Signal and System

Signals

Continuous-Time (CT) $x(t)$, $t \in \mathbb{R}$ (实数集)

Discrete-Time (DT) $x[n]$, $n \in \mathbb{Z}$ (所有整数)

Periodic Signals $x(t) = x(t+T)$ or $x[n] = x[n+N]$ * otherwise aperiodic

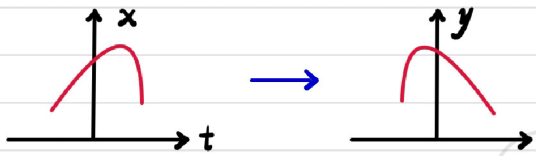
Even Signals $x(t) = x(-t)$ or $x[n] = x[-n]$

Odd Signals $x(t) = -x(-t)$ or $x[n] = -x[-n]$

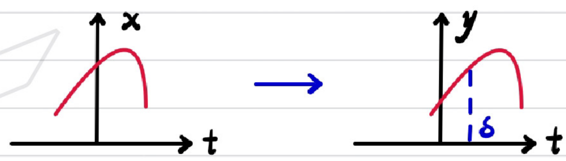
* every signal can be decomposed into sum of even and odd signals

$$x(t) = x_e(t) + x_o(t) = \frac{1}{2}[x(t) + x(-t)] + \frac{1}{2}[x(t) - x(-t)]$$

Time Reversal $y(t) = x(-t)$



Time Shift $y(t) = x(t-\delta)$



Complex Exponential $x(t) = Ce^{at}$ or $x[n] = Dz^n$

purely imaginary exponentials $x(t) = e^{j(\omega t + \phi)} = \cos(\omega t + \phi) + j\sin(\omega t + \phi)$

* signal is periodic $\longrightarrow e^{j\omega t} = e^{j\omega(t+T)} = e^{j\omega t} e^{j\omega T}$

$\longrightarrow e^{j\omega T} = 1 \longrightarrow T_0 = \frac{2\pi}{|\omega|}$

fundemental period

fundemental frequency

general complex exponentials

$$x(t) = Ce^{at} = (|c|e^{j\phi})e^{(\sigma + j\omega)t} = |c|e^{\sigma t} \cdot e^{j(\omega t + \phi)} = \overbrace{|c|e^{\sigma t}\underbrace{\cos(\omega t + \phi)}_{\text{periodic}}}^{\text{real}} + \overbrace{j|c|e^{\sigma t}\underbrace{\sin(\omega t + \phi)}_{\text{periodic}}}^{\text{imaginary}}$$

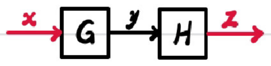
Systems

System device or process that receives input signals and produces output signals

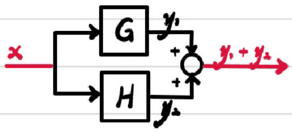
$$u(t) \xrightarrow{H} y(t) \quad \text{or} \quad u[n] \xrightarrow{H} y[n]$$

Block diagrams

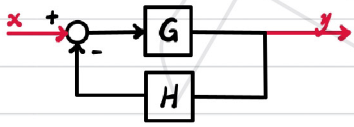
series connection



parallel connection



feedback connection



Memoryless for every t, output $y(t)$ is a fuction of the input $u(t)$ only

Causal 因果 the value of the output depends only on current and past input

Invertible $u(t) \xrightarrow{H} y(t) \xrightarrow{G} w(t) = u(t) \longrightarrow G = H^{-1}$

Time-Invarient if $x(t) \xrightarrow{H} y(t)$ then $x(t-\delta) \xrightarrow{H} y(t-\delta)$

Stable every bounded input produce bounded output $\longrightarrow$ if $|x(t)| < K$ then $|y(t)| < B$

Linear System

scaling $a u_1(t) \xrightarrow{H} a y_1(t)$

+ additivity $[u_1(t) + u_2(t)] \xrightarrow{H} [y_1(t) + y_2(t)]$

$\longrightarrow$ superposition $[a u_1(t) + b u_2(t)] \xrightarrow{H} [a y_1(t) + b y_2(t)]$

