Describing an Aircraft Dynamics

A Coordinate Free Approach

Assumption aircraft behaves dynamically like a rigid body of constant mass

* limit in application e.g. large bending and torsion of wing

Newton's Law $F = \frac{d(mv)}{dt} \approx m\frac{dv}{dt} \longrightarrow F$ and $\frac{dv}{dt}$ are in the same direction

* $F = m(\frac{dv}{dt} + a_{additional})$ e.g. Coriolis and centrifugal accelerations

The Rotational Dynamics (Euler's Equation)

$M = \frac{d}{dt}I(\omega) \xrightarrow[\text{angular momentum } I(\omega)]{\text{inertia tensor } I} M$ and $\frac{d\omega}{dt}$ not nessarily goes to the same direction

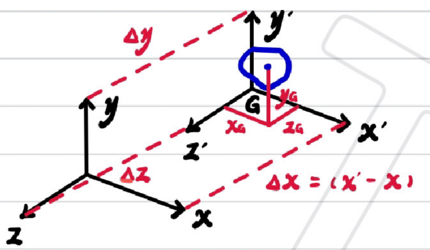
The Inertia Tensor

Tensor all tensor are linear operators $\longrightarrow I(cv) = cI(v)$ & $I(u+v) = I(u) + I(v)$

Inertia Tensor

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} = \begin{bmatrix} \int(y^2+z^2)\,dm & -\int xy\,dm & -\int xz\,dm \\ -\int xy\,dm & \int(x^2+z^2)\,dm & -\int yz\,dm \\ -\int xz\,dm & -\int yz\,dm & \int(x^2+y^2)\,dm \end{bmatrix}$$

Parallel Axis Theorem



- $I_{xx} = \int(y^2+z^2)\,dm = \int(y_G+\Delta y)^2 + (z_G+\Delta z)^2\,dm$

$= \int(y_G^2+z_G^2)\,dm + 2\Delta y\underbrace{\int y_G\,dm}_{0} + 2\Delta z\underbrace{\int z_G\,dm}_{0} + (\Delta y^2+\Delta z^2)\int dm$

$= I_{G,xx} + m(\Delta y^2+\Delta z^2)$

- $I_{yy} = I_{G,yy} + m(\Delta x^2+\Delta z^2)$ · $I_{zz} = I_{G,zz} + m(\Delta x^2+\Delta y^2)$

- $I_{xy} = I_{yx} = I_{G,xy} + m\Delta x\Delta y$ · $I_{xz} = I_{zx} = I_{G,xz} + m\Delta x\Delta z$ · $I_{yz} = I_{zy} = I_{G,yz} + m\Delta y\Delta z$

Choose of Axes

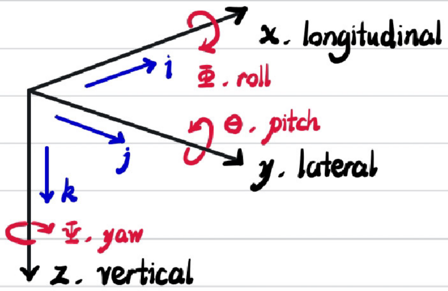
- Earth Fixed Inertia Axes $\longrightarrow$ fixed respected to ground $\longrightarrow$ I change with time

- Body-fixed axes ⟶ fixed respected to aircraft ⟶ I not change with time
- Principal Axes ⟶ $I_{xy} = I_{xz} = I_{yz} = 0$
- Stability Axes ⟶ x aligned to freestream ⟶ used

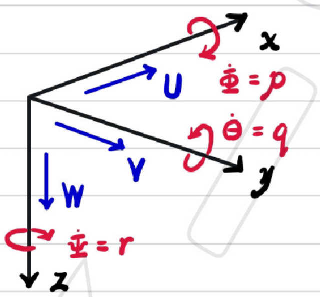
The Aircraft as a Dynamical System

Body-fixed Coordinate System

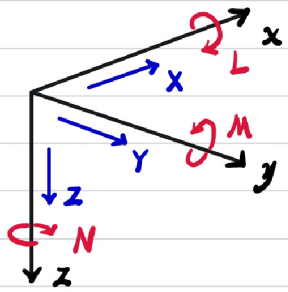
- displacement



- velocity



- force & moment



Equation of Motion

Assumption

- distribution of mass remains constant * ignore burning fuel
- angular momentum of the aircraft results solely from the rotation of CG
- mass distribution symmetry about lateral plane ⟶ $I_{xy} = I_{yz} = 0$

Body-fixed form of Newton's Equation

$$F = Xi + Yj + Zk = m\frac{dv}{dt} = m\frac{d}{dt}(Ui + Vj + Wk) = m(\dot{U}i + \dot{V}j + \dot{W}k + U\frac{di}{dt} + V\frac{dj}{dt} + W\frac{dk}{dt})$$

$$* \frac{di}{dt} = \omega \times i = \begin{vmatrix} i & j & k \\ p & q & r \\ 1 & 0 & 0 \end{vmatrix} = rj - qk \quad \& \quad \frac{dj}{dt} = \omega \times j = -ri + pk \quad \& \quad \frac{dk}{dt} = \omega \times k = qi - pj$$

$$\longrightarrow F = m[(\dot{U} - Vr + Wq)i + (\dot{V} + Ur - Wp)j + (\dot{W} - Uq + Vp)k]$$

$$\longrightarrow F = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = m\begin{bmatrix} \dot{U} - Vr + Wq \\ \dot{V} + Ur - Wp \\ \dot{W} - Uq + Vp \end{bmatrix}$$

Body-fixed form of Euler's Equation

$$\mathbf{M} = L\mathbf{i} + M\mathbf{j} + N\mathbf{k}$$

$$= \frac{d}{dt} I(\omega) = \frac{d}{dt}\begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \frac{d}{dt}\left[(I_{xx}p - I_{xz}r)\mathbf{i} + I_{yy}q\mathbf{j} + (-I_{xz}p + I_{zz}r)\mathbf{k}\right]$$

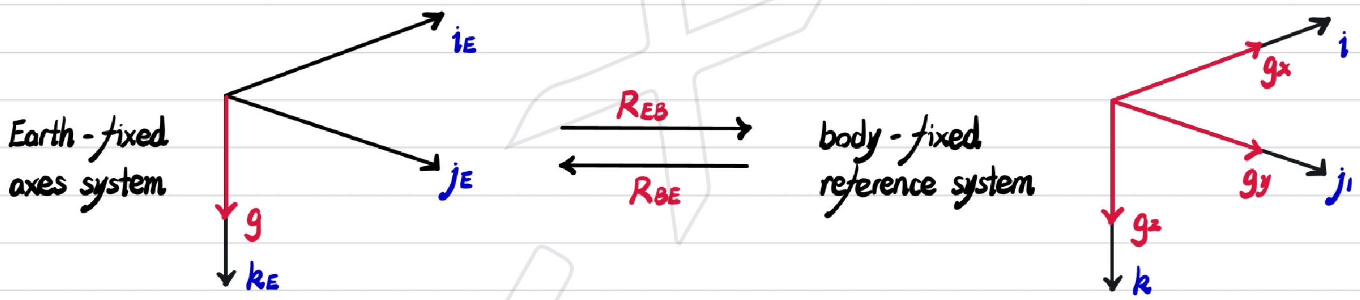
$$= (I_{xx}\dot{p} - I_{xz}\dot{r})\mathbf{i} + (I_{xx}p - I_{xz}r)\frac{d\mathbf{i}}{dt} + I_{yy}\dot{q}\mathbf{j} + I_{yy}q\frac{d\mathbf{j}}{dt} + (-I_{xz}\dot{p} + I_{zz}\dot{r})\mathbf{k} + (-I_{xz}p + I_{zz}r)\frac{d\mathbf{k}}{dt}$$

$$\longrightarrow \mathbf{M} = \begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{p} - I_{xz}\dot{r} - (I_{yy}q)r + (-I_{xz}p + I_{zz}r)q \\ I_{yy}\dot{q} + (I_{xx}p - I_{xz}r)r - (-I_{xz}p + I_{zz}r)p \\ -I_{xz}\dot{p} + I_{zz}\dot{r} + (I_{yy}q)p - (I_{xx}p - I_{xz}r)q \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{p} - I_{xz}\dot{r} + (I_{zz} - I_{yy})qr - I_{xz}pq \\ I_{yy}\dot{q} + I_{xz}(p^2 - r^2) + (I_{xx} - I_{zz})pr \\ -I_{xz}\dot{p} + I_{zz}\dot{r} + (I_{yy} - I_{xx})pq + I_{xz}qr \end{bmatrix}$$

Force and Moment Acting on the Aircraft

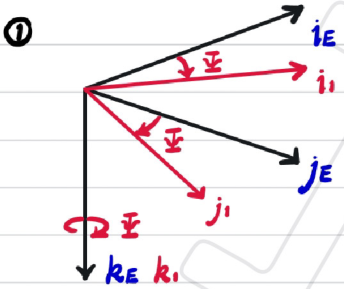
Gravitational Force

* gravity exerts no moment about the CG of the aircraft



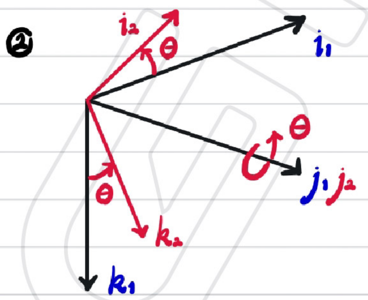
Convertion of Reference System

①



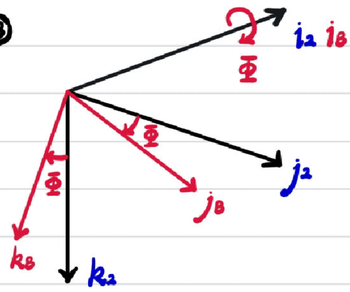
$$R_{1E} = \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \xleftrightarrow{R^{-1}} R_{E1} = \begin{bmatrix} \cos\Psi & -\sin\Psi & 0 \\ \sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

②



$$R_{21} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \xleftrightarrow{R^{-1}} R_{12} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

③



$$R_{B2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & \sin\Phi \\ 0 & -\sin\Phi & \cos\Phi \end{bmatrix} \xleftrightarrow{R^{-1}} R_{2B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & \sin\Phi & \cos\Phi \end{bmatrix}$$

· gravitational force

$$\begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = R_{BE}(\Phi, \Theta, \Psi) \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = R_{B2}(\Phi) R_{21}(\Theta) R_{1E}(\Psi) \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & \sin\Phi \\ 0 & -\sin\Phi & \cos\Phi \end{bmatrix} \begin{bmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ \sin\Theta & 0 & \cos\Theta \end{bmatrix} \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

$$= \begin{bmatrix} \cos\Theta\cos\Psi & \cos\Theta\sin\Psi & -\sin\Theta \\ \sin\Phi\sin\Theta\cos\Psi - \cos\Phi\sin\Psi & \sin\Phi\sin\Theta\sin\Psi + \cos\Phi\cos\Psi & \sin\Phi\cos\Theta \\ \cos\Phi\sin\Theta\cos\Psi + \sin\Phi\sin\Psi & \cos\Phi\sin\Theta\sin\Psi - \sin\Phi\cos\Psi & \cos\Phi\cos\Theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

$$= g \begin{bmatrix} -\sin\Theta \\ \sin\Phi\cos\Theta \\ \cos\Phi\cos\Theta \end{bmatrix}$$

· navigation equations

$$\begin{bmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{bmatrix} = R_{EB}(\Psi, \Theta, \Phi) \begin{bmatrix} U \\ V \\ W \end{bmatrix} \longrightarrow \text{integrate} \begin{bmatrix} x_E \\ y_E \\ z_E \end{bmatrix}$$

· angular velocity

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = R_{BE}(\Phi, \Theta, \Psi) \begin{bmatrix} 0 \\ 0 \\ \dot{\Psi} \end{bmatrix} + R_{B1}(\Phi, \Theta) \begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix} + R_{B2}(\Phi) \begin{bmatrix} \dot{\Phi} \\ 0 \\ 0 \end{bmatrix}$$

$$= \dot{\Psi} \begin{bmatrix} -\sin\Theta \\ \sin\Phi\cos\Theta \\ \cos\Phi\cos\Theta \end{bmatrix} + \dot{\Theta} \begin{bmatrix} 0 \\ \cos\Phi \\ -\sin\Phi \end{bmatrix} + \dot{\Phi} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & \sin\Phi\cos\Theta \\ 0 & -\sin\Phi & \cos\Phi\cos\Theta \end{bmatrix} \begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix}$$

$$* \begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & \sin\Phi\cos\Theta \\ 0 & -\sin\Phi & \cos\Phi\cos\Theta \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Aircraft Dynamics on State Space

State Space

Definition

· system of ODE $\dot{x}(t) = f(x(t), \delta)$

e.g. free-vibrate mass-spring system $m\ddot{r} + c\dot{r} + kr = 0$

$$\longrightarrow \dot{x}(t) = \begin{bmatrix} \dot{r} \\ \ddot{r} \end{bmatrix} = f(x(t), \delta) = \begin{bmatrix} \dot{r} \\ -\frac{k}{m}\dot{r} - \frac{c}{m}r \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}}_{\delta} \underbrace{\begin{bmatrix} r \\ \dot{r} \end{bmatrix}}_{x(t)}$$

