Overall Structural Analysis

Stiffness Method $Kr = R$

Stiffness matrix

$$\begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \cdots \\ r_n \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ \cdots \\ R_n \end{bmatrix}$$

Systematic method

① apply unit displacement to each degree of freedom $r_i = 1$ and $r_j = 0$ $(i \neq j)$

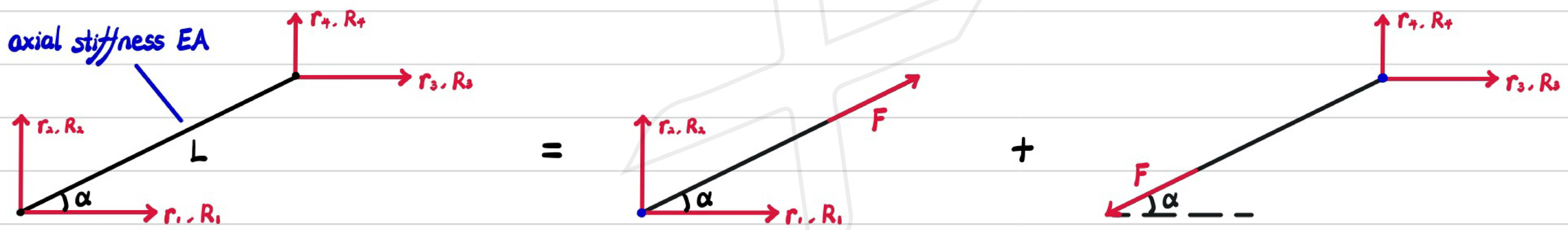
② find strain $\varepsilon = \frac{\Delta L}{L}$ $\quad * \ \varepsilon = \frac{\sigma}{E} + \alpha \Delta T$

③ using Hooke's law find stress $\sigma = E\varepsilon$ $\quad * \ \varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}) + \alpha\Delta T$

④ find force $F = \sigma A$

⑤ force equilibrium $\Sigma F + \Sigma R = 0$ $\quad * \ K_{ji} r_i = K_{ij} r_i = R_j$

Generic bar element



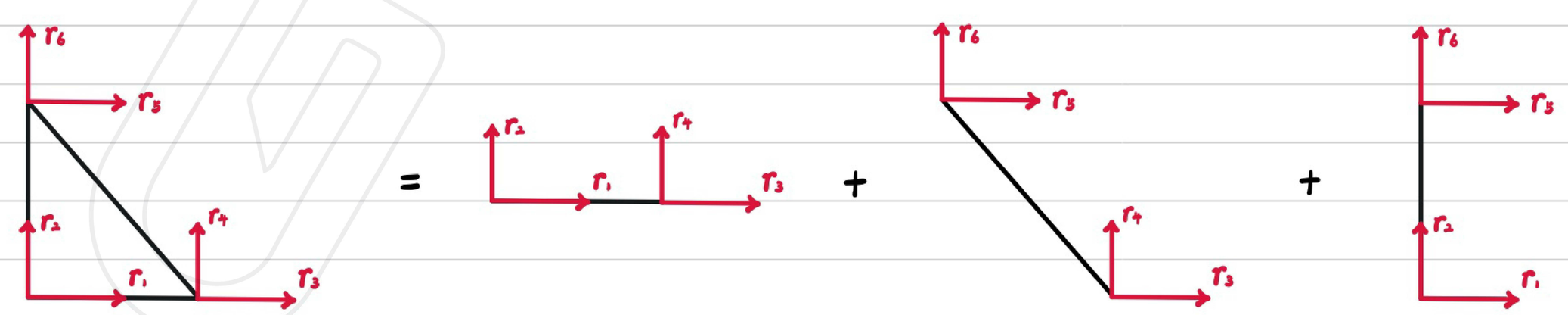
① $r_1 = 1,\ r_{2,3,4} = 0 \longrightarrow \varepsilon = -\frac{\cos\alpha}{L} \longrightarrow F = -\cos\alpha \frac{EA}{L}$

$\longrightarrow K_{11} = R_1 = \cos^2\alpha \frac{EA}{L},\ K_{12} = R_2 = \cos\alpha\sin\alpha \frac{EA}{L},\ K_{13} = R_3 = -\cos^2\alpha \frac{EA}{L},\ K_{14} = R_4 = -\cos\alpha\sin\alpha \frac{EA}{L}$

② $r_2 = 1,\ r_{1,3,4} = 0$ ..... ③ $r_3 = 1,\ r_{1,2,4} = 0$ ..... ④ $r_4 = 1,\ r_{1,2,3} = 0$ .....

$$\longrightarrow K = \frac{EA}{L} \begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha & -\cos^2\alpha & -\cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha & -\cos\alpha\sin\alpha & -\sin^2\alpha \\ -\cos^2\alpha & -\cos\alpha\sin\alpha & \cos^2\alpha & \cos\alpha\sin\alpha \\ -\cos\alpha\sin\alpha & -\sin^2\alpha & \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix}$$

Assemble bar element



Inclusion of support $r_{support} = 0$

Flexibility Method $FR = r$

Flexibility matrix

$$\begin{bmatrix} F_{11} & F_{12} & \dots & F_{1n} \\ F_{21} & F_{22} & \dots & F_{2n} \\ \dots & \dots & \dots & \dots \\ F_{n1} & F_{n2} & \dots & F_{nn} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ \dots \\ R_n \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \dots \\ r_n \end{bmatrix} \longrightarrow F = K^{-1} \text{ when } K \text{ is not singular}$$