LTI (Linear Time-Invariant) Systems

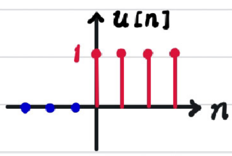
CT LTI Systems $H: \sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k u(t)}{dt^k}$ * better represent system by $\int$

DT LTI Systems $H: \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k u[n-k]$ * specify initial condition if $N \neq 0$

Unit Step

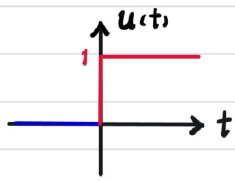
DT

$$u[n] = \sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



CT

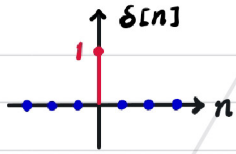
$$u(t) = \lim_{\varepsilon \to 0} u_\varepsilon(t) = \int_{-\infty}^{t} \delta_\varepsilon(\tau)\, d\tau = \begin{cases} 0 & t < 0 \\ \frac{t}{\varepsilon} & 0 \leq t < \varepsilon \\ 1 & t \geq \varepsilon \end{cases}$$



Unit Impulse

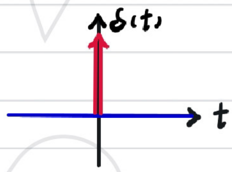
DT

$$\delta[n] = u[n] - u[n-1] = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases}$$



CT

$$\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t) = \frac{d}{dt} u_\varepsilon(t) = \begin{cases} 0 & t<0 \\ \frac{1}{\varepsilon} & 0 \le t < \varepsilon \\ 0 & t \ge \varepsilon \end{cases}$$



* value selector $\int_{-\infty}^{\infty} f(\tau)\delta(\tau - T)\, d\tau = f(T)$

Convolution

DT Convolution $x[n] * y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, y[n-k] = \sum_{k=-\infty}^{+\infty} x[n-k]\, y[k]$

CT Convolution $x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, y(\tau)\, d\tau$

* https://www.youtube.com/watch?v=KuXjwB4LzSA

Properties

commutative 可交换的 $x[n] * y[n] = y[n] * x[n]$ * same for CT

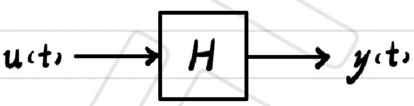
distributive $x[n] * (y_1[n] + y_2[n]) = x[n] * y_1[n] + x[n] * y_2[n]$ * same for CT

associative $x[n] * (y_1[n] * y_2[n]) = (x[n] * y_1[n]) * y_2[n]$ * same for CT

Fourier Series

System Function

CT



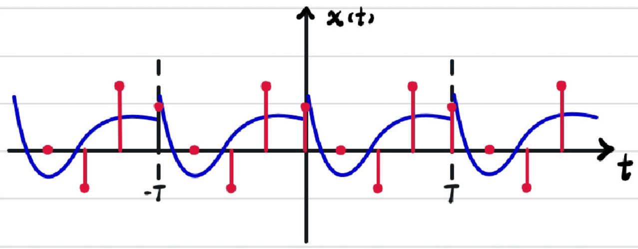
$$y(t) = u(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

$$\xrightarrow{u(t) = e^{st}} e^{st} \int_{-\infty}^{\infty} e^{-s\tau} h(\tau)\, d\tau \longrightarrow H(s) \triangleq \int_{-\infty}^{\infty} e^{-s\tau} h(\tau)\, d\tau$$

DT

$$y[n] = u[n] * h[n] = \sum_{k=-\infty}^{+\infty} u[n-k]\, h[k] \xrightarrow{u[n] = z^n} z^n \sum_{k=-\infty}^{+\infty} z^{-k} h[k] \longrightarrow H(z) \triangleq \sum_{k=-\infty}^{+\infty} z^{-k} h[k]$$

Basis Function



a family of periodic signal $p_k(t)$ / $p_k[n]$

- all periodic with period (multiple) T / N
- mutually orthogonal over any T / N long

Orthogonal Signals

CT

$$x(t) = \dots + C_{-2}p_{-2}(t) + C_{-1}p_{-1}(t) + C_0 p_0(t) + C_1 p_1(t) + C_2 p_2(t) + \dots$$ *coefficients $\{C_k\}$