· Jacobian

$$J_{x_0, \delta} = \begin{bmatrix} \frac{d}{dx_1} f_1(x,\delta) & \frac{d}{dx_2} f_1(x,\delta) & \cdots & \frac{d}{dx_n} f_1(x,\delta) \\ \frac{d}{dx_1} f_2(x,\delta) & \frac{d}{dx_2} f_2(x,\delta) & \cdots & \frac{d}{dx_n} f_2(x,\delta) \\ \cdots & \cdots & \cdots & \\ \frac{d}{dx_1} f_n(x,\delta) & \frac{d}{dx_2} f_n(x,\delta) & \cdots & \frac{d}{dx_n} f_n(x,\delta) \end{bmatrix}$$

· singular point $\dot{x}_0(t) = f(x_0(t), \delta) = 0 \longrightarrow$ equilibria of the system

· trajectary on state space $x + \dot{x}\Delta t = x + \dot{x} f(x, \delta)$ *never interesect except singular points

Derivation $Mx = R(\dot{x}, x, \delta)$

① at singular point $\longrightarrow M\dot{x}_0 = 0 \longrightarrow R(0, x_0, \delta) = 0 \longrightarrow$ find singular point(s) x_0

② apply small perturbation at singular point $\longrightarrow M(\dot{x}_0 + \dot{x}') = R(\dot{x}_0 + \dot{x}', x_0 + x', \delta)$

$\xrightarrow{\text{Taylor's expansion}} M(\dot{x}_0 + x') = R(\dot{x}_0, x_0, \delta) + \dot{x}' R_{\dot{x}_0} + x' R_{x_0} \longrightarrow M\dot{x}' = \dot{x}' R_{\dot{x}_0} + x' R_{x_0}$

$\longrightarrow (M - R_{\dot{x}_0})\dot{x}' = M_* \dot{x}' = x' R_{x_0}$

③ let $x' = \underbrace{x^*}_{\text{const}} e^{\lambda t} \longrightarrow \dot{x}' = \lambda x^* e^{\lambda t} = \lambda x' \longrightarrow (M^*\lambda - R_{x_0})x' = 0 \xrightarrow{x' \neq 0} |R - M^*\lambda| = 0$

$\longrightarrow$ find eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$

$\longrightarrow$ find eigenvectors $\xi_1, \xi_2, \dots, \xi_n$ *for repeated eigenvalues $\lambda_{1,2} \longrightarrow (A - \lambda I)\xi_2 = \xi_1$

$\longrightarrow$ general solution $x(t) = x_0 + \underbrace{k_1}_{\text{const}}\xi_1 e^{\lambda_1 t} + k_2\xi_2 e^{\lambda_2 t} + \dots + k_n\xi_n e^{\lambda_n t}$

Structure of Singular Points and Stability

Oscillatory

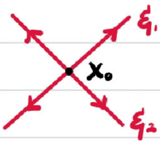
λ — real a $\longrightarrow$ non-osillatary
— pair of complex conjugate $a+ib$ $\longrightarrow$ oscillatory

Stability

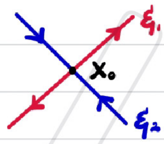
$Re(\lambda)$ — $<0 \longrightarrow$ linear stable
— $>0 \longrightarrow$ linear unstable

Elementary of Singular Points in 2-D Space

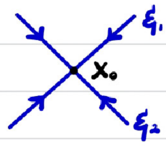
- $\lambda_1 = a > 0$ & $\lambda_2 = b > 0$



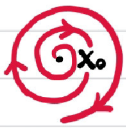
- $\lambda_1 = a > 0$ & $\lambda_2 = b < 0$



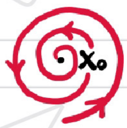
- $\lambda_1 = a < 0$ & $\lambda_2 = b < 0$



- $\lambda = a \pm ib$ & $a > 0$

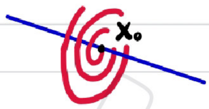


- $\lambda = a \pm ib$ & $a < 0$

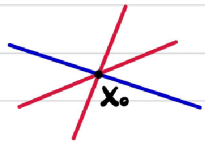


Elementary of Singular Points in 3-D Space

- $\lambda_1 = a$ & $\lambda_{2,3} = b \pm ic$

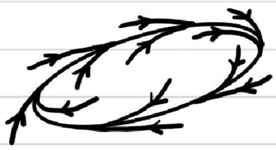


- $\lambda_1 = a$ & $\lambda_2 = b$ & $\lambda_3 = c$



Limit Cycle 极限环

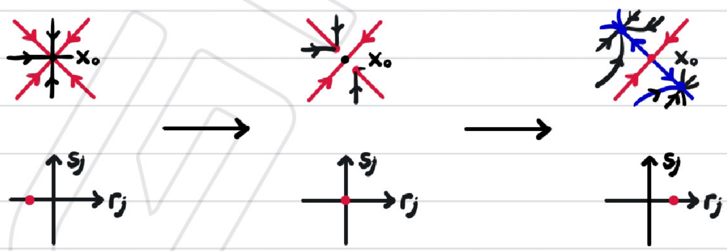
$J_{(x_0, \delta)} = 0 \longrightarrow$



Bifurcations

Pitchfork Bifurcation

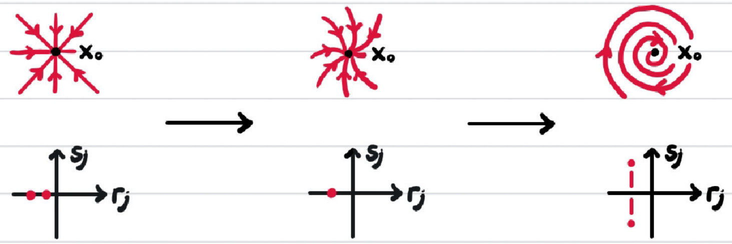
a real eigenvalue change sign from -ve to +ve



two new stable equilibria

* nose - slice high $\delta_E \longrightarrow$ small pertubation on pitch $\longrightarrow$ highly left & right (unpredictable) yawed final state

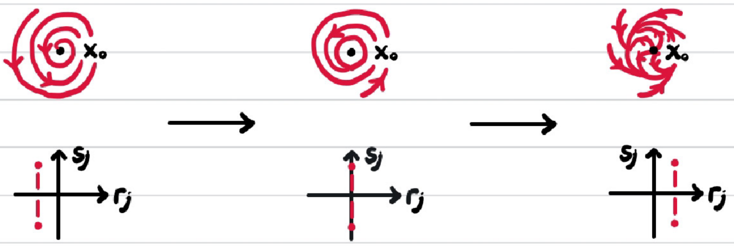
Two Real Eigenvalues Coalese to Form A Complex Conjugate Pair



impact the handling quality

Supercritical Hopf Bifurcation

a complex conjugate pair move across imaginary axis

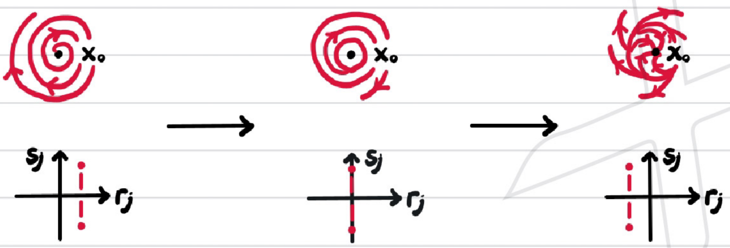


limit cycle oscillation

* wing-rock high δ_E $\longrightarrow$ oscillate about longitudinal axis

Subcritical Hopf Bifurcation

a complex conjugate pair move across imaginary axis



limit cycle oscillation

The Aircraft's State Equations

The Standard Form

Body-fixed form of Newton's Equation

$$F + mg = \begin{bmatrix} X^a \\ Y^a \\ Z^a \end{bmatrix} + g \begin{bmatrix} -\sin\Theta \\ \sin\Phi\cos\Theta \\ \cos\Phi\cos\Theta \end{bmatrix} = m\frac{dv}{dt} = m \begin{bmatrix} \dot{U} - Vr + Wq \\ \dot{V} + Ur - Wp \\ \dot{W} - Uq + Vp \end{bmatrix}$$

Body-fixed form of Euler's Equation

$$M = \begin{bmatrix} L^a \\ M^a \\ N^a \end{bmatrix} = \frac{d}{dt} I(\omega) = \begin{bmatrix} I_{xx}\dot{p} - I_{xz}\dot{r} + (I_{zz} - I_{yy})qr - I_{xz}pq \\ I_{yy}\dot{q} + I_{xz}(p^2 - r^2) + (I_{xx} - I_{zz})pr \\ -I_{xz}\dot{p} + I_{zz}\dot{r} + (I_{yy} - I_{xx})pq + I_{xz}qr \end{bmatrix}$$

Rate of Rotation

$$\omega = \begin{bmatrix} p\\ q\\ r\end{bmatrix} = R_{BE}(\Phi,\Theta,\Psi)\begin{bmatrix}0\\ 0\\ \dot\Psi\end{bmatrix} + R_{B1}(\Phi,\Theta)\begin{bmatrix}0\\ \dot\Theta\\ 0\end{bmatrix} + R_{B2}(\Phi)\begin{bmatrix}\dot\Phi\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}1&0&-\sin\Theta\\ 0&\cos\Phi&\sin\Phi\cos\Theta\\ 0&-\sin\Phi&\cos\Phi\cos\Theta\end{bmatrix}\begin{bmatrix}\dot\Phi\\ \dot\Theta\\ \dot\Psi\end{bmatrix}$$

$$\longrightarrow \begin{bmatrix}\dot\Phi\\ \dot\Theta\\ \dot\Psi\end{bmatrix} = \begin{bmatrix}1&0&-\sin\Theta\\ 0&\cos\Phi&\sin\Phi\cos\Theta\\ 0&-\sin\Phi&\cos\Phi\cos\Theta\end{bmatrix}^{-1}\begin{bmatrix}p\\ q\\ r\end{bmatrix} = \begin{bmatrix}1&\sin\Phi\tan\Theta&\cos\Phi\tan\Theta\\ 0&\cos\Phi&-\sin\Phi\\ 0&\sin\Phi\sec\Theta&\cos\Phi\sec\Theta\end{bmatrix}\begin{bmatrix}p\\ q\\ r\end{bmatrix}$$

State - Space Formulation

Array of Variables $x = [\underbrace{U\ W\ q\ \Theta}_{\text{longitudinal}}\ \underbrace{V\ p\ r\ \Phi\ \Psi}_{\text{lateral}}]$

Deriavation $M\dot{x} = N(x) + F(\dot{x}, x) + G(x)$

· inertia matrix

$$M\dot{x} = \begin{bmatrix} m&&&&&&&&\\ &m&&&&&&&\\ &&I_{yy}&&&&&&\\ &&&1&&&&&\\ &&&&m&&&&\\ &&&&&I_{xx}&-I_{xz}&&\\ &&&&&-I_{xz}&I_{zz}&&\\ &&&&&&&1&\\ &&&&&&&&1\end{bmatrix}\begin{bmatrix}\dot U\\ \dot W\\ \dot q\\ \dot\Theta\\ \dot V\\ \dot p\\ \dot r\\ \dot\Phi\\ \dot\Psi\end{bmatrix}$$

$F + mg = m\frac{dv}{dt}$

$M = \frac{d}{dt}I(\omega)$

$\frac{d}{dt}(\text{angle}) = \omega$

· matrix of non-linearities

$$N(x) = \begin{bmatrix} m(Vr - Wq)\\ m(Uq - Vp)\\ (I_{zz} - I_{xx})pr - I_{xz}(p^2 - r^2)\\ q\cos\Phi - r\sin\Phi\\ m(Wp - Ur)\\ (I_{yy} - I_{zz})qr + I_{xz}pq\\ (I_{xx} - I_{yy})pq - I_{xz}qr\\ p + q\sin\Phi\tan\Theta + r\cos\Phi\tan\Theta\\ -q\sin\Phi\sec\Theta + r\cos\Phi\sec\Theta\end{bmatrix}$$

$F + mg - m\frac{dv}{dt} = 0$

$M - \frac{d}{dt}I(\omega) = 0$

$\frac{d}{dt}(\text{angle}) = \omega$

· matrix of aerodynamic and propulsive forces acting on air-frame

* $F(x) \longrightarrow$ quazi-static aerodynamic model $\longrightarrow$ F change instantaneously

$F(\dot{x}, x) \longrightarrow$ time-shift in the aerodynamic response of the system

assume there are no other dynamic variables within the system besides x

$$F(\dot{x}, x) = \begin{bmatrix} X^a(\dot{x}, x) \\ Z^a(\dot{x}, x) \\ M^a(\dot{x}, x) \\ 0 \\ Y^a(\dot{x}, x) \\ L^a(\dot{x}, x) \\ N^a(\dot{x}, x) \\ 0 \\ 0 \end{bmatrix}$$