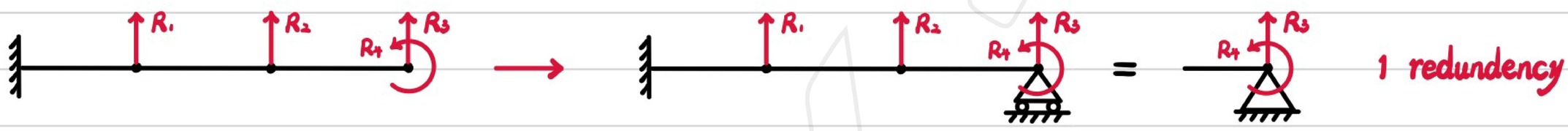
Systematic method

① apply unit force to each degree of freedom $R_i = 1$ and $R_j = 0$ $(i \neq j)$

② assuming Engineering Theory of Bending $r_{ij} = \int_V \varepsilon_i \bar{\sigma}_j \, dV$

③ find displacement $r_{ij} = \int_0^L \frac{M_i \bar{M}_j}{EI} dz$ $\quad$ * $F_{ji} R_i = r_{ij} = F_{ij} R_i = r_{ji}$

Indeterminate structure always ignore horizontal redundencies



1 redundency

① calculate F without redundency

②
$$\begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ 0 \\ r_4 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} F_{11} & F_{12} & F_{14} \\ F_{21} & F_{22} & F_{24} \\ F_{41} & F_{42} & F_{44} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_4 \end{bmatrix} + \begin{bmatrix} F_{13} \\ F_{23} \\ F_{43} \end{bmatrix} R_3 = \begin{bmatrix} r_1 \\ r_2 \\ r_4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} F_{31} & F_{32} & F_{34} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_4 \end{bmatrix} + F_{33} R_3 = 0$$

$$\longrightarrow \begin{bmatrix} F_{11} & F_{12} & F_{14} \\ F_{21} & F_{22} & F_{24} \\ F_{41} & F_{42} & F_{44} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_4 \end{bmatrix} + \begin{bmatrix} F_{13} \\ F_{23} \\ F_{43} \end{bmatrix} \left(-\frac{1}{F_{33}} \begin{bmatrix} F_{31} & F_{32} & F_{34} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_4 \end{bmatrix} \right) = \begin{bmatrix} G_{11} & G_{12} & G_{14} \\ G_{21} & G_{22} & G_{24} \\ G_{41} & G_{42} & G_{44} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_4 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_4 \end{bmatrix}$$

Pin-jointed frameworks

① apply unit force to each degree of freedom $R_i = 1$ and $R_j = 0$ $(i \neq j)$

② assuming Engineering Theory of Bending $r_{ij} = \int_V \varepsilon_i \bar{\sigma}_j \, dV$

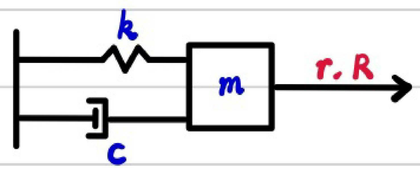
③ find displacement $r_{ij} = \sum_{b=1}^{n} \frac{N_b^i \bar{N}_b^j}{E_b A_b} L_b$ $\quad$ * $F_{ji} R_i = r_{ij} = F_{ij} R_i = r_{ji}$

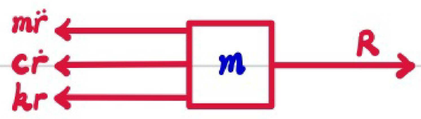
*④ for redundent structure $\delta_0 + N_{re} \bar{\delta}_{re} = \sum_{b=1}^{n} \frac{N_b^0 \bar{N}_b^{re}}{E_b A_b} L_b + N_{re} \sum_{b=1}^{n} \frac{N_b^{re} \bar{N}_b^{re}}{E_b A_b} L_b$

Structural Dynamics

Equation of Motion and Free Vibration

1-DoF motion



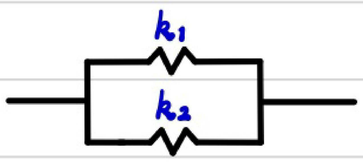


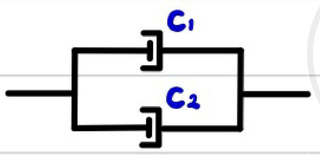
$m\ddot{r} + c\dot{r} + kr = R$

* series

$\frac{1}{k_{total}} = \frac{1}{k_1} + \frac{1}{k_2}$

$\frac{1}{C_{total}} = \frac{1}{C_1} + \frac{1}{C_2}$

* parallel 

$k_{total} = k_1 + k_2$ 

$C_{total} = C_1 + C_2$

1-DoF free vibration

$R(t) = 0 \longrightarrow m\ddot{r} + c\dot{r} + kr = 0 \longrightarrow r(t) = r_0 e^{pt} \longrightarrow \dot{r} = p r_0 e^{pt}$ and $\ddot{r} = p^2 r_0 e^{pt}$

$\longrightarrow mp^2 + cp + k = 0 \longrightarrow p = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$

for a lightly damped system, $c^2 - 4km < 0$

$\longrightarrow p = -\frac{c}{2m} \pm i\frac{\sqrt{4km - c^2}}{2m} = -\alpha \pm i\beta$ *β is damped natural frequency