$\longrightarrow p_k(t) = e^{jk\omega_0 t}$ where $\omega_0 \triangleq \frac{2\pi}{T}$ * $s = jk\omega_0$

$$\longrightarrow \langle p_k(t), p_l(t) \rangle = \langle e^{jk\omega_0 t}, e^{jl\omega_0 t} \rangle = \int_T e^{jk\omega_0 t}\overline{e^{jl\omega_0 t}}\,dt = \int_T e^{jk\omega_0 t} e^{-jl\omega_0 t}\,dt$$

$$= \int_T e^{j(k-l)\omega_0 t}\,dt = \int_T \cos[(k-l)\omega_0 t] + j\sin[(k-l)]\omega_0 t\,dt$$

$$= \begin{cases} 0 & \text{if } k \neq l \\ T & \text{if } k = l \end{cases}$$

DT

$$x[n] = \sum_{k=\langle N \rangle} C_k p_k[n]$$ *coefficients $\{C_k\}$

$\longrightarrow p_k[n] = e^{jk\omega_0 n}$ where $\omega_0 \triangleq \frac{2\pi}{N}$ * $z = e^{jk\omega_0}$

$$\longrightarrow \langle p_k[n], p_l[n] \rangle = \langle e^{jk\omega_0 n}, e^{jl\omega_0 n} \rangle = \sum_{n=\langle N \rangle} e^{jk\omega_0 n}\overline{e^{jl\omega_0 n}} = \sum_{n=\langle N \rangle} e^{jk\omega_0 n} e^{-jl\omega_0 n}$$

$$= \sum_{n=\langle N \rangle} e^{j(k-l)\omega_0 n} = \sum_{n=\langle N \rangle} \cos[(k-l)\omega_0 n] + j\sin[(k-l)\omega_0 n]$$

$$= \begin{cases} 0 & \text{if } k \neq l \\ N & \text{if } k = l \end{cases}$$

Fourier Series

CT

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k \phi_k(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}$$ * $x(t) \xleftrightarrow{FS} c_k$

$$\longrightarrow \int_T x(t) e^{-jl\omega_0 t} dt = \int_T \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t} e^{-jl\omega_0 t} = c_k \sum_{k=-\infty}^{+\infty} \langle \phi_k, \phi_l \rangle = T c_k$$

$$\longrightarrow c_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

CT Sin / Cosine Fourier Series

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t} = c_0 + \sum_{k=1}^{+\infty} [c_k e^{jk\omega_0 t} + c_{-k} e^{j(-k)\omega_0 t}]$$ * $c_{-k} = \overline{c_k} = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$

$$= c_0 + \sum_{k=1}^{+\infty} \{ c_k [\cos(k\omega_0 t) + j\sin(k\omega_0 t)] + \overline{c_k} [\cos(k\omega_0 t) - j\sin(k\omega_0 t)] \}$$

$$= c_0 + \sum_{k=1}^{+\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{+\infty} b_k \sin(k\omega_0 t)$$ * general formulae

a_k: $c_k + \overline{c_k} = Re(c_k)$ $\quad$ b_k: $j(c_k - \overline{c_k}) = Im(c_k)$

where $c_0 = \frac{1}{T}\int_0^T x(t) dt$

$$a_k = 2 \cdot Re(c_k) = \frac{1}{T}\int_T x(t) \cos(k\omega_0 t) dt$$

$$b_k = 2 \cdot Im(c_k) = \frac{1}{T}\int_T x(t) \sin(k\omega_0 t) dt$$

CT Dirichilet Conditions

to gurantee the Fourier Series will converge, $x(t)$ must

- be absolute integrable * $\int_{-\infty}^{\infty} |x(t)| dt < \infty$
- have finite number minima & maxima
- have finite number of finite discontinuous

DT

$$x[n] = \sum_{k=\langle N\rangle} c_k \phi_k = \sum_{k=\langle N\rangle} c_k e^{jk\omega_0 n}$$ * $x[n] \xleftrightarrow{FS} c_k$

$$\longrightarrow \sum_{n=\langle N\rangle} x[n] e^{-jl\omega_0 n} = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} c_k e^{jk\omega_0 n} e^{-jl\omega_0 n} = c_k \sum_{k=\langle N\rangle} \langle \phi_k, \phi_l \rangle = N c_k$$

$$\longrightarrow c_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-jk\omega_0 n}$$