$F + mg - m\frac{dv}{dt} = 0$

$M - \frac{d}{dt}I(\omega) = 0$

$\frac{d}{dt}(\text{angle}) - \omega = 0$

· matrix of gravitational force acting on air-frame

$$G(x) = \begin{bmatrix} -mg\sin\Theta \\ mg\cos\Phi\cos\Theta \\ 0 \\ 0 \\ mg\sin\Phi\cos\Theta \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$F + mg - m\frac{dv}{dt} = 0$

$M - \frac{d}{dt}I(\omega) = 0$

$\frac{d}{dt}(\text{angle}) - \omega = 0$

Behaviour Near Singular Points

$$M(\dot{x}+\dot{x}') = N(x+x') + F(\dot{x}+\dot{x}', x+x') + G(x+x') = N(x) + x'N_x + F(\dot{x}, x) + \dot{x}'F_{\dot{x}} + x'F_x + G(x) + x'G_x$$

at singular points $\longrightarrow M\dot{x}_0 = 0 \longrightarrow N(x_0) + F(0, x_0) + G(x_0) = 0$

$\longrightarrow M\dot{x}' = x'N_{x_0} + \dot{x}'F_{\dot{x}_0} + x'F_{x_0} + x'G_{x_0}$

$\longrightarrow (M - F_{\dot{x}_0})\dot{x}' = M_*\dot{x}' = (N_{x_0} + F_{x_0} + G_{x_0})x'$

Wings-Level Trimmed Flight

Trim

Conditions

$x_0 = [U_0 \;\; W_0 \;\; q_0 \;\; \Theta_0 \;\; V_0 \;\; p_0 \;\; r_0 \;\; \Phi_0 \;\; \Psi_0]^T$

$= [U_e \;\; 0 \;\; 0 \;\; \underset{\gamma}{\underline{\Theta_e}} \;\; 0 \;\; 0 \;\; 0 \;\; 0]$ where $U_e^2 = U_0^2 + V_0^2 + W_0^2$

Equation

· inertia matrix

$$M\dot{x}_0 = \begin{bmatrix} m &&&&&&&& \\ & m &&&&&&& \\ && I_{yy} &&&&&& \\ &&& 1 &&&&& \\ &&&& m &&&& \\ &&&&& I_{xx} & -I_{xz} && \\ &&&&& -I_{xz} & I_{zz} && \\ &&&&&&& 1 & \\ &&&&&&&& 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

· matrix of non-linearities

$$N(x_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

· matrix of aerodynamic and propulsive forces

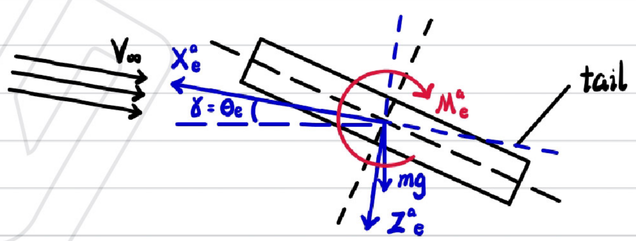
$$F(0, x_0) = \begin{bmatrix} X_e^a \\ Z_e^a \\ M_e^a \\ 0 \\ Y_e^a \\ L_e^a \\ N_e^a \\ 0 \\ 0 \end{bmatrix}$$

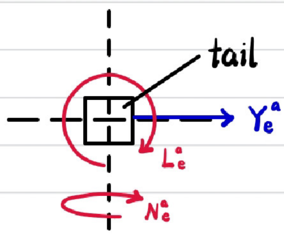
· matrix of gravitational force

$$G(x_0) = \begin{bmatrix} -mg\sin\theta_e \\ mg\cos\theta_e \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$M\dot{x}_0 = 0 \longrightarrow N(x_0) + F(\dot{x}_0, x_0) + G(x_0) = 0$$

$$\longrightarrow \begin{bmatrix} X_e^a - mg\sin\theta_e \\ Z_e^a + mg\cos\theta_e \\ M_e^a \\ 0 \\ Y_e^a \\ L_e^a \\ N_e^a \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$*$ $\longrightarrow$ 

& 

Linearised Form

$$M\dot{x}' = x' N_{x_0} + \dot{x}' F_{\dot{x}_0} + x' F_{x_0} + x' G_{x_0}$$

$$\longrightarrow (M - F_{\dot{x}_0})\dot{x}' = M_*\dot{x}' = (N_{x_0} + F_{x_0} + G_{x_0})\,x'$$

· matrix of non-linearities

$$N_{x_0} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} \\ \left[\begin{array}{cccc|ccccc} & & & & & & & & \\ & & mU_e & & & & & & \\ & & & & & & & & \\ & & 1 & & & & & & \\ \hline & & & & & & -mU_e & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & 1 & \tan\Theta_e & & \\ & & & & & & \sec\Theta_e & & \end{array}\right] \end{array}$$

· matrix of gravitational forces

$$G_{x_0} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} \\ \left[\begin{array}{cccc|ccccc} & & & -mg\cos\Theta_e & & & & & \\ & & & -mg\sin\Theta_e & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ \hline & & & & & & & mg\cos\Theta_e & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{array}\right] \end{array}$$

· matrix of aerodynamics and propulsive forces

$$F_{x_0} = \begin{bmatrix} X_u & X_w & X_q & X_\theta & X_v & X_p & X_r & X_\Phi & X_\Psi \\ Z_u & Z_w & Z_q & Z_\theta & Z_v & Z_p & Z_r & Z_\Phi & Z_\Psi \\ M_u & M_w & M_q & M_\theta & M_v & M_p & M_r & M_\Phi & M_\Psi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Y_u & Y_w & Y_q & Y_\theta & Y_v & Y_p & Y_r & Y_\Phi & Y_\Psi \\ L_u & L_w & L_q & L_\theta & L_v & L_p & L_r & L_\Phi & L_\Psi \\ N_u & N_w & N_q & N_\theta & N_v & N_p & N_r & N_\Phi & N_\Psi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{where} \quad X_u = \left[\frac{\partial X^a}{\partial U}\right]_{x = x_0,\ \dot{x} = 0}$$

Curie principle no asymmetric effect can results from symmetric case, vice versa

$$\longrightarrow F_{x_0} = \begin{bmatrix} X_u & X_w & X_q & X_\theta & & & & & \\ Z_u & Z_w & Z_q & Z_\theta & & & & & \\ M_u & M_w & M_q & M_\theta & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & Y_v & Y_p & Y_r & Y_\Phi & Y_\Psi \\ & & & & L_v & L_p & L_r & L_\Phi & L_\Psi \\ & & & & N_v & N_p & N_r & N_\Phi & N_\Psi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

assume the aerodynamic and propulsive force and moment are independent of the orientation of the aircraft with respect to the ground * neglect ground effect

$$\longrightarrow F_{x_0} = \begin{bmatrix} X_u & X_w & X_q & & & & & & \\ Z_u & Z_w & Z_q & & & & & & \\ M_u & M_w & M_q & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & Y_v & Y_p & Y_r & & \\ & & & & L_v & L_p & L_r & & \\ & & & & N_v & N_p & N_r & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_{\dot x_0} = \begin{bmatrix} X_{\dot u} & X_{\dot w} & X_{\dot q} & X_{\dot \theta} & X_{\dot v} & X_{\dot p} & X_{\dot r} & X_{\dot \Phi} & X_{\dot \Psi} \\ Z_{\dot u} & Z_{\dot w} & Z_{\dot q} & Z_{\dot \theta} & Z_{\dot v} & Z_{\dot p} & Z_{\dot r} & Z_{\dot \Phi} & Z_{\dot \Psi} \\ M_{\dot u} & M_{\dot w} & M_{\dot q} & M_{\dot \theta} & M_{\dot v} & M_{\dot p} & M_{\dot r} & M_{\dot \Phi} & M_{\dot \Psi} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Y_{\dot u} & Y_{\dot w} & Y_{\dot q} & Y_{\dot \theta} & Y_{\dot v} & Y_{\dot p} & Y_{\dot r} & Y_{\dot \Phi} & Y_{\dot \Psi} \\ L_{\dot u} & L_{\dot w} & L_{\dot q} & L_{\dot \theta} & L_{\dot v} & L_{\dot p} & L_{\dot r} & L_{\dot \Phi} & L_{\dot \Psi} \\ N_{\dot u} & N_{\dot w} & N_{\dot q} & N_{\dot \theta} & N_{\dot v} & N_{\dot p} & N_{\dot r} & N_{\dot \Phi} & N_{\dot \Psi} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{where} \quad X_{\dot u} = \left[\frac{\partial X^a}{\partial \dot u}\right]_{x = x_0,\ \dot x = 0}$$

$F_{\dot x_0}$ not affect by $\dot\Phi, \dot\Theta, \dot\Psi$

$$\longrightarrow F_{\dot x_0} = \begin{bmatrix} X_{\dot u} & X_{\dot w} & X_{\dot q} & & & X_{\dot p} & X_{\dot r} & & \\ Z_{\dot u} & Z_{\dot w} & Z_{\dot q} & & & Z_{\dot p} & Z_{\dot r} & & \\ M_{\dot u} & M_{\dot w} & M_{\dot q} & & & M_{\dot p} & M_{\dot r} & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Y_{\dot u} & Y_{\dot w} & Y_{\dot q} & & & Y_{\dot p} & Y_{\dot r} & & \\ L_{\dot u} & L_{\dot w} & L_{\dot q} & & & L_{\dot p} & L_{\dot r} & & \\ N_{\dot u} & N_{\dot w} & N_{\dot q} & & & N_{\dot p} & N_{\dot r} & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

experimental evidence* $\longrightarrow$ $F_{\dot{x}_0} = \begin{bmatrix} & X_{\dot{w}} & & & & & & & \\ & Z_{\dot{w}} & & & & & & & \\ & M_{\dot{w}} & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

*careful the vaildity of the simplification for compressibility effect and added-mass

$$\longrightarrow M_{*} = M - F_{\dot{x}_0} = \begin{bmatrix} m - X_{\dot{w}} & & & & & & & & \\ & m - Z_{\dot{w}} & & & & & & & \\ & & I_{yy} - M_{\dot{w}} & & & & & & \\ & & & 1 & & & & & \\ & & & & m & & & & \\ & & & & & I_{xx} & -I_{xz} & & \\ & & & & & -I_{xz} & I_{zz} & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix}$$

$$\longrightarrow \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} \\ \begin{bmatrix} m & -X_{\dot{w}} & & & & & & & \\ & m - Z_{\dot{w}} & & & & & & & \\ & -M_{\dot{w}} & I_{yy} & & & & & & \\ & & & 1 & & & & & \\ & & & & m & & & & \\ & & & & & I_{xx} & -I_{xz} & & \\ & & & & & -I_{xz} & I_{zz} & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix} \end{array} \begin{bmatrix} \dot{u}' \\ \dot{w}' \\ \dot{q}' \\ \dot{\theta}' \\ \dot{v}' \\ \dot{p}' \\ \dot{r}' \\ \dot{\Phi}' \\ \dot{\Psi}' \end{bmatrix}$$

$$= \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} \\ \begin{bmatrix} X_u & X_w & X_q & -mg\cos\theta_e & & & & & \\ Z_u & Z_w & Z_q + mU_e & -mg\sin\theta_e & & & & & \\ M_u & M_w & M_q & & & & & & \\ & & & & & & & & \\ & & & & Y_v & Y_p & Y_r - mU_e & mg\cos\theta_e & \\ & & & & L_v & L_p & L_r & & \\ & & & & N_v & N_p & N_r & & \\ & & & & & 1 & \tan\theta_e & & \\ & & & & & & \sec\theta_e & & \end{bmatrix} \end{array} \begin{bmatrix} u' \\ w' \\ q' \\ \theta' \\ v' \\ p' \\ r' \\ \Phi' \\ \Psi' \end{bmatrix}$$

Decoupling

· longitudinal, symmetric matrix 4×4

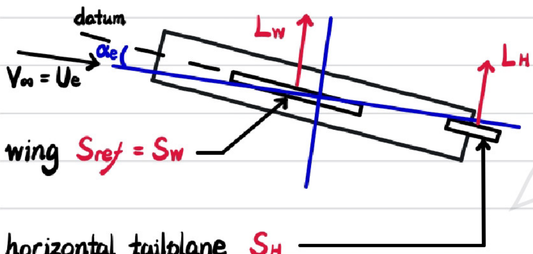
$$\begin{bmatrix} m & -X_{\dot w} & 0 & 0 \\ 0 & m-Z_{\dot w} & 0 & 0 \\ 0 & -M_{\dot w} & I_{yy} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot U' \\ \dot W' \\ \dot q' \\ \dot\Theta' \end{bmatrix} = \begin{bmatrix} X_u & X_w & X_q & -mg\cos\Theta_e \\ Z_u & Z_w & Z_q + mU_e & -mg\sin\Theta_e \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} U' \\ W' \\ q' \\ \Theta' \end{bmatrix}$$

