复习 2024-12-19

$$H(\omega)=\int_{-\infty}^{+\infty} f(\xi)\cdot e^{-j\omega\xi}\cdot G(\omega)\, d\xi = G(\omega)\cdot\int_{-\infty}^{+\infty} f(\xi)e^{-j\omega\xi} d\xi$$

$$= G(\omega)\cdot F(\omega)$$

第三章：随机过程的线性变换

线性系统：$y(t)=L[x(t)]$

线性系统的性质：叠加性，比例性，时不变性

叠加性：$L\{x_1(t)+x_2(t)\}=L\{x_1(t)\}+L\{x_2(t)\}$

比例性：$L\{Kx_1(t)\}=K\cdot L\{x_1(t)\}$

时不变性：$L\{x(t)\}=y(t)$ 则 $L\{x(t+\tau)\}=y(t+\tau)$

① 使用冲激响应法获得传递函数：

$$y(t)=L\{x(t)\}=L\left\{\int_{-\infty}^{+\infty} x(\xi)\delta(t-\xi)d\xi\right\}$$ 由于在L自来仅有t为变量

$$=\int_{-\infty}^{+\infty} x(\xi)\, L\{\delta(t-\xi)\} d\xi.$$ 设 $L\{\delta(t-\xi)\}=h(t-\xi)$

$$=\int_{-\infty}^{+\infty} x(\xi)\cdot h(t-\xi)\, d\xi = x(t)*h(t)$$

结论：若 $h(t)=L\{\delta(t)\}$ 为系统冲激响应

则 $L\{x(t)\}=x(t)*h(t)$

② 使用频率响应法获得传递函数

L只对t敏感

$$y(t)=L\{x(t)\}=L\left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\cdot e^{j\omega t} d\omega\right\}=\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\cdot L\{e^{j\omega t}\}d\omega$$

由①得知 $\forall x(t)$ $L\{x(t)\}=x(t)*h(t)$ 故 $L\{e^{j\omega t}\}=e^{j\omega t}*h(t)$

$$e^{j\omega t} * h(t) = \int_{-\infty}^{+\infty} e^{j\omega\xi}\cdot h(t-\xi)\,d\xi = \int_{-\infty}^{+\infty} e^{j\omega(t-\xi)}\cdot h(\xi)\,d\xi$$

$$= \int_{-\infty}^{+\infty} e^{j\omega t}\cdot e^{-j\omega\xi} h(\xi)\,d\xi = e^{j\omega t}\cdot H(\omega)$$

故 $y(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\cdot H(\omega)\cdot e^{j\omega t}\,d\omega$ 同时

$y(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(\omega)\, e^{j\omega t}\,d\omega$ 故 $\boxed{\begin{aligned} Y(\omega) &= X(\omega)\cdot H(\omega) \\ y(t) &= x(t) * h(t) \end{aligned}}$

~~传递函数~~ 频率响应函数

其中 $h(t)=L\{\delta(t)\}$

冲激响应

例题

设 $y(t)=x(t-T)-2x(t)+x(t+T)$ 求频率响应函数

① 先求冲激响应：

$$h(t)=L\{\delta(t)\}=\delta(t-T)-2\delta(t)+\delta(t+T)$$

② 再求频率响应函数：对冲激响应作傅里叶变换

$$H(j\omega)=\int_{-\infty}^{+\infty} e^{-j\omega t} h(t)\,dt = \int_{-\infty}^{+\infty}\{e^{-j\omega t}\delta(t-T)-2e^{-j\omega t}\delta(t)+e^{-j\omega t}\delta(t+T)\}\,dt$$

$$= e^{-j\omega T}-2e^{-j\omega\cdot 0}+e^{-j\omega(-T)} = e^{-j\omega T}+e^{j\omega T}-2$$

$$= 2\{\cos(\omega T)-1\}$$

均方收敛以及依概率收敛：

① 依概率收敛：$\lim_{n\to\infty} P\{|X_n-X|\geq\varepsilon\}=0$ 记作 $\lim_{n\to\infty} X_n \overset{P}{=} X$

