Introduction

What is Mechanics ?

Study of what happens to a object . when force apply on it

SI unit

International System of Units (SI)

Name	Length	Time	Mass	Force
International System of Units SI	meter m	second s	kilogram kg	newton* N $\left(\frac{kg \cdot m}{s^2}\right)$

Accuracy , Limits and Approximation .

- No measurement can be exact

- Accuracy : closeness between measure value and true value .

- Precision : degree of exactness (depend on method and device)

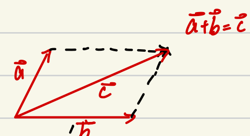
Significant Figures

- zeros do not count .

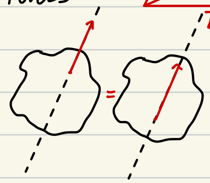
Decimal places

- number of significant figures after " . "

Newtonian Mechanics



(i) parallelogram law for the addition of Forces



(ii) The principle of transmissibility .

(iii) Newton's three laws of Motion

① if the resultant force acting on a particle is zero , the particle remain at rest (if originally at rest) or moves with constant speed in straight line (if originally in motion) .

② if a resultant force acting on a particle is not zero, the particle has an acceleration proportional to the magnitude of the resultant force and in the direction of the resultant force $F=ma$

③ The forces of action and reaction between bodies in contact have the same magnitude and same line of action and opposite direction

(iv) Newton's Law of Gravitation

$$F=\frac{GMm}{r^2}$$

Idealisation in Mechanics

a) Continuum $\longrightarrow$ Continuum distribution of matter

b) Rigid body $\longrightarrow$
- No deformation,
- All particles constant distance before and after
- Do not consider material

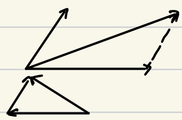
c) A particle $\longrightarrow$ Consider the difference in size, size can be neglected

d) The point force $\longrightarrow$ force assume at a point.

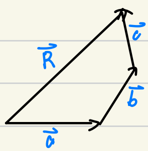
Scalar and 2D-vectors

Quantities
- Vector : single magnitude with unit without direction
- Scalor : a magnitude with unit and direction

Application of vector Addition
- Triangle Law
- Parallelogram law
- Commutative $A+B=B+A$

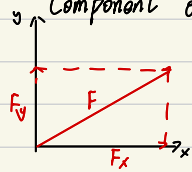


Vector addition :



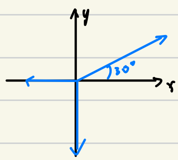
$\vec{R} = \vec{a} + \vec{b} + \vec{c}$

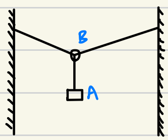
Component of Force



Equilibrium in 2-Dimensions

Equilibrium of a object in 2-Dimensions $\longrightarrow$ $\begin{cases} \Sigma F_x = 0 \\ \Sigma F_y = 0 \end{cases}$

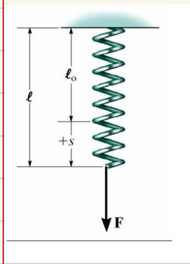




if whole system in equilibrium
then A is in equilibrium
B is in equilibrium

Free Body Diagram (受力分析图)

① Simple springs



Spring force

$F = k \cdot x$

k → spring constant

x → deformation of spring

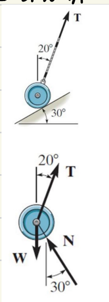
② Cables and Pulleys

Tension

$T_1 = T_2$

always on the direction of cable

③ smooth contact



Normal force

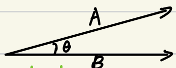
$N + W + T = 0$

alway in direction normal to grand

Dot product

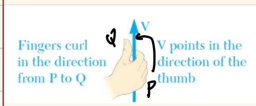
characteristic: { The result of dot product is scalor
The unit of dot product will be the product of units of A.B

$\vec{A} \cdot \vec{B} = |A| \cdot |B| \cdot \cos\theta$

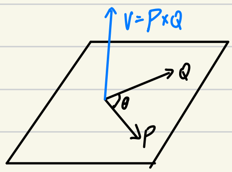


θ: smallest angle between 2 vectors

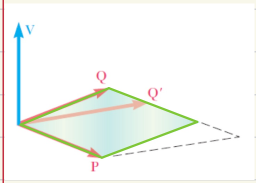
cross product (求垂直向量)



$\vec{P} \times \vec{Q} = |\vec{P}| \times |\vec{Q}| \times \sin\theta$



careful the order and direction



Area of parallelogram

$A = \vec{P} \times \vec{Q}$

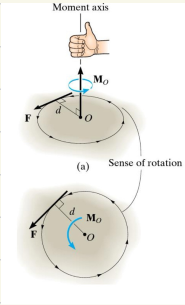
同理，若$\vec{Q}$变$\vec{Q'}$，$\vec{P} \times \vec{Q'}$不变，Area不变，叉乘结果不变

Moment of a force about a axis

sometimes called torque

Moment ⟶ tendency to rotate

$M_o = F \cdot d$



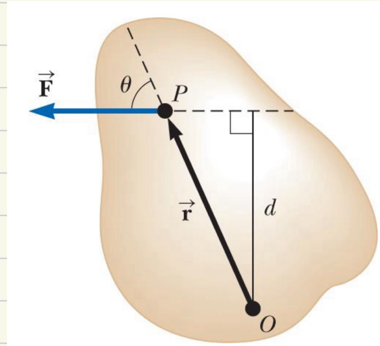
Equilibrium of Rigid Body

Conditions $\begin{cases}\text{The external force must be zero.} \\ \text{The external torque must be zero.}\end{cases}$ $\begin{cases}\sum \vec{F} = 0 \\ \sum \vec{\tau} = 0\end{cases}$

$\vec{\tau} = \vec{r} \cdot \vec{F}$

进一步 $\begin{cases}\sum \vec{F_x} = 0 \\ \sum \vec{F_y} = 0 \\ \sum \vec{F_z} = 0\end{cases}$

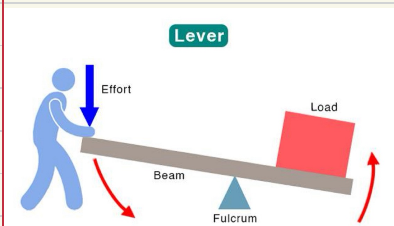
$\begin{cases}\sum \vec{\tau_x} = 0 \\ \sum \vec{\tau_y} = 0 \\ \sum \vec{\tau_z} = 0\end{cases}$



Centre of gravity

$$x_{cg} = \frac{\sum_i m x_i}{\sum_i m_i}$$

Lever



W inversely proportional to d

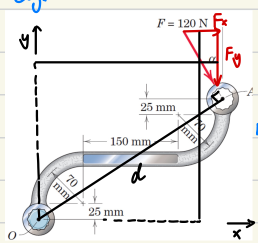
weight $W \propto \frac{1}{d}$ distant

$W_1 d_1 = W_2 d_2$

力到作用点的分量

or 力取相切分量

e.g.



method 1: 取 F_x, F_y，分别算再相加

method 2: 直接算距离

Moment of a couple

What is couple ?

