin\omega_0 t + Y\cos\omega_0 t]\,dt$.

$= \lim_{T\to\infty}\frac{1}{2T}\Big[\underbrace{-\frac{X}{\omega_0}\cdot\cos\omega_0 t + \frac{Y}{\omega_0}\sin\omega_0 t}_{有界}\Big]_{-T}^{T} = 0$ 故符合要求.

三.

$$1.\ E[X(t)] = \underbrace{E\{\cdots\}}_{=0} = \sum_{i=0}^{n}\left\{E\{\cancelto{0}{u_i+v_i}\}\cos w_i t + E\{\cancelto{0}{u_i-v_i}\}\cdot j\sin w_i t\right\}$$

$$R_X(t_1,t_2) = E\{X(t_1)X(t_2)\} = E\left\{\left(\sum_{i=1}^{n} X_i(t_1)\right)\left(\sum_{j=1}^{n} X_j^*(t_2)\right)\right\}$$

其中 $X_i(t) = (u_i+v_i)\cos w_i t + (u_i-v_i)j\sin w_i t$ 故 $i\neq j$ 时 $X_i(t_1)$ 与 $X_j(t_2)$ 正交

且 $E[X_i(t)]=0$. 故 $R_X(t_1,t_2) = E\left\{\sum_{i=1}^{n} X_i(t_1)X_i^*(t_2)\right\}$

$$= E\{\cancel{(u_i+v_i)^2\cos w_i t + (u_i-v_i)^2 \sin w_i t}\}$$

$$= E\left\{\sum_{i=1}^{n}\left\{(u_i+v_i)^2\cos w_i t_1\cos w_i t_2 + (u_i-v_i)^2\sin w_i t_1 \sin w_i t_2\right.\right.$$

$$\left.\left. - \cancelto{0}{(u_i^2-v_i^2)}\, j\cos w_i t_1 \sin w_i t_2 + \cancelto{0}{(u_i^2-v_i^2)}\cdot j\sin w_i t_1 \cos w_i t_2\right\}\right\}$$

其中 $E\{(u_i+v_i)^2\} = E\{u_i^2 + \cancelto{0}{2u_iv_i} + v_i^2\} = 2\sigma^2 = E\{(u_i-v_i)^2\}$

$$= \sum_{i=1}^{n} 2\sigma^2\cos w_i(t_1-t_2).\quad 故\quad R_X(t_1-t_2) = \sum_{i=1}^{n} 2\sigma^2 \cos w_i(t_1-t_2).$$

2. 由1我们得知. 均值不随时间变化. 自相关函数只与时间差有关.

只需再证明功率有限即可说明是广义平稳的.

$$E[X^2(t)] \le \underbrace{E\left\{\sum_{i=1}^{n}\left((u_i+v_i)^2 + (u_i-v_i)^2\right)\right\}}_{=4n\sigma^2} < \infty$$ 故功率有限. 得证.

3. 由于 $\cos w_0\tau \leftrightarrow \pi(\delta(\omega+w_0) + \delta(\omega-w_0))$.

故 $R_X(\tau) \leftrightarrow \sum_{i=1}^{n} 2\pi\sigma^2(\delta(\omega+w_i)+\delta(\omega-w_i)) = S_X(\omega)$.

即为自功率谱密度

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四

1. $E\{X(t)\cdot\dot{X}(t)\} = E\{X(t)\cdot\lim_{\Delta t\to 0}\frac{X(t+\Delta t)-X(t)}{\Delta t}\} =$

$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\cdot E\{\underbrace{X(t)X(t+\Delta t)}_{R_X(\Delta t)} - \underbrace{X^2(t)}_{R_X(0)}\} = R_X'(0)$. 由于 $R_X(\tau)$ 在 $\tau=0$ 处

取极大值. 故 $R_X'(0)=0$. 得证.

2. $E\{X(t)\hat{X}(t)\} = \cancel{E\{X(t)\cdot(X(t)*\frac{1}{\pi t})\}} = R_{X\hat{X}}(0) = -\hat{R}_X(0)$.