② 均方收敛：$\lim_{n\to\infty} E\{|X_n-X|^2\}=0$ 记作 $\underset{n\to\infty}{\mathrm{l.i.m}}\, X_n = X$

随机过程的均方连续

定义: $\lim\limits_{\Delta t\to 0} E\left\{\left|X(t+\Delta t)-X(t)\right|^2\right\}=0$

条件: 若 $R_X(t,t)$ 在此连续, 则 $X(t)$ 均方连续 (充要)

推导: $\lim\limits_{\Delta t\to 0} E\left\{\left(X(t+\Delta t)-X(t)\right)^2\right\}=\lim\limits_{\Delta t\to 0} E\left\{X^2(t+\Delta t)+X^2(t)-2X(t)X(t+\Delta t)\right\}$

$=\lim\limits_{\Delta t\to 0}\left\{R_X(t+\Delta t,t+\Delta t)+R_X(t,t)-2R_X(t,t+\Delta t)\right\}$ 若 $R_X$ 连续 $=0$

性质: $\lim\limits_{\Delta t\to 0} E\left\{X(t+\Delta t)\right\}=E\left\{\underset{\Delta t\to 0}{\mathrm{l.i.m}}\, X(t+\Delta t)\right\}=E[X(t)]$

验证: 若 $\underset{\Delta t\to 0}{\mathrm{l.i.m}}\, X(t+\Delta t)=X$ 则 $\lim\limits_{\Delta t\to 0} E\left\{X(t+\Delta t)\right\}=E[X]$

$\lim\limits_{\Delta t\to 0} E\left\{\left|X(t+\Delta t)-X\right|^2\right\}=0$

不知道.

已知 $f(t)\xrightarrow{F}F(\omega)$ 求 $f'(t)\xrightarrow{F}$ ? —设 $t\to\infty$ 时 $f(t)=0$

$\int_{-\infty}^{+\infty} f'(t)\cdot e^{-j\omega t}dt=\int_{-\infty}^{+\infty} e^{-j\omega t}df(t)=\left[f(t)e^{-j\omega t}\right]_{-\infty}^{+\infty}$ ? 0

$-\int_{-\infty}^{+\infty} f(t)\,de^{-j\omega t} = (j\omega)\int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt=(j\omega)F(\omega)$

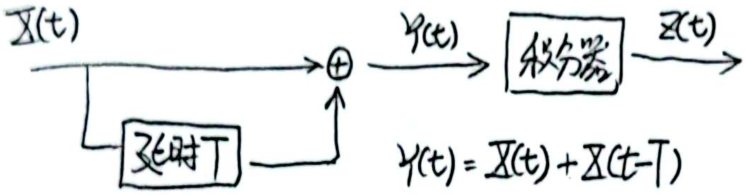
$\left[f(t)e^{-j\omega t}\right]_{-\infty}^{+\infty}$ 在何条件为零? 设 $f(t)$ 为偶函数

$\lim\limits_{T\to\infty} f(T)\cdot\left\{e^{-j\omega T}-e^{j\omega T}\right\}=\lim\limits_{T\to\infty} f(T)\cdot\left\{2j\sin(\omega T)\right\}$

$=f(\infty)\cdot 2\pi j\omega\delta(\omega)$

2024-12-21 复习

例: 求 $H(j\omega)$




$y(t) = x(t) + x(t-T)$

其中 $z(t) = \int_{-\infty}^{t} y(\xi)\,d\xi = \int_{-\infty}^{t}\{x(\xi)+x(\xi-T)\}\,d\xi$ 带入 $x(t)=\delta(t)$

$$h(t) = \int_{-\infty}^{t}\{\delta(\xi)+\delta(\xi-T)\}\,d\xi = u(t)+u(t-T)$$

故 $$H(j\omega) = \int_{-\infty}^{+\infty} h(t)\cdot e^{-j\omega t}\,dt = \underbrace{\int_{-\infty}^{+\infty} u(t)\cdot e^{-j\omega t}\,dt} + \underbrace{\int_{-\infty}^{+\infty} u(t-T)\cdot e^{-j\omega t}\,dt}$$

