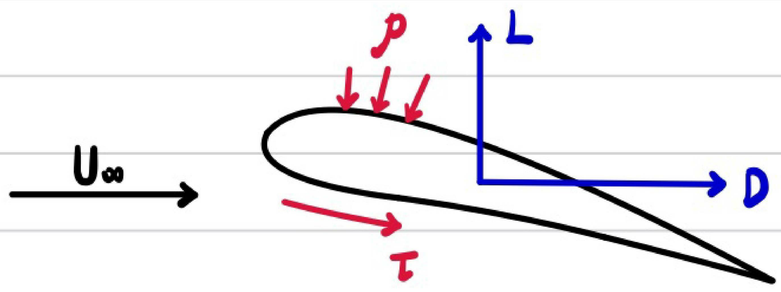
Fundenmentals

Basic definations

Aerodynamic forces



p pressure distribution

τ shear stress distribution (e.g. skin friction)

Independent of units

Reynolds number $Re = \frac{\rho_\infty U_\infty c}{\mu_\infty} = \frac{U_\infty c}{\nu_\infty}$

Mach number $M = \frac{U}{a} = \frac{U}{\sqrt{\gamma R T}}$

dynamic pressure $q_\infty = \frac{1}{2}\rho_\infty U_\infty^2$

pressure coefficient $C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2}$

skin friction coefficient $C_f = \frac{\tau}{\frac{1}{2}\rho_\infty U_\infty^2}$

lift coefficient $C_L = \frac{L}{\frac{1}{2}\rho_\infty U_\infty^2 S}$ (3-D) $= \frac{L}{\frac{1}{2}\rho_\infty U_\infty^2 c}$ (2-D) $= g(\alpha, Re)$ for incompressible flow

drag coefficient $C_D = \frac{D}{\frac{1}{2}\rho_\infty U_\infty^2 S}$ (3-D) $= \frac{D}{\frac{1}{2}\rho_\infty U_\infty^2 c}$ (2-D) $= 0$ for 2-D incompressible, invisid flow

Dynamic similarity

for scale model testing $\longrightarrow$ $Re = Re_M$ and $M = M_M$

for $M < 0.3$ $\longrightarrow$ affect of compressibility unlikely to be important $\longrightarrow$ $Re = Re_M$

Simple Vortex Motion

Vorticity $\omega = \nabla \times u$ = changing orientation of the fluid particle $\neq$ motion of the particle

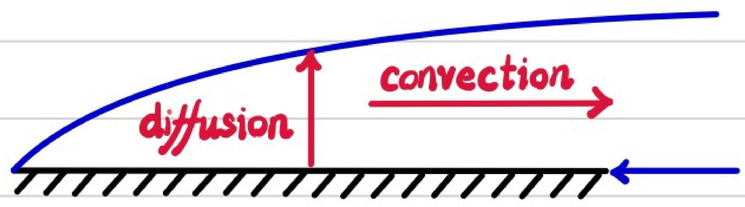
$\omega_{AB} \circlearrowleft = \frac{dv}{dx}$

$\omega_{CD} \circlearrowleft = -\frac{du}{dy}$

$\longrightarrow$ $\omega_{angular} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

$\longrightarrow$ $\omega = 2\omega_{angular} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ (2-D Cartesian) $= \frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta}$ (2-D polar) $\longrightarrow$ $\omega = \nabla \times u$ (3-D)

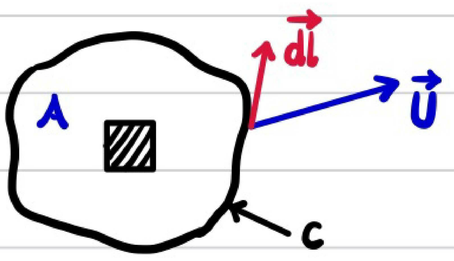
vorticity moves by



* flow with vorticity is rotational flow

vorticity generated at the wall

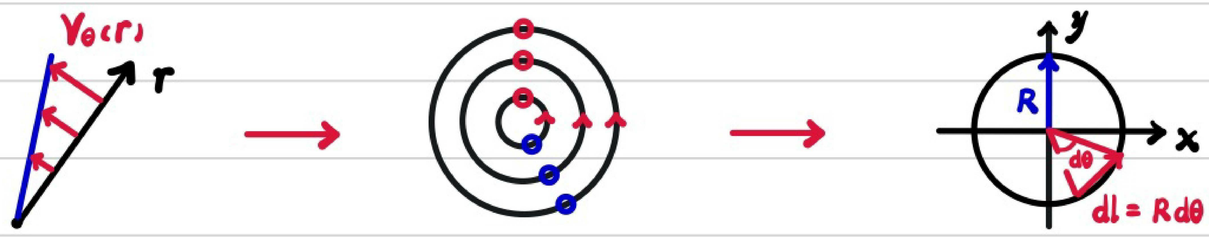
Circulation $\Gamma \circlearrowleft = \oint_c \vec{U} \cdot d\vec{l}$



* Stoke's theorem $\Gamma = \oint_c \vec{U} \cdot d\vec{l} = \iint_A (\nabla \times \vec{U}) \cdot k \, d\vec{A} = \iint_A \vec{\omega} \cdot d\vec{A}$

$L = -\rho U_\infty \Gamma$ (2-D flow per unit span)

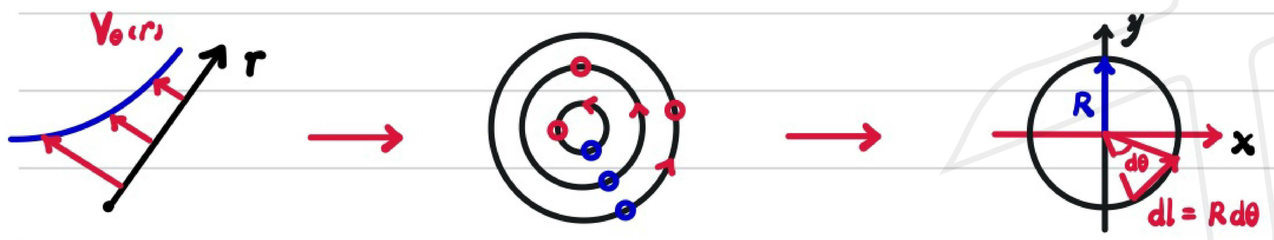
Forced vortex (soild body rotation) $\vec{V}_\theta = c \cdot r$ and $\vec{V}_r = 0$



$$\Gamma = \int \vec{U} \cdot d\vec{l} = \int_0^{2\pi} (V_\theta) \cdot dl = \int_0^{2\pi} (c \cdot R) \cdot R d\theta = 2c\pi R^2$$

$$\omega = \frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial r} = 2c \longrightarrow \Gamma = \iint_A \vec{\omega} \cdot d\vec{A} = 2c\pi R^2$$

Free vortex $\vec{V}_\theta = \frac{c}{r}$ and $\vec{V}_r = 0$

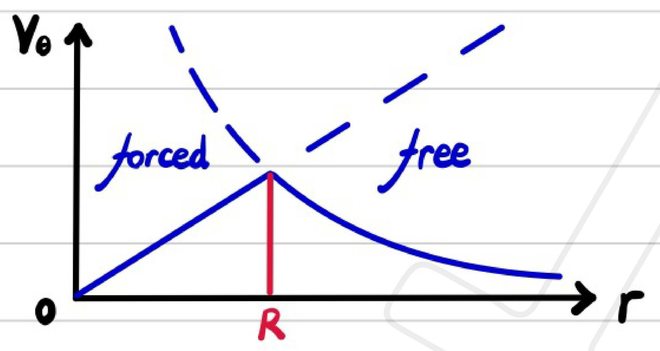


$$\Gamma = \int \vec{U} \cdot d\vec{l} = \int_0^{2\pi} (V_\theta) \cdot dl = \int_0^{2\pi} \left(\frac{c}{R}\right) \cdot R d\theta = 2c\pi$$

$$\omega = \frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial r} = 0 \longrightarrow \Gamma = \iint_A \vec{\omega} \cdot d\vec{A} = 0$$

→ "singularity" $\omega \to \infty$ at $r \to 0$

Rankine vortex a good vortex model



Equations of Motion

Euler approach fixed Control Volume ⟷ fluid moving through it

Lagrangian approach finite system moving with fluid

Mass conservation

rate of change of mass within C.V. $\frac{\partial \rho}{\partial t}V$ + net flux of mass out of C.V. $\rho u A = 0$

$$\longrightarrow \frac{\partial \rho}{\partial t}\Delta x \Delta y + \left[\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2} + O(\Delta x^2)\right]\Delta y - \left[\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2} + O(x^2)\right]\Delta y$$

$$+ \left[\rho v + \frac{\partial(\rho v)}{\partial y}\frac{\Delta y}{2} + O(\Delta y^2)\right]\Delta x + \left[\rho v - \frac{\partial(\rho v)}{\partial y}\frac{\Delta y}{2} + O(\Delta y^2)\right]\Delta x = 0$$

$$\longrightarrow \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad (2\text{-D})$$

for incompressible flow. $\rho = \text{const} \longrightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nabla \cdot u = 0$

Momentum conservation

rate of change of momentum within C.V. $\frac{\partial(\rho u)}{\partial t}V$ +

net flux of momentum out of C.V. $\rho u^2 A / \rho u v A$ = total force act on C.V. p

$$\longrightarrow \frac{\partial(\rho u)}{\partial t}\Delta x \Delta y + \left[\rho u^2 + \frac{\partial(\rho u^2)}{\partial x}\frac{\Delta x}{2} + O(\Delta x^2)\right]\Delta y - \left[\rho u^2 - \frac{\partial(\rho u^2)}{\partial x}\frac{\Delta x}{2} + O(\Delta x^2)\right]\Delta y$$

$$+ \left[\rho u v + \frac{\partial(\rho u v)}{\partial y}\frac{\Delta y}{2} + O(\Delta y^2)\right]\Delta x - \left[\rho u v - \frac{\partial(\rho u v)}{\partial y}\frac{\Delta y}{2} + O(\Delta y^2)\right]\Delta x$$

$$= \left[P - \frac{\partial P}{\partial x}\frac{\Delta x}{2} + O(\Delta x^2)\right]\Delta y - \left[P + \frac{\partial P}{\partial x}\frac{\Delta x}{2} + O(\Delta x^2)\right]\Delta y$$

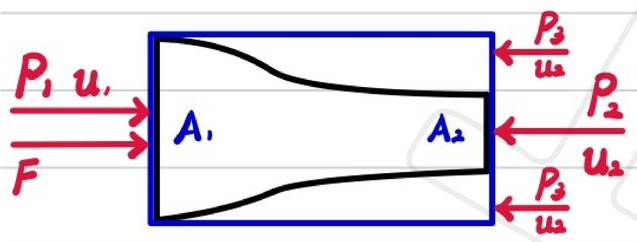
$$\longrightarrow \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho u v)}{\partial y} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + u\left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y}\right) = -\frac{\partial P}{\partial x} \quad (2\text{-D})$$

for conservation of mass. $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$

$$\longrightarrow \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

for steady flow. $u = \text{const} \longrightarrow u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$

Force on nozzle

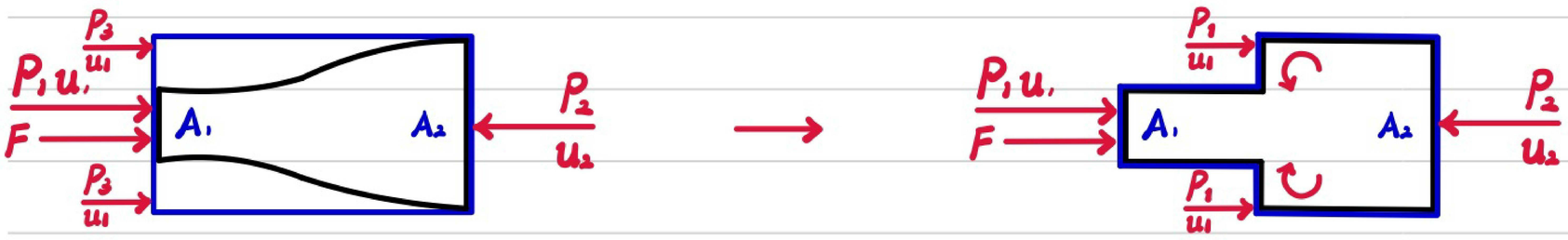


for steady flow $\longrightarrow \rho_2 u_2^2 A_2 - \rho_1 u_1^2 A_1 = P_2 A_2 + P_3(A_2 - A_1) - P_1 A_1 - F$

for incompressible flow $\longrightarrow \rho u A = \text{const}$ and $\rho = \text{const} \longrightarrow u_1 A_1 = u_2 A_2$

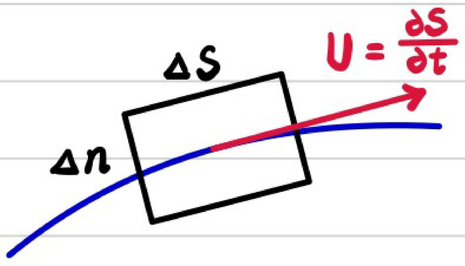
according to Bernoulli $\longrightarrow P_0 = P_1 + \frac{1}{2}\rho u_1^2 = P_2 + \frac{1}{2}\rho u_2^2 = P_3 + \frac{1}{2}\rho u_2^2 \longrightarrow P_2 = P_3$

Smooth and rapid expansion in ducts



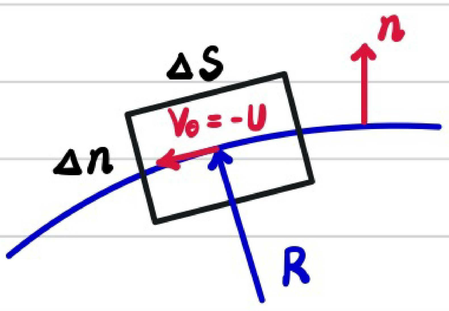
no Bernoulli due to losses $\longrightarrow$ Assume pressure $= P_1$ across left hand side of C.V.