$\cancel{= \int_{-\infty}^{+\infty} X(t)\left(\int_{-\infty}^{+\infty} X(\xi)\cdot\frac{1}{\pi(t-\xi)}\,d\xi\right)}$

由于 $R_X(\tau)$ 为偶函数. 故. $\hat{R}_X(\tau)$ 为奇函数 故 $\hat{R}_X(0) = -\hat{R}_X(-0)$

得到 $\hat{R}_X(0)=0$.

~~又由于 $\frac{\sin B\tau}{B\tau}$ 为偶函数. 故~~

五. 1. ~~$X(t)$~~ 根据. 希尔伯特过程的性质可知.

~~$\hat{X}_c(t) = X_c(t) =$~~

$\hat{X}(t) = X_c(t)\sin W_0 t + X_s(t)\cos W_0 t$.

故 $X_c(t) = \cos W_0 t\, X(t) + \sin W_0 t\,\hat{X}(t)$.

$R_c(\tau) = E\{X_c(\tau)X_c(0)\} = E\{(\cos W_0\tau X(\tau) + \sin W_0\tau\hat{X}(\tau)).X(0)\}$

$= \cos W_0\tau\cdot R_X(\tau) + \sin W_0\tau\cdot R_{\hat{X}X}(\tau) = \cos W_0\tau R_X(\tau) + \sin W_0\tau\hat{R}_X(\tau)$.

而 $\hat{R}_X(\tau) = \frac{\sin B\tau}{B\tau}\cdot\sin W_0\tau$　　故 $X_c(t) = \frac{\sin B\tau}{B\tau}$.

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2. $R_{\tilde{X}}(\tau)=E\{\tilde{X}(t)\tilde{X}^*(t)\}=E\{[X(t)+j\hat{X}(t)][X(t)-j\hat{X}(t)]\}$

$= R_X(\tau)-j\underbrace{R_{X\hat{X}}(\tau)}_{-\hat{R}_X(\tau)}+j\underbrace{R_{\hat{X}X}(\tau)}_{\hat{R}_X(\tau)}+\underbrace{R_{\hat{X}}(\tau)}_{R_X(\tau)}$.

$=2R_X(\tau)+2j\cdot\hat{R}_X(\tau)$. 相当于 $\sim(*\frac{1}{\pi t})$

$\therefore S_{\tilde{X}}(\omega)=2S_X(\omega)+2jS_X(\omega)\cdot(-j\,\mathrm{sgn}(\omega))=\begin{cases}4S_X(\omega) & \omega>0.\\ 2S_X(\omega) & \omega=0.\\ 0. & \omega<0.\end{cases}$

其中 $S_X(\omega)$ 由 $R_X(\tau)$ 做傅里叶变换得到

$\int_{-\infty}^{+\infty}\frac{\sin B\tau}{B\tau}\cdot e^{-j\omega\tau}d\tau \xrightarrow{\text{对}\omega\text{求导}} \int_{-\infty}^{+\infty}\frac{\sin B\tau}{jB}\cdot e^{-j\omega\tau}d\tau$

$=\int_{-\infty}^{+\infty}\frac{1}{jB}\cdot\frac{1}{2}(e^{jB\tau}-e^{-jB\tau})\cdot e^{-j\omega\tau}d\tau=\frac{1}{2jB}\cdot(\underset{\omega-B}{\delta(B-\omega)}-\underset{\omega+B}{\delta(B+\omega)})$.

$\xrightarrow{\text{对}\omega\text{积分}}\frac{1}{2jB}\{u(\omega-B)-u(\omega+B)\}$

六. 1. $h(t)=\int_{-\infty}^{t}[\delta(\alpha)-\delta(\alpha-T)]d\alpha$.

$= u(t)-u(t-T)$.

$\int_{-\infty}^{+\infty}u(t)\cdot e^{-j\omega t}dt$

$=[e^{-j\omega t}\cdot\frac{1}{-j\omega}]_{0}^{+\infty}$

$=\frac{1}{j\omega}+\pi\delta(\omega)$.