$$= \frac{1}{j\omega} + e^{-j\omega T}\cdot\frac{1}{j\omega}$$

$$e^{-j\omega T}\int_{-\infty}^{+\infty} u(t-T)\,e^{-j\omega(t-T)}\,dt$$

$$= \underbrace{\frac{1}{j\omega}\left(1+e^{-j\omega T}\right)}$$

$$\boxed{d e^{j\omega t} = jt\cdot e^{j\omega t}\,d\omega}$$

$$u(t) \xrightarrow{F} \frac{1}{j\omega}$$

$$\underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{j\omega}\cdot e^{j\omega t}\,d\omega} = \frac{1}{2\pi}\int \frac{1}{j\omega\cdot jt}\cdot d e^{j\omega t}$$

$$= \frac{1}{2\pi}\left[\underbrace{\frac{e^{j\omega t}}{\omega t}}_{=0}\right]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty}\frac{e^{j\omega t}}{\omega t}\,dt$$

$$u(t) \xrightarrow{\text{求导}} \delta(t)$$

$$\frac{1}{j\omega} \longleftarrow D(\omega) = \underbrace{\int_{-\infty}^{+\infty}\delta(t)\cdot e^{-j\omega t}\,dt} = 1$$

$$\delta(t) * e^{j\omega t}$$

$$\int_{-\infty}^{+\infty} u(t)\,e^{-j\omega t}\,dt = \int_{0}^{+\infty} e^{-j\omega t}\,dt = \left[-\frac{1}{j\omega}e^{-j\omega t}\right]_{0}^{+\infty}$$

R

$$= -\frac{1}{j\omega}e^{-j\omega(R)} - \left(-\frac{1}{j\omega}\right) = \underbrace{-\frac{1}{j\omega}e^{-j\omega R}} + \frac{1}{j\omega}$$

2024-12-21 复习

$$\int_{-\infty}^{+\infty} sgn(t)\cdot e^{-j\omega t}dt = \int_{-\infty}^{0} -e^{-j\omega t}dt + \int_{0}^{+\infty} e^{-j\omega t}dt$$

$$= \left[\frac{1}{j\omega}e^{-j\omega t}\right]_{-\infty}^{0} + \left[-\frac{1}{j\omega}e^{-j\omega t}\right]_{0}^{+\infty}$$

设 $f(t) = \frac{1}{j\omega}\cdot e^{-j\omega t}$

$$= \left[f(t)\right]_{-\infty}^{0} + \left[-f(t)\right]_{0}^{+\infty}.$$

$$= \lim_{R\to+\infty}\left\{\left[f(t)\right]_{-R}^{0} + \left[-f(t)\right]_{0}^{+R}\right\}$$

$$= \lim_{R\to+\infty}\left\{f(0) - f(-R) + (-f(R)) - (-f(0))\right\}$$

$$= \lim_{R\to+\infty}\left\{2f(0) - f(-R) - f(R)\right\}$$

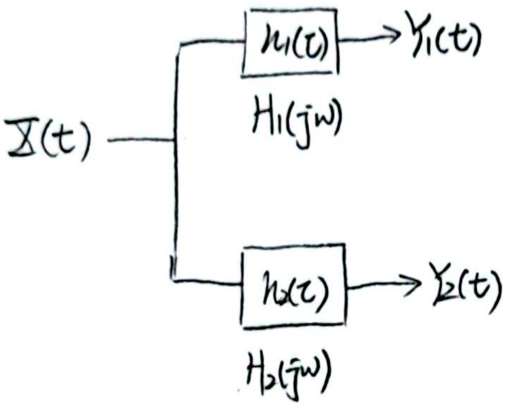
$$= \frac{2}{j\omega} - \lim_{R\to+\infty}\left\{f(R) + f(-R)\right\} \rightarrow C\delta(\omega).$$

$$f(R) + f(-R) = \frac{1}{j\omega}\left\{e^{-j\omega R} + e^{j\omega R}\right\} = \frac{1}{j\omega}\cdot 2\cos(\omega R) = \frac{1}{j\omega}\cdot 2\pi\omega\delta(\omega) = \frac{2\pi}{j}\delta(\omega)$$

