Basic Probability

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Notations

- ϕ & $\{\}$ empty set
- $\cup$ union
- $\cap$ intersection
- $\subset$ inclusion
- $\setminus$ set difference
- $\bar{A}$ & A^c complement of A (not contained)
- S & Ω set of all elements

$\longrightarrow$ $\bar{A} = S \setminus A$ and $S = A \cup \bar{A}$

- $|A|$ size of set A
- $A \times B$ Cartesian product of A and B
- $A \cap B = \phi$ disjoint

Definitions

- sample space can be discrete or continous
- event subset of sample space S
- certain event S & Ω
- impossible event ϕ

let S be a sample space and let $A, B \subset S$ be two events

$\longrightarrow$ $\cdot\ A \cup \phi = A$ $\quad \cdot\ A \cup A = A$ $\quad \cdot\ A \cup S = S$ $\quad \cdot\ A \cup B = B \cup A$

$\longrightarrow$ $\cdot\ (\bar{\bar{A}}) = A$ $\quad \cdot\ \bar{\phi} = S$ $\quad \cdot\ A \cup \bar{A} = S$ $\quad \cdot\ A \cap \bar{A} = \phi$

De Morgan's Laws

let A, B be events $\longrightarrow$ $\cdot\ \overline{A \cup B} = \bar{A} \cap \bar{B}$ $\quad \cdot\ \overline{A \cap B} = \bar{A} \cup \bar{B}$

let A, B, C be events

$\longrightarrow$ $\cdot\ A \cap (B \cup C) = (A \cap B) \cup (B \cap C)$ $\quad \cdot\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Principle of Inclusion-exclusion $|A \cup B| = |A| + |B| - |A \cap B|$

Basic Law of Probability Functions

$\cdot\ P(A) \leq 1$ $\quad \cdot\ P(A) = 1 - P(\bar{A})$ $\quad \cdot\ P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Equal Outcomes $P(A) = \frac{m}{n}$

event outcomes

sample space outcomes

Conditional Probability

Conditional Probability

Conditional Probability $P(A|B) = \frac{P(A\cap B)}{P(B)}$ * B happends then A happends

Partition Rule $P(A) = P(A\cap B) + P(A\cap \bar{B}) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})$

Properties of Conditional Probability

- $P(A|B) \geq 0$ · $P(S|B) = 1$
- if A and B disjoint $\longrightarrow P(A\cup B|C) = P(A|C) + P(B|C)$ *otherwise ... $- P(A\cap B|C)$
- $P(\bar{A}|B) = 1 - P(A|B)$

Independent Event $P(A|B) = P(A) \longrightarrow P(A\cap B) = P(A)P(B)$

Co-independent for $A_1, A_2, A_3, \dots, A_n$ $P(A_i \cap A_j \cap A_k) = P(A_i)P(A_j)P(A_k)$

$\longrightarrow P(A\cup B|C) = P(A|C) + P(B|C) - P(A|C)P(B|C)$

Law of Total Probability $P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$

Bayes' Theorem

Bayes' Theorem $P(A|B) = \frac{P(B|A)P(A)}{P(B)} \longrightarrow P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=0}^{n} P(B|A_i)P(A_i)}$

Screening

Random Variables

Discrete Random Variables

pmf (Probability Mass Function) $P(X = x_i) = p_i$

Binomial Distribution $X \sim B(\overset{\text{trials}}{n}, \underset{\text{probability of sucess}}{p})$

$P(X=x) = \underbrace{\binom{n}{x}}_{\text{number of possible sequences}} p^x (1-p)^{n-x}$ where $\binom{n}{x} = {}^nC_x = \frac{n!}{x!(n-x)!}$, $x = 0, 1, 2, \dots, n$

Sum of Independent Random Variables* convolution

* throwing two dices for $Z = X + Y$, $P(Z=n) = P(X=n) * P(Y=n) = \sum_{k=-\infty}^{\infty} P(X=k)P(Y=n-k)$