故 $H(j\omega)=$ ~~[illegible]~~

$=(1+e^{-j\omega T})(\frac{1}{j\omega}+\pi\delta(\omega))$

$\int_{-\infty}^{+\infty}u(t-T)\cdot e^{-j\omega t}dt = \int_{-\infty}^{+\infty}u(t)\cdot e^{-j\omega(t+T)}dt = e^{-j\omega T}\cdot(\cdots)$.

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2.

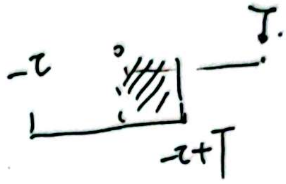
$$R_Y(\tau)=\underbrace{R_X(\tau)}_{=\frac{N_0}{2}\delta(\tau)}*R_h(\tau)\qquad \text{其中 } R_h(\tau)=h(\tau)*h(-\tau)$$

$0\le\xi\le T$

$$h(\tau)*h(-\tau)=\int_{-\infty}^{+\infty}h(\tau+\xi)h(\xi)\,d\xi=\int_{-\infty}^{+\infty}\big(u(\tau+\xi)-u(\tau+\xi-T)\big)\big(u(\xi)-u(\xi-T)\big)\,d\xi$$

$-\tau\le\xi\le-\tau+T$

$$\begin{cases}=T-|\tau|, & |\tau|\le T\\ =0 & \text{其他}\end{cases}$$



故 $R_Y(\tau)=$

$$\begin{cases}=\frac{N_0}{2}\cdot(T-|\tau|) & |\tau|\le T\\ =0. & \text{其他、}\end{cases}$$

故 $S_Y(\omega)=\int_{-\infty}^{+\infty}R_Y(\tau)\cdot e^{-j\omega\tau}d\tau=\int_{-T}^{T}\frac{N_0}{2}(T-|\tau|)e^{-j\omega\tau}d\tau.$

$$=\frac{N_0}{2}\int_{-T}^{0}(T+\tau)e^{-j\omega\tau}d\tau+\frac{N_0}{2}\int_{0}^{T}(T-\tau)e^{-j\omega\tau}d\tau.$$

= 能做. 要会积. 风暴后.

$\left[-\frac{1}{j\omega}e^{-j\omega\tau}\right]$

$\tau\cdot e^{-j\omega\tau}d\tau$

$\left[\tau\cdot\frac{1}{-j\omega}e^{-j\omega\tau}d(-j\omega\tau)\right]$

$\downarrow$

$$\frac{\tau}{-j\omega}\cdot de^{-j\omega\tau}\Rightarrow\left[\frac{\tau}{-j\omega}\cdot e^{-j\omega\tau}\right]_{-T}^{0}-\int_{0}^{-T}e^{-j\omega\tau}d\frac{\tau}{-j\omega}$$

$-\frac{T}{j\omega}\cdot e^{j\omega T}$

1. 求 $P(X_{n+2}=c \mid X_n=b)$

$$= P_{bc}^{(2)} = P_{ba}^{(1)} \cdot P_{ac}^{(1)} + P_{bb}^{(1)} \cdot P_{bc}^{(1)} + P_{bc}^{(1)} \cdot P_{cc}^{(1)}$$

$$= \frac{3}{4} \times \frac{1}{2} + 0 + 0 = \frac{3}{8}$$

2. $P(X_1=a, X_2=b, X_3=c) = P(X_1=a) \cdot P(X_2=b \mid X_1=a) \cdot P(X_3=c \mid X_2=b)$

$$= \frac{1}{3} \times \frac{1}{4} \times 0 = 0$$

六.2. 均值. $R_Y(\omega) = \frac{N_0}{2} \cdot T$

对 $\frac{N_0}{2}(T-|\tau|) \quad |\tau| \le T$ 做傅里叶变换.