$\longrightarrow r = r_{01} e^{p_1 t} + r_{02} e^{p_2 t} = r_{01} e^{-\alpha t} e^{i\beta t} + r_{02} e^{-\alpha t} e^{-i\beta t} \xlongequal{\text{real part}} r_{01} e^{-\alpha t} \sin(\beta t) + r_{02} e^{-\alpha t} \cos(\beta t)$

or $\longrightarrow \ddot{r} + \frac{c}{m}\dot{r} + \frac{k}{m} r = 0 \longrightarrow \ddot{r} + 2\omega_n \xi \dot{r} + \omega_n^2 r$ *$\omega_n = \sqrt{\frac{k}{m}}$ is undamped natural frequency

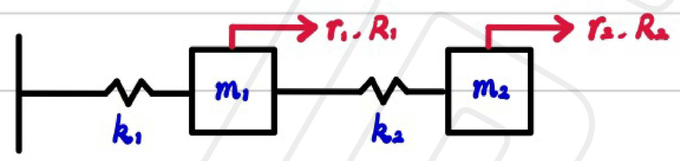
*$\xi = \frac{c}{2\sqrt{km}}$ is damping ratio

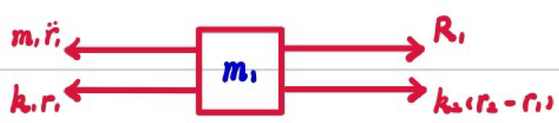
n-DoF free vibration

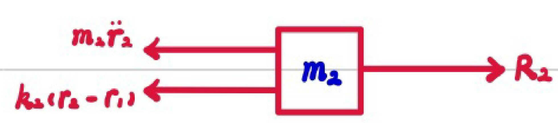
$M\ddot{r} + Kr = 0$ when $C = 0 \longrightarrow r = r_0 e^{i\omega t} \longrightarrow \ddot{r} = -\omega^2 r_0 e^{i\omega t} \longrightarrow (-\omega^2 M + K) r_0 e^{i\omega t} = 0$

$\longrightarrow -\omega^2 M + K = 0 \longrightarrow$ eigenvalue $\omega_i^2 \longrightarrow$ eigenvector $\Phi = [\psi_1, \psi_2, \dots, \psi_n]$

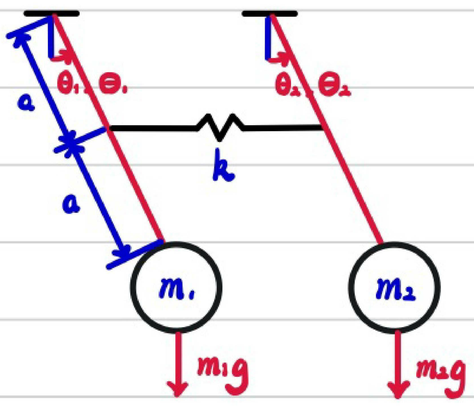
n-DoF motion

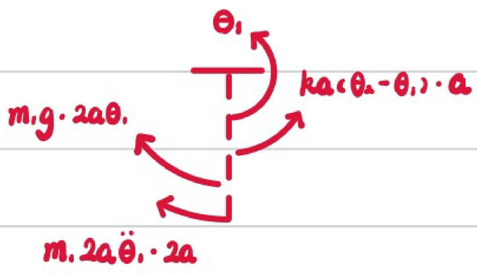


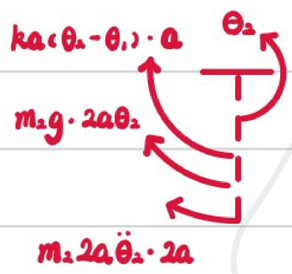




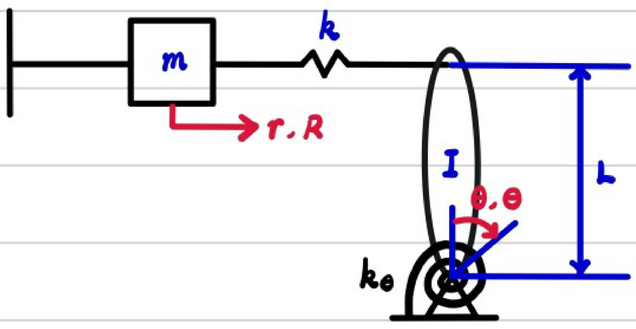
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{r}_1 \\ \ddot{r}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

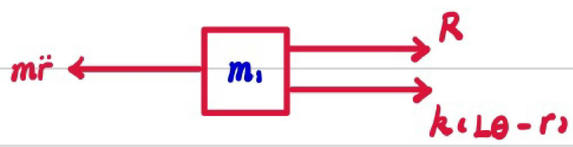


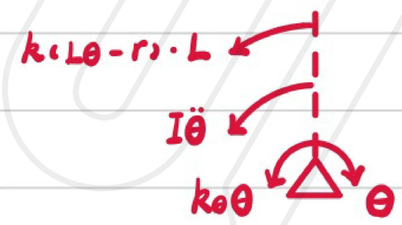




$$4a^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + a\begin{bmatrix} 2m_1g+ka & -ka \\ -ka & 2m_2g+ka \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}$$







$* \; k_\theta = \frac{M}{\theta}$

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k & -kL \\ -kL & kL^2 \end{bmatrix}\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} R \\ \Theta \end{bmatrix}$$

Modal Analysis

Orthogonal properties

$-w_j^2 M\psi_j + K\psi_j = 0 \longrightarrow -w_j^2\psi_j^T M + \psi_j^T K = 0 \longrightarrow -w_j^2\psi_j^T M\psi_i + \psi_j^T K\psi_i = 0$