LTI Systems

$$y(t) = x(t) * h(t) = \sum_{k=-\infty}^{+\infty} \{a_k H(jk\omega_0)\} e^{jk\omega_0 t}$$ or $$y[n] = x[n] * h[n] = \sum_{k=\langle N\rangle} \{a_k H(e^{jk\omega_0})\} e^{jk\omega_0 n}$$

$\longrightarrow$ concentrated around frequencies $k\omega_0$ $\longrightarrow$ act as filter

Time Shifts / Delays

CT

if $x(t) \xleftrightarrow{FS} a_k$, $y(t) = x(t-\delta) \xleftrightarrow{FS} b_k$

$\longrightarrow b_k = \frac{1}{T}\int_T y(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_T x(t-\delta) e^{-jk\omega_0 t} dt$

$\xrightarrow[d\xi = dt]{\xi = t-\delta} \frac{1}{T}\int_T x(\xi) e^{-jk\omega_0(\xi+\delta)} d\xi = \frac{1}{T} e^{-jk\omega_0\delta} \int_T x(\xi) e^{-jk\omega_0\xi} d\xi = e^{-jk\omega_0\delta} a_k$

DT

if $x[n] \xleftrightarrow{FS} a_k$, $y[n] = x[n-k] \xleftrightarrow{FS} b_k = e^{-jk\omega_0 n} a_k$

Periodic Convolution

CT

if $x(t) \xleftrightarrow{FS} a_k$, $y(t) \xleftrightarrow{FS} b_k$, $z(t) = \int_T x(\tau) y(t-\tau) d\tau \xleftrightarrow{FS} c_k$

$\longrightarrow z(t) = \int_T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0\tau} \sum_{l=-\infty}^{\infty} b_k e^{jl\omega_0(t-\tau)} dt = \sum_{l=-\infty}^{\infty} a_k b_k e^{jl\omega_0 t} \sum_{k=-\infty}^{\infty} \int_T e^{j(k-l)\omega_0\tau} d\tau = \sum_{k=-\infty}^{\infty} T a_k b_k e^{jl\omega_0 t}$

$\longrightarrow c_k = T a_k b_k$

DT

if $x[n] \xleftrightarrow{FS} a_k$, $y[n] \xleftrightarrow{FS} b_k$, $z[n] = \sum_{k=\langle N\rangle} x[k] y[n-k] \xleftrightarrow{FS} c_k = N a_k b_k$

Parseval's Theorem

Purpose

use to identify which frequencies are important $\longrightarrow$ filter design

CT $\quad P = \frac{1}{T}\int_T |x(t)|^2 dt$ where $|x(t)|^2 = x(t)\overline{x(t)}$ and $|c_k|^2 = c_k \overline{c_k}$

$= \frac{1}{T}\int_T \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right)\left(\sum_{l=-\infty}^{\infty} \overline{c_l} e^{-jl\omega_0 t}\right) dt = \sum_{k=-\infty}^{\infty} c_k \overline{c_k} = \sum_{k=-\infty}^{\infty} |c_k|^2$

DT $\quad P = \frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2$ where $|x[n]|^2 = x[n]\overline{x[n]}$ and $|c_k|^2 = c_k \overline{c_k}$

$= \frac{1}{N}\sum_{n=\langle N\rangle} \left(\sum_{k=\langle N\rangle} c_k e^{jk\omega_0 n}\right)\left(\sum_{l=\langle N\rangle} \overline{c_l} e^{-jl\omega_0 n}\right) = \sum_{k=\langle N\rangle} c_k \overline{c_k} = \sum_{k=\langle N\rangle} |c_k|^2$

Fourier Transform

Fourier Transform

CT

let $\omega = k\omega_0 = \frac{2k\pi}{T} \longrightarrow x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega t} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{T}\int_T x(t) e^{-j\omega t} dt\right) e^{j\omega t}$