The moment produced by two equal opposite and non-collinear forces is called couple

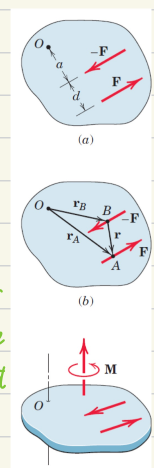
不共线的

$M_{net} = F(a+d) - Fa$

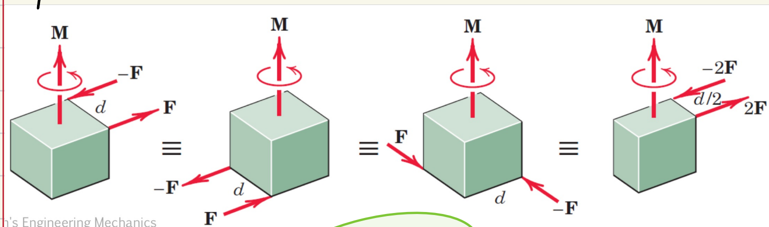
$= r_A \cdot F + r_B \cdot (-F) = (r_A - r_B) \cdot F$

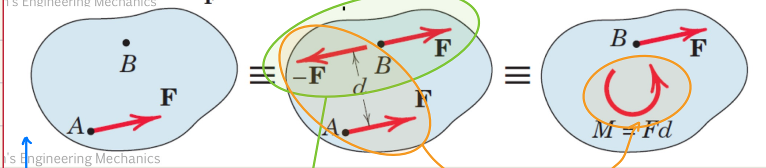
$r = r_A - r_B$, then $M = r \times F$

M always perpendicular to the plane of the forces which constitued the couple



Equivalent



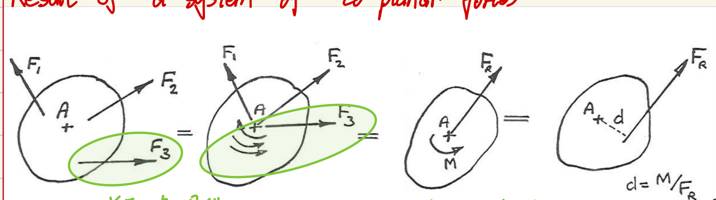


transfer moment

net force = 0
net moment = 0

合力为0, 提供大小为Fd moment

Result of a system of co-planar forces

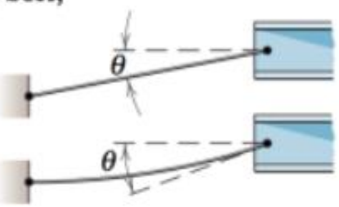
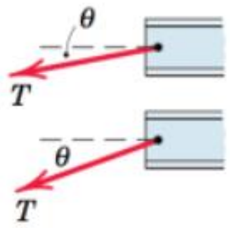
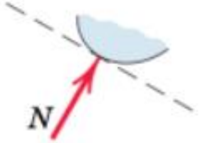
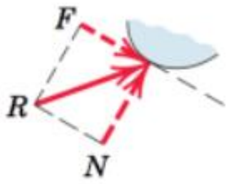
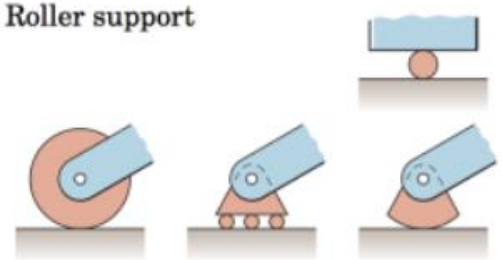
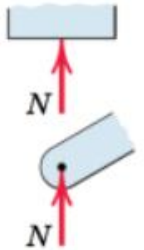
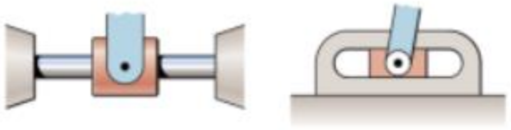
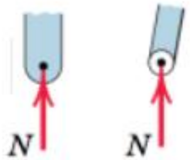


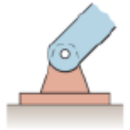
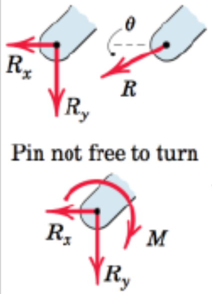
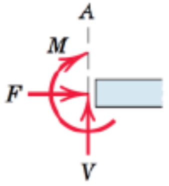
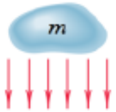
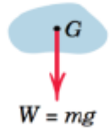
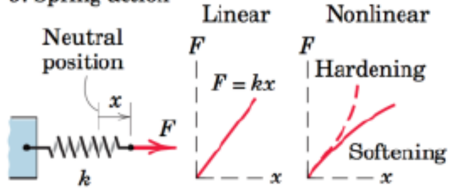
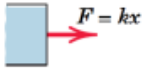
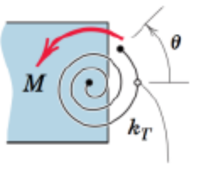
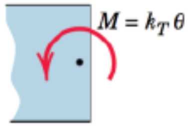
关于a点, 提供一个力与力矩

拆分成一个力 一个力矩

力与力矩分开算

再合并

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS		
Type of Contact and Force Origin	**Action on Body to Be Isolated**	
1. Flexible cable, belt, chain, or rope Weight of cable negligible Weight of cable not negligible 		Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.
2. Smooth surfaces 		Contact force is compressive and is normal to the surface.
3. Rough surfaces 		Rough surfaces are capable of supporting a tangential component F (frictional force) as well as a normal component N of the resultant contact force R.
4. Roller support 		Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.
5. Freely sliding guide 		Collar or slider free to move along smooth guides; can support force normal to guide only.