$-w_i^2 M\psi_i + K\psi_i = 0 \longrightarrow -w_i^2\psi_j^T M\psi_i + \psi_j^T K\psi_i = 0$

$\longrightarrow (w_i^2 - w_j^2)\,\psi_j^T M\psi_i = 0$

if $i \neq j$, $w_i^2 \neq w_j^2 \longrightarrow \psi_j^T M\psi_i = 0$ and $\psi_j^T K\psi_i = 0$

if $i = j$, $w_i^2 = w_j^2 \longrightarrow \psi_i^T M\psi_i = m_i$ ith modal mass and $\psi_i^T K\psi_i = k_i$ ith modal stiffness

Modal transformation

$r = \Phi q \longrightarrow \Phi^T M\Phi\ddot{q} + \Phi^T C\Phi\dot{q} + \Phi^T K\Phi q = \Phi^T R \longrightarrow m\ddot{q} + c\dot{q} + kq = Q$ fully uncoupled equation

Periodic Loading

1-DoF

$m\ddot{r} + kr = R \longrightarrow r = r_0 e^{iwt}$ and $R = R_0 e^{iwt} \longrightarrow (-w^2 m + k)\, r_0 = R_0$

$\longrightarrow r = \frac{1}{k - w^2 m} R_0 e^{iwt} \xlongequal{\text{real part}} r = \frac{1}{k - w^2 m} R_0 \cos(wt)$

$m\ddot{r} + c\dot{r} + kr = R \longrightarrow r = r_0 e^{i\omega t}$ and $R = R_0 e^{i\omega t} \longrightarrow (-\omega^2 m + i\omega c + k) r_0 = R_0$

$$\longrightarrow r = \frac{1}{k + i\omega c - \omega^2 m} R_0 e^{i\omega t} = \frac{(k-\omega^2 m) - i\omega c}{(k-\omega^2 m)^2 + (\omega c)^2} R_0 e^{i\omega t}$$

$$\xrightarrow{\text{real part}} r = \frac{k-\omega^2 m}{(k-\omega^2 m)^2 + (\omega c)^2} R_0 \cos(\omega t) + \frac{\omega c}{(k-\omega^2 m)^2 + (\omega c)^2} R_0 \sin(\omega t)$$

n-DoF

$$M\ddot{r} + Kr = R \longrightarrow r = (K - \omega^2 M)^{-1} R_0 \cos(\omega t) \xrightarrow{\text{modal space}} q_i = \frac{1}{k_i - \omega_i^2 m_i} Q_{0i} \cos(\omega_i t)$$

$M\ddot{r} + C\dot{r} + Kr = R \longrightarrow r = (K + i\omega c - \omega^2 M)^{-1} R_0 \cos(\omega t) = H(\omega) R_0 \cos(\omega t)$ * Rayleigh damping $C = \alpha M + \beta K$

$$\xrightarrow{\text{modal space}} q_i = \frac{k_i - \omega_i^2 m_i}{(k_i - \omega_i^2 m_i)^2 + (\omega_i c_i)^2} Q_{0i} \cos(\omega_i t) + \frac{\omega_i c_i}{(k_i - \omega_i^2 m_i)^2 + (\omega_i c_i)^2} Q_{0i} \sin(\omega_i t)$$

$$= D_i(\omega) Q_{0i} e^{i\omega t} \xrightarrow{\text{real part}} D_i(\omega) Q_{0i} \cos(\omega_i t) \longrightarrow H(\omega) = \Phi D(\omega) \Phi^T$$

Transient Loading

Convolution integral

* time t ** impluse applied time τ * $r = \Phi q$ in modal space

$r|_{\tau=0} = 0$ and $\dot{r}|_{\tau=0} = \frac{R(\tau)}{m} d\tau$ and $r = r_{01} e^{-\alpha t} \sin(\beta t) + r_{02} e^{-\alpha t} \cos(\beta t)$

$$\longrightarrow r|_{\tau=0} = r_{02} = 0 \longrightarrow r = r_{01} e^{-\alpha t} \sin(\beta t)$$

$$\longrightarrow \dot{r}|_{\tau=0} = \beta r_{01} = \frac{R(\tau)}{m} d\tau \longrightarrow r_{01} = \frac{R(\tau)}{\beta m} d\tau \longrightarrow r = \int_0^t \frac{1}{\beta m} e^{-\alpha t} \sin(\beta t) R(\tau) d\tau$$

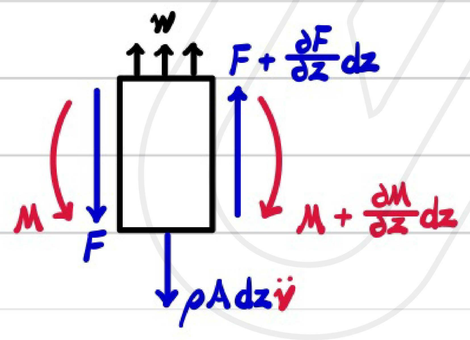
for $\tau \neq 0 \longrightarrow r = \int_0^t \frac{1}{\beta m} e^{-\alpha(t-\tau)} \sin(\beta(t-\tau)) R(\tau) dt = \int_0^t W(t-\tau) R(\tau) dt$

with free vibration $\longrightarrow r = \int_0^t \frac{1}{\beta m} e^{-\alpha(t-\tau)} \sin(\beta(t-\tau)) R(\tau) dt + r_0 e^{-\alpha t} \cos(\beta t) + \frac{\dot{r}_0 + \alpha r_0}{\beta} e^{-\alpha t} \sin(\beta t)$

when $c = 0 \longrightarrow \alpha = 0$ and $\beta = \omega_u \longrightarrow r = \int_0^t \frac{1}{\omega_u m} \sin(\omega_u (t-\tau)) R(\tau) dt + r_0 \cos(\omega_u t) + \frac{\dot{r}_0}{\beta} \sin(\omega_u t)$