$\xrightarrow{T\to\infty} \Delta\omega = \frac{2\pi}{T} \longrightarrow \frac{\Delta\omega}{2\pi} = \frac{1}{T} \to 0 \longrightarrow x(t) = \int_{-\infty}^{\infty} \left(\frac{\Delta\omega}{2\pi}\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt\right) e^{j\omega t}$

$\longrightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}(j\omega) e^{j\omega t} d\omega$ where $\hat{X}(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

CT Dirichilet Conditions

to gurantee the Fourier Transform will converge, $x(t)$ must

- be absolute integrable $\quad * \int_{-\infty}^{\infty} |x(t)| dt < \infty$
- have finite number minima & maxima
- have finite number of finite discontinuous
- above must apply over finite interval

DT

let $\omega = k\omega_0 = \frac{2k\pi}{N} \longrightarrow x[n] = \sum_{k=\langle N\rangle} c_k e^{j\omega n} = \sum_{k=\langle N\rangle} \left(\frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-j\omega n}\right) e^{j\omega n}$

$\xrightarrow{N\to\infty} \Delta\omega = \frac{2\pi}{N} \longrightarrow \frac{\Delta\omega}{2\pi} = \frac{1}{N} \to 0 \longrightarrow x[n] = \int_{2\pi} \left(\frac{\Delta\omega}{2\pi}\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}\right) e^{j\omega n}$

$\longrightarrow x[n] = \frac{1}{2\pi}\int_{2\pi} \hat{X}(e^{j\omega}) e^{j\omega n} d\omega$ where $\hat{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

Linearity

CT $\quad z(t) = ax(t) + by(t) \xleftrightarrow{F} \hat{Z}(j\omega) = a\hat{X}(j\omega) + b\hat{Y}(j\omega)$

DT $\quad z[n] = ax[n] + by[n] \xleftrightarrow{F} \hat{Z}(e^{j\omega}) = a\hat{X}(e^{j\omega}) + b\hat{Y}(e^{j\omega})$

Time Shifts / Delays

CT

if $x(t) \xleftrightarrow{F} X(j\omega)$, $y(t) = x(t-\delta) \xleftrightarrow{F} \hat{Y}(j\omega)$

$\longrightarrow \hat{Y}(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t-\delta) e^{-j\omega t} dt$

$\xrightarrow[d\xi = dt]{\xi = t-\delta} \hat{Y}(j\omega) = \int_{-\infty}^{\infty} x(\xi) e^{-j\omega(\xi+\delta)} d\xi = e^{-j\omega\delta} \int_{-\infty}^{\infty} x(\xi) e^{-j\omega\xi} d\xi = e^{-j\omega\delta} \hat{X}(j\omega)$

DT

if $x[n] \xleftrightarrow{F} X(e^{j\omega}),\ y[n] = x[n-k] \xleftrightarrow{F} e^{-j\omega k} \hat{X}(e^{j\omega})$

Parseval's Theorem

CT $E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{X}(j\omega)} e^{-j\omega t} d\omega \right) dt$

$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{X}(j\omega)} \left(\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{X}(j\omega)} X(j\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{X}(j\omega)|^2 d\omega$

DT $E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{1}{2\pi} \int_{2\pi} \overline{\hat{X}(e^{j\omega})} e^{-j\omega n} d\omega \right)$

$= \frac{1}{2\pi} \int_{2\pi} \overline{\hat{X}(e^{j\omega})} \left(\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right) d\omega = \frac{1}{2\pi} \int_{2\pi} \overline{X(e^{j\omega})} X(e^{j\omega}) d\omega = \frac{1}{2\pi} \int_{2\pi} |\hat{X}(e^{j\omega})|^2 d\omega$

Convolution

CT

if $x(t) \xleftrightarrow{F} \hat{X}(j\omega),\ h(t) \xleftrightarrow{F} \hat{H}(j\omega),\ y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \xleftrightarrow{F} \hat{Y}(j\omega)$

$\longrightarrow Y(j\omega) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \right) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h(t-\tau) e^{-j\omega t} dt \right) d\tau$

$= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} H(j\omega) d\tau = X(j\omega) \cdot H(j\omega)$