Bernoulli equation



by system approach $\longrightarrow$ $F = (-\frac{\partial P}{\partial s}\Delta s)\Delta n = ma = \rho \Delta s \Delta n \cdot \frac{\partial U}{\partial t}$

$\longrightarrow -\frac{\partial P}{\partial s} = \rho \frac{\partial U}{\partial t} = \rho \frac{\partial U}{\partial s}\cdot\frac{\partial s}{\partial t} = \rho U \frac{\partial U}{\partial s} = \frac{\partial}{\partial s}(\frac{1}{2}\rho U^2) \longrightarrow P_o = P + \frac{1}{2}\rho U^2$



by system approach $\longrightarrow$ $F = (-\frac{\partial P}{\partial n}\Delta n)\Delta s = ma = \rho \Delta s \Delta n \cdot (-\frac{U^2}{R})$

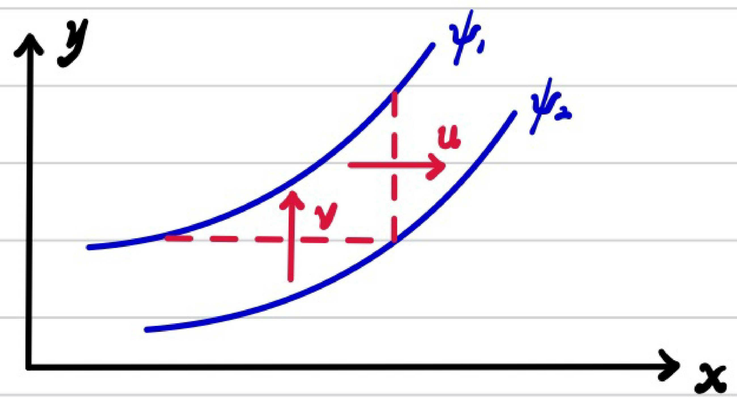
$\longrightarrow -\frac{\partial P}{\partial n} = -\frac{\partial P_o}{\partial n} + \frac{1}{2}\frac{\partial \rho U^2}{\partial n} = -\frac{\rho U^2}{R} \longrightarrow \frac{\partial P_o}{\partial n} - (\rho U \frac{\partial U}{\partial n} + \frac{\rho U^2}{R}) = \frac{\partial P_o}{\partial n} + \rho U \omega = 0$

$\longrightarrow$ total pressure will not change when $\omega = 0$

Steady, Inviscid, Incompressible 2-D Flow

Stream function ψ

* no flow through the streamline or the boundary



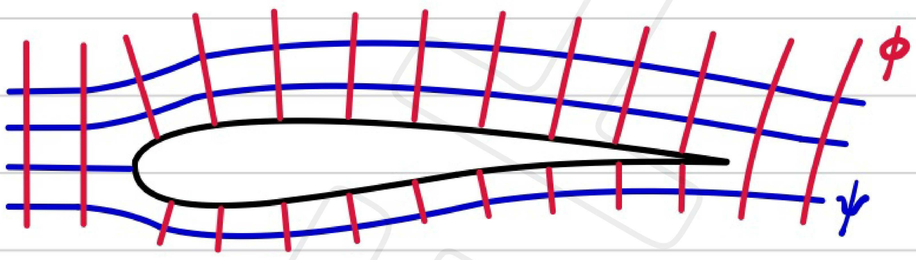
$u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$

$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$ and $V_\theta = -\frac{\partial \psi}{\partial r}$

$\longrightarrow \omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$ Laplace's equation

Velocity potential ϕ

$\vec{\nabla}\psi \cdot \vec{\nabla}\phi = 0 \longrightarrow \psi \perp \phi$



$u = \frac{\partial \phi}{\partial x}$ and $v = \frac{\partial \phi}{\partial y}$

$V_r = \frac{\partial \phi}{\partial r}$ and $V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$

$\longrightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ Laplace's equation

Complex potential $F(z) = \phi + i\psi$ where $z = x + iy = re^{i\theta}$

* $\tan\theta = \frac{y}{x}$

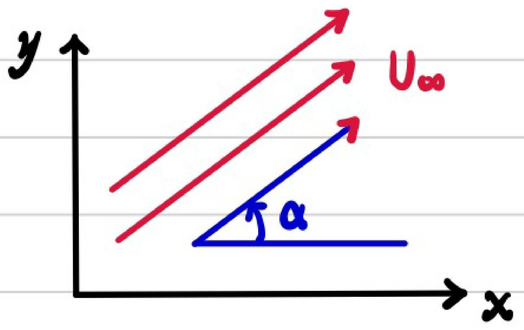
$\longrightarrow \frac{dF}{dz}dz = \frac{dF}{dx}dx + \frac{dF}{dy}dy \longrightarrow \frac{dF}{dz}(dx + idy) = (\frac{d\phi}{dx} + i\frac{d\psi}{dx})dx + (\frac{d\phi}{dy} + i\frac{d\psi}{dy})dy$

$\longrightarrow \frac{dF}{dz} = u - iv$

2-D Incompressible, Inviscid Flow

Basic Solutions

Uniform stream $F(z) = U_\infty z e^{-i\alpha}$



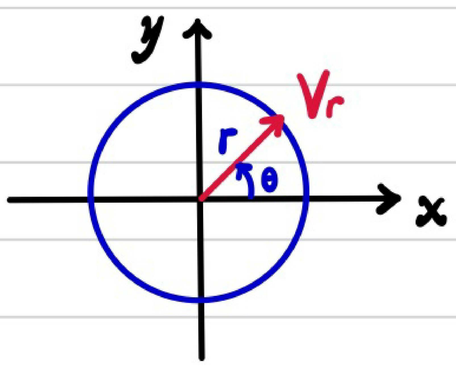
$$F(z) = \phi + i\psi = U_\infty z e^{-i\alpha} = U_\infty (x+iy)e^{-i\alpha} = U_\infty (x+iy)[\cos(-\alpha) - i\sin(-\alpha)]$$

$$= U_\infty [[x\cos(-\alpha) - y\sin(-\alpha)] - i[x\sin(-\alpha) + y\cos(-\alpha)]$$

$$= U_\infty [(x\cos\alpha + y\sin\alpha) - i(y\cos\alpha - x\sin\alpha)]$$

$\longrightarrow$ $u = \frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x} = U_\infty \cos\alpha$ and $v = -\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial y} = U_\infty \sin\alpha$

Source $F(z) = \frac{m}{2\pi}\ln z$



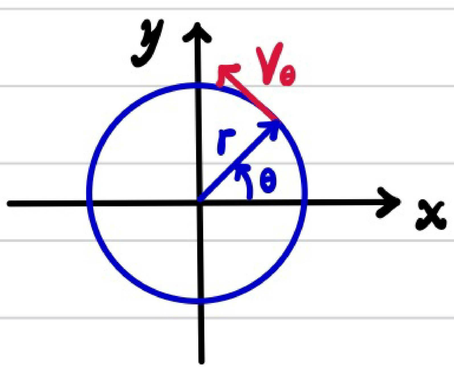
$$F(z) = \phi + i\psi = \frac{m}{2\pi}\ln(re^{i\theta}) = \frac{m}{2\pi}(\ln r + i\theta)$$

$\longrightarrow$ $V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r} = \frac{m}{2\pi r}$ and $V_\theta = -\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = 0$

* shift to right by z_1 $\longrightarrow$ $F(z) = \frac{m}{2\pi}\ln(z - z_1)$

Sink $F(z) = -\frac{m}{2\pi}\ln z$

Free (irrotational) vortex $F(z) = -\frac{i\Gamma}{2\pi}\ln z$



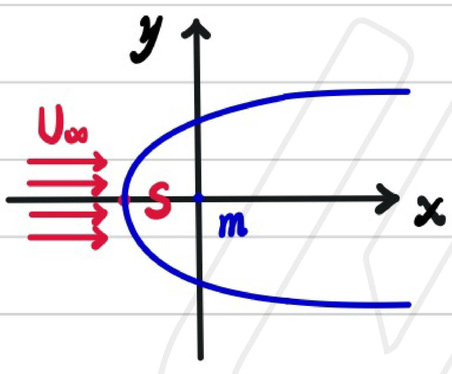
$$F(z) = \phi + i\psi = -\frac{i\Gamma}{2\pi}\ln re^{i\theta} = -\frac{i\Gamma}{2\pi}(\ln r + i\theta) = \frac{\Gamma}{2\pi}\theta - i\frac{\Gamma}{2\pi}\ln r$$

$\longrightarrow$ $V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r} = 0$ and $V_\theta = -\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\Gamma}{2\pi r}$

* shift to right by z_1 $\longrightarrow$ $F(z) = -\frac{i\Gamma}{2\pi}\ln(z - z_1)$

Superposition

Rankine half-body $F(z) = U_\infty z + \frac{m}{2\pi}\ln z$



$$F(z) = \phi + i\psi = U_\infty z + \frac{m}{2\pi}\ln re^{i\theta} = (U_\infty x + \frac{m}{2\pi}\ln r) + i(U_\infty y + \frac{m}{2\pi}\theta)$$

for $U_\infty(x+iy)$ $\longrightarrow$ $u = U_\infty$ and $v = 0$

for $\frac{m}{2\pi}(\ln r + i\theta)$ $\longrightarrow$ $V_r = \frac{m}{2\pi r}$ and $V_\theta = 0$

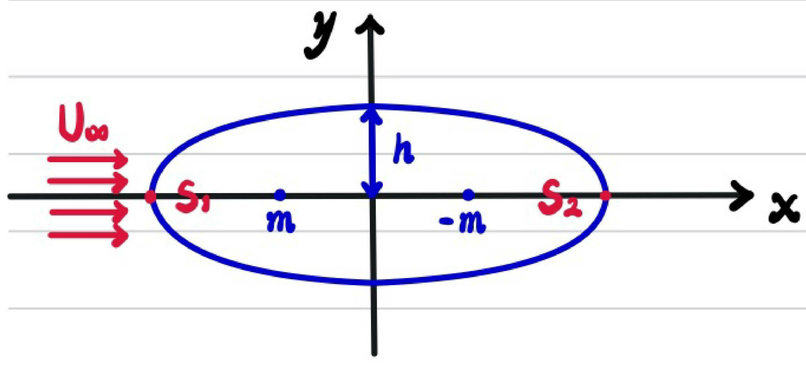
$\longrightarrow$ $V_r = U_\infty\cos\theta + \frac{m}{2\pi r}$ and $V_\theta = -U_\infty\sin\theta$

at stagnation point, $V_r = V_\theta = 0 \longrightarrow \theta = \pi \longrightarrow -U_\infty + \frac{m}{2\pi r_s} = 0 \longrightarrow x_s = -\frac{m}{2\pi U_\infty}$

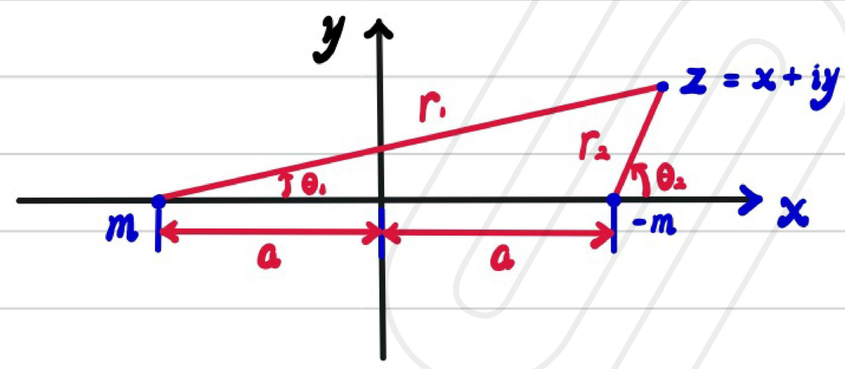
for streamline function $\longrightarrow$ at s, $\psi = \frac{m}{2} \longrightarrow U_\infty y + \frac{m}{2\pi}\theta = \frac{m}{2} \longrightarrow y = \frac{1}{U_\infty}\left(\frac{m}{2} - \frac{m}{2\pi}\theta\right)$

$\longrightarrow$ at $x \to \infty$, $\theta \to 0 \longrightarrow y|_{x\to\infty} = \frac{m}{2U_\infty}$

Rankine oval $\quad F(z) = U_\infty z + \frac{m}{2\pi}\ln(z+a) - \frac{m}{2\pi}\ln(z-a)$



$\longrightarrow$



$$F(z) = \phi + i\psi = U_\infty z + \frac{m}{2\pi}\ln r_1 e^{i\theta_1} - \frac{m}{2\pi}\ln r_2 e^{i\theta_2} = \left[U_\infty x + \frac{m}{2\pi}\ln\left(\frac{r_1}{r_2}\right)\right] + i\left[U_\infty y + \frac{m}{2\pi}(\theta_1 - \theta_2)\right]$$

$$= \left[U_\infty x + \frac{m}{2\pi}\ln\left(\frac{r_1}{r_2}\right)\right] + i\left[U_\infty y + \frac{m}{2\pi}\left[\tan^{-1}\left(\frac{y}{x+a}\right) - \tan^{-1}\left(\frac{y}{x-a}\right)\right]\right]$$

* $y = \tan^{-1}(x) \longrightarrow x = \tan y \longrightarrow \frac{dx}{dy} = \sec^2 y = \frac{1}{\cos^2 y} = \frac{\cos^2 y + \sin^2 y}{\cos^2 y} = 1 + x^2 \longrightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$

$$\longrightarrow u = \frac{\partial\psi}{\partial y} = U_\infty + \frac{m}{2\pi}\left[\frac{1}{x+a}\frac{1}{1+\left(\frac{y}{x+a}\right)^2} - \frac{1}{x-a}\frac{1}{1+\left(\frac{y}{x-a}\right)^2}\right] = U_\infty + \frac{m}{2\pi}\left[\frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}\right]$$

and $$v = -\frac{\partial\psi}{\partial x} = \frac{m}{2\pi}\left[\frac{y}{(x+a)^2}\frac{1}{1+\left(\frac{y}{x+a}\right)^2} - \frac{y}{(x-a)^2}\frac{1}{1+\left(\frac{y}{x-a}\right)^2}\right] = \frac{m}{2\pi}\left[\frac{y}{(x+a)^2+y^2} - \frac{y}{(x-a)^2-y^2}\right]$$