Vibration of Continuous Beam

Equation of motion



$$\Sigma F_v = 0 \longrightarrow \frac{\partial F}{\partial z} dz + w dz = \rho A dz \ddot{v} \longrightarrow \frac{\partial F}{\partial z} + w = \rho A \ddot{v}$$

$$F = \frac{\partial M}{\partial z} \text{ and } v = \iint -\frac{M}{EI} dz^2 \longrightarrow \frac{\partial^2 M}{\partial z^2} + w = -EI \frac{\partial^4 v}{\partial z^4} + w = \rho A \ddot{v}$$

$\longrightarrow \quad EI\frac{\partial^4 v}{\partial z^4} + \rho A\frac{\partial^2 v}{\partial t^2} = w$

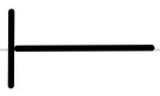
General solution for free vibration

$v(z,t) = V(z)\cos(\omega t)$ and $EI\frac{\partial^4 v}{\partial z^4} + \rho A\frac{\partial^2 v}{\partial t^2} = 0$

$\longrightarrow \quad V = C_1\sinh(\lambda z) + C_2\cosh(\lambda z) + C_3\sin(\lambda z) + C_4\cos(\lambda z)$

$\longrightarrow \quad \frac{\partial^4 v}{\partial z^4} = \frac{\partial^4 V}{\partial z^4}\cos(\omega t) = \lambda^4 V\cos(\omega t)$ and $\frac{\partial^2 v}{\partial t^2} = -\omega^2 V\cos(\omega t) \longrightarrow \lambda^4 = \frac{\rho A\omega^2}{EI}$

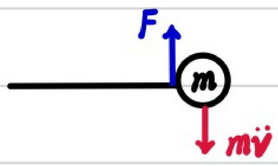
Boundary conditions



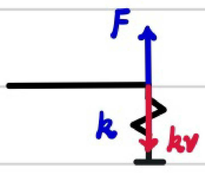
fixed end $\quad V = 0, \quad \frac{dV}{dz} = 0$

or simply supported end $\quad V = 0, \quad \frac{d^2V}{dz^2} = 0$

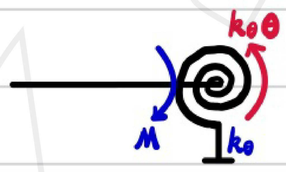
free end $\quad \frac{d^2V}{dz^2} = 0, \quad \frac{d^3V}{dz^3} = 0 \quad * F = 0$



concentrated mass end $\quad \frac{d^2V}{dz^2} = 0, \quad EI\frac{d^3V}{dz^3} = m\ddot{v} \quad * F = -m\ddot{v}$



translational spring end $\quad \frac{d^2V}{dz^2} = 0, \quad EI\frac{d^3V}{dz^3} = kv$



rotational spring end $\quad EI\frac{d^2V}{dz^2} = k_\theta\theta, \quad \frac{d^3V}{dz^3} = 0$

Variational Mechanics

Calculus variations

for functional, $S(f) = \int_{t_1}^{t_2} F(f, \dot{f}, t)\,dt$ and $f(t) = u(t) + \varepsilon\eta(t)$

$\longrightarrow \quad \left.\frac{dS(u(t)+\varepsilon\eta(t))}{d\varepsilon}\right|_{\varepsilon=0} = 0 \longrightarrow \frac{\partial F}{\partial u} - \frac{\partial}{\partial u}\frac{\partial F}{\partial \dot{u}} = 0$

Hamilton principle

$\vec{F}_i - \frac{d\vec{p}_i}{dt} = \vec{0} \quad * \vec{p}_i = m_i\dot{\vec{u}}_i \quad **$ displacement vector $\vec{u}_i(t)$

$\longrightarrow$ assume all force conservative $\delta\int_{t_1}^{t_2} L\,dt = 0 \quad *$ Lagrangian $L = T - U = K.E. - P.E.$

$\longrightarrow \int_{t_1}^{t_2}(\delta W^{ext} + \delta W^{nc} + \delta T - \delta U)\,dt = 0 \quad *$ non-conservative internal force (e.g. damping) δW^{nc}

Lagrange's equation of motion

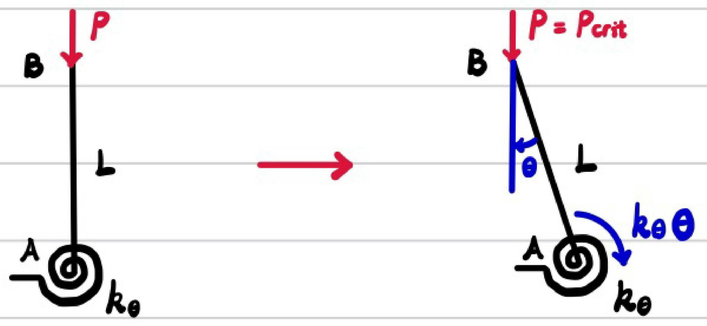
$\frac{\partial F}{\partial u} - \frac{\partial}{\partial u}\frac{\partial F}{\partial \dot{u}} = 0 \longrightarrow$ assume all force conservative $\frac{\partial L}{\partial r_i} - \frac{\partial}{\partial r_i}\frac{\partial L}{\partial \dot{r}_i} = 0$