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS (*cont.*)		
Type of Contact and Force Origin	**Action on Body to Be Isolated**	
6. Pin connection 	Pin free to turn Pin not free to turn 	A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. We may either show two components R_x and R_y or a magnitude R and direction θ. A pin not free to turn also supports a couple M.
7. Built-in or fixed support 		A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.
8. Gravitational attraction 		The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center of gravity G.
9. Spring action 		Spring force is tensile if the spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.
10. Torsional spring action 		For a linear torsional spring, the applied moment M is proportional to the angular deflection θ from the neutral position. The stiffness k_T is the moment required to deform the spring one radian.

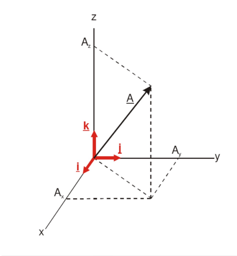
3D vectors

Basics of vector mechanics

Cartesian Vector form

unit Vector of A is defined as $U_A = \frac{A}{A}$

- magnitude = 1
- dimensionless
- points same direction with origin



Representation of 3D vector

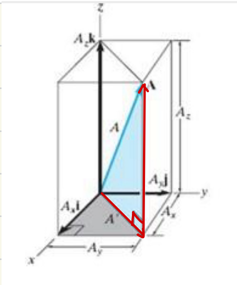
$\underline{A} = a\underline{i} + b\underline{j} + c\underline{k} = (a, b, c)$

magnitude $|\underline{A}| = \sqrt{a^2+b^2+c^2}$

unit vectors $|\underline{i}| = |\underline{j}| = |\underline{k}| = 1$

$|\underline{i}| = (1,0,0)$ $|\underline{k}| = (0,0,1)$

$|\underline{j}| = (0,1,0)$



Find position of A

$P = \sqrt{a^2+b^2+c^2}$

Direction of Cartesian Vector

$\cos\alpha = \frac{A_x}{A}$ $\cos\beta = \frac{A_y}{A}$ $\cos\gamma = \frac{A_z}{A}$

x y z

与x, y, z axis 夹角, range 0-180°

satisfy $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

for unit vector $U_A = \frac{A}{A} = \frac{A_x}{A} i + \frac{A_y}{A} j + \frac{A_z}{A} k$

$U_A = \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k$

Addition of Cartesian Vectors

e.g.

$A = ai + bj + ck$ $B = di + ej + fk$

$A \pm B = (a \pm d)i + (b \pm e)j + (c \pm f)k$

Vector product: Application in 3D Equilibrium

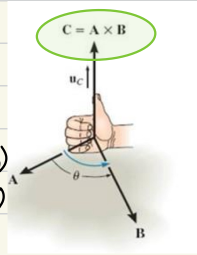
Cross product : $\vec{A} \times \vec{B} = |A| \cdot |B| \cdot \sin\theta$

$A = (A_x, A_y, A_z)$ smallest angle between

$B = (B_x, B_y, B_z)$ 2 vectors

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z \cdot k) + (A_zB_x \cdot i) + (A_xB_y \cdot j) - (A_yB_x \cdot k) - (A_zB_y \cdot i) - (A_xB_z \cdot j) = \begin{bmatrix} A_yB_z - A_zB_y \\ A_xB_z - A_zB_x \\ A_xB_y - A_yB_x \end{bmatrix} \cdot \begin{bmatrix} i \\ j \\ k \end{bmatrix}$$

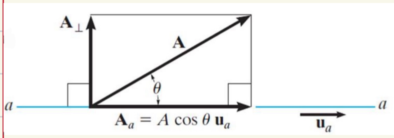
蓝-红



注意C方向正交

Geometrical and Physical quantity defined in terms of products of vectors

① Determine projection of a vector (向量分解)



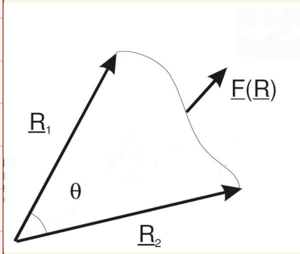
step 1 : 算 V_a ⟶ 关于要求方向的 unit vector

step 2 : Find projection

$A = A_\parallel + A_\perp$ ⟶ 分解成 求的方向和垂直于它的方向

$A_\parallel = A \cdot V_a = \underbrace{\cos\alpha A_x + \cos\beta A_y + \cos\gamma A_z}_{\text{取每个方向分量}}$

② Work done by a force



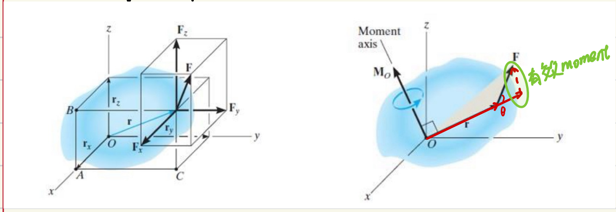
$W_{1-2} = \int_{R_1}^{R_2} \underbrace{F \cdot dR}_{\text{Scalar product}}$

$= F \cdot R_2 - F \cdot R_1$

$= F(R_2 - R_1)$

$= F \cdot \Delta R$

③ Moment of a force — Vector Formulation (沿一个点力矩)



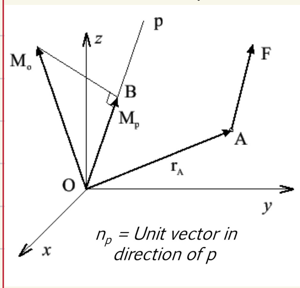
$M_o = r \times F$

r ↓ 从 O 指向力作用点距离

F ↓ 力

moment $= \sin\theta\,|F|\,|r| = r \cdot F$

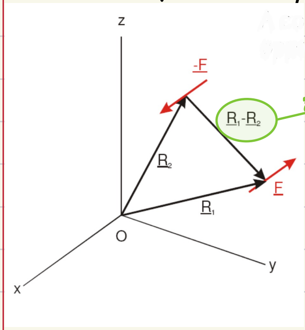
④ moment of a force about a axis (沿一个轴力矩)