DT

if $x[n] \xleftrightarrow{F} \hat{X}(e^{j\omega}),\ h[n] \xleftrightarrow{F} \hat{H}(e^{j\omega}),\ y[n] = x[n] * h[n] = \sum_{n=-\infty}^{\infty} x[k] h[n-k] \xleftrightarrow{F} \hat{Y}(e^{j\omega}) = \hat{X}(e^{j\omega}) \cdot \hat{H}(e^{j\omega})$

DT Periodicity $X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$

Differentiation

CT $\frac{d}{dt} x(t) \xleftrightarrow{F} \int_{-\infty}^{\infty} \frac{d}{dt} x(t) e^{-j\omega t} dt = [x(t) e^{-j\omega t}]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -j\omega x(t) e^{-j\omega t} dt = j\omega \hat{X}(j\omega)$

$\frac{d}{d\omega} X(j\omega) \xleftrightarrow{F} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d}{d\omega} \hat{X}(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} ([\hat{X}(j\omega) e^{j\omega t}]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} jt \hat{X}(j\omega) e^{j\omega t} d\omega = -jt\, x(t)$

Constant Coefficient Linear Differential Equation

DT $\quad H: \sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k u(t)}{dt^k} \longrightarrow \hat{H}(j\omega) = \frac{\hat{Y}(j\omega)}{\hat{X}(j\omega)} \longrightarrow$ impulse response $h(t) = \dots u(t)$

CT $\quad H: \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k u[n-k] \longrightarrow \hat{H}(e^{j\omega}) = \frac{\hat{Y}(e^{j\omega})}{\hat{X}(e^{j\omega})} \longrightarrow$ impulse response $h[n] = \dots u[n]$

Laplace Transform

Unilateral Laplace Transform

Definition

H is an CT LTI system that is causal $\longrightarrow$ impulse response $h(t) = 0$ for all $t < 0$

for any complex number s and apply an input $u(t) = e^{st}$

$\longrightarrow y(t) = u(t) * h(t) = \int_{-\infty}^{\infty} u(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} u(t-\tau) h(\tau) d\tau = e^{st} \int_0^{\infty} e^{-s\tau} h(\tau) d\tau = H(s) e^{st}$

Relation to CT Fourier Transform

$s = \sigma + j\omega \longrightarrow H(s) = \int_0^{\infty} h(t) e^{-(\sigma + j\omega)t} dt = \int_0^{\infty} [h(t) e^{-\sigma t}] e^{-j\omega t} dt = \mathcal{F}\{h(t) e^{-\sigma t}\}$

Region of Convergence (ROC)

the set of value s for which Laplace transform is well-defined $\longrightarrow$ integrable

Inverse Laplace Transform $\quad x(t) = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} X(s) e^{-st} ds \qquad$ * rarely used

System Poles and Zeros

Rational Transfer Function $\quad H(s) = \frac{\prod_{i=1}^{N_z} (s - z_i)}{\prod_{i=1}^{N_p} (s - p_i)}$ — z_i: zeros, p_i: poles $\qquad$ * #poles $\geq$ #zeros

Stability of LTI System $\quad$ ROC starting from (not including) the right-most pole

poles in LHP $\longrightarrow$ stable

poles on the imaginary axis —
- distinct $\longrightarrow$ marginally stable
- repeated $\longrightarrow$ unstable

poles in RHP $\longrightarrow$ unstable

Property

Linearity $z(t) = ax(t) + by(t) \xrightarrow{L} \hat{Z}(s) = a\hat{X}(s) + b\hat{Y}(s)$

Convolution $x(t) * h(t) \xrightarrow{L} \hat{X}(s) \cdot \hat{H}(s)$

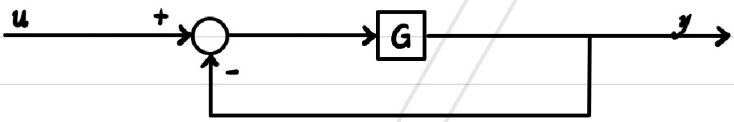
Differentiation $\frac{d}{dt}x(t) \xrightarrow{L} s\hat{X}(s) - x(0)$

Integration $\int_{-\infty}^{t} x(\tau)\, d\tau \xrightarrow{L} \frac{1}{s}\hat{X}(s)$

Block Diagrams

Black's Formula Transfer Function $= \dfrac{\text{Direct Path}}{1 - \text{Loop Path}}$