$\cos(\omega R) \pm j\sin(\omega R)$

$$\boxed{\begin{aligned}&\cos\left(\omega\left(R + \frac{\pi}{2\omega}\right)\right)\\ &= \cos\omega R\,\underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0} + \sin\omega R\cdot\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} = \sin\omega R.\end{aligned}}$$

2024-12-21 复习




求 $S_{Y_1Y_2}(\omega)$

$Y_1(t) = X(t) * h_1(t) \rightarrow Y_1(\omega) = X(\omega)\cdot H_1(\omega)$

$Y_2(t) = X(t) * h_2(t) \rightarrow Y_2(\omega) = X(\omega)\cdot H_2(\omega)$

$$R_{Y_1Y_2}(\tau) = E\{Y_1(\tau)Y_2(0)\} = E\{(X(\tau)*h_1(\tau))\cdot(X(0)*h_2(0))\}$$

$$= E\left\{\int_{-\infty}^{+\infty} X(\xi_1)\cdot h_1(\tau-\xi_1)\,d\xi_1 \cdot \int_{-\infty}^{+\infty} X(\xi_2)\cdot h_2(-\xi_2)\,d\xi_2\right\}$$

$$= E\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} X(\xi_1)X(\xi_2)h_1(\tau-\xi_1)h_2(-\xi_2)\,d\xi_1 d\xi_2\right\}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_X(\xi_1-\xi_2)\,h_1(\tau-\xi_1)h_2(-\xi_2)\,d\xi_1 d\xi_2$$

误 $\int_{-\infty}^{+\infty} (R_X * h_1)(\tau-\xi_2)\,h_2(-\xi_2)\,d\xi_2$

$$= R_X(\tau)*h_1(\tau)*h_2(-\tau)$$

$$|J| = \frac{\partial(\xi_1,\xi_2)}{\partial(u,v)} = \begin{vmatrix} \frac12 & \frac12 \\ \frac12 & -\frac12 \end{vmatrix} = \frac12$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_X(v)\,h_1\left(\tau-\tfrac12(u+v)\right)h_2\left(-\tfrac12(u-v)\right)\tfrac12\,du\,dv$$

$$= R_X(\tau)*h_1(\tau)*h_2(-\tau)$$

故 $S_{Y_1Y_2}(\omega) = \frac12 S_X(\omega)\cdot H_1(j\omega)\cdot H_2^*(j\omega)$

2024-12-21 复习 

求线性系统输出的协方差函数：已知 $\begin{cases} h(t) \\ H(j\omega) \end{cases}$ 求 $Y(t)=X(t)*h(t)$ 的

$C_X(\tau)$

$$C_Y(\tau)=E\{[(Y(\tau)-m_Y)(Y(0)-m_Y)]\}=E\{Y(\tau)Y(0)\}-m_Y^2=R_Y(\tau)-m_Y^2$$

$$R_Y(\tau)=E\{(X(\tau)*h(\tau))\cdot(X(0)*h(0))\}$$

$$=E\left\{\int_{-\infty}^{+\infty}X(\xi_1)h(\tau-\xi_1)d\xi_1\cdot\int_{-\infty}^{+\infty}X(\xi_2)h(-\xi_2)d\xi_2\right\}$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}h(\tau-\xi_1)h(-\xi_2)R_X(\xi_1-\xi_2)d\xi_1 d\xi_2$$

设 $\begin{cases} u=\xi_1-\xi_2 \\ v=-\xi_2 \end{cases} \rightarrow \begin{cases} \xi_1=u+v \\ \xi_2=v \end{cases}$ $\quad |J|=\dfrac{\partial(\xi_1,\xi_2)}{\partial(u,v)}=\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}=1$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}h(\tau-u-v)h(v)\cdot R_X(u)\,du\cdot dv$$

$$=R_X(\tau)*h(\tau)*h(-\tau)$$

$$=\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}h(\tau-u-v)\cdot h(v)dv\right\}\cdot R_X(u)\cdot du$$

$$(h*h)(\tau-u)\cdot R_X(u)$$

$=$

2024-12-23 复习：白噪声通过RC积分器

$S_X(\omega)=\frac{1}{2}N_0$. $h(t)=ae^{-at}\ (t>0) \longleftrightarrow \frac{2a^2}{a^2+w^2}$ 设 $a=\frac{1}{RC}$