在点的基础上取与轴垂直的

$$M_p = M_o \cdot n_p$$
$$= \underbrace{(r_A \times F)}_{\text{沿点moment}} \cdot \underbrace{n_p}_{\cos\alpha}$$

⑤ moment of a couple



$$M_{net} = M_1 + M_2$$
$$= R_1 \times F_1 + R_2 \times (-F_1)$$
$$\xrightarrow{\text{距离}} = \underbrace{(R_1 - R_2) \cdot F_1}$$

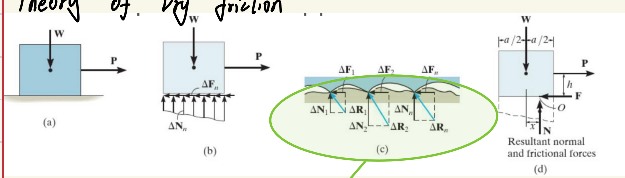
cross product 可以分配率

不可 { 交换
结合

Friction

Types of friction {
- Fluid friction (流体) : friction in moving layer of viscous fluid
- Internal friction (内部) : dissipates energy of materials
- Dry friction (库仑) : friction along $\underbrace{\text{non-lubricated}}_{\text{无润滑的}}$ contact surfaces

Theory of Dry friction

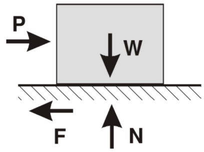


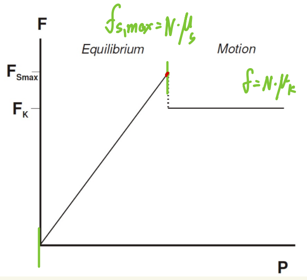
每个微小形变造成的 shear force 合成 friction.

Conditions { rough surface
deformable → 非理想 rigid body

Coulomb Friction

Block on Horizontal Plane

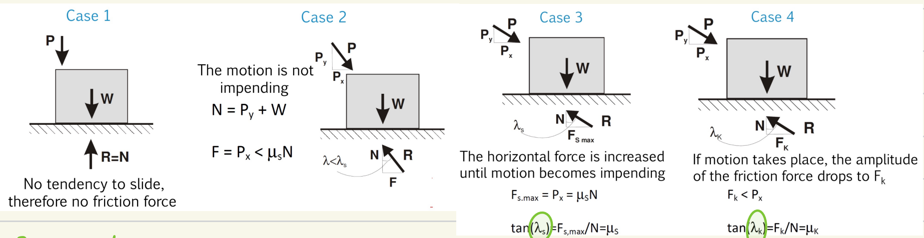




Pushing Block case, when $P \le F_{s,max}$. stationary, $f = P$

when $P > F_{s,max}$ moving, $f = N \cdot \mu_k$

Angles of friction



$P \perp$ Ground

no friction

P 有角度，有滑的趋势

但力不够，$P_x < \mu_s N$

equilibrium $P_x = \mu_{s\,max}$

再大一些就滑

angle of friction

$\lambda_s = \tan^{-1}\left(\frac{F_{s,max}}{N}\right)$

开始滑

$P_x > F_k$

angle of friction

$\lambda_f = \tan^{-1}\left(\frac{F_k}{N}\right)$

Kinematics

What is Kinematics?

Mechanics: The study of how bodies react to the forces acting on them.

Statics: The study of bodies in equilibrium.

Dynamics:

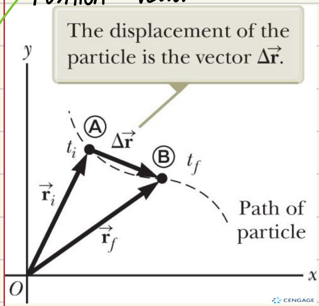
考虑怎么动

1. **Kinematics** - concerned with the geometric aspects of motion

2. **Kinetics** - concerned with the forces causing the motion

考虑为什么动

Position Vector



Describe change of position

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i$$

Final position − Initial position

求导

Velocity Vector

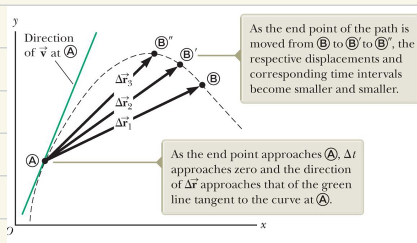
$$\vec{v} = \frac{\Delta\vec{r}}{\Delta t} \qquad \vec{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \qquad v = |\vec{v}|$$

speed = magnitude of velocity

e.g.

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

$$\dot{x} = \frac{dx}{dt} \quad \dot{y} = \frac{dy}{dt} \quad \dot{z} = \frac{dz}{dt}$$



Acceleration Vector

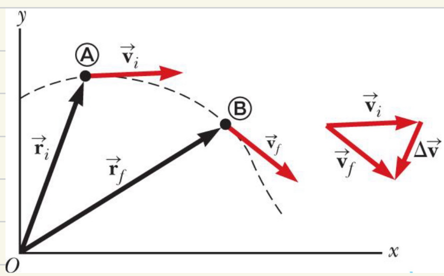
(Rate of change of Velocity Vector)

$$\vec{a}_{avg} \equiv \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} \qquad \vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{dv}{dt}$$

e.g.

$$\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

$$\ddot{x} = \frac{d^2x}{dt^2} \quad \ddot{y} = \frac{d^2y}{dt^2} \quad \ddot{z} = \frac{d^2z}{dt^2}$$



Motion in straight line

−Rectilinear motion

Differentiate:

$$v = \frac{ds}{dt} \qquad a = \frac{dv}{dt} \text{ or } v\frac{dv}{ds}$$

$$v = v_0 + at$$

Integrate:

$$v = \int_{v_0}^{v} dv \text{ or } \int_{t_0}^{t} a\,dt \qquad s = \int_{s_0}^{s} ds \text{ or } \int_{t_0}^{t} v\,dt$$