Feedback System



Limit Theorems

Final Value Theorem (FVT)

if $h(t) \overset{L}{\longleftrightarrow} H(s) \longrightarrow \lim_{t\to\infty} h(t) = \lim_{s\to 0} sH(s)$

Initial Value Theorem (IVT)

if $h(t) \overset{L}{\longleftrightarrow} H(s) \longrightarrow \lim_{t\to 0^+} h(t) = \lim_{s\to\infty} sH(s)$

z-Transform

z-Transform

Definition

H is an DT LTI system that is causal $\longrightarrow$ impulse response $h[n] = 0$ for all $n < 0$

for any complex number z and apply an input $x[n] = z^n$

$$\longrightarrow y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = z^n \sum_{k=0}^{\infty} z^{-k} h[k] = H(z) z^n$$

Relation to DT Fourier Transform

$z = re^{j\omega} \longrightarrow H(z) = \sum_{n=0}^{\infty} r^{-n} e^{-j\omega n} h[n] = \sum_{n=0}^{\infty} [h[n] r^{-n}] e^{-j\omega n} = \mathcal{F}\{h[n] r^{-n}\}$

System Poles and Zeros

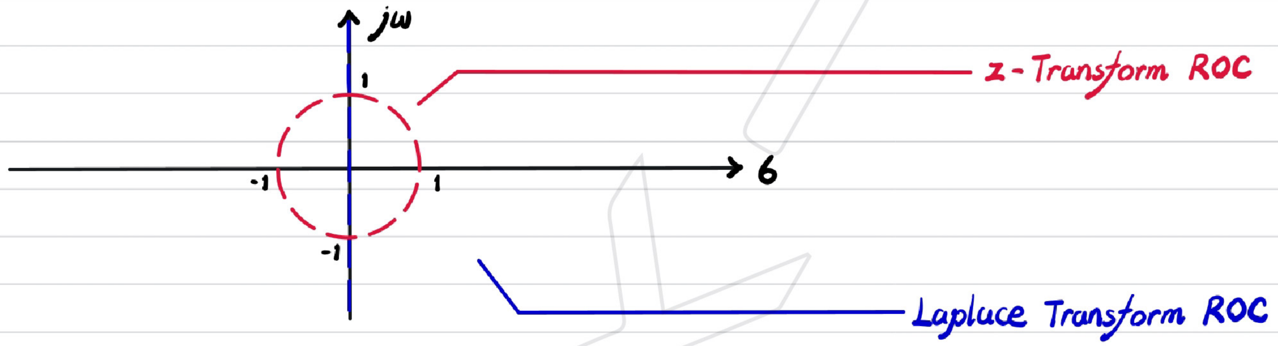
Stability for LTI System

ROC starting from (not including) the outer-most pole

poles in interior of the unit circle $\longrightarrow$ stable

poles on the unit circle — distinct $\longrightarrow$ marginally stable; repeated $\longrightarrow$ unstable

poles in annulus of the unit circle $\longrightarrow$ unstable



Limit Theorems

Final Value Theorem (FVT)

if $h[n] \xleftrightarrow{z} H[n] \longrightarrow \lim_{n\to\infty} h[n] = \lim_{z\to 1}\{(z-1)H(z)\}$

Initial Value Theorem (IVT)

if $h[n] \xleftrightarrow{z} H[n] \longrightarrow h[0] = \lim_{z\to\infty} H(z)$

Bode Diagrams

Frequency Response of LTI Systems

System Response to Sinusoidal Inputs

$x(t) = \cos(\omega t) = Re\{e^{j\omega t}\}$

$\longrightarrow y(t) = Re\{H(j\omega)e^{j\omega t}\} = Re\{|H(j\omega)|e^{j\phi}e^{j\omega t}\}$ * $\phi = \angle H(j\omega)$ $= |H(j\omega)|\cos(\omega t + \angle H(j\omega))$

* $|H(j\omega)| \longrightarrow$ amplification of sinuisodal inputs as a function of frequency ω

$\angle H(j\omega) \longrightarrow$ phase shift of sinuisodal inputs as a function of frequency ω