① $R_X(\tau)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{2}N_0\cdot e^{+j\omega\tau}d\omega=\frac{N_0}{2}\delta(\tau)$

② $R_Y(\tau)=R_X(\tau)*h(\tau)*h(-\tau)=\frac{N_0}{2}R_h(\tau)=\frac{N_0 a}{4}\cdot e^{-a|\tau|}$

$\updownarrow$ $\updownarrow$

$S_Y(\omega)=S_X(\omega)\cdot H(\omega)\cdot H^*(\omega)=\frac{N_0}{2}\cdot 1 \cdots\cdots \frac{a}{2}\cdot\frac{2a}{a^2+w^2}$

$R_h(\tau)=\int_{0}^{+\infty}ae^{-a(\xi)}\cdot ae^{-a(\tau+\xi)}d\xi$

※

$=ae^{-a\tau}\int_{0}^{+\infty}ae^{-2a\xi}d\xi=-\frac{1}{2}ae^{-a\tau}\int_{0}^{+\infty}(-2a)e^{-2a\xi}d\xi$

$=-\frac{1}{2}ae^{-a\tau}\cdot\left[e^{-2a\xi}\right]_{0}^{+\infty}=\frac{a}{2}e^{-a|\tau|}$

※ RC 积分器 $t<0$ 时 $h(t)=0$

噪声等效通频带 $S_Y(\omega_0)\Delta W_e=\int_0^{\infty}S_Y(\omega)d\omega$. 功率谱密度在正半轴积分

最大功率谱密度值

白噪声通过理想低通网络：$|H(j\omega)|=\begin{cases}K_0, & -\Delta\omega<\omega<\Delta\omega \\ 0\end{cases}$

白噪声：$S_X(\omega)=\frac{N_0}{2}$

① 求 $R_X(\tau)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{N_0}{2}\cdot e^{+j\omega\tau}d\omega=\frac{N_0}{2}\cdot\delta(\tau)$

② 求 $R_h(\tau)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}H^2(j\omega)e^{+j\omega\tau}d\omega=\frac{1}{2\pi}\int_{-\Delta\omega}^{+\Delta\omega}K_0^2\cdot e^{j\omega\tau}d\omega$

$\left[\frac{1}{j\tau}e^{j\omega\tau}\right]_{-\Delta\omega}^{+\Delta\omega}$

$=\frac{1}{2\pi}\left[\frac{K_0^2}{j\tau}e^{j\omega\tau}\right]_{-\Delta\omega}^{+\Delta\omega}=\frac{1}{j\tau}\cdot 2j\sin\Delta\omega\tau\cdot\frac{1}{2\pi}=\frac{2\sin\Delta\omega\tau}{2\pi\cdot\tau}\cdot K_0^2$

$=\frac{K_0^2\sin\Delta\omega\tau}{\pi\tau}$

修改：$\boxed{\frac{N_0}{2}R_h(\tau)=\frac{N_0K_0^2}{2\pi}\cdot\frac{\sin\Delta\omega\tau}{\tau}}$

当 $\Delta\omega\to\infty$ 时 $R_h(\tau)\to 2\pi K_0^2\delta(\tau)$

③ 求 $R_Y(\tau)=\underbrace{R_X(\tau)}_{\frac{N_0}{2}\cdot\delta(\tau)}*R_h(\tau)=\int_{-\infty}^{+\infty}R_X(\xi)\cdot R_h(\tau-\xi)d\xi=\frac{N_0}{2}\cdot R_h(0)=\frac{N_0}{2}\cdot\frac{K_0^2\Delta\omega}{\pi}$ ?

$\boxed{※\ \delta(\tau)*h(\tau)=h(\tau)}$

这种做法与PPT结果不同.