$$s = s_0 + v_0 t + \frac{1}{2}at^2$$

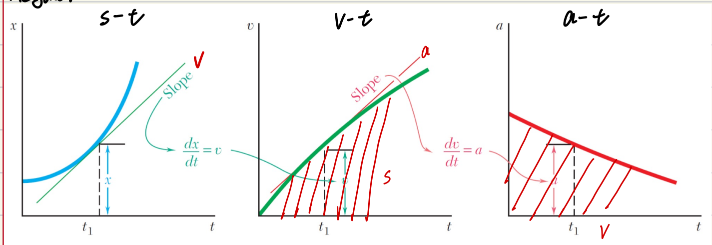
$$v^2 = v_0^2 + 2a(s - s_0)$$

v_0 = Initial velocity

t_0 = Initial time

Graphic Solutions

Regular



$$v = \frac{dx}{dt} \quad \text{and} \quad a = \frac{dv}{dt}$$

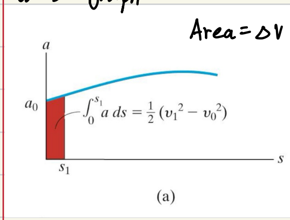
gradient/Area of each graph

$a = v\frac{dv}{ds} \longrightarrow a\,ds = v\,dv$

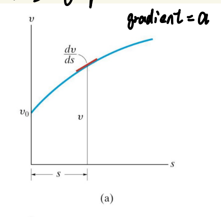
$\int a\,ds = \int v\,dv$

Specials

a-s graph

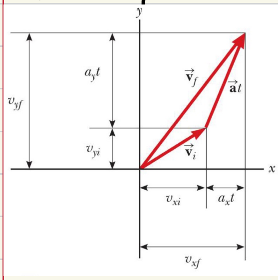


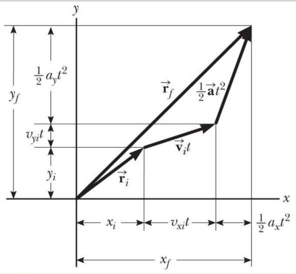
v-s graph



Two-Dimensional Motion with constant Acceleration

$\vec{r} = x\hat{i} + y\hat{j}$





$$\begin{cases} V_{xf} = V_{xi} + a_x t \\ V_{yf} = V_{yi} + a_y t \\ \vec{r_f} = (x_i\hat{i} + y_i\hat{j}) + (V_{xi}\hat{i} + V_{yi}\hat{j})t + \frac{1}{2}(a_x\hat{i} + a_y\hat{j})t^2 \end{cases}$$

用 vector 表示

projectile motion

$V_h = u\cos\theta$

$V_v = u\sin\theta - gt^2$

$S_h = V_h \cdot t = u\cdot\cos\theta\cdot t$

$S_v = u\sin\theta t - \frac{1}{2}gt^2$

V, u, t 关系

求 H, x

$y = x\tan\theta - \frac{gx^2}{2u^2}\sec^2\theta$ u, x, y, θ 四个未知数

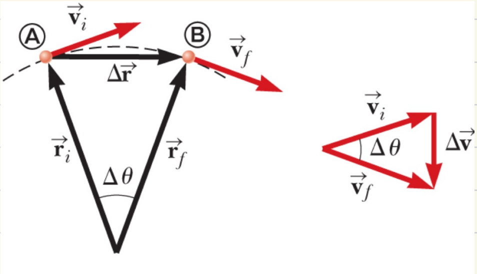
$y = u\sin\theta t - \frac{1}{2}gt^2$ $x = u\cos\theta t$

let $t = \frac{x}{u\cos\theta}$ 得

$y = u\sin\theta\cdot\frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x^2}{u^2\cos^2\theta}\right)$

$y = x\tan\theta - \frac{gx^2}{2u^2}\sec^2\theta$

Uniform Circular Motion



用相似三角做

$$\theta = \frac{l}{r} \approx \frac{\Delta V}{V}$$

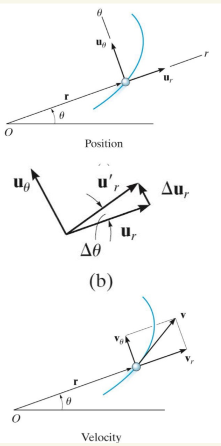
$$\frac{\Delta V}{V} = \theta$$

$$\frac{\Delta V}{\Delta t} = \frac{\theta V}{\Delta t}$$

$$a = V \cdot w$$

$$a = \frac{V^2}{R} \leftarrow w = \frac{V}{r}$$

Cylindrical Coordinates (using in 3-D curve motion)



S – In polar system, define 其中 $r = r(t)$

$$r = r u \qquad \theta = \theta(t)$$

scalar 距离   unit vector, 离心方向

远离圆轴方向

$$u_r = \cos\theta \cdot \hat{i} + \sin\theta \cdot \hat{j} = r\text{方向}$$

$$u_\theta = -\sin\theta \cdot \hat{i} + \cos\theta \cdot \hat{j}$$

与 u_r 相切

V – For instantaneous velocity

$$V = \frac{dr}{dt} = \frac{d(ru)}{dt}$$

$$V = \dot{r} u_r + \frac{du_r}{dt} \cdot r$$

$$u_r = \cos\theta \cdot \hat{i} + \sin\theta \cdot \hat{j}$$

$$\frac{du_r}{dt} = -\sin\theta \cdot \frac{d\theta}{dt} \cdot \hat{i} + \cos\theta \cdot \frac{d\theta}{dt} \cdot \hat{j}$$

$$\frac{du_r}{dt} = \frac{d\theta}{dt}\underbrace{(\cos\theta \cdot \hat{j} - \sin\theta \cdot \hat{i})}_{\text{切向方向 } u_\theta}$$

$$\frac{du_r}{dt} = u_\theta \cdot \frac{d\theta}{dt} \qquad \dot{u}_r = u_\theta \cdot \dot{\theta}$$

$$V = \underbrace{\dot{r} u_r}_{\text{径向速度}} + \underbrace{u_\theta \cdot \dot{\theta} \cdot r}_{\text{切向速度}}$$

$$\dot{r}\theta - r\dot{\theta}^2$$

$$\ddot{r}\theta - 2\dot{r}\dot{\theta}$$

A – instantaneous acceleration

$$a = \frac{dv}{dt} = \frac{d}{dt}(\dot{r} u_r + r \cdot u_\theta \cdot \dot{\theta})$$

$$= (\ddot{r} u_r + \dot{r}\dot{\theta} u_\theta + \dot{r}(u_\theta \cdot \dot{\theta}) + \dot{u}_\theta (r \cdot \dot{\theta}) + \ddot{\theta}(r \cdot u_\theta))$$