$\boxed{6}\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1} \longrightarrow$

$\longleftarrow \boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{6}$ $\longrightarrow P(2) = \frac{1}{36}$, $P(3) = \frac{2}{36}$, ..., $P(7) = \frac{6}{36}$, ..., $P(12) = \frac{1}{36}$

Sum of Multiple Random Variables*

convolution $\xrightarrow{\text{Central Limit Theorem}}$ Normal distribution

Poisson Distribution

for events occur repeatly but randomly and independently

$X \sim \text{Poisson}(\mu)$ where $\mu = \lambda t$, λ: rate

$P(X=x) = \frac{e^{-\mu}\mu^x}{x!}$, $x = 0, 1, 2, \ldots$

* $P(X+Y)$ for $X \sim \text{Poisson}(\mu)$ and $Y \sim \text{Poisson}(\lambda)$

$\xrightarrow{\text{convolution}} P(X+Y)$ with $(X+Y) \sim \text{Poission}(\mu + \lambda)$

Continuous Random Variables

cdf (Cumulative Distribution Function)

$F(x) = P(X \le x)$

* $P(a \le X \le b) = P(X \le b) - P(X \le a) = F(b) - F(a)$

· $F(-\infty) = 1$ · $F(\infty) = 1$ · $0 \le F(x) \le 1$ · $F(x)$ is an increasing function

pdf (Probability Density Function)

$f(x) = \frac{dF(x)}{dx} \longrightarrow F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$ $\quad$ * $\int_{-\infty}^{\infty} f(t)\,dt = F(\infty) - F(-\infty) = 1$

The pdf of a Sum of Random Variables*

convolution

for $Z = X + Y$, $f_Z(z) = f_X(x) * f_Y(y) = \int_{-\infty}^{\infty} f_X(\tau) f_Y(t-\tau)\,d\tau$

Transformation of Random Variables

$Y = g(X) \longrightarrow P(Y \le y) = P(g(X) \le y) = P(X \ge g^{-1}(y)) = 1 - P(X \le g^{-1}(y))$

$\longrightarrow F_Y(y) = 1 - F_X(g^{-1}(y)) \longrightarrow f_Y(y) = f_X(\underbrace{g^{-1}(y)}_{x}) \left| \frac{dx}{dy} \right|$

* in 2-D, $f_{UV}(u,v) = f_{XY}(x,y) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$

Exponential Distribution

memoryless $\longrightarrow P(T > t_2 \mid T > t_1) = P(t > (t_2 - t_1))$

$X \sim Exp(\underbrace{\lambda}_{rate})$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & 0 \le x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Uniform Distribution

$X \sim U(a, b)$

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Normal Distribution

$X \sim N(\underbrace{\mu}_{mean}, \underbrace{6^2}_{variance}) \longrightarrow P(X \le a) \longrightarrow \Phi(\frac{x-\mu}{6})$ with $N(0, 1)$

$^* f_{x \pm Y} = f_x * f_Y \longrightarrow (X \pm Y) \sim N(\mu_x \pm \mu_Y, 6_x^2 + 6_Y^2) \longrightarrow (aX + b) \sim N(a\mu_x + b, a^2 6_x^2)$

Weibull Distribution

$\lambda, \beta > 0$ $\quad f(x) = \lambda\beta(\lambda x)^{\beta-1} e^{-\lambda^\beta x^\beta}$

Expectation and Variance

Expectation, Varience and Function of a Random Variable

Expectation

· discrete $E(x) = \sum_{i=1}^{n} x_i P(X = x_i)$

$\longrightarrow$ · Binomial np · Poission μ

· continuous $E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$

$\longrightarrow$ · Exponential $\frac{1}{\lambda}$ · Uniform $\frac{a+b}{2}$ · Normal μ