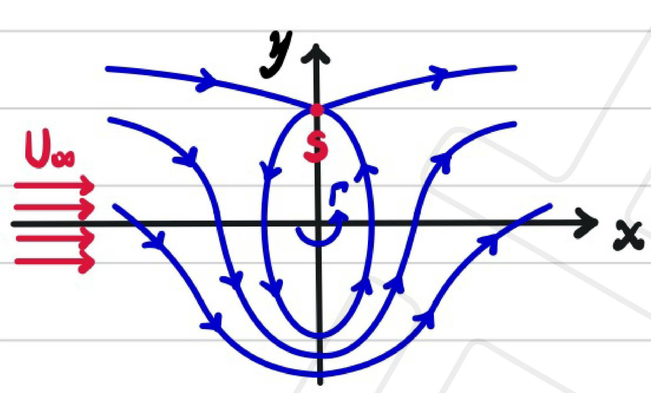
at stagnation points, $u = v = 0 \longrightarrow y = 0 \longrightarrow U_\infty + \frac{m}{2\pi}\left(\frac{1}{x+a} - \frac{1}{x-a}\right) = 0 \longrightarrow x = \left(a^2 + \frac{ma}{\pi U_\infty}\right)^{\frac{1}{2}}$

for streamline function $\longrightarrow$ at s, $\psi = 0$

$\longrightarrow$ at $x = 0$, $U_\infty h + \frac{m}{2\pi}\left[\tan^{-1}\left(\frac{h}{a}\right) - \tan^{-1}\left(-\frac{h}{a}\right)\right] = U_\infty h + \frac{m}{2\pi}\left[2\tan^{-1}\left(\frac{h}{a}\right)\right] = 0$

* Rankine oval is symmetric $\longrightarrow L = D = 0$ $\qquad$ ** for $z_1 = 0 \longrightarrow$ form a circle doublet

Stationary vortex 'held' in a uniform stream $\qquad F(z) = U_\infty z - \frac{i\Gamma}{2\pi}\ln z$



$$F(z) = \phi + i\psi = U_\infty z - \frac{i\Gamma}{2\pi}(\ln r + i\theta) = U_\infty(x+iy) + \frac{\Gamma}{2\pi}(\theta - i\ln r)$$

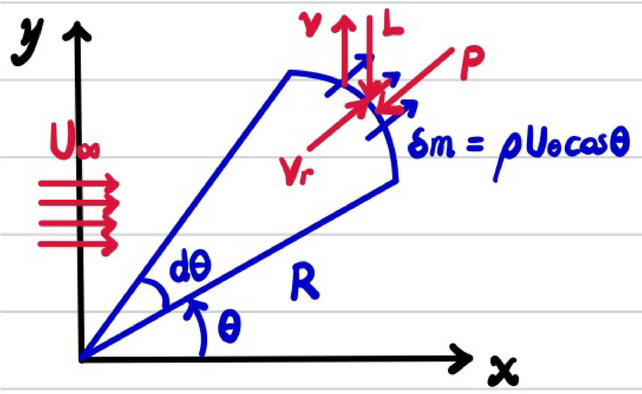
$$\longrightarrow V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r} = U_\infty\cos\theta$$

and $$V_\theta = -\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U_\infty\sin\theta + \frac{\Gamma}{2\pi r}$$

at stagnation points, $V_r = V_\theta = 0 \longrightarrow \theta = \frac{\pi}{2} \longrightarrow -U_\infty + \frac{\Gamma}{2\pi r_s} = 0 \longrightarrow r_s = \frac{\Gamma}{2\pi U_\infty}$

* symmetric to y-axis $\longrightarrow D = 0$ $\qquad$ ** asymmetric to x-axis $\longrightarrow L = -\rho U_\infty \Gamma$

Relationship between lift and circulation $L = -\rho U_\infty \Gamma$



for conservation of momentum in direction of y

$$\longrightarrow 0 + (\rho V_r v) R d\theta = (\rho U_\infty \cos\theta)(V_\theta \cos\theta) R d\theta = -L + (-P\sin\theta) R d\theta$$

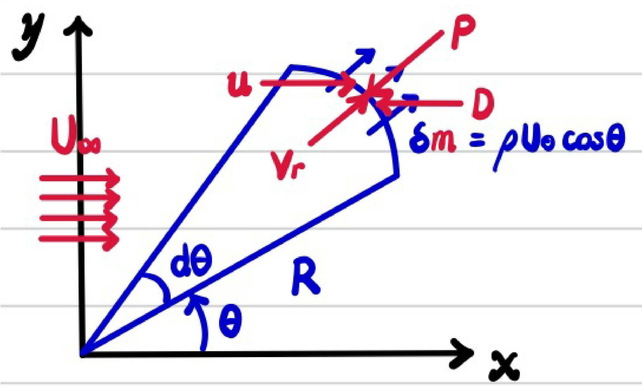
$$\longrightarrow L = \int_0^{2\pi} (-\rho U_\infty V_\theta \cos^2\theta - P\sin\theta) R d\theta$$

* according to Bernoulli $\longrightarrow P_0 = P_\infty + \frac{1}{2}\rho U_\infty^2 = P + \frac{1}{2}\rho U^2 = P + \frac{1}{2}\rho[(U_\infty - V_\theta \sin\theta)^2 + (V_\theta \cos\theta)^2]$

$$= P + \frac{1}{2}\rho U_\infty^2 + \frac{1}{2}\rho V_\theta^2 - \rho U_\infty V_\theta \sin\theta \longrightarrow P = P_\infty - \frac{1}{2}\rho V_\theta^2 + \rho U_\infty V_\theta \sin\theta$$

$$\longrightarrow L = -\int_0^{2\pi} [(P_\infty - \frac{1}{2}\rho V_\theta^2)\sin\theta + \rho U_\infty V_\theta] R d\theta = 0 - \int_0^{2\pi} \rho U_\infty V_\theta R d\theta = -\rho U_\infty \int_0^{2\pi} \frac{\Gamma}{2\pi R} R d\theta = -\rho U_\infty \Gamma$$

Relationship between drag and circulation $D = 0$



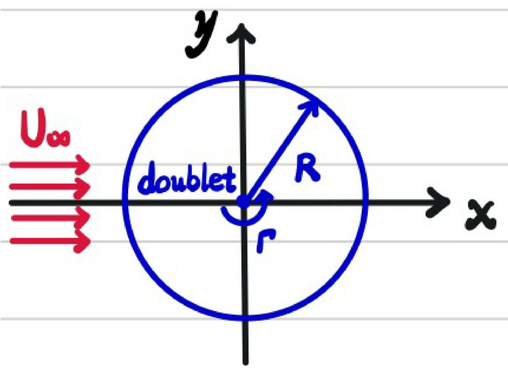
for conservation of momentum in direction of x

$$\longrightarrow 0 + (\rho V_r u) R d\theta = (\rho U_\infty \cos\theta)(U_\infty - V_\theta \sin\theta) R d\theta = -D + (-P\cos\theta) R d\theta$$

$$\longrightarrow D = \int_0^{2\pi} (\rho U_\infty V_\theta \cos\theta \sin\theta - \rho U_\infty^2 \cos\theta - P\cos\theta) R d\theta$$

$$= -\int_0^{2\pi} [(P_\infty - \frac{1}{2}\rho V_\theta^2 + \rho U_\infty^2)\cos\theta] R d\theta = 0$$

Flow around a circular cylinder $F(z) = U_\infty (z + \frac{R^2}{z}) - \frac{i\Gamma}{2\pi}\ln z$



$$F(z) = \phi + i\psi = U_\infty (re^{i\theta} + \frac{R^2}{r}e^{-i\theta}) + \frac{\Gamma}{2\pi}(\theta - i\ln r)$$

$$= U_\infty [(r + \frac{R^2}{r})\cos\theta + i(r - \frac{R^2}{r})\sin\theta] + \frac{\Gamma}{2\pi}(\theta - i\ln r)$$

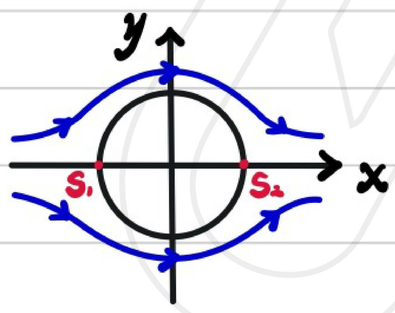
$$\longrightarrow V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r} = U_\infty (1 - \frac{R^2}{r^2})\cos\theta$$

and $V_\theta = -\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U_\infty (1 + \frac{R^2}{r^2})\sin\theta + \frac{\Gamma}{2\pi R}$

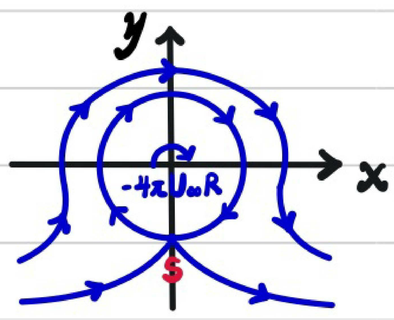
at surface of cylinder. $r = R \longrightarrow V_r = 0$ and $V_\theta = -2U_\infty \sin\theta + \frac{\Gamma}{2\pi R}$

at stagnation points. $V_r = V_\theta = 0 \longrightarrow -2U_\infty \sin\theta + \frac{\Gamma}{2\pi R} = 0 \longrightarrow \theta = \sin^{-1}(\frac{\Gamma}{4\pi U_\infty R})$

* for $\Gamma = 0 \longrightarrow \theta = 0$ or π



** for $\Gamma = -4\pi U_\infty R \longrightarrow \theta = \frac{3}{2}\pi$

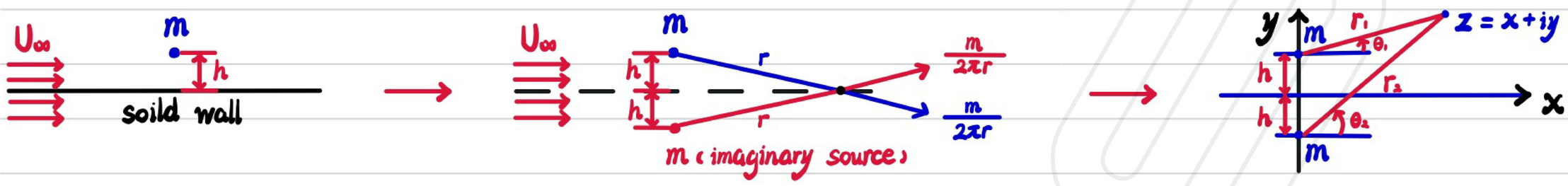


*** for $\Gamma = 0 \longrightarrow C_p = \dfrac{P - P_\infty}{\frac{1}{2}\rho U_\infty^2} = \dfrac{\frac{1}{2}\rho U_\infty^2 - \frac{1}{2}\rho U^2}{\frac{1}{2}\rho U_\infty^2} = 1 - 4\sin^2\theta$

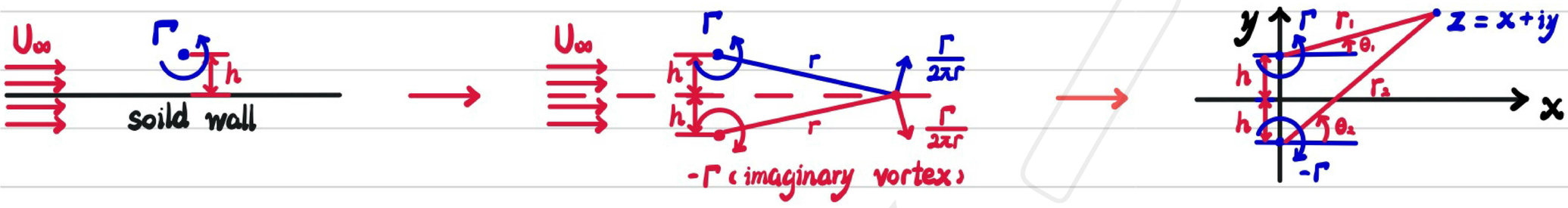
$\longrightarrow$ see circular cylinder lab for accuracy of theory solutions

Rigid Boundaries and the Method of Images

Source above rigid wall $\quad F(z) = U_\infty z + \frac{m}{2\pi}\ln(z - ih) + \frac{m}{2\pi}\ln(z + ih)$