Transient Response to Sinsoidal Inputs

assume input starts at some fixed point in time $\longrightarrow x(t) = e^{j\omega_0 t} \cdot u(t) \xleftrightarrow{L} X(s) = \frac{1}{s - j\omega_0}$

$$\longrightarrow Y(s) = H(s) \cdot X(s) = H(s) \cdot \frac{1}{s - j\omega_0} = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \frac{A_3}{s-p_3} + \dots + \frac{A_N}{s-p_N} + H(j\omega_0)\frac{1}{s-j\omega_0}$$

$$\longrightarrow y(t) = \underbrace{A_1 e^{p_1 t} + A_2 e^{p_2 t} + A_3 e^{p_3 t} + \dots + A_N e^{p_N t}}_{\text{transient response} \quad \text{stable} \longrightarrow \text{decay to } 0} + \underbrace{H(j\omega_0) e^{j\omega_0 t}}_{\text{steady-state response}}$$

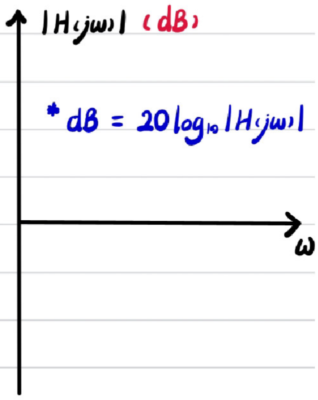
Bode Diagrams

Drawing the Bode Diagram

1. rewrite transfer function $H(s) = K \frac{(1+T_{z1}s)\dots(1+T_{zm}s)}{(1+T_{p1}s)\dots(1+T_{pn}s)}$

2. mark break frenquencies $\{\frac{1}{T_{z1}}, \dots, \frac{1}{T_{zm}}\}$ and $\{\frac{1}{T_{p1}}, \dots, \frac{1}{T_{pm}}\}$ on the horizontal axis ω (rad/s)

3. ① determine the initial slope at $\omega = 0$



* dB = $20\log_{10}|H(j\omega)|$

1 zero $\longrightarrow$ 20 dB/dec / 1 pole $\longrightarrow$ -20 dB/dec

* extend initial slope up to the lowest breaking frequency

② no poles/zeros at origin $\longrightarrow |H(j\omega)|_{\omega \to 0} = 20\log_{10}(|K|)$

have poles/zeros at origin $\longrightarrow$ manually calculate $|H(j\omega)|$

e.g. let $s = j\omega$, $\omega = 0.1$, $|H(0.1j)| = \dots$ if plot start from $\omega = 10^{-1}$ rad/s

③ when pass the break frequency

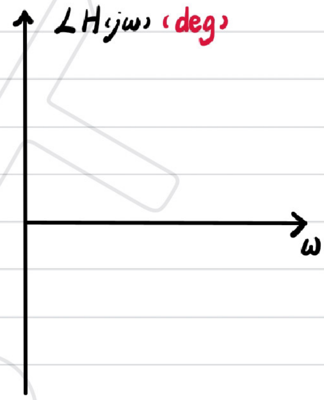
1 zero $\longrightarrow$ 20 dB/dec / 1 pole $\longrightarrow$ -20 dB/dec

④ for complex pole or zero pairs

pole $\longrightarrow H(s) = \frac{1}{1 \pm 2\xi Ts + T^2 s^2} \longrightarrow$ peak at $\frac{1}{T}$

zero $\longrightarrow H(s) = 1 \pm 2\xi Ts + T^2 s^2 \longrightarrow$ trough at $\frac{1}{T}$

4. ① determine the phase at $\omega = 0$ by starting from $\angle H(j\omega) = 0$



1 zero $\longrightarrow$ $+90°$ / 1 pole $\longrightarrow$ $-90°$ / negative K $\longrightarrow$ $+180°$

* extend initial phase one decade below the lowest break frequency

② when pass the break frequency

1 LHP zero $\longrightarrow$ $+90°/2$ dec / 1 LHP pole $\longrightarrow$ $-90°/2$ dec

1 RHP zero $\longrightarrow$ $-90°/2$ dec / 1 RHP pole $\longrightarrow$ $+90°/2$ dec

③ for complex pole and zero pairs

$\longrightarrow$ range of frequencies over phase change $\propto \xi$