代入 $\frac{du_r}{dt} = u_\theta \cdot \dot{\theta}$ $\dot{u}_\theta = -\dot{\theta} \cdot r \longrightarrow \dot{u}_\theta = -\dot{\theta}\,(u_r)$

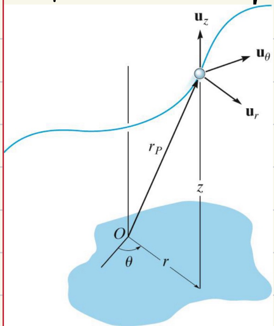
$$u_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\dot{u}_\theta = -\cos\theta \cdot \dot{\theta} \cdot \hat{i} - \sin\theta \cdot \dot{\theta}\,\hat{j}$$

$$a = u_r\underbrace{(\ddot{r} - r\dot{\theta}^2)}_{\text{radical acceleration } a_r} + u_\theta\underbrace{(r\ddot{\theta} + 2\dot{r}\dot{\theta})}_{\text{transverse acceleration } a_t}$$

$$\text{magnitude} = \sqrt{a_r^2 + a_t^2} = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2}$$

If particle moves in space curve



$r_p = r u_r + z u_z$

$v_p = \dot r u_r + r\dot\theta u_\theta + \dot z u_z$ 考虑z轴

$a_p = \underbrace{(\ddot r - r\dot\theta^2)}_{\text{radical}} u_r + \underbrace{(r\ddot\theta + 2\dot r\dot\theta)}_{\text{transverse}} u_\theta + \ddot z u_z$

Some special cases

① constant circular speed

$\dot\theta = w \quad \dot z = 0$

$a = (\ddot r - r w^2) u_r + (2\dot r w) u_\theta$

② constant Circular speed with Constant radius (r=constant)

$a = -r w^2 u_r$

③ No force in tangential direction

$a = (\ddot r - \dot r\dot\theta^2) u_r + \underbrace{(r\ddot\theta + 2\dot r\dot\theta)}_{=0} u_\theta + \ddot z u_z$

$r\ddot\theta + 2\dot r\dot\theta = \frac{d}{dt}(r^2\dot\theta)\cdot\frac{1}{r}$ apply differential equation

So $\frac{1}{r}\cdot\frac{d}{dt}(r^2\cdot\dot\theta) = 0$

therefore $r^2\cdot\dot\theta = \text{constant}$

fictitious force 伪力 在旋转观察中无切向力，轨道偏移

※ Coriolis Acceleration ⟶ $\underbrace{\text{deflection}}_{\text{偏移}}$ of moving object when viewed in rotation reference frame

$a_{cor} = 2m\dot r \times w$

why 2

两次 product rule

产生两项

由于 $\begin{cases} u_r \text{ 方向变化，产生的切向 acceleration} \\ u_\theta \text{ 大小变化,} \end{cases}$

e.g. 扔小球

以扔球者视角

(切向速度，径向力为 standard)

球扔出去带速但不受力

因此轨道不同

差异就是 a_{cor}

Kinetics of a particle

Concern about the force cause the motion

e.g. We can't prove Newton's law, It's nature not a experimental analysis

Key points
- Mass is measure of resistance to a change in velocity of the object
- Weight depends on local gravitational field
- Unbalanced force cause the acceleration of objects

Consider F = constant

e.g. Newton's second law

$$m \cdot \ddot{r} = F$$

$\downarrow$ 积分

$$\dot{r} = \frac{Ft}{m} + C_1$$

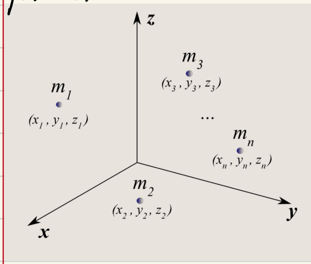
求 C_1, C_2 需有两个 r 在不同 time 然后代回.

$\downarrow$ 积分

$$r = \frac{Ft^2}{2m} + C_1 t + C_2$$

Centre of Mass

particles



assume all mass act at a single points

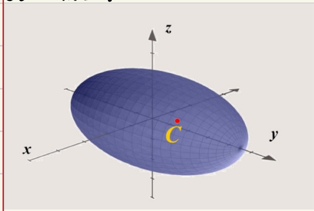
for a object, the centre of mass in a space x.y.z can be calculated by:

$$\begin{cases} \bar{x} = \frac{1}{M}\sum_{i=1}^{n} x_i m_i \\ \bar{y} = \frac{1}{M}\sum_{i=1}^{n} y_i m_i \\ \bar{z} = \frac{1}{M}\sum_{i=1}^{n} z_i m_i \end{cases}$$

centre of mass $(\bar{x}, \bar{y}, \bar{z})$

concrete $\downarrow$ continuous

Continuum



对每个 v

$$M = \int_V dm = \int_V \rho \, dV = \iiint_V \rho(x,y,z) \, dx\,dy\,dz$$

$$\bar{x} = \frac{1}{M}\int_V x\rho \, dV$$

$$\bar{y} = \frac{1}{M}\int_V y\rho \, dV$$

$$\bar{z} = \frac{1}{M}\int_V z\rho \, dV$$

if we only want centre of volume then, ignore the ρ.

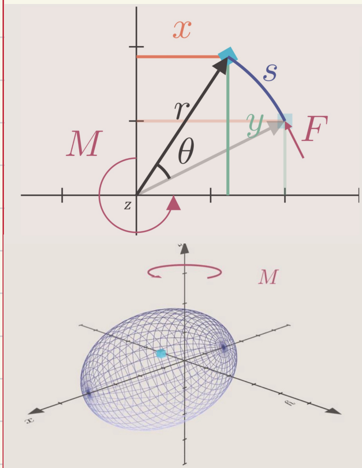
$$\bar{x} = \frac{1}{V}\int_V x \, dV$$

$\longrightarrow$ for Area 同理

$$\bar{x} = \frac{1}{A}\int_A x \, dA$$

转动惯量
moments of Inertia

Describe rotation about a axis



$r=\sqrt{x^2+y^2}$

$s=r\theta$

$F=ma$ $s=r\theta \longrightarrow \dot{s}=r\dot{\theta}$ $\ddot{s}=r\ddot{\theta}$