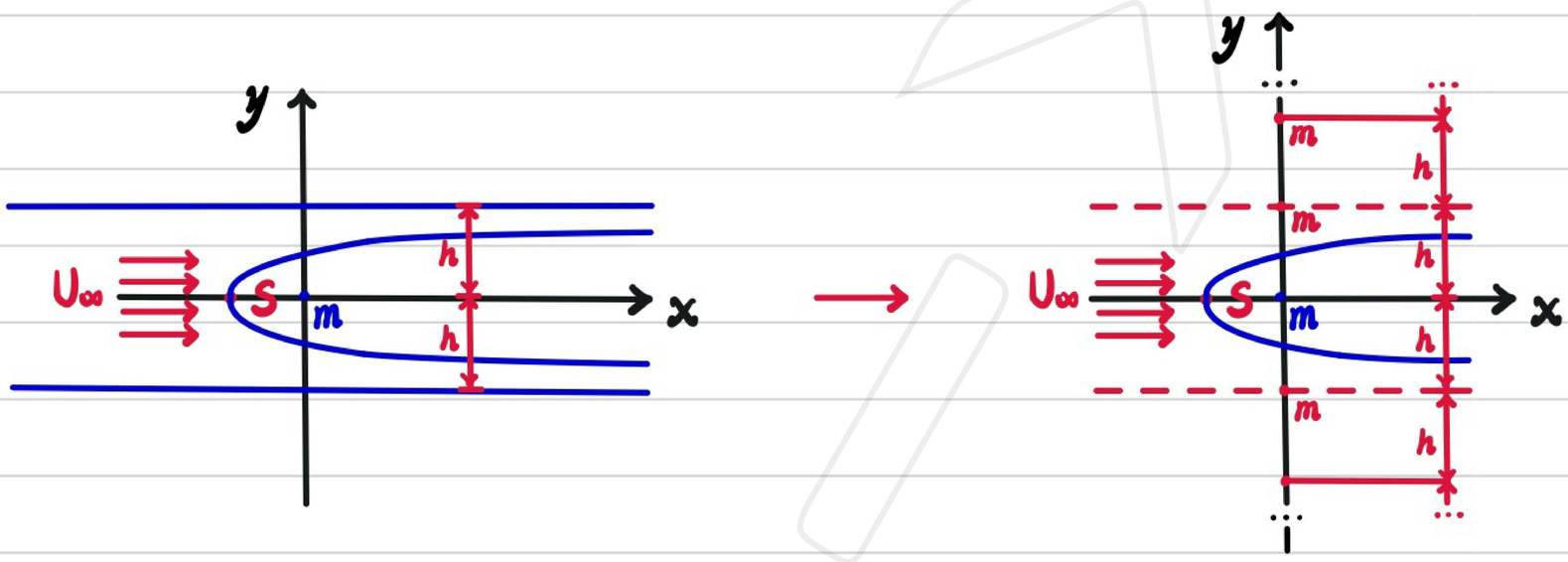


Vortex above rigid wall $\quad F(z) = U_\infty z - \frac{i\Gamma}{2\pi}\ln(z - ih) + \frac{i\Gamma}{2\pi}\ln(z + ih)$



Circular wall

e.g. Rankine half-body in wind tunnel

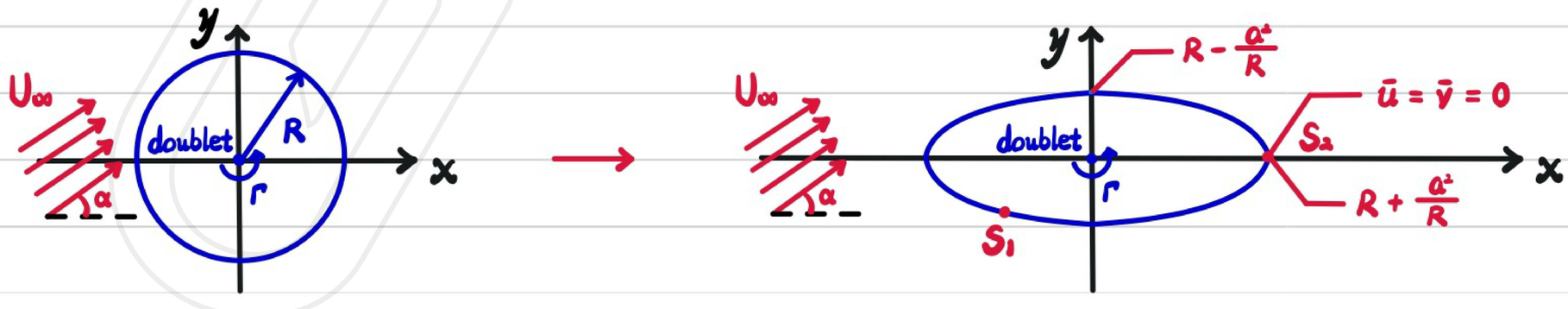


Conformal Transformations

Joukowski Transformations

Joukowski Transformations $\quad \bar{z} = z + \frac{a^2}{z}$

Ellipse $\quad F(z) = U_\infty\left(\bar{z}e^{-i\alpha} + \frac{R^2 e^{i\alpha}}{\bar{z}}\right) - \frac{i\Gamma}{2\pi}\ln z$



* $\bar{z} = z + \frac{R^2}{z} = re^{i\theta} + \frac{a^2}{r}e^{-i\theta} = (r + \frac{a^2}{r})\cos\theta + (r - \frac{a^2}{r})\sin\theta \longrightarrow$ at $r = R$, $(\frac{\bar{x}}{R+\frac{a^2}{R}})^2 + (\frac{\bar{y}}{R-\frac{a^2}{R}})^2 = 1$

$\frac{dF}{dz} = u - iv \longrightarrow \frac{dF}{d\bar{z}} = \frac{dF}{dz}\frac{dz}{d\bar{z}} = \frac{dF}{dz}(\frac{d\bar{z}}{dz})^{-1} = \frac{dF}{dz}(1 + \frac{a^2}{z^2})^{-1} = \bar{u} - i\bar{v}$

$\longrightarrow \bar{u} - i\bar{v} = [U_\infty(e^{-i\alpha} - \frac{R^2}{z^2}e^{i\alpha}) - i\frac{\Gamma}{2\pi z}](1 - \frac{a^2}{z^2})^{-1}$

at stagnation points, $\bar{u} - i\bar{v} = 0 \longrightarrow U_\infty(e^{-i\alpha} - \frac{R^2}{z^2}e^{i\alpha}) - i\frac{\Gamma}{2\pi z} = 0$

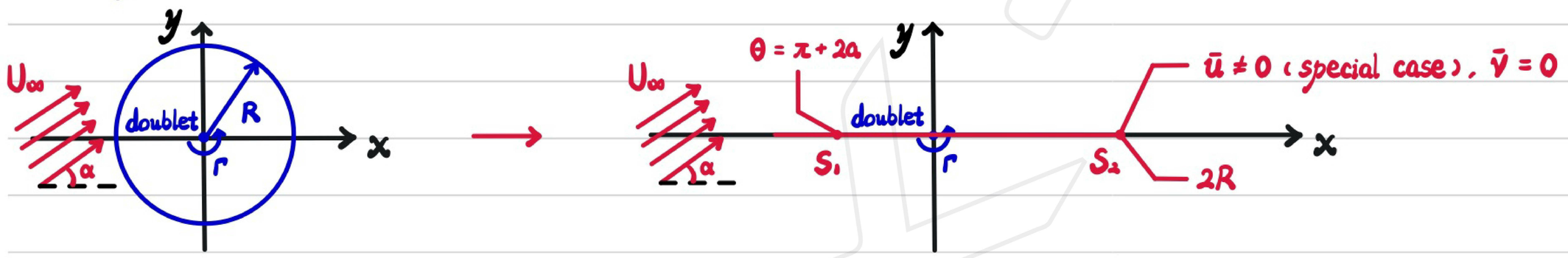
$\longrightarrow$ at $r = R$, $U_\infty(e^{-i\alpha} - e^{-2i\theta}e^{i\alpha}) - i\frac{\Gamma}{2\pi R}e^{-i\theta} = 0$

for Kutta condition, $\theta = 0 \longrightarrow U_\infty(e^{-i\alpha} - e^{i\alpha}) - i\frac{\Gamma}{2\pi R} = 0$

$\longrightarrow -2iU_\infty\sin\alpha - i\frac{\Gamma}{2\pi R} = 0 \longrightarrow \Gamma = -4\pi R U_\infty \sin\alpha$

$\longrightarrow C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 - (\frac{\bar{u}}{U_\infty})^2$ and $C_L = \frac{L}{\frac{1}{2}\rho U_\infty^2 c} = \frac{-\rho U_\infty(-4\pi R U_\infty \sin\alpha)}{\frac{1}{2}\rho U_\infty^2[2(R + \frac{a^2}{R})]} = \frac{4\pi\sin\alpha}{1 + \frac{a^2}{R^2}}$

Flat plate $a = R$



at stagnation points for Kutta condition $\longrightarrow \Gamma = -4\pi R U_\infty \sin\alpha \longrightarrow C_L = 2\pi\sin\alpha$

at stagnation points $\longrightarrow U_\infty(e^{-i\alpha} - e^{-2i\theta}e^{i\alpha}) + i(2U_\infty\sin\alpha)e^{-i\theta} = 0$

$\longrightarrow U_\infty e^{-i\theta}(e^{i\theta}e^{-i\alpha} - e^{-i\theta}e^{i\alpha}) + 2iU_\infty\sin\alpha e^{-i\theta} = 0$

$\longrightarrow iU_\infty e^{-i\theta}(2\sin(\theta - \alpha) + 2\sin\alpha) = 0 \longrightarrow \theta = \pi + 2\alpha$ or 0

$\longrightarrow \bar{u} - i\bar{v} = [U_\infty(e^{-i\alpha} - \frac{R^2}{z^2}e^{i\alpha}) - i\frac{\Gamma}{2\pi z}](1 - \frac{a^2}{z^2})^{-1} = [2iU_\infty e^{-i\theta}(\sin(\theta - \alpha) + \sin\alpha)](1 - e^{-2i\theta})^{-1}$

$= [2iU_\infty(\sin(\theta - \alpha) + \sin\alpha)](e^{i\theta} - e^{-i\theta})^{-1} = [2iU_\infty(\sin(\theta - \alpha) + \sin\alpha)](2i\sin\theta)^{-1}$

$= U_\infty\frac{\sin\theta\cos\alpha - \sin\alpha\cos\theta + \sin\alpha}{\sin\theta} = U_\infty(\cos\alpha + \sin\alpha\frac{1 - \cos\theta}{\sin\theta}) = U_\infty(\cos\alpha \pm \sin\alpha\frac{1 - \cos\theta}{[(1 + \cos\theta)(1 - \cos\theta)]^{\frac{1}{2}}})$

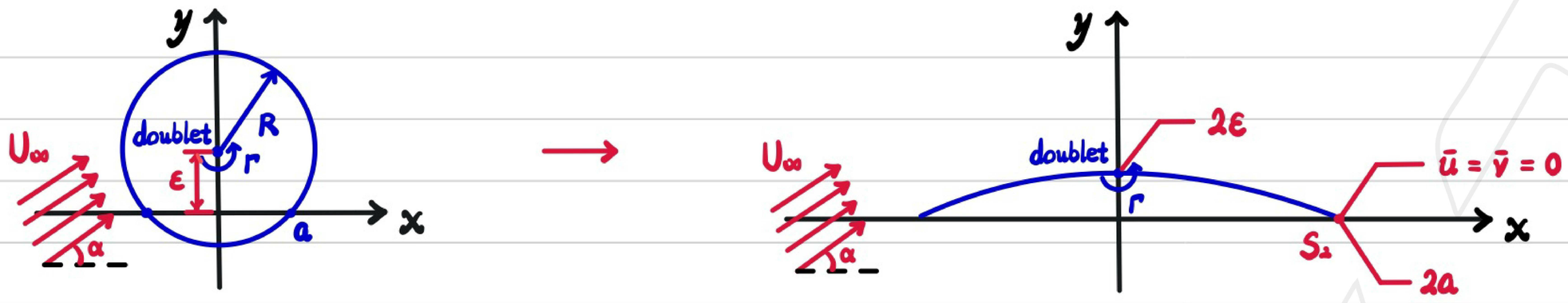
* $\sin^2\theta = (1 + \cos\theta)(1 - \cos\theta)$ when $0 \leq \theta \leq \pi$ and $\sin^2\theta = -(1 + \cos\theta)(1 - \cos\theta)$ when $\pi \leq \theta \leq 2\pi$

$\longrightarrow \bar{u} = U_\infty(\cos\alpha \pm \sin\alpha\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}})$ where $\bar{x} = 2R\cos\theta \longrightarrow C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 - (\frac{\bar{u}}{U_\infty})^2$

* at L.E., $\theta = \pi \longrightarrow \bar{u} \longrightarrow \pm\infty$ and $C_p \longrightarrow -\infty$

** at T.E., $\theta = 0 \longrightarrow \bar{u} = U_\infty\cos\alpha$ and $C_p = 1 - \cos^2\alpha = \sin^2\alpha$

Cambered flat plate $a = R$ and $F(z) = U_\infty\left[(z - i\varepsilon)e^{-i\alpha} + \frac{R^2 e^{i\alpha}}{z - i\varepsilon}\right] - \frac{i\Gamma}{2\pi}\ln(z - i\varepsilon)$



$$* \quad \bar{z} = (z - i\varepsilon) + \frac{R^2}{z - i\varepsilon} = (re^{i\theta} - i\varepsilon) + \frac{R^2}{re^{i\theta} - i\varepsilon} = [r\cos\theta + i(r\sin\theta - \varepsilon)] + \frac{R^2}{r\cos\theta + i(r\sin\theta - \varepsilon)}$$

$$= \left[r\cos\theta + \frac{R^2 r\cos\theta}{(r\cos\theta)^2 + (r\sin\theta - \varepsilon)^2}\right] + i\left[(r\sin\theta - \varepsilon) - \frac{R^2(r\sin\theta - \varepsilon)}{(r\cos\theta)^2 + (r\sin\theta - \varepsilon)^2}\right] = \bar{x} + i\bar{y}$$