$\longrightarrow \frac{\partial}{\partial r}\frac{\partial T}{\partial \dot{r}_i} - \frac{\partial T}{\partial r_i} + \frac{\partial U}{\partial r_i} + \frac{\partial D}{\partial \dot{r}_i} = R_i$

Buckling

Equilibrium Method

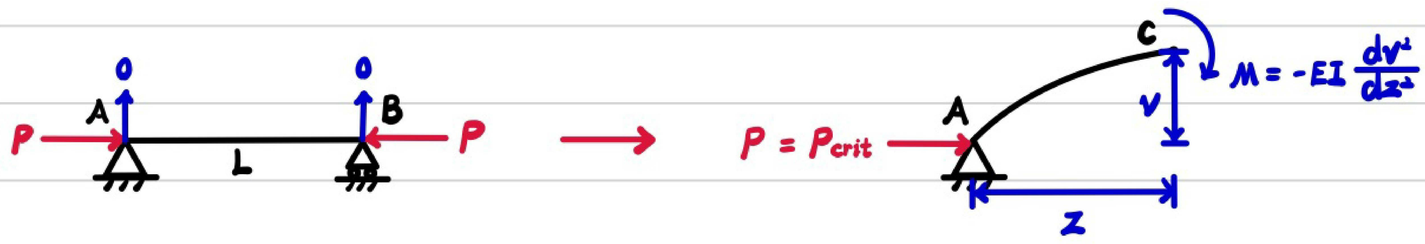
Rigid struct with a torsion spring



$$\Sigma M_A = P_{crit} L\theta - k_\theta \theta = 0$$

$$\longrightarrow P_{crit} = \frac{k_\theta}{L}$$

Pin - pin end struct



$$\Sigma M_c = Pv - M = 0 \longrightarrow Pv + EI\frac{d^2v}{dz^2} = \frac{P}{EI}v + \frac{d^2v}{dz^2} = 0 \longrightarrow \mu^2 v + \frac{d^2v}{dz^2} = 0$$

$$\longrightarrow v = v_{01}\sin(\mu z) + v_{02}\cos(\mu z)$$

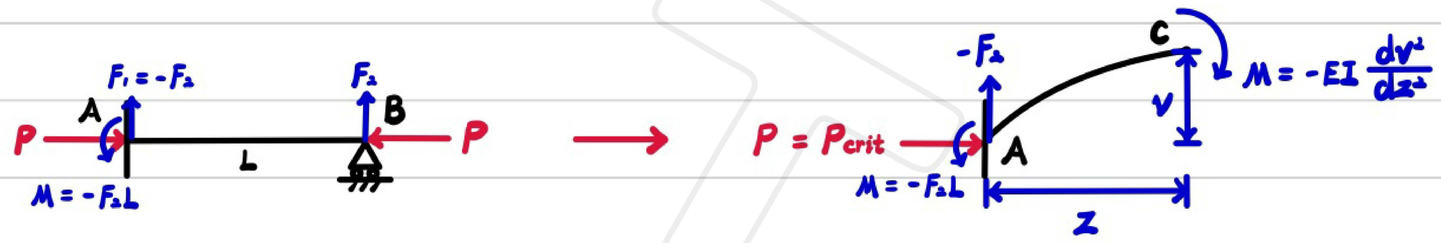
for $v = 0$ at $z = 0 \longrightarrow v_{02} = 0 \longrightarrow v = v_{01}\sin(\mu z)$

for $v = 0$ at $z = L \longrightarrow v_{01}\sin(\mu L) = 0 \longrightarrow \sin(\mu L) = 0 \longrightarrow \mu L = n\pi$, where $n = 0, 1, 2, 3 \ldots$

$$\longrightarrow P_{crit} = \frac{(n\pi)^2 EI}{L^2} = \frac{\pi^2 EI}{L^2}$$

* $n = 1$, $n = 2$, $n = 3$, ...

Fixed - pin end struct



$$\Sigma M_c = Pv + (-F_2L) - (-F_2 z) - \left(-EI\frac{d^2v}{dz^2}\right) = 0 \longrightarrow \mu^2 v + \frac{d^2v}{dz^2} = \frac{F_2}{EI}(L - z)$$

$$\longrightarrow v = v_{01}\sin(\mu z) + v_{02}\cos(\mu z) + Cz \longrightarrow \mu^2 Cz = \frac{F_2}{EI}(L - z) \longrightarrow Cz = \frac{F_2}{\mu^2 EI}(L - z) = \frac{F_2}{P}(L - z)$$

$$\longrightarrow v = v_{01}\sin(\mu z) + v_{02}\cos(\mu z) + \frac{F_2}{P}(L - z)$$