· lateral, asymmetric matrix 5×5

$$\begin{bmatrix} m & 0 & 0 & 0 & 0 \\ 0 & I_{xx} & -I_{xz} & 0 & 0 \\ 0 & -I_{xz} & I_{zz} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot V' \\ \dot p' \\ \dot r' \\ \dot\Phi' \\ \dot\Psi' \end{bmatrix} = \begin{bmatrix} Y_v & Y_p & Y_r - mU_e & mg\cos\Theta_e & 0 \\ L_v & L_p & L_r & 0 & 0 \\ N_v & N_p & N_r & 0 & 0 \\ 0 & 1 & \tan\Theta_e & 0 & 0 \\ 0 & 0 & \sec\Theta_e & 0 & 0 \end{bmatrix}\begin{bmatrix} V' \\ p' \\ r' \\ \Phi' \\ \Psi' \end{bmatrix}$$

Flight in Equilibrium

Lift $L = L_W + L_H = \frac{1}{2}\rho V_\infty^2 S_{ref} C_L \longrightarrow C_L = \underbrace{a}_{\frac{dC_L}{d\alpha}}(\alpha - \underbrace{\alpha_0}_{[\alpha]_{C_L=0}}) = C_{LW} + \frac{S_H}{S_{ref}} C_{LH}$



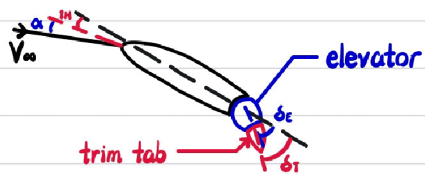
$\longrightarrow C_{LW} = a_W(\alpha + i_W - \alpha_{0W})$ * $C_{L0W} = a_W(i_W - \alpha_{0W})$

$\longrightarrow C_{LH} = a_H(\alpha_H + i_H - \alpha_{0H}) + a_E\delta_E + a_T\delta_T$

$= a_H[(\alpha - \underbrace{\varepsilon}_{\text{downwash}}) + i_H - \alpha_{0H}] + a_E\delta_E + a_T\delta_T$

$= a_H[(1 - \frac{d\varepsilon}{d\alpha})\alpha - \varepsilon_0 + i_H - \alpha_{0H}] + a_E\delta_E + a_T\delta_T$

where $\varepsilon = \varepsilon_0 + \frac{d\varepsilon}{d\alpha}\alpha$ * $C_{L0H} = a_H(-\varepsilon_0 + i_H - \alpha_{0H})$

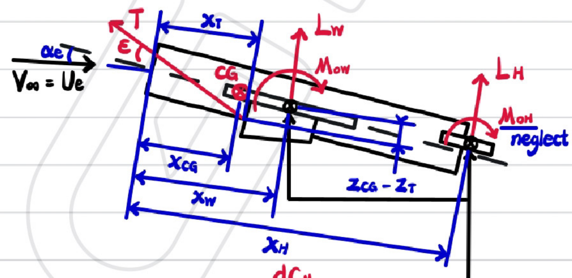


Drag $D = D_W + D_H = \frac{1}{2}\rho V_\infty^2 S_{ref} C_D$

$\longrightarrow C_D = \frac{C_L^2}{\pi \mathcal{AR} e}$ or $\frac{kC_L^2}{\pi \mathcal{AR}} = \underbrace{C_{D0}}_{C_{D0W} + \frac{S_H}{S_{ref}} C_{D0H}} + \frac{C_{LW}^2}{\pi \mathcal{AR}_W e_W} + \frac{S_H}{S_{ref}} \frac{C_{LH}^2}{\pi \mathcal{AR}_H e_H}$

$\longrightarrow C_D = C_{D0} + \frac{1}{\pi \mathcal{AR}_W e_W}[a_W(\alpha + i_W - \alpha_{0W})]^2 + \frac{S_H}{S_{ref}} \frac{1}{\pi \mathcal{AR}_H e_H}[\{a_H[(1 - \frac{d\varepsilon}{d\alpha})\alpha - \varepsilon_0 + i_H - \alpha_{0H}]\}^2 + d_E\delta_E + d_T\delta_T]$

Pitch Moment $M = Tl_T + M_{0W} + (x_{CG} - x_W)L_W + (x_{CG} - x_H)L_H = Tl_T + \frac{1}{2}\rho V_\infty^2 S_{ref} \bar c C_M$



aerodynamic centre $\frac{dC_M}{d\alpha} = 0$

$\longrightarrow l_T = \cos(\varepsilon + \alpha_e)(z_{CG} - z_T) + \sin(\varepsilon + \alpha_e)(x_{CG} - x_T)$

$\longrightarrow C_M = C_{M0W} + \frac{x_{CG} - x_W}{\bar c} C_{LW} + \frac{S_H}{S_{ref}} \frac{x_{CG} - x_H}{\bar c} C_{LH}$

$= C_{M0W} + \frac{x_{CG} - x_W}{\bar c} C_L + \frac{S_H}{S_{ref}} \underbrace{\frac{x_W - x_H}{\bar c}}_{-\frac{l_H}{\bar c}} C_{LH}$

Speed Stability $\frac{dV_\infty}{dx} < 0$ $X_u < 0$ & $\frac{dZ}{dw} < 0$ $Z_w < 0$ & $\frac{dY}{dv} < 0$ $Y_v < 0$

Pitch Stability $\frac{dM}{d\alpha} < 0$ $M_w < 0$ $\longrightarrow \frac{dC_M}{d\alpha} < 0 \longrightarrow \frac{dC_M}{dC_L}\frac{dC_L}{d\alpha} < 0 \xrightarrow{\frac{dC_L}{d\alpha} > 0 \text{ for small } \alpha} \frac{dC_M}{dC_L} < 0$

Stick-fixed Static Stability

$\frac{dC_M}{dC_L} = \frac{x_{cg} - x_w}{\bar c} - \frac{S_H}{S_{ref}}\frac{l_H}{\bar c}\frac{dC_{LH}}{dC_L}$ where $\frac{dC_{LH}}{dC_{LW}} = \frac{dC_{LH}}{d\alpha}\frac{d\alpha}{dC_L} = \frac{a_H}{a}(1 - \frac{d\epsilon}{d\alpha})$

$= \frac{x_{cg} - x_w}{\bar c} - \bar V_H[\frac{a_H}{a}(1 - \frac{d\epsilon}{d\alpha})] = 0$ at neutrally stable $\longrightarrow x_{cg} = x_{np}$

$\longrightarrow \frac{x_{np} - x_w}{\bar c} - \bar V_H[\frac{a_H}{a}(1 - \frac{d\epsilon}{d\alpha})] = 0 \longrightarrow x_{np} = x_w + \bar V_H \bar c \frac{a_H}{a}(1 - \frac{d\epsilon}{d\alpha})$

$\longrightarrow \frac{dC_M}{dC_L} = \frac{x_{cg} - x_{np}}{\bar c} = \bar x_{cg} - \bar x_{np} \longrightarrow K_n = -\frac{dC_M}{dC_L} = \frac{x_{np} - x_{cg}}{\bar c}$

$\longrightarrow M = T l_T + \frac{1}{2}\rho V_\infty^2 S_{ref} \bar c (\underbrace{C_{M0}}_{[C_M]_{C_L=0}} - K_n C_L)$

Stick-free Static Stability* assume $\frac{dC_T}{dC_L} = 0$

$C_M = \frac{x_{cg} - x_w}{\bar c} C_L - \bar V_H C_{LH}$ where $C_{LH} = a_H[(1 - \frac{d\epsilon}{d\alpha})\alpha - \epsilon_0 + i_H - \alpha_{0H}] + a_E\delta_E + a_T\delta_T$

$\longrightarrow \frac{dC_M}{dC_L} = (\bar x_{cg} - \bar x_w) - \bar V_H \frac{dC_{LH}}{dC_L}$

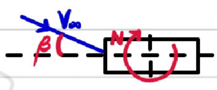
where $\frac{dC_{LH}}{dC_L} = \frac{dC_{LH}}{d\alpha}\frac{d\alpha}{dC_L} = [a_H(1 - \frac{d\epsilon}{d\alpha}) + a_E\frac{d\delta_E}{d\alpha}]\frac{1}{a}$

$\longrightarrow C_H$ elevator hinge moment coefficient $= b_0 + b_H\underbrace{\alpha_H}_{(1 - \frac{d\epsilon}{d\alpha})\alpha - \epsilon_0} + b_E\delta_E + b_T\delta_T = 0$ at equilibrium

$\longrightarrow \delta_E = -\frac{1}{b_E}\{b_0 + b_H[(1 - \frac{d\epsilon}{d\alpha})\alpha - \epsilon_0] + b_T\delta_T\} \longrightarrow \frac{d\delta_E}{d\alpha} = -\frac{b_H}{b_E}(1 - \frac{d\epsilon}{d\alpha})$

$\longrightarrow \frac{dC_M}{dC_L} = (\bar x_{cg} - \bar x_w) - \bar V_H \frac{a_H}{a}(1 - \frac{d\epsilon}{d\alpha})(1 - \frac{a_E}{a_H}\frac{b_H}{b_E}) \longrightarrow x_{np} = x_w + \bar V_H c \frac{a_H}{a}(1 - \frac{d\epsilon}{d\alpha})(1 - \frac{a_E}{a_H}\frac{b_H}{b_E})$

Directional Static Stability



$\frac{dN}{d\beta} > 0$ $N_v > 0$

Lateral Static Stability

$\xrightarrow{V_\infty \sin\beta = V}$

$\frac{dL}{d\beta} < 0$ $L_v < 0$

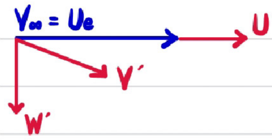
for wing root at upper fuselage / diheral $\Gamma > 0$ / swept $\Lambda > 0$

* $L_v = (\frac{\partial L}{\partial v})_e = (\frac{\partial L}{\partial \beta})_e(\frac{\partial \beta}{\partial v})_e = \frac{1}{U_e}(\frac{\partial L}{\partial \beta})_e$

Trim, Straight, Level Flight $L + T\sin(\epsilon + \alpha_e) = W = mg$ & $T\cos(\epsilon + \alpha_e) = D$ & $M = 0$

Longitudinal Stability Derivatives

Small Perturbation in Velocity



$$V_\infty(U_e+U',\ 0+V',\ 0+W') \xrightarrow{\text{Taylor's expansion}} U_e + U'\left(\frac{\partial V_\infty}{\partial U}\right)_e + V'\left(\frac{\partial V_\infty}{\partial V}\right)_e + W'\left(\frac{\partial V_\infty}{\partial W}\right)_e$$

$$\&\ V_\infty^2 = U^2+V^2+W^2 \longrightarrow V_\infty^2 = (U_e+U')^2 + (V')^2 + (W')^2$$

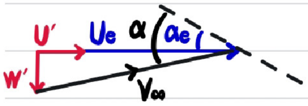
$$\longrightarrow \left[U_e + U'\left(\frac{\partial V_\infty}{\partial U}\right)_e + V'\left(\frac{\partial V_\infty}{\partial V}\right)_e + W'\left(\frac{\partial V_\infty}{\partial W}\right)_e\right]^2 = (U_e+U')^2 + (V')^2 + (W')^2$$

$$\xrightarrow{(U')^2=(V')^2=(W')^2=0} U_e^2 + 2U_eU'\left(\frac{\partial V_\infty}{\partial U}\right)_e + 2U_eV'\left(\frac{\partial V_\infty}{\partial V}\right)_e + 2U_eW'\left(\frac{\partial V_\infty}{\partial W}\right)_e = U_e^2 + 2U_eU'$$

$$\longrightarrow \left(\frac{\partial V_\infty}{\partial U}\right)_e = 1 \ \&\ \left(\frac{\partial V_\infty}{\partial V}\right)_e = 0 \ \&\ \left(\frac{\partial V_\infty}{\partial W}\right)_e = 0$$

Small Perturbation in AoA

perturbation in U and W

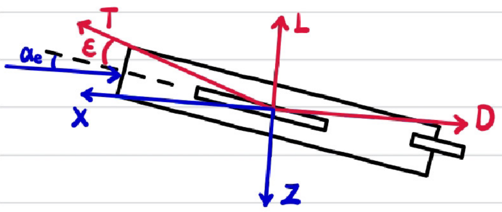


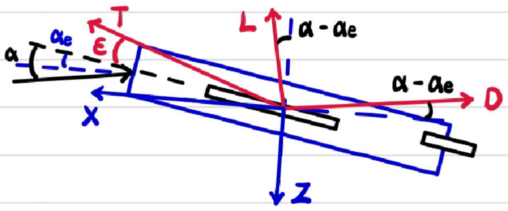
$$\alpha(U,W) \xrightarrow{\text{Taylor's expansion}} \alpha_e + U'\left(\frac{\partial \alpha}{\partial U}\right)_e + W'\left(\frac{\partial \alpha}{\partial W}\right)$$

$$\&\ \alpha = \alpha_e + \tan^{-1}\left(\frac{W'}{U_e+U'}\right) \approx \alpha_e + \frac{W'}{U_e+U'} \approx \alpha_e + \frac{W'}{U_e}$$

$$\longrightarrow \left(\frac{\partial \alpha}{\partial U}\right)_e = 0 \ \&\ \left(\frac{\partial \alpha}{\partial V}\right)_e = 0 \ \&\ \left(\frac{\partial \alpha}{\partial W}\right)_e = \frac{1}{U_e} \qquad * \left(\frac{\partial}{\partial V}\right)_e = \frac{1}{U_e}\left(\frac{\partial}{\partial \beta}\right)_e$$