$dF=dm\,\ddot{s}$ 同时求导

$M=Fr$

$\underbrace{dM}_{\text{moment}}=r^2 dm\,\ddot{\theta}$

$M=(\underbrace{\int_V r^2 dm}_{})\ddot{\theta}$ = moments of Inertia

$M=I_{zz}\ddot{\theta}$

So, $I=\int r^2 dm=\int_V r^{\boxed{2}}\rho\, dV$ 比centre of mass 多个 r

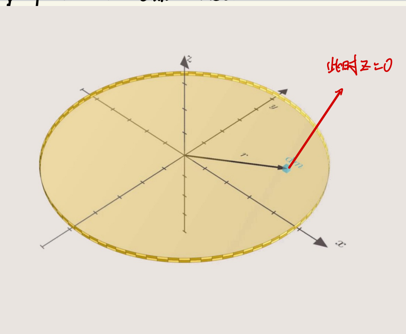
$$\begin{cases} I_{xx}=\int (y^2+z^2)\,dm \\ I_{yy}=\int (x^2+z^2)\,dm \\ I_{zz}=\int (x^2+y^2)\,dm \end{cases}$$

(关于x轴转动) y z

Interesting compare

$F=m\ddot{x}$ $M=I\ddot{\theta}$

perpendicular axis theorem



此时 z=0

$$I_{xx}=\int (y^2+\cancel{z^2})\,\mathrm{d}m$$

$$I_{yy}=\int (x^2+\cancel{z^2})\,\mathrm{d}m$$

$$I_{zz}=\int (x^2+y^2)\,\mathrm{d}m$$

$$\boxed{I_{zz}=I_{xx}+I_{yy}}$$

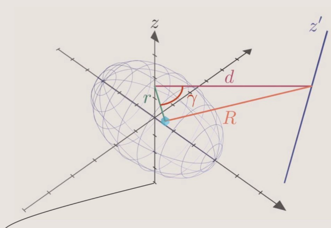
Perpendicular axis theorem

-当质心在一个平面内时

???

Parallel axis theorem

Parallel axis theorem



The axis z **must pass** through the **centre of mass**

condition

$$I_{z'}=\int R^2\ \mathrm{d}m$$

$$R^2=r^2+d^2-2rd\cos(\gamma)$$

$\vec{R}=\vec{r}-\vec{d} \xrightarrow{\text{平方}} R^2=r^2+d^2-\underbrace{2\vec{r}\cdot\vec{d}}_{\text{vector product}}$

$$I_{z'}=\int r^2\ \mathrm{d}m+\int d^2\ \mathrm{d}m-\underbrace{\cancel{\int 2rd\cos(\gamma)\ \mathrm{d}m}}_{r\cos\gamma=x}$$

$$I_{z'}=I_{zz}+d^2M$$

$$\boxed{I_{z'}=I_{zz}+d^2M}$$

Parallel axis theorem

-已知 I_{zz}，且 $I_{z'} \parallel I_{zz}$

then $I_{z'}=I_{zz}+d^2M$

Condition:

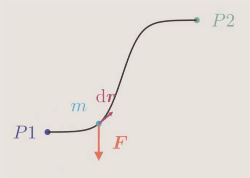
y-axis cross through centre of mass

质心在z轴上不意味 $I_{zz}=0$

只是对称分布

Energy

Work and kinetic energy



$$W_{1-2} = \int_{P1}^{P2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

$$\downarrow$$

$$W_{1-2} = \frac{1}{2} m \left(v_2^2 - v_1^2\right)$$

$$W_{1-2} = \Delta \mathrm{KE}_{1-2}$$

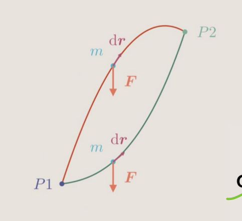
$KE = \frac{1}{2} m v_2$

$Work_{2-1} = \frac{1}{2} m (v_2^2 - v_1^2)$

加速/减速所需能量

Conservative System

- A dynamic system is conservative if the work done moving a particle between two points in space is independent of the path taken



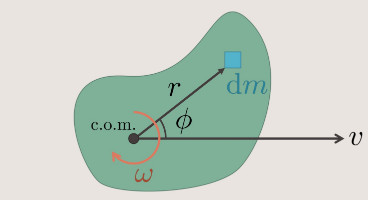
$\underbrace{\int_{P_1}^{P_2} F \cdot dr}_{\text{rout 1}} = \underbrace{\int_{P_1}^{P_2} F dr}_{\text{rout 2}}$

$= \oint F dr$

Conservation of energy > PE + KE = constant in a given system

Systems are conservative when energy is not lost due to damping effects, thermal effects, acoustic effects, etc. We can often assume this to be the case. no energy lost

Kinetic energy of a rotating body



$$\mathrm{KE}_{\text{total}} = \underbrace{\frac{1}{2} m v^2}_{\text{translation}} + \underbrace{\frac{1}{2} I \omega^2}_{\text{rotation}}$$

$E = \frac{1}{2} m v^2$, $v = r \cdot w$

$= \frac{1}{2} \cdot dm \cdot (r \cdot w)^2$

$= \frac{1}{2} w^2 \cdot \int r^2 dm$

$= \frac{1}{2} w^2 \cdot I$

速度=0 KE不一定为0

Impulse: change of momentum

momentum = m·v vector

force = rate of change of momentum

Impulse = change of momentum $J = \int_{t_2}^{t_1} F dt = F \Delta t$

e.g. 安全气囊

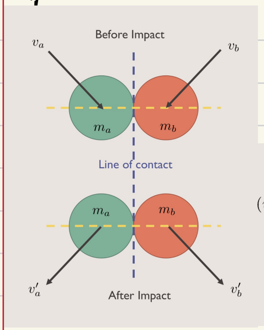
Conservation of momentum

$$\sum_{i=1}^{n} I_i = \sum_{i=1}^{n} m_i v_{i2} - \sum_{i=1}^{n} m_i v_{i1}$$

if impulse $\sum_{i=1}^{n} I_i = 0$, We have $\sum_{i=1}^{n} m_i v_{i2} = \sum_{i=1}^{n} m_i v_{i1}$

external force = 0, conservation of momentum

Impact

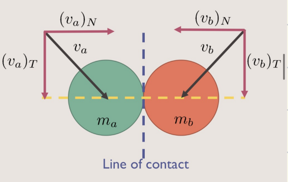


$$\begin{cases} m_a v_a + m_b v_b = m_a v_a' + m_b v_b' \\ e = \dfrac{v_b' + v_a'}{v_a + v_b} \\ K_{E\text{-}B} = K_{E\text{-}A} \end{cases}$$