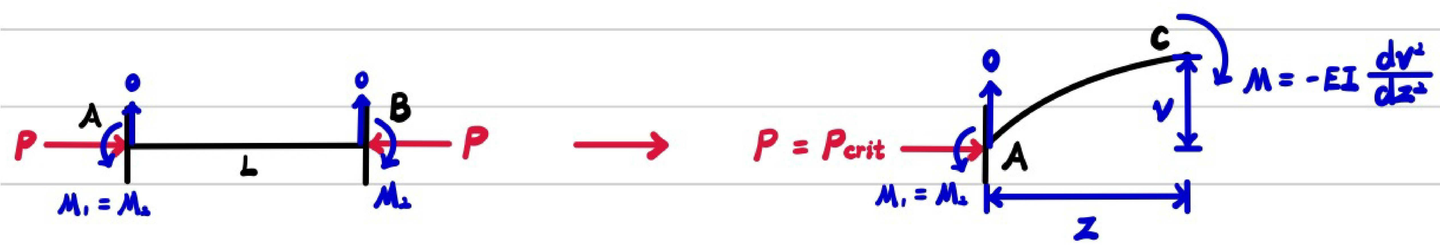
for $v = 0$ at $z = 0 \longrightarrow v_{02} = -\frac{F_2 L}{P} \longrightarrow v = v_{01}\sin(\mu z) - \frac{F_2 L}{P}\cos(\mu z) + \frac{F_2}{P}(L - z)$

for $\frac{dv}{dz} = 0$ at $z = 0 \longrightarrow v_{01} = \frac{F_2}{\mu P} \longrightarrow v = \frac{F_2}{\mu P}\sin(\mu z) - \frac{F_2 L}{P}\cos(\mu z) + \frac{F_2}{P}(L - z)$

for $v = 0$ at $z = L$ $\longrightarrow$ $\frac{F_2}{\mu P}\sin(\mu L) - \frac{F_2 L}{P}\cos(\mu L) = 0$ $\longrightarrow$ $\sin(\mu L) - \mu L\cos(\mu L) = 0$ $\longrightarrow$ $\tan(\mu L) = \mu L$

$\longrightarrow$ $\mu L = 4.49\,\text{rad}, \ldots$ $\longrightarrow$ $P_{crit} = \frac{2.05\pi^2 EI}{L^2}$

Fixed-fixed end struct



$\Sigma M_c = Pv + M_2 - (-EI\frac{d^2v}{dz^2}) = 0$ $\longrightarrow$ $\mu^2 v + \frac{d^2v}{dz^2} = -\frac{M_2}{EI}$

$\longrightarrow$ $v = v_{01}\sin(\mu z) + v_{02}\cos(\mu z) - \frac{M_2}{P}$

for $v = 0$ at $z = 0$ $\longrightarrow$ $v_{02} = \frac{M_2}{P}$ $\longrightarrow$ $v = v_{01}\sin(\mu z) + \frac{M_2}{P}\cos(\mu z) - \frac{M_2}{P}$

for $\frac{dv}{dz} = 0$ at $z = 0$ $\longrightarrow$ $v_{01} = 0$ $\longrightarrow$ $v = \frac{M_2}{P}\cos(\mu z) - \frac{M_2}{P}$

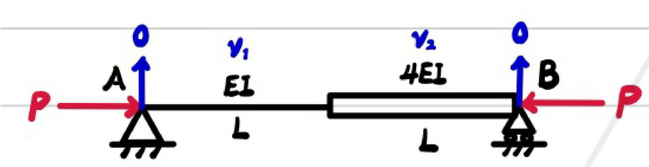
for $v = 0$ at $z = L$ $\longrightarrow$ $\frac{M_2}{P}\cos(\mu z) - \frac{M_2}{P} = 0$ $\longrightarrow$ $\cos(\mu L) = 1$ $\longrightarrow$ $\mu L = 2n\pi$, where $n = 0, 1, 2, 3 \ldots$

$\longrightarrow$ $P_{crit} = \frac{(2n\pi)^2 EI}{L^2} = \frac{4\pi^2 EI}{L^2}$

Fixed-free end struct

$\longrightarrow$ $P_{crit} = \frac{(n\pi)^2 EI}{(2L)^2} = \frac{\pi^2 EI}{4L^2}$

Non-uniform struct

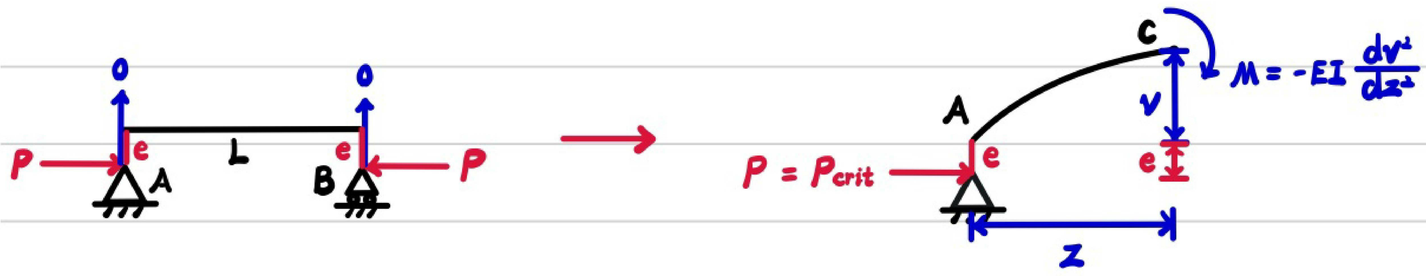


$\longrightarrow$ $v_1(0) = v_2(2L) = 0$ $\longrightarrow$ $v_1(L) = v_2(L)$ and $\frac{dv_1}{dz}(L) = \frac{dv_2}{dz}(L)$