U-derivatives (X_u, Z_u, M_u)





* for $M > 0.3$, considering compressibility effect $\longrightarrow \dfrac{\partial}{\partial V_\infty} = \dfrac{\partial}{\partial M_\infty}\dfrac{\partial M_\infty}{\partial V_\infty} = \dfrac{1}{C_s}\dfrac{\partial}{\partial M_\infty}$

$$\cdot\ X_u = \left(\frac{\partial X}{\partial U}\right)_e = \left(\frac{\partial X}{\partial V_\infty}\right)_e\left(\frac{\partial V_\infty}{\partial U}\right)_e = \left(\frac{\partial X}{\partial V_\infty}\right)_e$$

$$= \frac{\partial}{\partial V_\infty}\left[\underbrace{T\cos(\alpha_e+\varepsilon)}_{T_x} + L\sin(\alpha-\alpha_e) - D\cos(\alpha-\alpha_e)\right]_e \longrightarrow V_\infty = U_e \ \&\ \alpha = \alpha_e$$

$$= \frac{\partial}{\partial V_\infty}\left\{T_x + \frac{1}{2}\rho V_\infty^2 S_{ref}\left[C_L\sin(\alpha-\alpha_e) - C_D\cos(\alpha-\alpha_e)\right]\right\}_e$$

$$= \left\{\frac{\partial T_x}{\partial V_\infty} + \rho V_\infty S_{ref}\left[C_L\sin(\alpha-\alpha_e) - C_D\cos(\alpha-\alpha_e)\right] + \frac{1}{2}\rho V_\infty^2 S_{ref}\left[\frac{\partial C_L}{\partial V_\infty}\sin(\alpha-\alpha_e) - \frac{\partial C_D}{\partial V_\infty}\cos(\alpha-\alpha_e)\right]\right\}_e$$

$$= \left(\frac{\partial T_x}{\partial V_\infty}\right)_e - \rho U_e S_{ref}(C_D)_e - \frac{1}{2}\rho U_e^2 S_{ref}\left(\frac{\partial C_D}{\partial V_\infty}\right)_e$$

$$= \frac{1}{C_s}\left(\frac{\partial T_x}{\partial M_\infty}\right)_e - \rho U_e S_{ref}(C_D)_e - \frac{1}{2}\rho U_e^2 S_{ref}\frac{1}{C_s}\left(\frac{\partial C_D}{\partial M_\infty}\right)_e$$

$$\cdot\ Z_u = \frac{\partial}{\partial V_\infty}\left\{-\underbrace{T\sin(\alpha_e+\varepsilon)}_{T_z} - L\cos(\alpha-\alpha_e) - D\sin(\alpha-\alpha_e)\right\}_e$$

$$= -\left(\frac{\partial T_z}{\partial V_\infty}\right)_e - \rho U_e S_{ref}(C_L)_e - \frac{1}{2}\rho U_e^2 S_{ref}\left(\frac{\partial C_L}{\partial V_\infty}\right)_e$$

$$= -\frac{1}{c_s}\left(\frac{\partial T_z}{\partial M_\infty}\right)_e - \rho U_e S_{ref} (C_L)_e - \frac{1}{2}\rho U_e^2 S_{ref} \frac{1}{c_s}\left(\frac{\partial C_L}{\partial M_\infty}\right)_e$$

$$\cdot\ M_u = \frac{\partial}{\partial V_\infty}\left[T l_T + \frac{1}{2}\rho V_\infty^2 S_{ref} \bar{c} (C_{M0} - K_n C_L) \right]$$

$$= \left(\frac{\partial T}{\partial V_\infty}\right)_e l_T + \underbrace{\rho U_e S_{ref} \bar{c} (C_{M0} - K_n C_L)_e}_{M_e = 0} + \frac{1}{2}\rho U_e^2 S_{ref} \bar{c} \left[\left(\frac{\partial C_{M0}}{\partial V_\infty}\right)_e - \frac{\partial K_n}{\partial V_\infty}(C_L)_e - K_n \left(\frac{\partial C_L}{\partial V_\infty}\right)_e \right]$$

$$= \frac{1}{c_s}\left(\frac{\partial T}{\partial M_\infty}\right)_e l_T + \frac{1}{2}\rho U_e^2 S_{ref} \bar{c} \frac{1}{c_s}\left[\left(\frac{\partial C_{M0}}{\partial M_\infty}\right)_e - \underbrace{\frac{\partial K_n}{\partial M_\infty}}_{= 0 \text{ when } M < 1}(C_L)_e - K_n \left(\frac{\partial C_L}{\partial M_\infty}\right)_e \right]$$

W - derivatives (X_w, Z_w, M_w)

$$\cdot\ X_w = \left(\frac{\partial X}{\partial W}\right)_e = \left(\frac{\partial X}{\partial \alpha}\right)_e \left(\frac{\partial \alpha}{\partial W}\right)_e = \frac{1}{U_e}\left(\frac{\partial X}{\partial \alpha}\right)_e$$

$$= \frac{1}{U_e}\frac{\partial}{\partial \alpha}\left[T_x + L\sin(\alpha - \alpha_e) - D\cos(\alpha - \alpha_e) \right]_e$$

$$= \frac{1}{U_e}\left\{ \left(\frac{\partial T_x}{\partial \alpha}\right) + \frac{1}{2}\rho V_\infty^2 S_{ref} \left[C_L \cos(\alpha - \alpha_e) - \left(\frac{\partial C_D}{\partial \alpha}\right)\cos(\alpha - \alpha_e) \right] \right\}_e$$

$$= \frac{1}{U_e}\left(\frac{\partial T_x}{\partial \alpha}\right)_e + \frac{1}{2}\rho U_e S_{ref} \left[(C_L)_e - \left(\frac{\partial C_D}{\partial \alpha}\right)_e \right]$$

$$\text{where } \left(\frac{\partial C_D}{\partial \alpha}\right)_e = \frac{\partial}{\partial \alpha}\left(C_{D0} + \frac{C_L^2}{\pi A\!R e} \right)_e = \frac{1}{\pi A\!R e}\left(\frac{\partial C_L^2}{\partial \alpha}\right)_e = \frac{2(C_L)_e}{\pi A\!R e}\left(\frac{\partial C_L}{\partial \alpha}\right)_e$$

$$\text{where } \left(\frac{\partial C_L}{\partial \alpha}\right)_e = a = a_w + \frac{S_H}{S_{ref}} a_H \left(1 - \frac{d\epsilon}{d\alpha}\right)$$

$$\cdot\ Z_w = \frac{1}{U_e}\frac{\partial}{\partial \alpha}\left[-T_z - L\cos(\alpha - \alpha_e) - D\sin(\alpha - \alpha_e) \right]$$

$$= -\frac{1}{U_e}\left(\frac{\partial T_z}{\partial \alpha}\right)_e - \frac{1}{2}\rho U_e S_{ref} \left[\left(\frac{\partial C_L}{\partial \alpha}\right)_e + (C_D)_e \right]$$

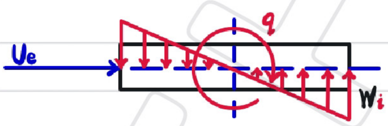
$$\cdot\ M_w = \frac{1}{U_e}\frac{\partial}{\partial \alpha}\left[T l_T + \frac{1}{2}\rho V_\infty^2 S_{ref} \bar{c} (C_{M0} - K_n C_L) \right]_e$$

$$= \frac{1}{U_e}\left(\frac{\partial T}{\partial \alpha}\right)_e l_T + \frac{1}{2}\rho U_e S_{ref} \bar{c} \left[\underbrace{\left(\frac{\partial C_{M0}}{\partial \alpha}\right)_e}_{= 0} - \underbrace{\frac{\partial K_n}{\partial \alpha}}_{= 0}(C_L)_e - K_n \left(\frac{\partial C_L}{\partial \alpha}\right)_e \right]$$

$$= \frac{1}{U_e}\left(\frac{\partial T}{\partial \alpha}\right)_e l_T - \frac{1}{2}\rho U_e S_{ref} \bar{c} K_n \left(\frac{\partial C_L}{\partial \alpha}\right)_e$$

q - derivatives (X_q, Z_q, M_q)

major contribute by horizontal tailplane * *also canard*



induced $U_i = -(z - z_{CG})q$

induced $W_i = (x - x_{CG})q$

$$\cdot\ X_q = \left(\frac{\partial X}{\partial q}\right)_e = X_u \left(\frac{\partial U_i}{\partial q}\right)_e + X_w \left(\frac{\partial W_i}{\partial q}\right)_e \longrightarrow \underbrace{-X_{uH}}_{\text{negiligible}}(z_H - z_{CG}) + X_{wH}\underbrace{(x_H - x_{CG})}_{l_H'}$$

$$\text{where } X_{wH} = \frac{1}{U_e}\frac{\partial}{\partial \alpha}\left[L_H \sin(\alpha - \alpha_e) - D_H \cos(\alpha - \alpha_e) \right]_e = \frac{1}{2}\rho U_e S_H \left[(C_{LH})_e - \left(\frac{\partial C_{DH}}{\partial \alpha}\right)_e \right]$$

$$\text{where } \left(\frac{\partial C_{DH}}{\partial \alpha}\right)_e = \frac{\partial}{\partial \alpha}\left(C_{D0} + \frac{C_L^2}{\pi A\!R e} \right)_e = \frac{2C_L}{\pi A\!R e}\left(\frac{\partial C_{LH}}{\partial \alpha}\right)_e$$

where $(\frac{\partial C_{LH}}{\partial \alpha})_e = a_H(1-\frac{d\epsilon}{d\alpha})$

· $Z_q = Z_{UH}(\frac{\partial U_i}{\partial q})_e + Z_{WH}(\frac{\partial W_i}{\partial q})_e = \underbrace{-Z_{UH}(Z_H - Z_{CG})}_{\text{negligible}} + Z_{WH}\underbrace{(X_H - X_{CG})}_{l'_H}$

where $Z_{WH} = \frac{1}{U_e}\frac{\partial}{\partial\alpha}[-L_H\cos(\alpha-\alpha_e) - D_H\sin(\alpha-\alpha_e)] = -\frac{1}{2}\rho U_e S_H[(\frac{\partial C_{LH}}{\partial\alpha})_e + (C_{DH})_e]$

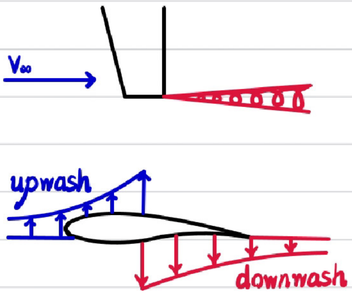
· $M_q = M_U(\frac{\partial U_i}{\partial q})_e + M_W(\frac{\partial W_i}{\partial q})_e + (\frac{\partial M}{\partial X})_e X_q + (\frac{\partial M}{\partial Z})_e Z_q$

$\longrightarrow \underbrace{-M_{UH}(Z_H - Z_{CG})}_{\text{negligible}} + M_{WH}\underbrace{(X_H - X_{CG})}_{l'_H} + X_q\underbrace{(Z_{CG} - Z_H)}_{-h_H} + W_q\underbrace{(X_H - X_{CG})}_{l'_H}$ always negative

where $M_{WH} = \frac{1}{2}\rho U_e S_H \bar{c}_H(\frac{\partial C_{MH}}{\partial\alpha})_e$

$\dot{W}$ - derivative ($X_{\dot w}$, $Z_{\dot w}$, $M_{\dot w}$) major contribute by horizontal tailplane by downwash

① downwash lag model



trailing vortex sheet at x_W formed $\longrightarrow \begin{bmatrix} 0 \\ 0 \\ -U_e\epsilon(t) \end{bmatrix}$

② $\alpha_H\downarrow = \alpha - \epsilon\uparrow + i_H = (1-\frac{d\epsilon}{d\alpha}\uparrow)\alpha - \epsilon_0 + i_H$

③ $\begin{bmatrix} 0 \\ 0 \\ W_H \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -U_e\epsilon_H(t) \end{bmatrix}$

④ $\epsilon_H(t) = \epsilon(t-\Delta t) = \epsilon(t - \frac{x_H - x_W}{U_e}) = \epsilon(t - \frac{l_H}{U_e})$

⑤ $\epsilon_H(t) = \epsilon[\alpha(t-\frac{l_H}{U_e})] \xrightarrow{\text{Taylor's expansion}} (\epsilon)_e + \alpha(t-\frac{l_H}{U_e})(\frac{d\epsilon}{d\alpha})_e$