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$S_Y(\omega)=S_X(\omega)\cdot|H(j\omega)|^2=\begin{cases}\frac{N_0K_0^2}{2}, & -\Delta\omega<\omega<\Delta\omega \\ 0\end{cases}$

$\therefore R_Y(\tau)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}S_Y(\omega)e^{+j\omega\tau}d\omega=\frac{1}{2\pi}\int_{-\Delta\omega}^{+\Delta\omega}\frac{N_0K_0^2}{2}\cdot e^{j\omega\tau}\cdot d\omega$

$=\frac{N_0K_0^2}{4\pi}\cdot\left[\frac{1}{j\tau}e^{j\omega\tau}\right]_{-\Delta\omega}^{+\Delta\omega}=\frac{N_0\cdot K_0^2}{4\pi}\cdot\frac{2j\sin\Delta\omega\tau}{j\tau}=\boxed{\frac{N_0K_0^2}{2\pi}\cdot\frac{\sin\Delta\omega\tau}{\tau}}$

这种做法也与PPT不同.

符合教材.

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看下教材吧：算出来一致的结果但与PPT不同.

基本方法 $\begin{cases}R_Y(\tau)=R_X(\tau)*R_h(\tau), & \boxed{R_h(\tau)=h(\tau)*h(-\tau)} \\ \updownarrow\qquad\updownarrow\qquad\updownarrow\qquad\updownarrow \\ S_Y(\omega)=S_X(\omega)\cdot H(j\omega)\cdot H^*(j\omega)\end{cases}$

2024-12-28 复习：设 $X(t)$ 为确定信号

$X(t)=A\cos(\omega_0 t+\Theta)$ 求 FT：

$$X(\omega)=\int_{-\infty}^{+\infty}X(t)e^{-j\omega t}dt=\int_{-\infty}^{+\infty}A\cos(\omega_0 t+\Theta)e^{-j\omega t}dt$$

拆.

$$=A\int_{-\infty}^{+\infty}\{\cos(\omega_0 t)\cos\Theta-\sin(\omega_0 t)\sin\Theta\}\cdot e^{-j\omega t}dt$$

$$=A\cos\Theta\int_{-\infty}^{+\infty}\cos\omega_0 t\,e^{-j\omega t}dt-A\sin\Theta\int_{-\infty}^{+\infty}\sin\omega_0 t\,e^{-j\omega t}dt$$

$$\frac{1}{2}(e^{j\omega_0 t}+e^{-j\omega_0 t}) \qquad \frac{1}{2}(e^{j\omega_0 t}-e^{-j\omega_0 t}).$$

$$=A\pi\cos\Theta\cdot(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))-A\pi\sin\Theta(\delta(\omega-\omega_0)-\delta(\omega+\omega_0))$$

这说明任意一个余弦信号，在频谱上总是只有两个点处的贡献

① $\omega-\omega_0$ 处贡献：$A\pi\cos\Theta-A\pi\sin\Theta$

② $\omega+\omega_0$ 处贡献：$A\pi\cos\Theta+A\pi\sin\Theta$

因此，任意给定一个频谱图，我们可以将其分解成 $\sum_i P_i\delta_i$ 的形式

其中 $\delta_i=\delta(\omega-\omega_i)$，成对出现，可以分解成无数个 $\cos(\omega_i t+\Theta)$ 叠加的形式

设 $X(t)=A\cos(\omega_0 t+\Theta)$ 是随机信号，A 为常量，$\Theta\in[0,2\pi]$ 均匀

$$R_X(\tau)=E\{A\cos(\omega_0\tau+\Theta)\cdot A\cos(\Theta)\}=A^2\cdot E\{\frac{1}{2}[\cancel{\cos(\omega_0\tau+2\Theta)}+\cos\omega_0\tau]\}$$

$$=\frac{1}{2}A^2\cos\omega_0\tau$$

故 $S_X(\omega)=\int_{-\infty}^{+\infty}R_X(\tau)e^{-j\omega\tau}d\tau=\frac{1}{2}A^2\int_{-\infty}^{+\infty}\cos\omega_0\tau\,e^{-j\omega\tau}d\tau=\frac{2\pi}{4}A^2\cdot(\delta(\omega+\omega_0)+\delta(\omega-\omega_0))$

$$=\frac{A^2\pi}{2}\cdot(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$$