Rigid struct $EI \longrightarrow \infty$

Imperfect pin-ended struct

Eccentricity



$$\Sigma M_c = P(v+e) - M = 0 \longrightarrow \mu^2 v + \frac{d^2v}{dz^2} = -\mu^2 e \longrightarrow v = v_{01}\sin(\mu z) + v_{02}\cos(\mu z) - e$$

for $v = 0$ at $z = 0 \longrightarrow v_{02} = e \longrightarrow v = v_{01}\sin(\mu z) + e[\cos(\mu z) - 1]$

for $v = 0$ at $z = L \longrightarrow v_{01}\sin(\mu L) + e[\cos(\mu L) - 1] = 0 \longrightarrow \sin(\mu L) = \cos(\mu L) - 1 = 0 \longrightarrow P_{crit} = \frac{\pi^2 EI}{L^2}$

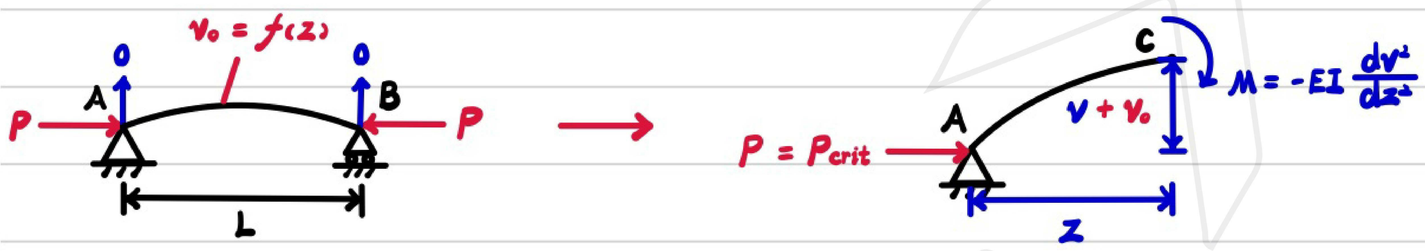
$$\longrightarrow v_{01} = e\frac{\cos(\mu L) - 1}{\sin(\mu L)} \longrightarrow v = e\left[\frac{\cos(\mu L) - 1}{\sin(\mu L)}\sin(\mu z) + \cos(\mu z) - 1\right]$$

at $z = \frac{L}{2}$, $v_{max} \longrightarrow v_{max} = e\left[\frac{1}{\cos(\frac{1}{2}\mu L)} - 1\right] = e\left[\frac{1}{\cos(\frac{\pi}{2}(\frac{P}{P_{crit}})^{\frac{1}{2}})} - 1\right]$

$$* \ \mu^2 = \frac{P}{EI} = \frac{P_{crit}}{EI}\frac{P}{P_{crit}} = \frac{\frac{\pi^2 EI}{L^2}}{EI}\frac{P}{P_{crit}} = \frac{\pi^2}{L^2}\frac{P}{P_{crit}}$$

for small $e \longrightarrow \frac{P}{P_{crit}} = \left[\frac{2}{\pi}\cos^{-1}\left(\frac{e}{v_{max}}\right)\right]^2$

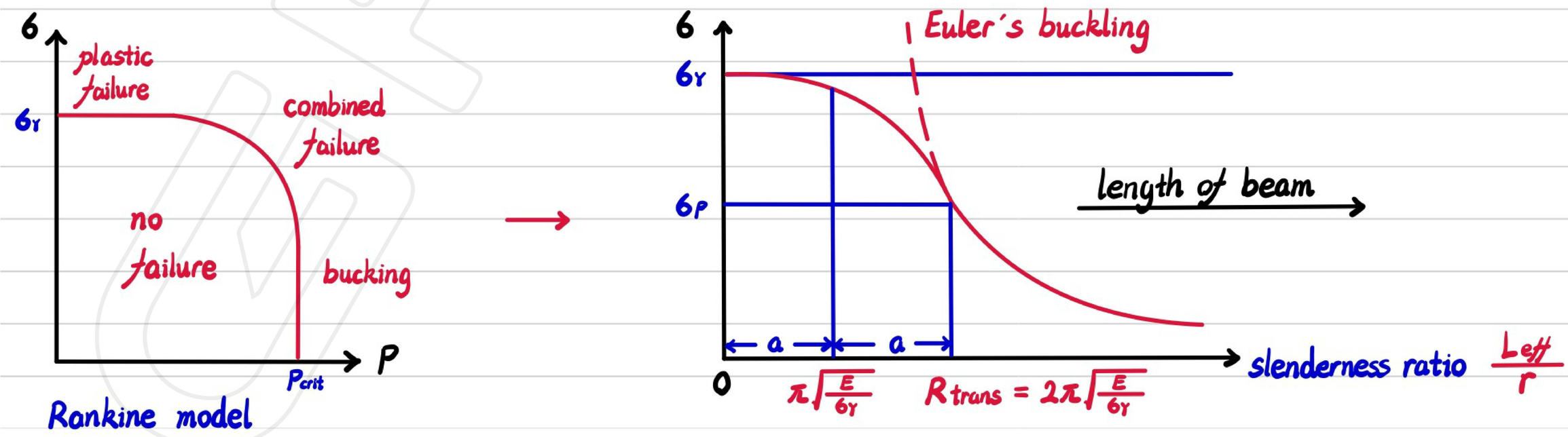
Misaligned beam



Compressive Failure Analysis

Critical stress

$$\sigma_{crit} = \frac{P_{crit}}{A} = \frac{\pi^2 EI}{L_{eff}^2}\frac{r^2}{I} = \frac{\pi^2 E}{(L_{eff}/r)^2}$$

* radius of gyration $r = \sqrt{\frac{I}{A}}$

Combination of failure mechanics