$\xrightarrow{\text{Taylor's expansion}} (\epsilon)_e + (\frac{d\epsilon}{d\alpha})_e[\alpha - \frac{l_H}{U_e}(\frac{d\alpha}{dt})]$

where $\frac{d\alpha}{dt} = \frac{d\alpha}{dw}\frac{dw}{dt} = \frac{1}{U_e}\dot w \longrightarrow (\epsilon)_e + (\frac{d\epsilon}{d\alpha})_e[\alpha - \frac{l_H}{U_e^2}\dot w]$

$\longrightarrow \frac{\partial\epsilon_H}{\partial\dot w} = -\frac{l_H}{U_e^2}(\frac{d\epsilon}{d\alpha})_e$

· $X_{\dot w} = (\frac{\partial X}{\partial\epsilon_H})_e(\frac{\partial\epsilon_H}{\partial\dot w}) = X_{WH}(\frac{\partial W_H}{\partial\epsilon_H})(\frac{\partial\epsilon_H}{\partial\dot w}) = X_{WH}(-U_e)[-\frac{l_H}{U_e^2}(\frac{d\epsilon}{d\alpha})_e] = X_{WH}\frac{l_H}{U_e}(\frac{d\epsilon}{d\alpha})_e$

· $Z_{\dot w} = Z_{WH}\frac{l_H}{U_e}(\frac{d\epsilon}{d\alpha})_e$

$\cdot\ M_{\dot{w}} = \left(\frac{\partial M}{\partial \varepsilon_H}\right)_e \left(\frac{\partial \varepsilon_H}{\partial \dot{w}}\right)_e + \left(\frac{\partial M}{\partial X}\right)_e \left(\frac{\partial X}{\partial \dot{w}}\right)_e + \left(\frac{\partial M}{\partial Z}\right)_e \left(\frac{\partial Z}{\partial \dot{w}}\right)_e$

$= M_{WH} \frac{l_H}{U_e} \left(\frac{d\varepsilon}{d\alpha}\right)_e + (x_{CG} - x_H) X_{\dot{w}} + (z_H - z_{CG}) Z_{\dot{w}}$

* for lighter-than-air vehicles $\longrightarrow$ added-mass effect $\longrightarrow$ $\dot{U}$ & $\dot{W}$-derivatives

Laterial Stability Derivatives *

Typical Stability Derivatives for Commerical Aircraft

Longitudinal, Symmetric Derivatives

- $X_u < 0$ $U\uparrow \longrightarrow D\uparrow \longrightarrow X\downarrow$
- $X_w > 0$ $\alpha\uparrow \longrightarrow L\uparrow \longrightarrow L\sin(\alpha-\alpha_e)\uparrow \longrightarrow X\uparrow$
- $X_q > 0$ $q\uparrow \longrightarrow W_i\uparrow \xrightarrow{X_{WH}>0} X\uparrow$
- $X_{\dot{w}} > 0$ $\frac{d\varepsilon}{d\alpha}$ & $\frac{l_H}{U_e}$ & $X_{WH} > 0$
- $Z_u < 0$ $U\uparrow \longrightarrow L\uparrow \longrightarrow Z\downarrow$
- $Z_w < 0$ $\alpha\uparrow \longrightarrow L\uparrow \longrightarrow Z\downarrow$
- $Z_q < 0$ $q\uparrow \longrightarrow W_i\uparrow \xrightarrow{Z_{WH}<0} Z\downarrow$
- $Z_{\dot{w}} < 0$ $\frac{d\varepsilon}{d\alpha}$ & $\frac{l_H}{U_e} > 0$ while $Z_{WH} < 0$
- $M_u < 0$ $U\uparrow \longrightarrow L\uparrow \longrightarrow M\downarrow$
- $M_w < 0$ $\alpha\uparrow \longrightarrow L\uparrow \longrightarrow M\downarrow$
- $M_q < 0$ $q\uparrow \longrightarrow W_i\uparrow \xrightarrow{M_{WH}<0} M\downarrow$
- $M_{\dot{w}} < 0$ $\frac{d\varepsilon}{d\alpha}$ & $\frac{l_H}{U_e}$ & $X_{WH} > 0$ while Z_{WH} & $M_{WH} < 0$

Lateral, Asymmetric Derivatives

- $Y_v < 0$ $V\uparrow \longrightarrow D_{fuselage}\uparrow$ & $L_{fin}\uparrow$ in direction opposite to $Y \longrightarrow Y\downarrow$
- $Y_p < 0$ $p\uparrow \longrightarrow V_i\uparrow \longrightarrow L_{fin}\uparrow$ in direction opposite to $Y \longrightarrow Y\downarrow$
- $Y_r > 0$ $r\uparrow \longrightarrow L_{fin}\uparrow$ in same direction with $Y \longrightarrow Y\downarrow$
- $L_v < 0$ Λ (sweep angle) & Γ (dihedral angle) > 0 but low-wing configuration
- $L_p < 0$ wing & fin generate opposing L respect to change in AoA

- $L_r > 0$ $(V_\infty)_{left} > (V_\infty)_{right} \longrightarrow L_{left} > L_{right} \longrightarrow L \uparrow$
- $N_v > 0$ $V \uparrow \longrightarrow L_{fin} \uparrow$ in direction opposite to $Y \longrightarrow N \uparrow$
- $N_p < 0$ wing & fin generate opposing $D_{induced}$ respect to change in AoA
- $N_r < 0$ $(V_\infty)_{left} > (V_\infty)_{right} \longrightarrow D_{left} > D_{right} \longrightarrow N \downarrow$

Typical Aircraft Dynamic Modes

The State-Space Structure

Longitudinal, Symmetric Matrix $X_S = [\,U \;\; W \;\; q \;\; \theta\,]^T$

$$M_S \dot{X}'_S = (N_S + F_S + G_S) X'_S = R_S X'_S \longrightarrow \dot{X}'_S = M_S^{-1} R_S X'_S \longrightarrow |R_S - M_S \lambda| = 0$$

$$\longrightarrow \begin{vmatrix} X_u - m\lambda & X_w + X_{\dot{w}}\lambda & X_q & -mg\sin\theta_e \\ Z_u & Z_w + (Z_{\dot{w}} - m)\lambda & Z_q + mU_e & -mg\cos\theta_e \\ M_u & M_w + M_{\dot{w}}\lambda & M_q - I_{yy}\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$\longrightarrow A_S\lambda^4 + B_S\lambda^3 + C_S\lambda^2 + D_S\lambda + E_S = 0$$

$$\xrightarrow{\text{usually } D_S \;\&\; E_S \ll C_S} \underbrace{(a\lambda^2 + b\lambda + c)}_{\text{heavy damped}}\underbrace{(\lambda^2 + \gamma\lambda + \delta)}_{\text{light damped}} = 0$$

$$\longrightarrow A_S = a \;\&\; B_S = a\gamma + b \approx b \;\&\; C_S = a\delta + b\gamma + c \approx c \;\&\; D_S = b\delta + c\gamma \;\&\; E = c\delta$$

$$\longrightarrow \delta = \frac{E_S}{C_S} \longrightarrow \gamma = \frac{C_S D_S - B_S E_S}{C_S^2}$$

$$\longrightarrow (A_S\lambda^2 + B_S\lambda + C_S)(\lambda^2 + \frac{C_S D_S - B_S E_S}{C_S^2}\lambda + \frac{E_S}{C_S}) = 0$$

Lateral, Asymmetrix Matrix $X_A = [\,V, p, r, \Phi, \Psi\,]^T$

$$M_A \dot{X}'_A = (N_A + F_A + G_A) X'_A = R_A X'_A \longrightarrow \dot{X}'_A = M_A^{-1} R_A X'_A \longrightarrow |R_A - M_A \lambda| = 0$$

$$\longrightarrow \begin{vmatrix} Y_v - m\lambda & Y_p & Y_r - mU_e & mg\cos\theta_e & 0 \\ L_v & L_p - I_{xx}\lambda & L_r + I_{xz}\lambda & 0 & 0 \\ N_v & N_p + I_{xz}\lambda & N_r - I_{zz}\lambda & 0 & 0 \\ 0 & 1 & \tan\theta_e & -\lambda & 0 \\ 0 & 0 & \sec\theta_e & 0 & -\lambda \end{vmatrix} = 0$$

$$\longrightarrow \lambda(A_A\lambda^4 + B_A\lambda^3 + C_A\lambda^2 + D_A\lambda + E_A) = 0$$

① $\lambda_{A1} = 0$

② λ_{A2} typically very large $\longrightarrow A_A\lambda_{A2}^4 + B_A\lambda_{A2}^3 + C_A\lambda_{A2}^2 + D_A\lambda_{A2} + E_A \approx A_A\lambda_{A2}^4 + B_A\lambda_{A2}^3 = 0$

$\longrightarrow \lambda_{A2} = -\frac{B_A}{A_A}$

③ λ_{A3} typically very small $\longrightarrow A_A\lambda_{A3}^4 + B_A\lambda_{A3}^3 + C_A\lambda_{A3}^2 + D_A\lambda_{A3} + E_A \approx D_A\lambda_{A3} + E_A = 0$

$\longrightarrow \lambda_3 = -\frac{E_A}{D_A}$

④ $\lambda_{A4,A5} \longrightarrow \lambda_{A1}(\lambda_{A2} + \frac{B_A}{A_A})(\lambda_{A3} + \frac{E_A}{D_A})(\alpha\lambda^2 + \beta\lambda + \gamma) = \lambda(A_A\lambda^4 + B_A\lambda^3 + C_A\lambda^2 + D_A\lambda + E_A)$

$\xrightarrow{\lambda_{A2} \gg \lambda_{A3} \longrightarrow \frac{B_A}{A_A} \gg \frac{E_A}{D_A}} A_A = \alpha$ & $B_A = \alpha(\frac{B_A}{A_A} + \frac{E_A}{D_A}) + \beta$

& $C_A = \alpha\frac{B_A}{A_A}\frac{E_A}{D_A} + \beta(\frac{B_A}{A_A} + \frac{E_A}{D_A}) + \gamma \approx \alpha\frac{B_A}{A_A}\frac{E_A}{D_A} + \beta\frac{B_A}{A_A} + \gamma$

& $D_A = \beta\frac{B_A}{A_A}\frac{E_A}{D_A} + \gamma(\frac{B_A}{A_A} + \frac{E_A}{D_A}) \approx \beta\frac{B_A}{A_A}\frac{E_A}{D_A} + \gamma\frac{B_A}{A_A}$

& $E_A = \gamma\frac{B_A}{A_A}\frac{E_A}{D_A}$

$\longrightarrow \gamma = \frac{A_A D_A}{B_A}$ & $\beta\frac{E_A}{D_A} + \gamma = \frac{A_A D_A}{B_A} \longrightarrow \beta \approx \frac{C_A A_A}{B_A}$

$\longrightarrow \lambda(\lambda + \frac{B_A}{A_A})(\lambda + \frac{E_A}{D_A})(\lambda^2 + \frac{C_A}{B_A}\lambda + \frac{D_A}{B_A}) = 0$

Routh's Discriminant

rearrange to let $A > 0$

$\longrightarrow$ only if B & C & D & $E < 0$ and $R = BCE - AD^2 - B^2E > 0$

$\longrightarrow$ negative $Re(\lambda)$

* for 3rd order polynomial $A\lambda^3 + B\lambda^2 + C\lambda + D = 0$

$\longrightarrow$ only if B & C & $D = 0$ and $R = AD - BC > 0 \longrightarrow$ negative $Re(\lambda)$

Representing Data

· damped natural frequency $\omega_d = Im(\lambda)$ rad/s

· undamped natural frequency $\omega_n = \sqrt{Re(\lambda)^2 + Im(\lambda)^2}$ rad/s

· time to half $Re(\lambda) < 0$ / double $Re(\lambda) > 0$ frequency $t_{\frac{1}{2}} / t_2 = \frac{\ln 2}{|Re(\lambda)|}$

· damping ratio $\zeta = \frac{Re(\lambda)}{\omega_n}$

· general damped equation of motion $\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$

Longitudinal Dynamic Mode

SPPO (Short Period Pitch Oscillation) $A_s\lambda^2 + B_s\lambda + C = 0$ AoA, θ, q

$$\begin{bmatrix} m & -X_{\dot w} & 0 & 0 \\ 0 & m-Z_{\dot w} & 0 & 0 \\ 0 & -M_{\dot w} & I_{yy} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot U' \\ \dot W' \\ \dot q' \\ \dot\theta' \end{bmatrix} = \begin{bmatrix} X_u & X_w & X_q & -mg\cos\theta_e \\ Z_u & Z_w & Z_q + mU_e & -mg\sin\theta_e \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U' \\ W' \\ q' \\ \theta' \end{bmatrix}$$

$\longrightarrow$ neglect x-force & U-derivatives $\begin{bmatrix} m-Z_{\dot w} & 0 & 0 \\ -M_{\dot w} & I_{yy} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot W' \\ \dot q' \\ \dot\theta' \end{bmatrix} = \begin{bmatrix} Z_w & Z_q + mU_e & -mg\sin\theta_e \\ M_w & M_q & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} W' \\ q' \\ \theta' \end{bmatrix}$

