Everything about e (in relation to derivatives)

Layout of what each named equation shows

1: Derivative of e^x is plainly e^x
2: Derivative of $e^{f(x)}$ is $e^{f(x)} * f'(x)$
3: $a^{f(x)}$ is equivalent to $e^{f(x)*\ln(x)}$
4: Derivative of $a^{f(x)}$ is $a^{f(x)} * \ln(a) * f'(x)$
5: Derivative of natural log ($\ln(f(x))$) is $\frac{f'(x)}{f(x)}$
6: Derivative of $x^{f(x)}$ is $x^{f(x)} * [f(x) * \ln(x) + \frac{f(x)}{x}]$
7: Derivative of $f(x)^{g(x)}$ is $f(x)^{g(x)} * [g'(x) * \ln(f(x)) + \frac{f'(x)}{f(x)} * g(x)]$

To start, we must find the derivative of e^x. To do this we can look at the power series of the "natural exponential function" (e^x) which is as follows:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

First few terms written out

$$= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

which simplifies to

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

This, though initially looking more complex, will much more easily allow us to find the derivative.

$$\frac{d}{dx}[e^x] = \frac{d}{dx}[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots]$$

$$= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \ldots$$

Re-write to make it more clear what will cancel

$$= 0 + 1 + \frac{2x}{2*1!} + \frac{3x^2}{3*2!} + \frac{4x^3}{4*3!} + \ldots$$

$$= 0 + 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$= \quad 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Reference our earlier definition of e^x

$$= e^x$$

Therefore,

$$\frac{d}{dx}[e^x] = e^x \tag{1}$$

This relationship is why e is so useful (and subsequently ln is as well)
Knowing this as well as the chain rule, we can easily show that

$$\frac{d}{dx}[e^{f(x)}] = e^{f(x)} * f'(x) \tag{2}$$

Next, we can look at the equation $a^{f(x)}$ and how it can be re-arranged to work more nicely in combination with the knowledge

$$a^{f(x)} = u$$

$$\ln(a^{f(x)}) = \ln(u)$$

By power rule of logs ($log(x^y) = y * log(x)$)

$$f(x) * \ln(a) = \ln(u)$$

By definition of logs $z = log_x(y); x^z = y$)

$$e^{f(x)*\ln(a)} = u$$

Therefore,

$$a^{f(x)} = e^{f(x)*\ln(a)} \tag{3}$$

Now we can solve for the derivative of $a^{f(x)}$.

$$\frac{d}{dx}[a^{f(x)}]$$

Start by using the equality from equation (3)

$$\frac{d}{dx}[e^{f(x)*\ln(a)}]$$

Then we can reference the equality from equation (2) with pretty much any other log base, this wouldn't work out as nicely as it does with e

$$e^{f(x)*\ln(a)} * \frac{d}{dx}[f(x) * \ln(a)]$$

Here, $\ln(a)$ is a constant and can be treated as such
Also, the equality from equation (3) can be utilized

$$a^{f(x)} * \ln(a) * \frac{d}{dx}[f(x)]$$

$$a^{f(x)} * \ln(a) * f'(x)$$

Therefore,

$$\frac{d}{dx}[a^{f(x)}] = a^{f(x)} * \ln(a) * f'(x) \tag{4}$$

Finally, we can use implicit differentiation to solve for the derivative of natural log ($\ln(f(x))$)

$$\ln(f(x)) = y$$

Again, utilize definition of derivatives

$$e^y = x$$

Then, get the derivative of both sides

$$\frac{d}{dx}[e^y] = \frac{d}{dx}[x]$$

$$\frac{dy}{dx} * e^y = 1$$

Use the equilivency provided at the start of this equation

$$\frac{dy}{dx} * e^{ln(f(x))} = f'(x)$$

Use the fact that $b^{log_b(x)} = x$

$$\frac{dy}{dx} * f(x) = f'(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

Therefore,

$$\frac{d}{dx}[ln(f(x))] = \frac{f'(x)}{f(x)} \tag{5}$$

Further expansions (written but not explained)

$$\begin{aligned}
x^{f(x)} &= e^{f(x)*\ln(x)} \\
\frac{d}{dx}[x^{f(x)}] &= \frac{d}{dx}[e^{f(x)*\ln(x)}] \\
&= e^{f(x)*\ln(x)} * \frac{d}{dx}[f(x)*\ln(x)] \\
&= e^{f(x)*\ln(x)} * [f'(x)*\ln(x) + \frac{1}{x}*f(x)] \\
&= e^{f(x)*\ln(x)} * [f'(x)*\ln(x) + \frac{f(x)}{x}] \\
&= x^{f(x)} * [f(x)*\ln(x) + \frac{f(x)}{x}]
\end{aligned}$$

Therefore,

$$\frac{d}{dx}[x^{f(x)}] = x^{f(x)} * [f(x)*\ln(x) + \frac{f(x)}{x}] \tag{6}$$

Further expansion (written but not explained)

$$\begin{aligned}
f(x)^{g(x)} &= e^{g(x)*\ln(f(x))} \\
\frac{d}{dx}[f(x)^{g(x)}] &= \frac{d}{dx}[e^{g(x)*\ln(f(x))}] \\
&= e^{g(x)*\ln(f(x))} * \frac{d}{dx}[g(x)*\ln(f(x))] \\
&= f(x)^{g(x)} * [g'(x)*\ln(f(x)) + \frac{f'(x)}{f(x)}*g(x)]
\end{aligned}$$

Therefore,

$$\frac{d}{dx}[f(x)^{g(x)}] = f(x)^{g(x)} * [g'(x)*\ln(f(x)) + \frac{f'(x)}{f(x)}*g(x)] \tag{7}$$