$\longrightarrow$ at $r = a$ and $\theta = 0$, $\bar{z} = 2a$ $\longrightarrow$ $\bar{x} = 2a$ $\longrightarrow$ $\lambda = \frac{\varepsilon}{2a}$ maximum camber to chord ratio

$$\bar{u} - i\bar{v} = \frac{dF}{d\bar{z}} = \frac{dF}{dz}\left(\frac{d\bar{z}}{dz}\right)^{-1} = \left[U_\infty\left(e^{-i\alpha} - \frac{R^2 e^{i\alpha}}{(z - i\varepsilon)^2}\right) - \frac{i\Gamma}{2\pi(z - i\varepsilon)}\right]\left(1 - \frac{R^2}{z^2}\right)$$

at stagnation point for Kutta condition, $u - iv = 0$ at $(a, 0)$ $\longrightarrow$ $U_\infty\left(e^{-i\alpha} - \frac{R^2 e^{i\alpha}}{(a - i\varepsilon)^2}\right) - \frac{i\Gamma}{2\pi(a - i\varepsilon)} = 0$

$$\longrightarrow \quad U_\infty\left[(a - i\varepsilon)e^{-i\alpha} - \frac{R^2 e^{i\alpha}}{(a - i\varepsilon)}\right] - \frac{i\Gamma}{2\pi} = 0 \quad \longrightarrow \quad i\Gamma = 2\pi U_\infty\left[(a - i\varepsilon)e^{-i\alpha} - \frac{R^2 e^{i\alpha}}{(a - i\varepsilon)}\right]$$

$$\longrightarrow \quad \Gamma = -2i\pi U_\infty\left[(a - i\varepsilon)e^{-i\alpha} - \frac{(a^2 + \varepsilon^2)e^{i\alpha}}{(a - i\varepsilon)}\right] = -2i\pi U_\infty\left[(a - i\varepsilon)e^{-i\alpha} - (a + i\varepsilon)e^{i\alpha}\right]$$

$$\longrightarrow \quad \Gamma = -2i\pi U_\infty(-2ia\sin\alpha - 2i\varepsilon\cos\alpha) = -4\pi U_\infty(a\sin\alpha + \varepsilon\cos\alpha)$$

$$\longrightarrow \quad \Gamma = -4\pi a U_\infty(2\lambda\cos\alpha + \sin\alpha)$$

$$\longrightarrow \quad C_L = \frac{L}{\frac{1}{2}\rho U_\infty^2 c} = \frac{-\rho U_\infty \Gamma}{\frac{1}{2}\rho U_\infty^2 c} = 2\pi(2\lambda\cos\alpha + \sin\alpha) \quad \text{where } c = 4a$$

Thick aerofoil $F(z) = U_\infty\left[(z + \varepsilon)e^{-i\alpha} + \frac{R^2 e^{i\alpha}}{z + \varepsilon}\right] - \frac{i\Gamma}{2\pi}\ln(z + \varepsilon)$

$$* \quad \bar{z} = (z + \varepsilon) + \frac{R^2}{z + \varepsilon} = (re^{i\theta} + \varepsilon) + \frac{R^2}{re^{i\theta} + \varepsilon} = [(r\cos\theta + \varepsilon) + ir\sin\theta] + \frac{R^2}{(r\cos\theta + \varepsilon) + ir\sin\theta}$$

$$= \left[(r\cos\theta + \varepsilon) + \frac{R^2(r\cos\theta + \varepsilon)}{(r\cos\theta + \varepsilon)^2 + (r\sin\theta)^2}\right] + i\left[r\sin\theta - \frac{R^2 r\sin\theta}{(r\cos\theta + \varepsilon)^2 + (r\sin\theta)^2}\right] = \bar{x} + i\bar{y}$$

$\longrightarrow$ at $r = a$ and $\theta = 0$, $\bar{z} = 2(a + \varepsilon)$ $\longrightarrow$ $\bar{x} = 2a$ for small ε

$\longrightarrow$ $\bar{y} = r\sin\theta - \frac{R^2 r\sin\theta}{(r\cos\theta + \varepsilon)^2 + (r\sin\theta)^2} = \left(r + \frac{R^2}{r}\right)\sin\theta$ for small ε $\longrightarrow$ $\bar{y} \approx 2\varepsilon(1 - \cos\theta)\sin\theta$ for small ε

$$\bar{u} - i\bar{v} = \frac{dF}{d\bar{z}} = \frac{dF}{dz}\left(\frac{d\bar{z}}{dz}\right)^{-1} = \left[U_\infty\left(e^{-i\alpha} - \frac{R^2 e^{i\alpha}}{(z + \varepsilon)^2}\right) - \frac{i\Gamma}{2\pi(z + \varepsilon)}\right]\left(1 - \frac{R^2}{z^2}\right)$$

at stagnation point for Kutta condition, $u - iv = 0$ at $(a, 0)$ $\longrightarrow$ $U_\infty\left(e^{-i\alpha} - \frac{R^2 e^{i\alpha}}{(a+\varepsilon)^2}\right) - \frac{i\Gamma}{2\pi(a+\varepsilon)} = 0$

$\longrightarrow$ $U_\infty\left[(a+\varepsilon)e^{-i\alpha} - \frac{R^2 e^{i\alpha}}{(a+\varepsilon)}\right] - \frac{i\Gamma}{2\pi} = 0$ $\longrightarrow$ $i\Gamma = 2\pi U_\infty\left[(a+\varepsilon)e^{-i\alpha} - \frac{R^2 e^{i\alpha}}{(a+\varepsilon)}\right]$

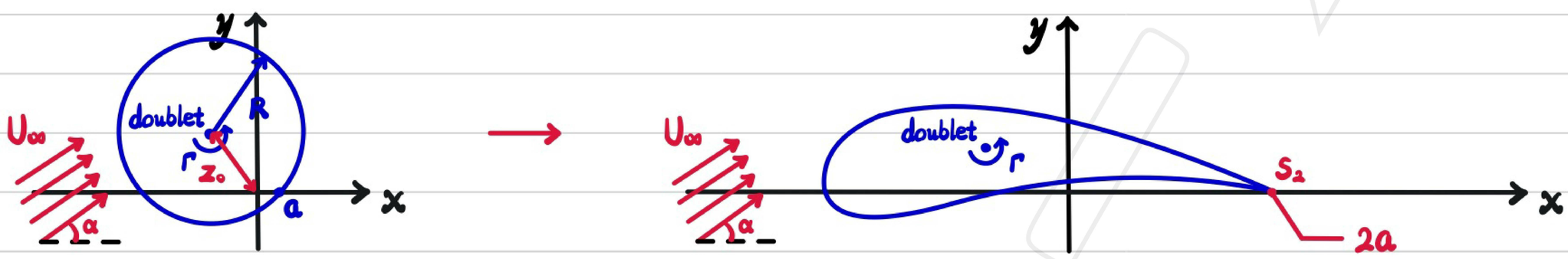
$\longrightarrow$ $\Gamma = -2i\pi U_\infty\left[(a+\varepsilon)e^{-i\alpha} - \frac{(a+\varepsilon)^2 e^{i\alpha}}{(a+\varepsilon)}\right] = -2i\pi U_\infty\left[(a+\varepsilon)e^{-i\alpha} - (a+\varepsilon)e^{i\alpha}\right]$

$\longrightarrow$ $\Gamma = -2i\pi U_\infty\left(-2i(a+\varepsilon)\sin\alpha\right) = -4\pi U_\infty (a+\varepsilon)\sin\alpha$

$\longrightarrow$ $\Gamma = -4\pi U_\infty \sin\alpha\,(a+\varepsilon)$

$\longrightarrow$ $C_L = \frac{L}{\frac{1}{2}\rho U_\infty^2 c} = \frac{-\rho U_\infty \Gamma}{\frac{1}{2}\rho U_\infty^2 c} = 2\pi\sin\alpha\,(a+\varepsilon) = 2\pi\sin\alpha\left(1 + \frac{4}{3\sqrt{3}}\frac{t_{max}}{c}\right)$ where $c = 4a$

Cambered aerofoil $F(z) = U_\infty\left[(z - z_0)e^{-i\alpha} + \frac{R^2 e^{i\alpha}}{z - z_0}\right] - \frac{i\Gamma}{2\pi}$



$\bar{u} - i\bar{v} = \frac{dF}{d\bar{z}} = \frac{dF}{dz}\left(\frac{d\bar{z}}{dz}\right)^{-1} = \left[U_\infty\left(e^{-i\alpha} - \frac{R^2 e^{i\alpha}}{(z-z_0)^2}\right) - \frac{i\Gamma}{2\pi(z-z_0)}\right]\left(1 - \frac{R^2}{z^2}\right)$

at stagnation point for Kutta condition, $u - iv = 0$ at $(a, 0)$ $\longrightarrow$ $U_\infty\left(e^{-i\alpha} - \frac{R^2 e^{i\alpha}}{(a+z_0)^2}\right) - \frac{i\Gamma}{2\pi(a+z_0)} = 0$

$\longrightarrow$ $U_\infty\left[(a+z_0)e^{-i\alpha} - \frac{R^2 e^{i\alpha}}{(a+z_0)}\right] - \frac{i\Gamma}{2\pi} = 0$ $\longrightarrow$ $i\Gamma = 2\pi U_\infty\left[(a-z_0)e^{-i\alpha} - \frac{R^2 e^{i\alpha}}{(a-z_0)}\right]$

$\longrightarrow$ $\Gamma = -2i\pi U_\infty\left[(a-z_0)e^{-i\alpha} - \frac{R^2 e^{i\alpha}}{(a-z_0)}\right]$

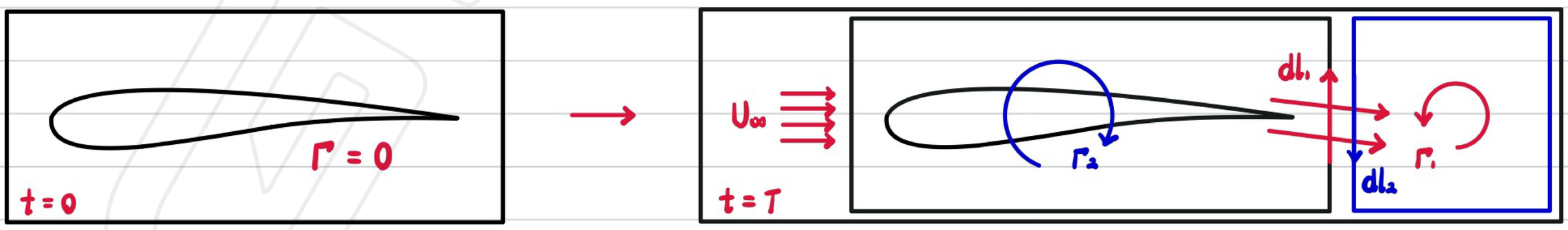
Kutta Joukouski Condition

Kutta condition $\Gamma = -4\pi R U_\infty \sin\alpha$

* use appropriate inviscid options to predict realistic flow

flow leaves trailing edge in smooth manner $\longrightarrow$ no seperation of boundary layer at T.E.

Kelvin's theorem $\frac{D\Gamma}{Dt} = 0$ in an inviscid flow with uniform density around a close loop



$\frac{D\Gamma}{Dt} = 0$ $\longrightarrow$ $\Gamma = \Gamma_1 + \Gamma_2 = 0$ $\longrightarrow$ $\int \vec{u}\cdot dl_1 + \int \vec{u}\cdot dl_2 = 0$ $\longrightarrow$ $\Gamma_2 = -\Gamma_1$

Compressible Flow

Thermodynamic Review

Ideal, perfect gas $p = \rho R T$ where $R = 287\,J/kgK$ for standard air condition

Speed of sound $a = \sqrt{\gamma R T} = \sqrt{\frac{\gamma p}{\rho}}$ where $\gamma = \frac{c_p}{c_v} = \frac{c_v + R}{c_v}$

Isentrophic process

isentrophic = reversible $\Delta S = \oint \frac{dQ}{T} = 0$ + adabatic $\Delta Q = 0$

for adabatic flow $\longrightarrow$ $\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2$

for isentrophic flow $\longrightarrow$ $\frac{p}{\rho^\gamma} = \text{constant}$

$\longrightarrow$ $\frac{p_0}{p} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}}$ and $\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{1}{\gamma-1}}$

Static and stagnation values

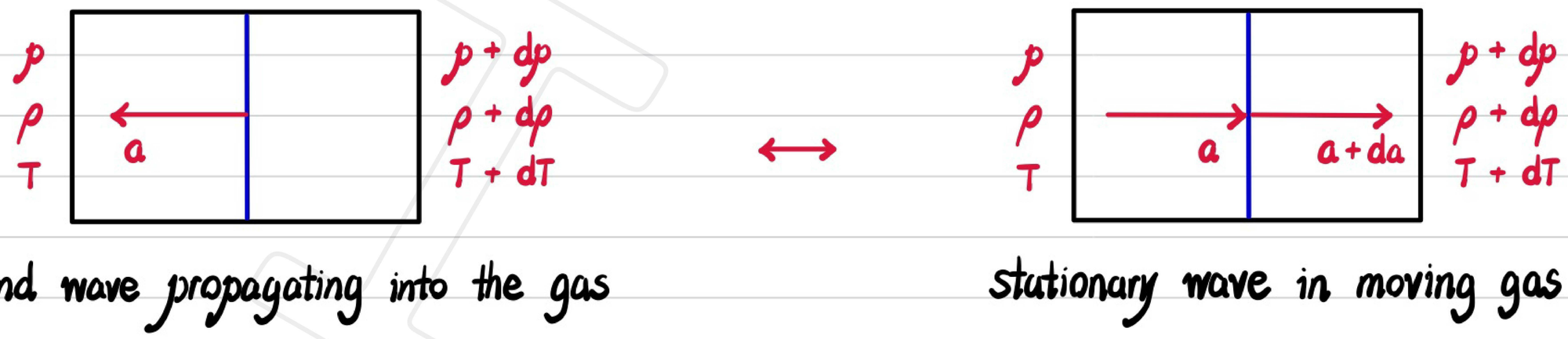
stagnation (total) $\longrightarrow$ $U = 0$ $\longrightarrow$ quantities are constant in an isentrophic flow

static $\longrightarrow$ $U \neq 0$ i.e. p_∞

Compressibility $\tau = -\frac{1}{v}\frac{dv}{dp} = \frac{1}{\rho}\frac{d\rho}{dp} \propto \frac{1}{a}$

Propagation of Sound Waves

Control volume over a sound wave isentrophic



sound wave propagating into the gas $\longleftrightarrow$ stationary wave in moving gas

for continuity, $\rho u A = \text{const}$ $\longrightarrow$ $\rho a = (\rho + d\rho)(a + da)$ $\longrightarrow$ $a = -\rho \frac{da}{d\rho}$

for conservation of momentum $\longrightarrow$ $(\rho + d\rho)(a + da)^2 - \rho a^2 = -dp$