$\longrightarrow$ effect of aerodynamic force >> gravitational $\begin{bmatrix} m-Z_{\dot w} & 0 \\ -M_{\dot w} & I_{yy} \end{bmatrix} \begin{bmatrix} \dot W' \\ \dot q' \end{bmatrix} = \begin{bmatrix} Z_w & Z_q + mU_e \\ M_w & M_q \end{bmatrix} \begin{bmatrix} W' \\ q' \end{bmatrix}$

$$\longrightarrow \begin{vmatrix} Z_w + (Z_{\dot w} - m)\lambda & Z_q + mU_e \\ M_w + M_{\dot w}\lambda & M_q - I_{yy}\lambda \end{vmatrix} = 0$$

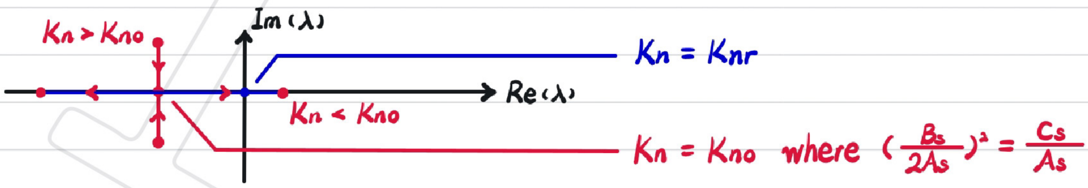
$\xrightarrow{Z_q \ll mU_e \ \& \ Z_{\dot w} \ll m}$ $A_s = (m - Z_{\dot w})I_{yy} \simeq m\underbrace{I_{yy}}_{>0}$

& $B_s = (Z_{\dot w} - m)M_q - I_{yy} - M_{\dot w}(Z_q + mU_e) \simeq -m\underbrace{M_q}_{\substack{\text{stable} < 0 \\ \text{large}}} - I_{yy}\underbrace{Z_w}_{\text{stable} < 0} - mU_e\underbrace{M_{\dot w}}_{\substack{\text{typically} < 0 \\ \text{smaller}}}$

& $C_s = Z_w M_q - (Z_q + mU_e)M_w \simeq \underbrace{Z_w}_{<0}\underbrace{M_q}_{<0} - mU_e\underbrace{M_w}_{\substack{<0 \text{ if } K_n > 0 \\ >0 \text{ if } K_n < 0}}$

$$\longrightarrow \lambda = \frac{-B_s \pm \sqrt{B_s^2 - 4A_sC_s}}{2A_s} = -\frac{B_s}{2A_s} \pm \sqrt{\left(\frac{B_s}{2A_s}\right)^2 - \frac{C_s}{A_s}}$$

*



Phugoid $\lambda^2 + \delta\lambda + \gamma = 0$ altitude, V_∞, θ

$$\begin{cases} m\dot U' - X_{\dot w}\dot W' = X_u U' + X_w W' + X_q q' - mg\cos\theta_e\theta' = X_u U' + X_w W' + X_q\dot\theta' - mg\cos\theta_e\theta' \\ (m - Z_{\dot w})\dot W' = Z_u U' + Z_w W' + (Z_q + mU_e)q' - mg\sin\theta_e\theta' = Z_u U' + Z_w W' + (Z_q + mU_e)\dot\theta' - mg\sin\theta_e\theta' \end{cases}$$

$\longrightarrow$ set $\dot{q} = 0$ & $\dot{W}' \approx W' \approx 0$ for this mode $\begin{bmatrix} m & -X_q \\ 0 & -Z_q - mU_e \end{bmatrix} \begin{bmatrix} \dot{U}' \\ \dot{\theta}' \end{bmatrix} = \begin{bmatrix} X_u & -mg\cos\theta_e \\ Z_u & -mg\sin\theta_e \end{bmatrix} \begin{bmatrix} U' \\ \theta' \end{bmatrix}$

$$\longrightarrow \begin{vmatrix} X_u - m\lambda & -mg\cos\theta_e + X_q\lambda \\ Z_u & -mg\sin\theta_e + (Z_q + mU_e)\lambda \end{vmatrix} = 0$$

$$\xrightarrow{Z_q \ll mU_e \ \& \ X_q \approx 0} \gamma = \frac{Z_u X_q - m^2 g\sin\theta_e - X_u(Z_q + mU_e)}{m(Z_q + mU_e)} \approx -\frac{X_u}{m} \xrightarrow{\text{negiligible thrust effect } \& \ \frac{\partial}{\partial U_e} \approx 0} -\frac{2g}{U_e}\left(\frac{C_D}{C_L}\right)_e$$

$$\delta = \frac{X_u\sin\theta_e - Z_u\cos\theta_e}{Z_q + mU_e} g \approx -\frac{g}{mU_e} Z_u \xrightarrow{\text{negiligible thrust effect } \& \ \frac{\partial}{\partial U_e} \approx 0} -\frac{2g^2}{U_e^2}$$

$$\longrightarrow \lambda = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \delta} \xrightarrow{\gamma^2 \ll 4\delta} \lambda = \pm\sqrt{-\delta} = \pm i\sqrt{2}\frac{g}{U_e}$$

$$\longrightarrow \omega_d = \sqrt{2}\frac{g}{U_e} \longrightarrow T = \frac{2\pi}{\omega_d} = \frac{\sqrt{2}\pi U_e}{g} \longrightarrow \zeta \approx \frac{1}{\sqrt{2}(L/D)_e}$$

Lateral Dynamic Mode

Yaw Independence $\lambda = 0$ linear term change in heading

neutral stable $\longrightarrow$ lateral equation remain entirely unchanged $\longrightarrow$ navigation

Roll Subsidence $\lambda + \frac{B_A}{A_A} = 0$ aerodynamic damping to change in roll rate

completely dominate by $\dot{p}$ $\longrightarrow I_{xx}\dot{p}' = L_p p \longrightarrow \lambda = \frac{L_p}{I_{xx}}$ — <0 for all conventional aircraft

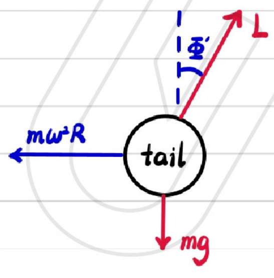
Spiral $\lambda + \frac{E_A}{D_A} = 0$ coupling sideslip with roll rate

$$\begin{bmatrix} m & 0 & 0 & 0 & 0 \\ 0 & I_{xx} & -I_{xz} & 0 & 0 \\ 0 & -I_{xz} & I_{zz} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{V}' \\ \dot{p}' \\ \dot{r}' \\ \dot{\Phi}' \\ \dot{\Psi}' \end{bmatrix} = \begin{bmatrix} Y_v & Y_p & Y_r - mU_e & mg\cos\theta_e & 0 \\ L_v & L_p & L_r & 0 & 0 \\ N_v & N_p & N_r & 0 & 0 \\ 0 & 1 & \tan\theta_e & 0 & 0 \\ 0 & 0 & \sec\theta_e & 0 & 0 \end{bmatrix} \begin{bmatrix} V' \\ p' \\ r' \\ \Phi' \\ \Psi' \end{bmatrix}$$

kimatic conditions

neglect y-force & $\dot{p}$ & $\dot{V}$ & $\dot{r}$ $\longrightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ L_v & L_p & L_r \\ N_v & N_p & N_r \end{bmatrix} \begin{bmatrix} V' \\ p' \\ r' \end{bmatrix}$

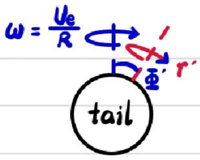
* rolling and yawing DoF are coupled



$\cdot \ L\cos\Phi' = \frac{1}{2}\rho V_\infty^2 S_{ref} C_L \cos\Phi' = mg$

$\cdot \ L\sin\Phi' = \frac{1}{2}\rho V_\infty^2 S_{ref} C_L \sin\Phi' = m\omega^2 R$

$$\longrightarrow \tan\Phi' = \frac{\omega^2 R}{g} \longrightarrow \Phi' = \frac{\omega^2 R}{g}$$



$\cdot\ \omega = r' \cos\Phi' \approx r' \longrightarrow \Phi' = \frac{U_e}{g}\omega = \frac{U_e}{g} r'$

$\cdot\ \dot{\Phi}' = p' \longrightarrow p' = \frac{U_e}{g}\dot{r}'$

$$\longrightarrow \begin{bmatrix} 0 & 0 & \frac{U_e}{g} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{v}' \\ \dot{p}' \\ \dot{r}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ L_v & L_p & L_r \\ N_v & N_p & N_r \end{bmatrix}\begin{bmatrix} v' \\ p' \\ r' \end{bmatrix}$$

$$\longrightarrow \begin{vmatrix} 0 & 1 & -\frac{U_e}{g}\lambda \\ L_v & L_p & L_r \\ N_v & N_p & N_r \end{vmatrix} = 0$$

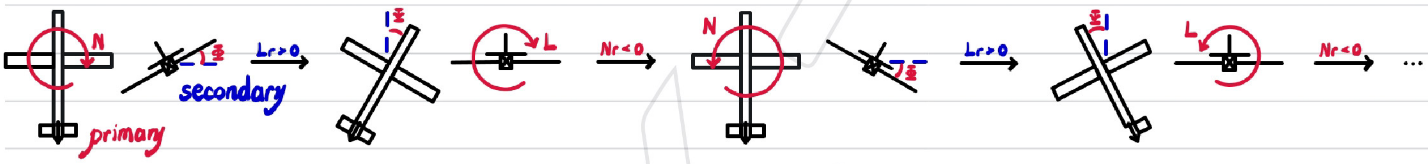
$$\longrightarrow \lambda = -\frac{g}{U_e}\,\frac{\overset{<0}{L_v N_r} - \overset{>0}{L_r}\,\overset{\text{stable}\,>0}{N_v}}{\underset{\text{stable}\,<0}{L_v}\,\underset{\text{typically}\,>0}{N_p} - \underset{<0}{L_p} N_v} < 0 \longrightarrow Re(\lambda) < 0$$

$* \ L_v = \left(\frac{\partial L}{\partial v}\right)_e = \left(\frac{\partial L}{\partial \beta}\right)_e \left(\frac{\partial \beta}{\partial v}\right)_e = \frac{1}{U_e}\left(\frac{\partial L}{\partial \beta}\right)_e$

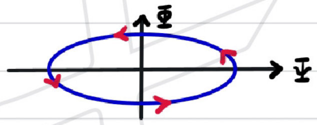
Dutch Roll

$\lambda^2 + \frac{C_s}{B_s}\lambda + \frac{E_s}{B_s}$

lightly damped oscillation between yawing, rolling and sideslipping



$\longrightarrow$ right - wing tip path



$$\begin{bmatrix} m & 0 & 0 & 0 & 0 \\ 0 & I_{xx} & -I_{xz} & 0 & 0 \\ 0 & -I_{xz} & I_{zz} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{v}' \\ \dot{p}' \\ \dot{r}' \\ \dot{\Phi}' \\ \dot{\Psi}' \end{bmatrix} = \begin{bmatrix} Y_v & Y_p & Y_r - mU_e & mg\cos\Theta_e & 0 \\ L_v & L_p & L_r & 0 & 0 \\ N_v & N_p & N_r & 0 & 0 \\ 0 & 1 & \tan\Theta_e & 0 & 0 \\ 0 & 0 & \sec\Theta_e & 0 & 0 \end{bmatrix}\begin{bmatrix} v' \\ p' \\ r' \\ \Phi' \\ \Psi' \end{bmatrix}$$

neglect $\dot{p}$ & p & Φ $\longrightarrow \begin{bmatrix} m & 0 \\ 0 & I_{zz} \end{bmatrix}\begin{bmatrix} \dot{v}' \\ \dot{r}' \end{bmatrix} = \begin{bmatrix} Y_v & Y_r - mU_e \\ N_v & N_r \end{bmatrix}\begin{bmatrix} v' \\ r' \end{bmatrix}$

$$\longrightarrow \begin{vmatrix} Y_v - m\lambda & Y_r - mU_e \\ N_v & N_r - I_{zz}\lambda \end{vmatrix} = 0$$

$$\longrightarrow m I_{zz}\lambda^2 - (m\underset{<0}{N_r} + I_{zz}\underset{<0}{Y_v})\lambda + (Y_v N_r - \underset{>0}{Y_r}\,\overset{\text{stable}\,>0}{N_v} + mU_e N_v) = 0$$

$\frac{1}{m^2}$ & $\frac{1}{mI_{zz}} \ll \frac{1}{I_{zz}} \longrightarrow \lambda = \frac{1}{2}\left(\frac{N_r}{I_{zz}} + \frac{Y_v}{m}\right) \pm i\sqrt{\frac{U_e N_v}{I_{zz}}}$