$E(X + Y) = E(X) + E(Y)$ & $E(aX + b) = aE(X) + b$

$E(XY) = E(X)E(Y)$ if X and Y are independent random variables

Function of Random Variables

· discrete $E(g(X)) = \sum_{i=1}^{n} g(x_i) P(X = x_i)$

· continuous $\int_{-\infty}^{\infty} E(g(x)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$

Variance $Var(X) = E[(X-E(X))^2] = E(X^2) - E(X)^2$

· discrete $Var(X) = E(X^2) - E(X)^2$

$\longrightarrow$ · Binomial $np(1-p)$ · Poission μ

· continuous $Var(X) = E(X^2) - E(X)^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left(\int_{-\infty}^{\infty} x f(x)\,dx\right)^2$

$\longrightarrow$ · Exponential $\frac{1}{\lambda^2}$ · Uniform $\frac{1}{12}(b-a)^2$ · Normal σ^2

$Var(X+Y) = Var(X) + Var(Y)$

$Var(aX+b) = a^2 Var(X)$

The Gamma Function $\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x}\,dx \longrightarrow \Gamma(z+1) = z\Gamma(z) = z!$

* to evaluate $\int_{-\infty}^{\infty} e^{-ax^2}\,dx$

$$\longrightarrow \int_{-\infty}^{\infty} e^{-ax^2}dx \cdot \int_{-\infty}^{\infty} e^{-ay^2}dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)}\,dxdy = \int_0^{2\pi}\int_0^{\infty} e^{-ar^2}\, r\, dr d\theta = \frac{\pi}{a}$$

$$\longrightarrow \int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$$

Expectation of Weibull Distribution

$$E(X) = \int_{-\infty}^{\infty} x\left[\lambda\beta(\lambda x)^{\beta-1} e^{-\lambda^\beta x^\beta}\right]dx = \int_{-\infty}^{\infty} x\left(a\beta x^{\beta-1} e^{-ax^\beta}\right)dx = \int_{-\infty}^{\infty} a\beta x^\beta e^{-ax^\beta}\,dx$$

let $u = ax^\beta$, $\frac{du}{dx} = a\beta x^{\beta-1}$

$$\longrightarrow \int_{-\infty}^{\infty} x e^{-u}\,du = \int_{-\infty}^{\infty} \left(\frac{u}{a}\right)^{\frac{1}{\beta}} e^{-u}du = \left(\frac{1}{a}\right)^{\frac{1}{\beta}}\int_{-\infty}^{\infty} u^{\frac{1}{\beta}} e^{-u}\,du = \left(\frac{1}{a}\right)^{\frac{1}{\beta}}\Gamma\left(1+\frac{1}{\beta}\right)$$

$$Var(X) = E(X^2) - [E(X)]^2 = \int_{-\infty}^{\infty} x^2\left(a\beta x^{\beta-1}e^{-ax^\beta}\right)dx - \left(\frac{1}{a}\right)^{\frac{2}{\beta}}\Gamma^2\left(1+\frac{1}{\beta}\right)$$

$$= \int_{-\infty}^{\infty} x^2 e^{-u}\,du - \left(\frac{1}{a}\right)^{\frac{2}{\beta}}\Gamma^2\left(1+\frac{1}{\beta}\right) = \left(\frac{1}{a}\right)^{\frac{2}{\beta}}\left[\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{1}{\beta}\right)\right]$$

Weak Law of Large Number

Chebychev's Inequality $P(X \geq r) \leq \frac{E(X)}{r}$

for $x \geq r$, $E(X) = \sum_{x_i \geq r}^{N} x_i (P = x_i) \geq \sum_{x_i \geq r}^{N} r(P = x_i) \longrightarrow P(X \geq r) \leq \frac{E(X)}{r}$