$\longrightarrow$ $\rho a (a + da) - \rho a^2 = -dp$ $\longrightarrow$ $dp = -\rho a\, da$

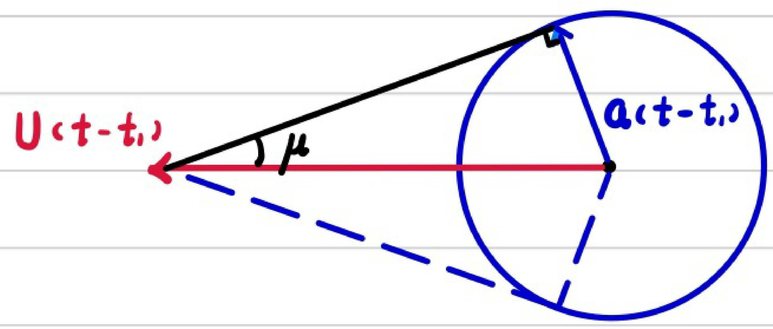
$\longrightarrow$ $a = -\rho \frac{1}{d\rho}\left(-\frac{dp}{\rho a}\right)$ $\longrightarrow$ $a^2 = \frac{dp}{d\rho}$ $\longrightarrow$ $a^2 = \frac{\gamma p}{\rho} = \frac{1}{\rho \tau}$

Pressure coefficient

* Bernoulli equation only vaild for incompressible flow

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} \longrightarrow C_p = \frac{p - p_\infty}{\frac{1}{2}\frac{p_\infty}{RT_\infty}(M_\infty a_\infty)^2} = \frac{p - p_\infty}{\frac{1}{2}\frac{p_\infty}{RT_\infty}M_\infty \gamma R T_\infty} = \frac{2\left(\frac{p}{p_\infty} - 1\right)}{\gamma M_\infty}$$

Mach wave



$$\longrightarrow \mu = \sin^{-1}\left(\frac{a}{U}\right) = \sin^{-1}\left(\frac{1}{M}\right)$$

* M_{crit} critical Mach number M_∞ which sonic flow first achieved on the aerofoil surface

** M_{DD} or M_{DIV} drag divergence Mach number M_∞ which D increase sharply due to shock wave

Shock Waves

Formation

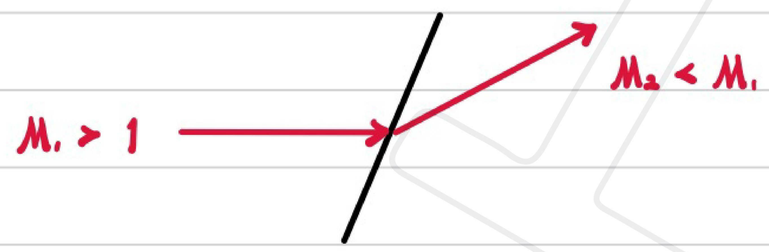
speed of the flow relative to body is too fast for information to propagate upstream

→ shock wave formed → considered as a discontinuity → adabatic process

Normal shock wave

- $u \downarrow$ · $p \uparrow$ · $\rho \uparrow$ · $T \uparrow$ · $s \uparrow$
- $p_0 \downarrow$ · $h_0 =$ → $T_0 =$

Oblique shock wave



- $V \downarrow$ · $p \uparrow$ · $\rho \uparrow$ · $T \uparrow$ · $s \uparrow$
- $p_0 \downarrow$ · $h_0 =$ → $T_0 =$

Normal Shocks

Normal shock waves

Rankine - Hugoniot relations

for continuity, $\rho u A = \text{const} \longrightarrow \rho_1 u_1 = \rho_2 u_2$

for conservation of momentum $\longrightarrow \rho_1 u_1^2 = \rho_2 u_2^2$

for conservation of energy $\longrightarrow E_1 + \frac{1}{2} m_1 v_1^2 = E_2 + \frac{1}{2} m_2 v_2^2 \longrightarrow C_{v,1} T_1 + \frac{1}{2} u_1^2 = C_{v,2} T_2 + \frac{1}{2} u_2^2$

$$\longrightarrow \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \frac{1}{2}u_1^2 = \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} + \frac{1}{2}u_2^2$$

$\longrightarrow M_2^2 = \frac{2+(\gamma-1)M_1^2}{2\gamma M_1^2-(\gamma-1)}$ and $\frac{p_2}{p_1} = \frac{2\gamma M_1^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1}$ and $\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}$

$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} \longrightarrow \frac{T_2}{T_1} = [2+(\gamma-1)M_1^2]\frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)^2 M_1^2} \longrightarrow M_1 > 0$ and $M_2 < M_1$

Moving vs stationary shock wakes

$\longrightarrow M_s$ absolute velocity $= M_1 - M_2$

Vary Area Ducts

1-D nozzle flow

for continuity, $\rho u A = \text{const} \longrightarrow \rho u A = (\rho+d\rho)(u+du)(A+dA) \longrightarrow \rho u dA + \rho A du + u A d\rho = 0$

$$\longrightarrow \frac{d\rho}{\rho} + \frac{dU}{U} + \frac{dA}{A} = 0$$

for conservation of momentum $\longrightarrow (\rho+d\rho)(u+du)^2(A+dA) - \rho u^2 A = pA - (p+dp)(A+dA)$

$$\longrightarrow U(\rho u dA + \rho A du + u A d\rho) + \rho A u du = -A dp$$

$$\longrightarrow dp = -\rho u du$$

$$\longrightarrow \frac{dA}{A} = (M^2-1)\frac{du}{u}$$

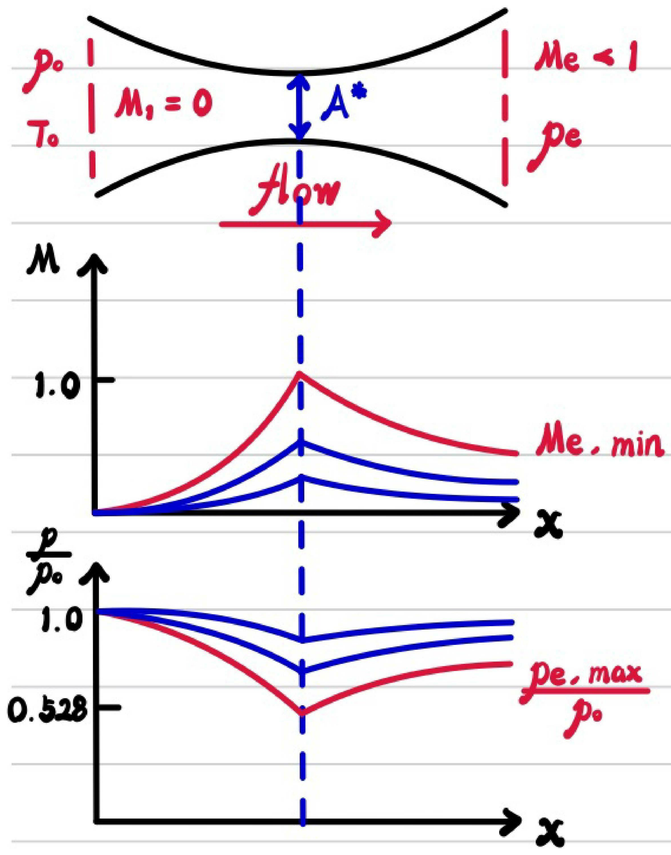
* for $M < 1$ subsonic $\longrightarrow \frac{dA}{A} = -\frac{du}{u}$ ** for $M > 1$ supersonic $\longrightarrow \frac{dA}{A} = \frac{du}{u}$

Chocking

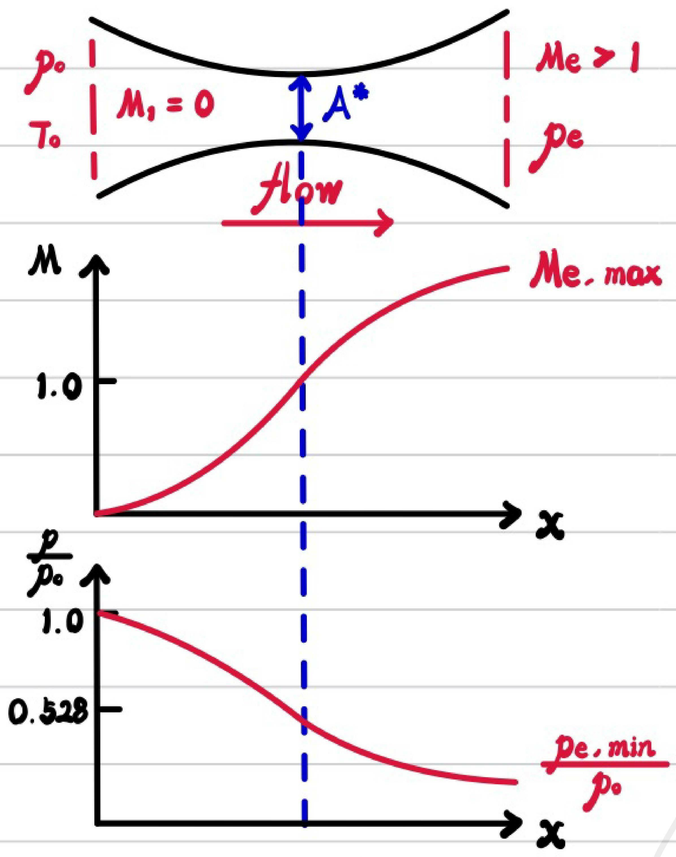
$M = 1$ at throat $\longrightarrow$ mass flow rate $\dot{m} = \rho u A$ cannot be increased

$$\longrightarrow \left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{2}{\gamma+1}M^2\right)\right]$$

The convergent-divergent nozzle

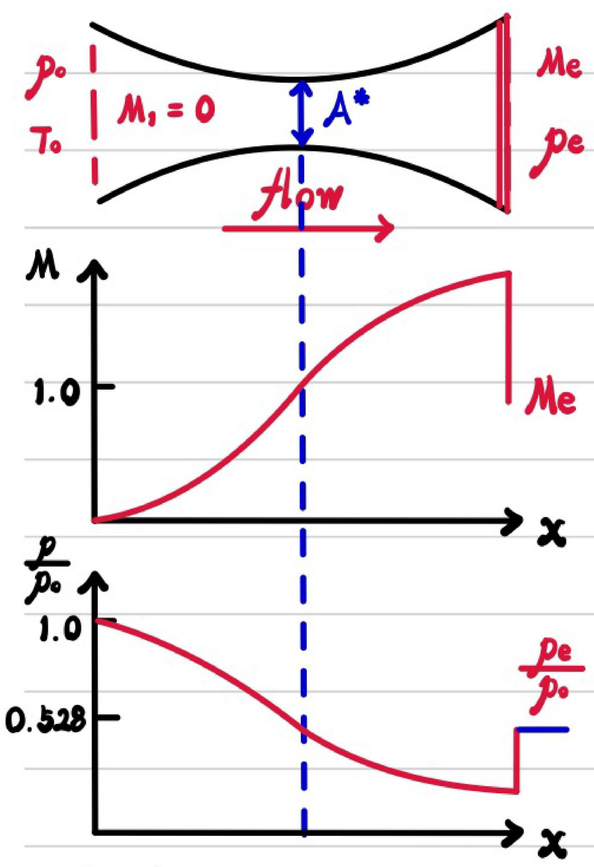


maximum back pressure

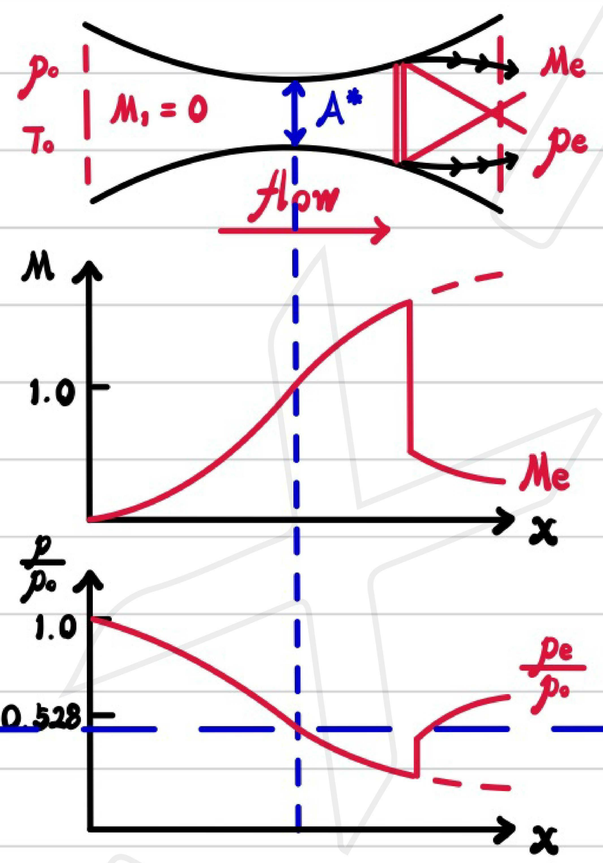


minimum back pressure

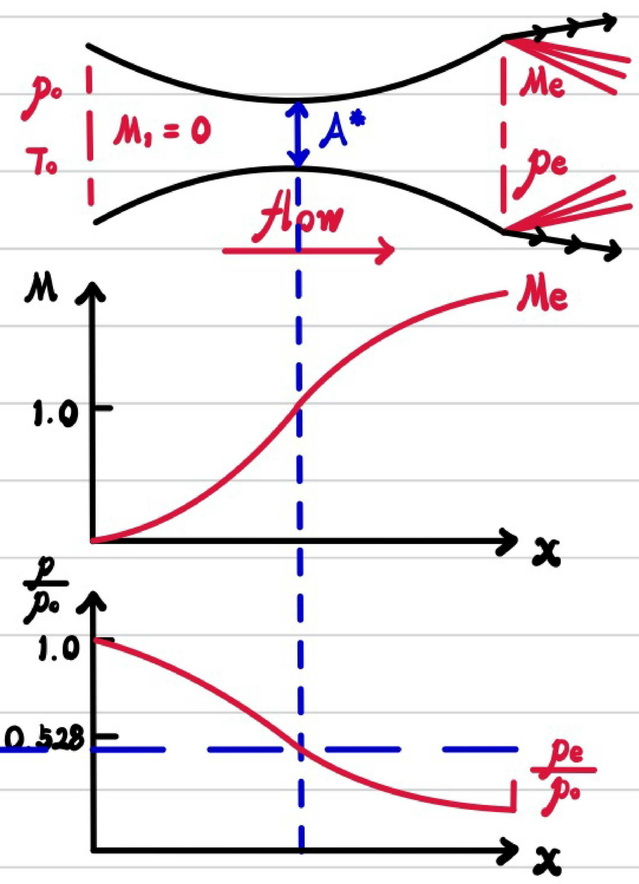
when forming normal shock wave, solution become non-isentrophic but adabatic