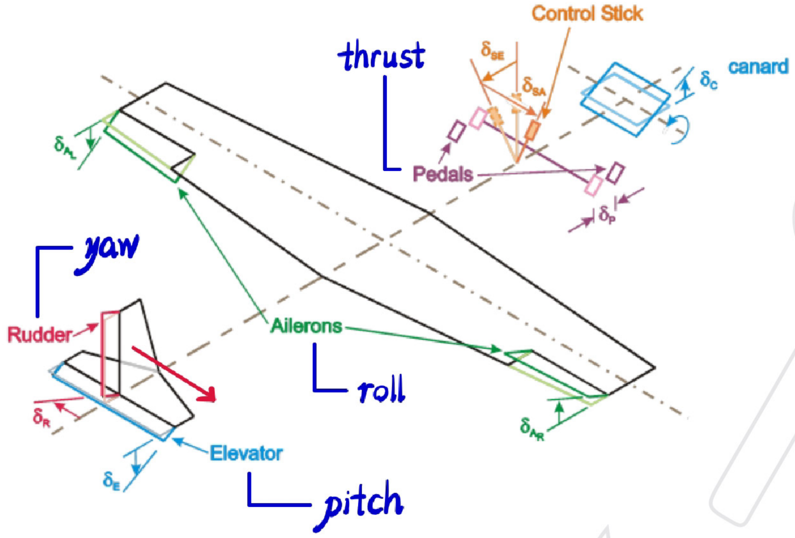
* poor approximation since Dutch Roll is truly a 3-DoF model where p & L have substantial influence

reduce damping ① increasing size of the tail fin $\longrightarrow N_v \downarrow \longrightarrow$ spiral stability $\downarrow$ ② yaw damper

Control System

Response to Control System

Basic Aircraft Control Layout



Modelling Control Effects

$$M_* \dot{X}' = R X' + Bu' = (N_{x_0} + F_{x_0} + G_{x_0}) X' + Bu'$$

$$Bu' = \begin{bmatrix} X_{\delta E} & X_{C_T} & X_{\delta A} & X_{\delta R} \\ Z_{\delta E} & Z_{C_T} & Z_{\delta A} & Z_{\delta R} \\ M_{\delta E} & M_{C_T} & M_{\delta A} & M_{\delta R} \\ 0 & 0 & 0 & 0 \\ Y_{\delta E} & Y_{C_T} & Y_{\delta A} & Y_{\delta R} \\ L_{\delta E} & L_{C_T} & L_{\delta A} & L_{\delta R} \\ N_{\delta E} & N_{C_T} & N_{\delta A} & N_{\delta R} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_E' \\ C_T' \\ \delta_A' \\ \delta_R' \end{bmatrix} \longrightarrow \left[\begin{array}{cc|cc} X_{\delta E} & X_{C_T} & & \\ Z_{\delta E} & Z_{C_T} & & \\ M_{\delta E} & M_{C_T} & & \\ 0 & 0 & & \\ \hline & & Y_{\delta A} & Y_{\delta R} \\ & & L_{\delta A} & L_{\delta R} \\ & & N_{\delta A} & N_{\delta R} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \delta_E' \\ C_T' \\ \delta_A' \\ \delta_R' \end{bmatrix}$$

$\longrightarrow$ decoupling

$$B_S u_S' = \begin{bmatrix} X_{\delta E} & X_{C_T} \\ Z_{\delta E} & Z_{C_T} \\ M_{\delta E} & M_{C_T} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_E' \\ C_T' \end{bmatrix} \quad \& \quad B_A u_A' = \begin{bmatrix} Y_{\delta A} & Y_{\delta R} \\ L_{\delta A} & L_{\delta R} \\ N_{\delta A} & N_{\delta R} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_A' \\ \delta_R' \end{bmatrix}$$

δ_E - deriavatives ($X_{\delta E}$, $Z_{\delta E}$, $M_{\delta E}$)

- $X_{\delta E} = \frac{\partial}{\partial \delta_E} [L_H \sin(\alpha - \alpha_e) - D_H \cos(\alpha - \alpha_e)]_e = \frac{1}{2}\rho V_e^2 S_H \left(\frac{\partial C_{DH}}{\partial \delta_E}\right)_e$

where $\left(\frac{\partial C_{DH}}{\partial \delta_E}\right)_e = \frac{\partial}{\partial \delta_E}\left\{ a_H\left[\left(1 - \frac{d\varepsilon}{d\alpha}\right)\alpha - \varepsilon_0 + i_H - \alpha_{0H}\right] + a_E \delta_E + d_E \delta_E \right\}_e = d_E$

$\cdot\ Z_{\delta E} = \frac{\partial}{\partial \delta_E}[L_H \cos(\alpha - \alpha_e) - D_H \sin(\alpha - \alpha_e)]_e = \frac{1}{2}\rho U_e^2 S_H (\frac{\partial C_L}{\partial \delta_E})_e$

where $(\frac{\partial C_L}{\partial \delta_E})_e = \frac{\partial}{\partial \delta_E}\{a_H[(1-\frac{d\varepsilon}{d\alpha})\alpha - \varepsilon_0 + i_H - \alpha_{0H}] + a_E \delta_E + a_T \delta_T\}_e = a_E$

$\cdot\ M_{\delta E} = (\frac{\partial M_H}{\partial \delta_E})_e + (\frac{\partial M_H}{\partial X})_e X_{\delta E} + (\frac{\partial M_H}{\partial Z})_e Z_{\delta E} = \frac{1}{2}\rho U_e^2 S_H \bar{c}_H (\frac{\partial C_{MH}}{\partial \delta_E})_e + (z_{CG} - z_H) X_{\delta E} + (x_H - x_{CG}) Z_{\delta E}$

C_T - derivatives (X_{C_T}, Z_{C_T}, M_{C_T})

$\cdot\ X_{C_T} = \frac{\partial}{\partial C_T}[T_x + L\sin(\alpha - \alpha_e) - D\cos(\alpha - \alpha_e)] = \frac{1}{2}\rho U_e^2 S_{ref}[\cos(\alpha_e + \varepsilon) - (\frac{C_D}{C_T})_e]$

$\cdot\ Z_{C_T} = \frac{\partial}{\partial C_T}[-T_z - L\cos(\alpha - \alpha_e) - D\sin(\alpha - \alpha_e)] = -\frac{1}{2}\rho U_e^2 S_{ref}[\sin(\alpha_e + \varepsilon) + (\frac{C_L}{C_T})_e]$

$\cdot\ M_{C_T} = \frac{\partial}{\partial C_T}[Tl_T + \frac{1}{2}\rho V_\infty^2 S_{ref}\bar{c}(C_{M0} + \frac{x_{CG} - x_W}{\bar{c}} C_{LW} + \frac{x_{CG} - x_H}{\bar{c}}\frac{S_H}{S_{ref}} C_{LH})]_e$

$= \frac{1}{2}\rho U_e^2 S_{ref}[l_T + (x_{CG} - x_W)(\frac{\partial C_{LW}}{\partial C_T})_e + (x_{CG} - x_H)\frac{S_H}{S_{ref}}(\frac{\partial C_{LH}}{\partial C_T})_e]$

δ_A - derivatives ($Y_{\delta A}$, $L_{\delta A}$, $N_{\delta A}$)* use strip - integral method

δ_R - derivatives ($Y_{\delta R}$, $L_{\delta R}$, $N_{\delta R}$)

$\cdot\ Y_{\delta R} = (\frac{\partial Y_V^a}{\partial \delta_R})_e = \frac{\partial}{\partial \delta_R}(\frac{1}{2}\rho V_\infty^2 S_V C_{LV})_e = \frac{1}{2}\rho V_\infty^2 S_V (\frac{\delta C_{LV}}{\delta R})_e$

where $(\frac{\partial C_{LV}}{\partial \delta_R})_e = \frac{\partial}{\partial \delta_R}(-a_V\beta + a_R\delta_R)_e = a_R$

$\cdot\ L_{\delta R} = (\frac{\partial L_V^a}{\partial \delta_R})_e = \frac{\partial}{\partial \delta_R}[(z_V - z_{CG})L_V^a] = \frac{1}{2}\rho U_e^2 S_V (z_V - z_{CG})(\frac{\partial C_{LV}}{\partial \delta_R})_e$

$\cdot\ N_{\delta R} = (\frac{\partial N_V^a}{\partial \delta_R})_e = \frac{\partial}{\partial \delta_R}[(x_{CG} - x_V)N_V^a] = \frac{1}{2}\rho U_e^2 S_V (x_{CG} - x_V)(\frac{\partial C_{LV}}{\partial \delta_R})_e$

Transfer Functions

$M_* \dot{x}' = Rx' + Bu' \xleftrightarrow{L} M_* s X'(s) = RX'(s) + BU'(s)$

$\longrightarrow (sM_* - R)X'(s) = BU'(s)$

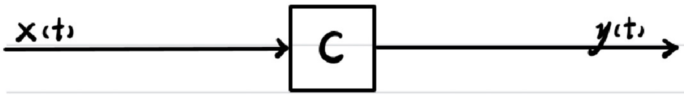
$\longrightarrow X'(s) = (sM_* - R)^{-1} BU'(s)$

$\longrightarrow G(s) = \frac{X'(s)}{U'(s)} = (sM_* - R)^{-1} B$

Feedback Control

General Principle

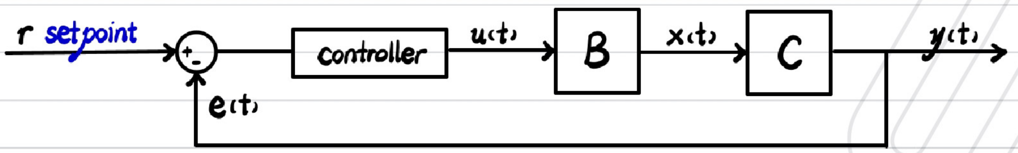
$y(t) = Cx(t)$



* for example, $y(t) = [V_\infty]$ and $x(t) = [U \;\; W \;\; q \;\; \theta]^T$

$$\longrightarrow C = \left[\frac{\partial V_\infty}{\partial U} \;\; \frac{\partial V_\infty}{\partial W} \;\; \frac{\partial V_\infty}{\partial q} \;\; \frac{\partial V_\infty}{\partial \theta}\right] = [\,1 \;\; 0 \;\; 0 \;\; 0\,]$$

$$e(t) = r - y(t) = r - Cx(t)$$



P Controller $u(t) = K_p e(t) = K_p (r - Cx)$ yaw damper

$$M_* \dot{x} = Rx + Bu = Rx + BK_p (r - Cx) = (R - BK_p C)x + BK_p r$$

at singular point, $0 = (R - BK_p C)x_0 + BK_p r \longrightarrow x_0 = -(R - BK_p C)^{-1} BK_p r$

$$\longrightarrow e_0 = r - Cx_0 = [I + C(R - BK_p C)^{-1} BK_p] r$$

for small perturbation, $M_* \dot{x}' = Rx' + Bu' = (R - BK_p C)x' + BK_p r$

$$\longrightarrow |R - BK_p C - M_* \lambda| = 0$$

PI Controller $u(t) = K_p e(t) + K_i \int_0^t e(\tau)\,d\tau$ roll controller

let $z = \int_0^t e(\tau)\,d\tau \longrightarrow \dot{z} = e(t) = r - Cx$

$$\longrightarrow \begin{cases} M_* \dot{x} = Rx + Bu = Rx + BK_p(r - Cx) + BK_i z \\ \dot{z} = e = r - Cx \end{cases}$$

$$\longrightarrow \begin{bmatrix} M_* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} R - BK_p C & BK_i \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} BK_p \\ I \end{bmatrix} r$$

PD Controller $u(t) = K_p e(t) + K_d \frac{d}{dt} e(t)$ advanced yaw damper

$$M_* \dot{x} = Rx + Bu = Rx + BK_p (r - Cx) + BK_d (\dot{r} - C\dot{x})$$

$$\longrightarrow (M_* + BK_d C)\dot{x} - BK_d \dot{r} = (R - BK_p C)x + BK_p r$$

PID Controller $u(t) = K_p e(t) + K_i \int_0^t e(\tau)\,d\tau + K_d \frac{d}{dt} e(t)$ U & θ controller

let $z = K_I \int_0^t e(\tau)\, d\tau \longrightarrow \dot{z} = e(t) = r - Cx$

$$\longrightarrow \begin{cases} M_* \dot{x} = Rx + BK_p(r - Cx) + BK_i z + BK_d(\dot{r} - C\dot{x}) \\ \dot{z} = r - Cx \end{cases}$$

$$\longrightarrow \begin{bmatrix} M_* + BK_dC & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} - \begin{bmatrix} BK_d \\ 0 \end{bmatrix} \dot{r} = \begin{bmatrix} R - BK_pC & BK_i \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} BK_p \\ I \end{bmatrix} r$$

Tuning PID

Typical effect of gains on the closed loop response of the system

	Rise Time	Overshoot	Settling Time	Steady State Error	Stability
K_p	Decrease	Increase	Little effect	Decrease	Degrade
K_i	Decrease	Increase	Increase	Eliminate	Degrade
K_d	Little effect	Decrease	Decrease	No effect	Improve