$$\int_{-T}^{T} \frac{N_0}{2}(T-|\tau|) \cdot e^{-j\omega\tau} d\tau \qquad \int_{-T}^{T} |\tau| e^{-j\omega\tau} d\tau = \int_0^T \tau e^{-j\omega\tau} d\tau + \int_{-T}^{0} -\tau e^{-j\omega\tau} d\tau$$

$$\frac{N_0}{2} T \cdot \left[-\frac{1}{j\omega} e^{-j\omega\tau} \right]_{-T}^{T} = \frac{\frac{N_0}{2} T}{-j\omega} \cdot \left(e^{-j\omega T} - e^{j\omega T} \right)$$

$$= \frac{N_0 T}{-2j\omega} \cdot \left(e^{-j\omega T} - e^{j\omega T} \right).$$

慢慢做吧. 應對了.

$$\int_0^T \tau e^{-j\omega\tau} d\tau \qquad \frac{1}{j\omega}\left(\frac{e^{-j\omega T}}{-j\omega} - \frac{1}{-j\omega} \right)$$

$$= \int_0^T \frac{\tau}{-j\omega} \underbrace{e^{-j\omega\tau} d(-j\omega\tau)}_{d(e^{-j\omega\tau})}$$

$$= \underbrace{\left[\frac{\tau}{-j\omega} e^{-j\omega\tau} \right]_0^T}_{\frac{T}{-j\omega} e^{-j\omega T}} - \underbrace{\int_0^T e^{-j\omega\tau} d\left(\frac{\tau}{-j\omega} \right)}_{\frac{1}{j\omega} \cdot \left[\frac{1}{-j\omega} e^{-j\omega\tau} \right]_0^T}$$

3.

$$R_{YX}(\tau) = E\{Y(\tau)\cdot X(0)\}$$
$$= E\{(X*h)(\tau)\cdot X(0)\}$$
$$= E\left\{\int_{-\infty}^{+\infty} X(\xi)\cdot h(\tau-\xi)\,d\xi\cdot X(0)\right\}$$
$$= \int_{-\infty}^{+\infty} E\{X(\xi)X(0)\}\cdot h(\tau-\xi)\,d\xi.$$
$$= R_X(\tau)*h(\tau). = \frac{N_0}{2}\delta(\tau)*(u(\tau)-u(\tau-T))$$
$$= \frac{N_0}{2}(u(\tau)-u(\tau-T))$$

故 $S_{YX}(\omega) = \frac{N_0}{2}\cdot(1+e^{-j\omega T})\cdot\left(\frac{1}{j\omega}+\pi\delta(\omega)\right)$

$S_{XY}(\omega) = \frac{N_0}{2}(1+e^{j\omega T})\left(-\frac{1}{j\omega}+\pi\delta(\omega)\right).$

4.
求 Y 的二维概率密度

$R_Y(0) = \frac{N_0}{2}T = E\{Y^2(t)\}$ 由于 $Y(t)$ 零均值，故 $\frac{N_0}{2}T$ 为方差.

协方差 $Cov\{Y(t_1),Y(t_2)\} = E\{Y(t_1)Y(t_2)\} = R_Y(t_1-t_2) = T-|\tau|,\quad |\tau|\le T.$

故 $r(t_1,t_2) = \dfrac{T-|\tau|}{\frac{N_0}{2}T} = \frac{2}{N_0}\cdot\left(1-\frac{|\tau|}{T}\right)$ 为相关系数

$|\tau|\le T$

故. ① 当 $|\tau|\le T$ 时. $f(y_1,y_2;t_1,t_2) = \dfrac{1}{2\pi\cdot\sigma(\frac{N_0}{2}T)}e^{-\frac{1}{2}\left\{\frac{y_1^2}{\frac{N_0}{2}T}+\frac{y_2^2}{\frac{N_0}{2}T}-\frac{1}{2r}\cdot\frac{y_1y_2}{\frac{N_0}{2}T}\right\}}$

② 当 $|\tau|>T$ 时. 二者独立 $f(y_1,y_2;t_1,t_2) = f(y_1;t_1)f(y_2;t_2)$