shock wave at exit

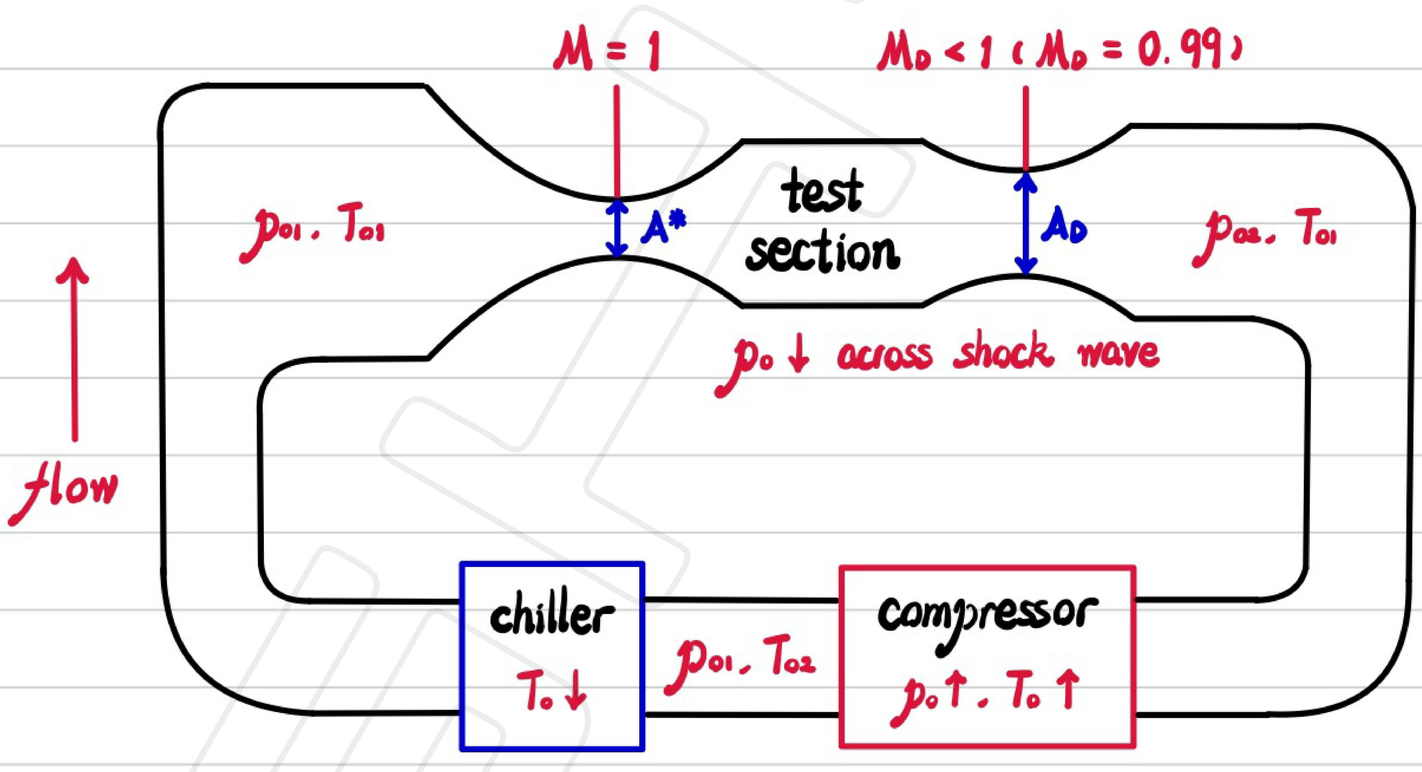


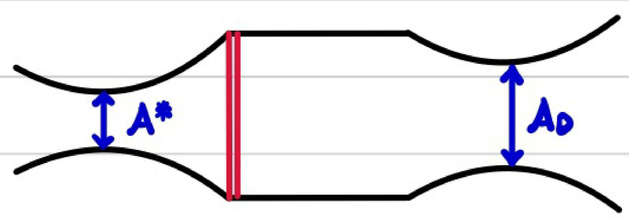
overexpanded



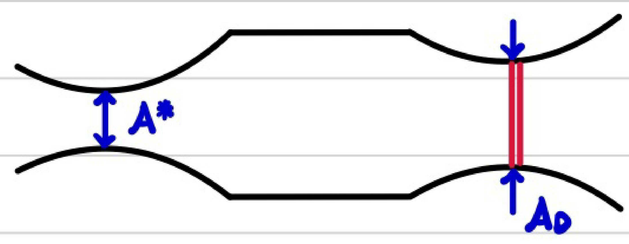
underexpanded

Supersonic wind tunnel





starting condition



running condition

$\longrightarrow$ $W = W_{compressor} = \dot{m} C_p \Delta T = \dot{m} C_p (T_{02} - T_{01})$ $\quad$ * $\frac{T_{02}}{T_{01}} = \left(\frac{p_{01}}{p_{02}}\right)^{\frac{\gamma-1}{\gamma}}$

Rocket engines $\quad F = \dot{m} V_e + (p_e - p_B) A_e$

Oblique Shocks

Oblique Shock Waves

Defination

where $V_1 = V_2$

Equations

change across oblique wave are governed only by component of velocity normal to the wave

$\longrightarrow$ $M_{n,1} = M_1 \sin\beta$ and $M_{n,2} = M_2 \sin(\beta-\theta)$

$\longrightarrow$ $M_2^2 \sin^2(\beta-\theta) = \dfrac{2+(\gamma-1)M_1^2\sin^2\theta}{2\gamma M_1^2\sin^2\theta-(\gamma-1)}$ and $\dfrac{p_2}{p_1} = \dfrac{2\gamma M_1^2\sin^2\theta}{\gamma+1} - \dfrac{\gamma-1}{\gamma+1}$

and $\dfrac{\rho_2}{\rho_1} = \dfrac{u_1}{u_2} = \dfrac{(\gamma+1)M_1^2\sin^2\theta}{2+(\gamma-1)M_1^2\sin^2\theta}$ and $\dfrac{T_2}{T_1} = [2+(\gamma-1)M_1^2\sin^2\theta]\dfrac{2\gamma M_1^2\sin^2\theta-(\gamma-1)}{(\gamma+1)^2M_1^2\sin^2\theta}$

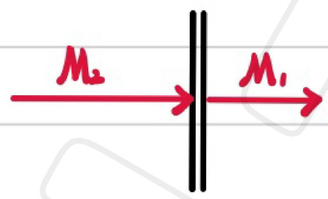
$\longrightarrow$ $\tan\theta = 2\cot\beta \dfrac{M_1^2\sin^2\beta-1}{M_1^2(\gamma+\cos(2\beta))+2}$

* θ_{max} maximum possible deflection angle for any M with only one β solution

** $\theta < \theta_{max}$ $\longrightarrow$ small β weak (common) solution and large β strong (rare) solution

Mach wave

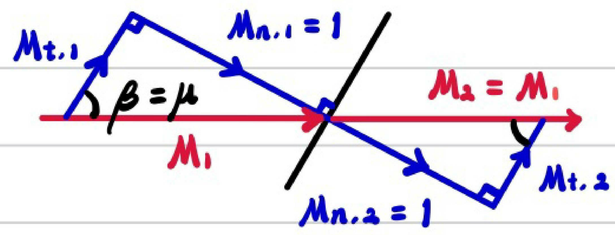
$\theta = 0^\circ$ $\longrightarrow$ $\beta = 90^\circ$



normal shock wave

or

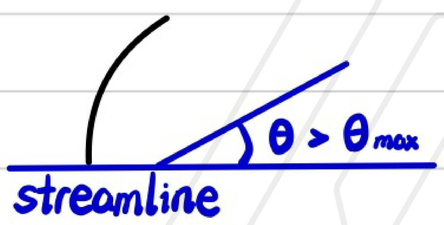
$\beta = \mu = \sin^{-1}\left(\frac{1}{M_1}\right)$



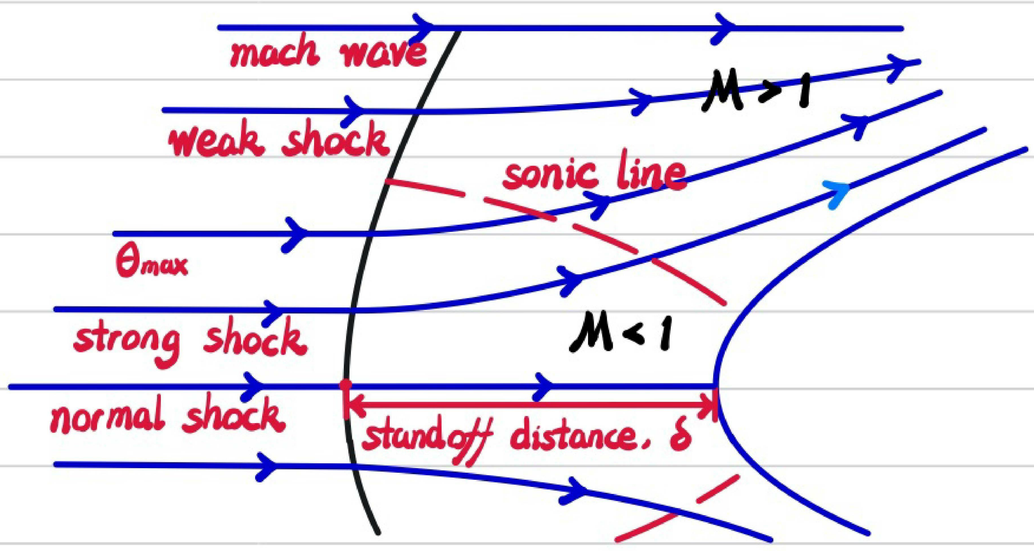
mach wave

Detached shocks

$\theta > \theta_{max}$

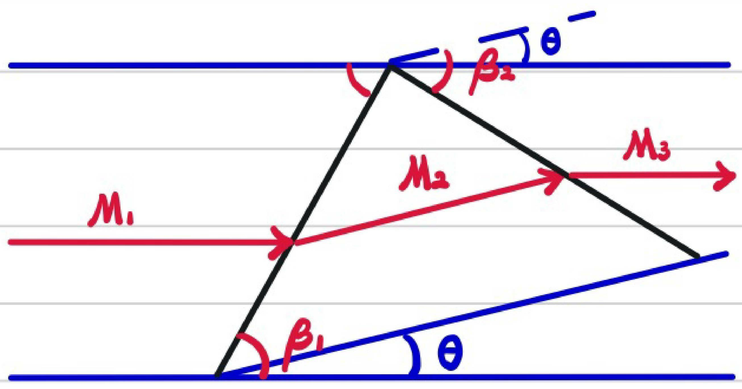


$\longrightarrow$ form a bow shock

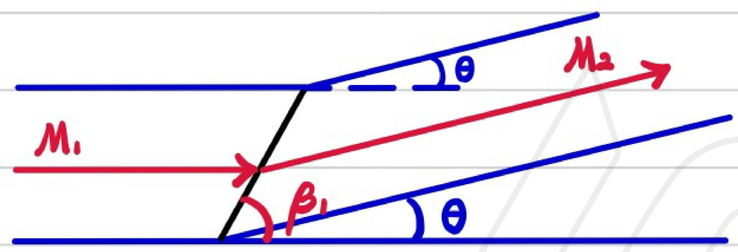


Shock Wave Reflection

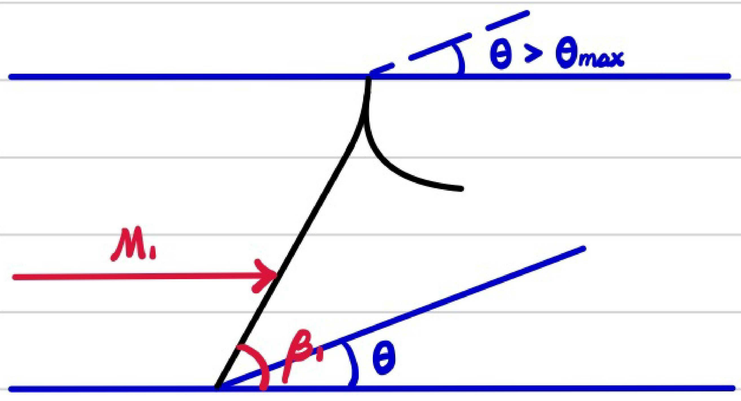
Regular wave reflection



* shock could be absorbed for one value of M_1



Mach reflection

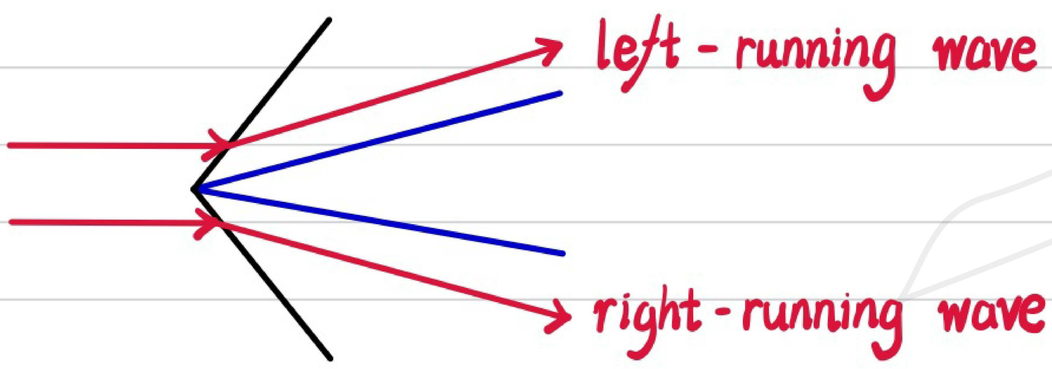


find θ for Mach reflection at M_1

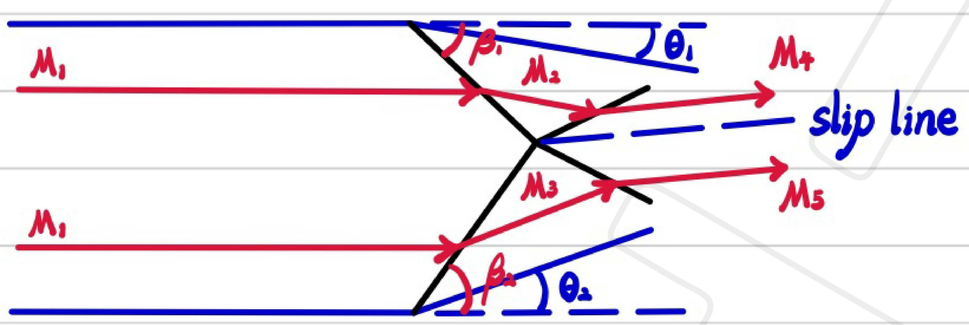
→ first guess $\theta = 20°$ → find β_1 → find M_2

→ if $\theta < \theta_{max}$ for M_2 → second guess → ...