Markov's Inequality $P(|X-\mu| \geq r\sigma) \leq \frac{1}{r^2}$

for $Y = (X - E(X))^2 = (X-\mu)^2 \longrightarrow E(Y) = Var(X)$

$$\longrightarrow P(Y \geq s^2) \leq \frac{E(Y)}{s^2} = \frac{Var(X)}{s^2} = \frac{\sigma^2}{s^2}$$

$$\longrightarrow P(|X-\mu| \geq s) \leq \frac{\sigma^2}{s^2}$$

$$\xrightarrow{s = r\sigma} P(|X-\mu| \geq r\sigma) \leq \frac{1}{r^2}$$

Weak Law of Large Number

$\bar{X}_n = \frac{S_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$ for $E(X_i) = \mu \longrightarrow$ for any ε, $P(|\bar{X}_n - \mu| < \varepsilon) \to 1$, $n \to \infty$

Central Limit Theorem

$Y_n = \sum_{i=1}^{n} X_i$ has Normal Distribution $Y_n \sim N(n\mu, n\sigma^2)$

$\longrightarrow \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ has Normal Distribution $\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$

Approximating Distribution

- Binomial by Poission for large n ($n \gg 20$), small $p \longrightarrow \mu = np$ ($np < 5$)
- Binomial by Normal

for large n and fixed p ($np > 5$, $n(1-p) > 5$) $\longrightarrow \mu = np$, $\sigma^2 = np(1-p)$

Continuity Correction

discrete $\longrightarrow$ continuous

- $P(X = a) \longrightarrow P(a - 0.5 < Y < a + 1.5)$
- $P(X < a) \longrightarrow P(Y < a - 0.5)$
- $P(X > a) = P(Y > a + 0.5)$
- $P(X \leq a) \longrightarrow P(Y < a + 0.5)$
- $P(X \geq a) = P(Y > a - 0.5)$

Survival Analysis

Survival (reliability) Analysis

Failure Time Density pdf $f(t)$

Failure Time Distribution $F(t) = P(T \leq t) = \int_0^t f(x)\,dx$ * $F(0) = 0$

Survivor (realiability) Function $S(t) = R(t) = P(T > t) = 1 - F(t)$

Hazard Function $h(t) = P(t<T<t+\delta t \mid T>t) = P(A|B) = \frac{P(A)}{P(B)} = \frac{f(t)}{S(t)}$

$$\longrightarrow \frac{dH(t)}{dt} = \frac{f(t)}{1-F(t)} = \frac{1}{1-F(t)}\frac{dF(t)}{dt} \longrightarrow H(t) = -\ln(1-F(t)) = -\ln(S(t))$$

Function to Model Survival (reliability)

Exponential Distribution

$$f(t)=\begin{cases}\lambda e^{-\lambda t} & \text{for } t\ge 0\\ 0 & \text{for } t<0\end{cases} \longrightarrow R(t)=1-F(t)=e^{-\lambda t} \longrightarrow H(t)=\lambda t \longrightarrow h(t)=\lambda$$

Wellbull Distribution

$$f(t)=\begin{cases}\lambda\beta(\lambda t)^{\beta-1}e^{-\lambda^{\beta} t^{\beta}} & \text{for } t\ge 0\\ 0 & \text{for } t<0\end{cases} \longrightarrow F(t)=\int_0^t \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}dx = [-e^{-\alpha x^{\beta}}]_0^t = -e^{-\alpha t^{\beta}}+1$$

$$\longrightarrow S(t)=1-F(t)=e^{-\alpha t^{\beta}} \longrightarrow H(t)=\alpha t^{\beta} \longrightarrow h(t)=\alpha\beta t^{\beta-1}=\beta\lambda^{\beta} t^{\beta-1}\propto t^{\beta-1}$$

Mean Time to Failure

$$E(T)=\int_0^\infty t\cdot f(t)\,dt = \int_0^\infty t\cdot -S'(t)\,dt = [-tS(t)]_0^\infty + \int_0^\infty S(t)\,dt$$

System of Components

Component in Series

$$S(t)=P(T_1>t \cap T_2>t) = S_1(t)S_2(t)$$

Component in Parallel

$$S(t)=P(T_1>t\cup T_2>t) = 1-(1-S_1(t))(1-S_2(t))$$