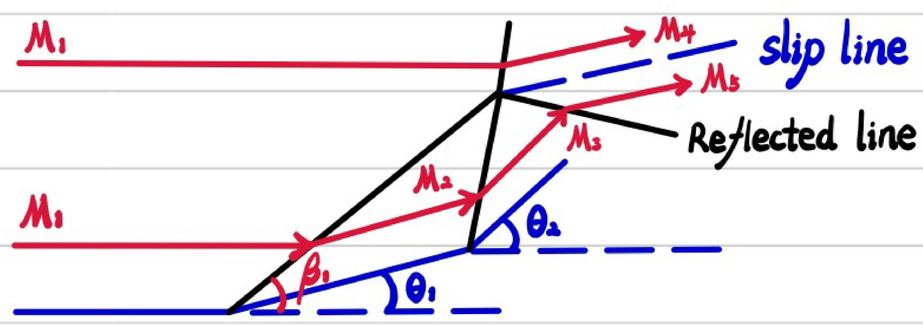
Left-running and right-running wave



Intersection of waves



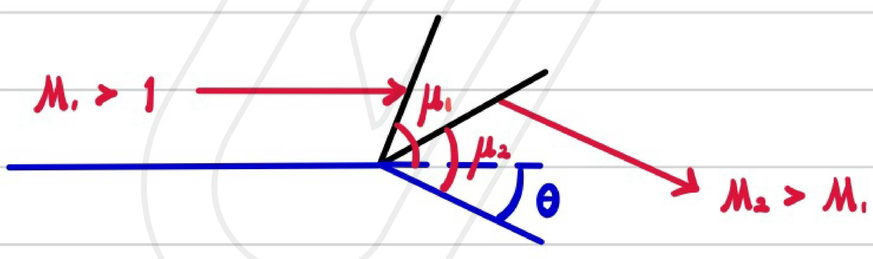
or



* different entrophy and speed but same direction and pressure both side of slip line

Expansion Fans

Prandtl-Meyer expansion wave



- $V \uparrow$
- $p \downarrow$
- $\rho \downarrow$
- $T \downarrow$
- $s =$
- $p_0 =$
- $h_0 =$ → $T_0 =$

* isentrophic process $\longrightarrow$ $\frac{p_2}{p_1} = \frac{p_2}{p_{02}}\frac{p_{01}}{p_1}$ and $\frac{T_2}{T_1} = \frac{T_2}{T_{02}}\frac{T_{01}}{T_1}$

Prandtl - Meyer function

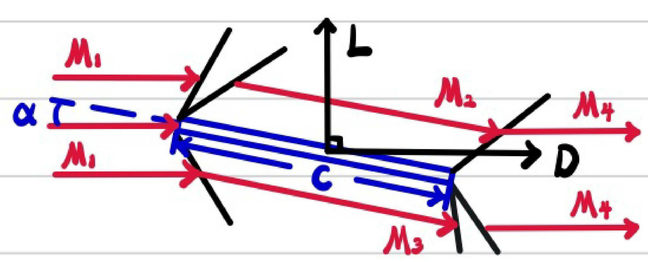
$\tan\theta = 2\cot\beta \frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma + \cos(2\beta)) + 2}$ where $\beta = \mu_1 = \sin^{-1}\left(\frac{1}{M_1}\right)$ $\longrightarrow$ $\theta \approx \tan\theta = 2\sqrt{M_1^2 - 1}\,\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma + 1)}$

$\longrightarrow$ $d\theta \pm \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M} = 0$ $\longrightarrow$ $\theta \pm \nu(M) = \text{constant}$

$\longrightarrow$ $\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\left(\frac{\gamma-1}{\gamma+1}\right)(M^2-1)} - \tan^{-1}\sqrt{M^2-1}$

$\longrightarrow$ $\theta + \nu(M) = \text{constant}$ for left-running wave and $\theta - \nu(M) = \text{constant}$ for right-running wave

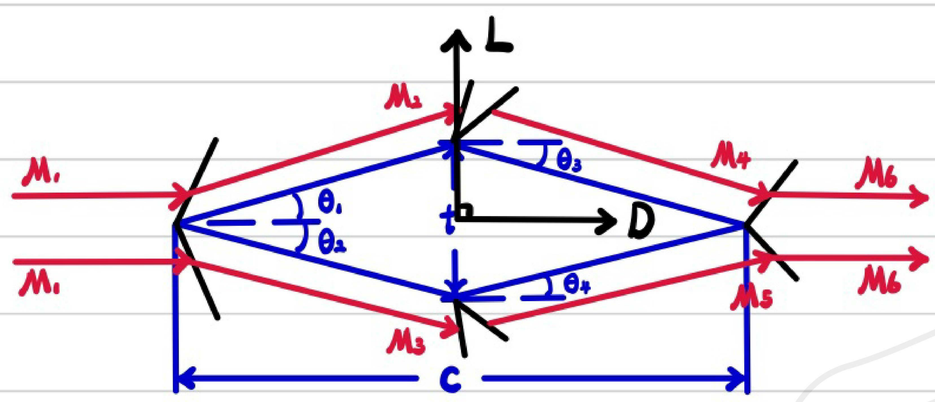
Lift and drag



$L = (p_2 - p_3)\cos\alpha \cdot c$

$D = (p_3 - p_2)\sin\alpha \cdot c$

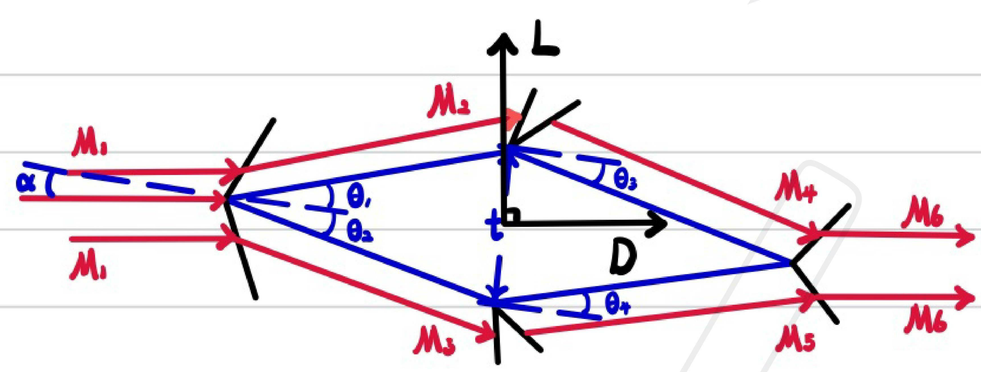
$\longrightarrow$



$L = N = \Delta\bar{p}_N \cdot c$ force normal to chord

$D = P = \Delta\bar{p}_P \cdot t$ force perpendicular to chord

$\longrightarrow$



$L = N\cos\alpha - P\sin\alpha$

$D = N\sin\alpha + P\sin\alpha$ * wave drag

Linear Supersonic Theory

Pressure coefficient

when θ is small $\longrightarrow$ $\theta \approx \tan\theta = 2\sqrt{M_1^2-1}\,\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma+1)}$ and $\frac{p_2}{p_1} = \frac{2\gamma M_1^2\sin^2\theta}{\gamma+1} - \frac{\gamma-1}{\gamma+1}$

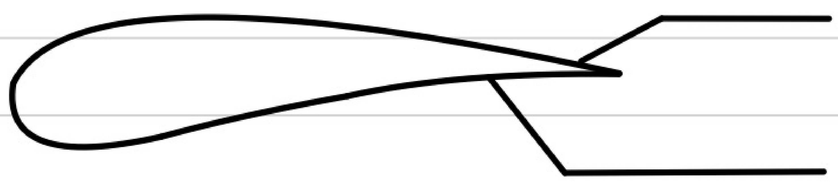
$\longrightarrow$ $\frac{p_2 - p_1}{p_1} = \frac{p_2}{p_1} - 1 = \frac{2\gamma}{\gamma+1}\frac{\theta(\gamma+1)M_1^2}{2\sqrt{M_1^2-1}} = \frac{\gamma\theta M_1^2}{\sqrt{M_1^2-1}}$

$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty u_\infty^2} = \frac{2\left(\frac{p}{p_\infty} - 1\right)}{\gamma M_\infty^2}$

$\longrightarrow$ $C_p = \frac{2\theta}{\sqrt{M_1^2-1}}$ for left-running wave and $C_p = -\frac{2\theta}{\sqrt{M_1^2-1}}$ for right-running wave

for flow past a curved body $\longrightarrow$ $C_{p,x} = C_{p,1} \pm \frac{2(\theta_x - \theta_1)}{\sqrt{M_1^2-1}} = \pm\frac{2\theta}{\sqrt{M_1^2-1}}$ for 1 being freestream

Lift and drag



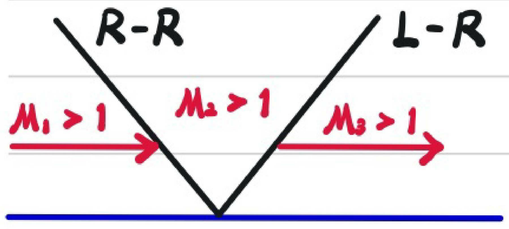
$$y_u(x) = y_c(x) + y_t(x)$$

$$y_l(x) = y_c(x) - y_t(x)$$

$$\longrightarrow \quad \theta_c = \frac{dy_c}{dx} \qquad \theta_t = \frac{dy_t}{dx}$$

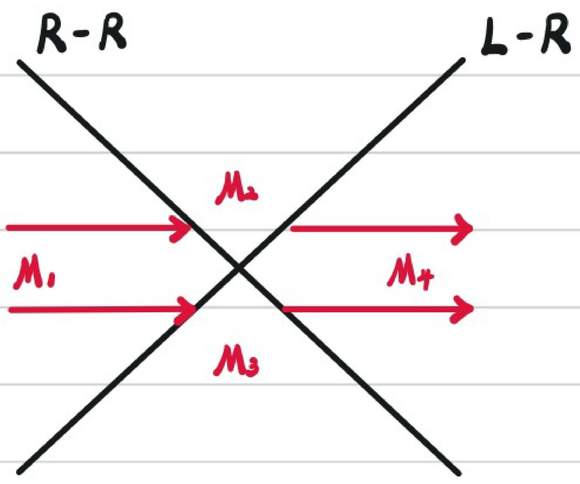
$$\longrightarrow \quad C_L = \frac{4\alpha}{\sqrt{M_1^2 - 1}} \quad \text{and} \quad C_D = \frac{4}{\sqrt{M_1^2 - 1}}\left(\alpha^2 + \int_0^1 (\theta_c^2 + \theta_t^2)\, d\left(\frac{x}{c}\right)\right)$$

Weak wave reflection



$$\longrightarrow \quad C_{p,2} = C_{p,1} - \frac{2(\theta_2 - \theta_1)}{\sqrt{M_1^2 - 1}} \quad \longrightarrow \quad C_{p,3} = \left(C_{p,1} - \frac{2(\theta_2 - \theta_1)}{\sqrt{M_1^2 - 1}}\right) + \frac{2(\theta_3 - \theta_2)}{\sqrt{M_1^2 - 1}}$$

Weak wave intersection



$$\longrightarrow \quad C_{p,2} = C_{p,1} - \frac{2(\theta_2 - \theta_1)}{\sqrt{M_1^2 - 1}} \quad \text{and} \quad C_{p,3} = C_{p,1} - \frac{2(\theta_3 - \theta_1)}{\sqrt{M_1^2 - 1}}$$

$$\longrightarrow \quad C_{p,4} = \left(C_{p,1} - \frac{2(\theta_2 - \theta_1)}{\sqrt{M_1^2 - 1}}\right) + \frac{2(\theta_4 - \theta_2)}{\sqrt{M_1^2 - 1}}$$

$$= \left(C_{p,1} + \frac{2(\theta_3 - \theta_1)}{\sqrt{M_1^2 - 1}}\right) - \frac{2(\theta_4 - \theta_3)}{\sqrt{M_1^2 - 1}}$$

$$\longrightarrow \quad \theta_4 - \theta_2 = \theta_3 - \theta_1 \quad \longleftrightarrow \quad \theta_4 - \theta_3 = \theta_2 - \theta_1$$