$e = 1$ means perfectly elastic

$e = 0$ means body remains in contact

只有 $e = 1$ 能用



斜撞考虑分解方向

Angular momentum

$H_A = m \cdot \dot{r} \times r$

angular momentum

$\frac{dr}{dt} = v$

$\int_{t_1}^{t_2} M_A \, dt = H_{A_2} - H_{A_1}$

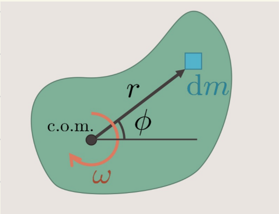
change in angular momentum

= angular impulse

$M_A = \dot{H}_A$

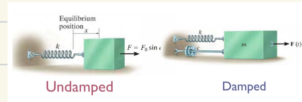
moment = rate of change of momentum

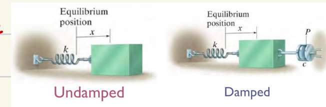
线运动 (Translation)	公式	转动 (Rotation)	公式
位移	x	角位移	θ
速度	$v = \dot{x}$	角速度	$\omega = \dot{\theta}$
加速度	$a = \dot{v}$	角加速度	$\alpha = \dot{\omega}$
质量	m	转动惯量	$I = \int r^2 dm$
动量	$p = mv$	角动量	$L = I\omega$
牛顿第二定律	$F = \frac{dp}{dt} = ma$	转动定律	$M = \frac{dL}{dt} = I\alpha$
冲量	$J = \int F dt$	角冲量	$J_\theta = \int M dt$
冲量定理	$\int F dt = p_2 - p_1$	角冲量定理	$\int M dt = L_2 - L_1$
功	$W = Fs$	转动功	$W = M\theta$
动能	$T = \frac{1}{2}mv^2$	转动动能	$T = \frac{1}{2}I\omega^2$
功能定理	$W = \Delta(\frac{1}{2}mv^2)$	转动功能定理	$W = \Delta(\frac{1}{2}I\omega^2)$
动量守恒	$\sum p = \text{const}$	角动量守恒	$\sum L = \text{const}$



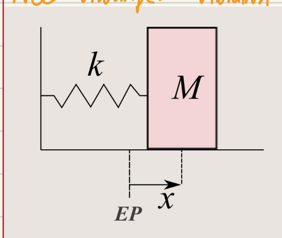
One degree of freedom Vibrations

Types of Vibration:
- Free Vibration $\longrightarrow$ no external force
- Forced Vibration $\longrightarrow$ under external force





Free Undamped Vibration



$F=-kx$

Step1: Draw free-body diagram

Step2: Use Newton's 2nd Law $F=ma$ to obtain equation of motion

Step3: Set up differential equation

e.g. $-kx = ma$

$-kx = m\ddot{x}$

注: damped 与否在于是否考虑摩擦,空气阻力

forced 与否在于是否有外部力持续供能

$kx + m\ddot{x} = 0$ $\quad \omega_n = \sqrt{\frac{k}{m}}$ $\longleftarrow$ $x = A\sin(\omega_n t) + B\cos(\omega_n t)$

$x(t) = A\cos\omega_n t + B\sin\omega_n t$ $\quad \ddot{x} = -A\omega^2\sin(\omega_n t) - B\omega_n^2\cos(\omega_n t)$

$x(t) = C\cos(\omega_n t + \phi)$ $\quad$ 代入原式 $0 = (A\frac{k}{m} - A\omega_n^2)\sin(\omega_n t)$

$+ (B\frac{k}{m} - A\omega_n^2)\sin(\omega_n t)$

$\omega_n = \sqrt{\frac{k}{m}}$

then, $\ddot{x} + \omega_n^2 x = 0$

Free damped Vibration

- different kind of damping

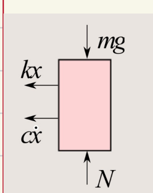
- Coulomb/friction damping — due to surface contact
- Hydraulic damping — due to turbulent fluid motion
- Viscous damping — our focus on this course
- Internal damping — due to viscous material properties
- Etc.

① Viscous Damping

Viscous damping load is proportional to the velocity.

$F = -c\frac{dx}{dt} = -c\dot{x}$ $\quad M = -c\frac{d\theta}{dt} = -c\dot{\theta}$ $\quad$ c: Viscous damping coefficient

linear damper $\quad$ angular damper



$\Sigma F_x = ma$

$-kx - c\dot{x} = ma$

$m\ddot{x} + c\dot{x} + kx = 0$

Natural frequency: $\omega_n = \sqrt{\frac{k}{m}}$

Damping ratio $\zeta = \frac{c}{2m\omega_n}$

$\ddot{x} + 2\zeta\omega_n\dot{x} + \underbrace{\omega_n^2 x}_{\text{remain to } \omega_n^2 x} = 0 \longrightarrow x(t) = A_1 e^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + A_2 e^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t}$

Specicial cases

$$x(t) = A_1 e^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + A_2 e^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t}$$

$\zeta = 1$ Critically-damped vibration.
Occurs when $c = c_0 = 2\sqrt{km}$ (Critical damping coefficient)

$\zeta > 1$ Over-damped vibration

$\zeta < 1$ Under-damped vibration

Damping ratio can be rewritten as $\zeta = \frac{c}{c_0}$

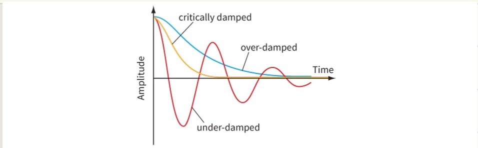
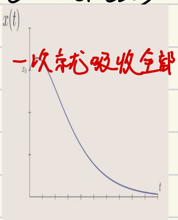


Figure 18.35: Critical damping is just enough to ensure that a damped system returns to equilibrium without oscillating.

Critically-damped vibration

$\zeta = 1$

$x(t) = e^{-\omega t}(C_1 + C_2 t)$

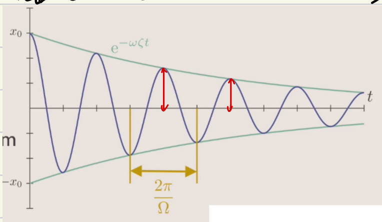


一次就吸收全部能量

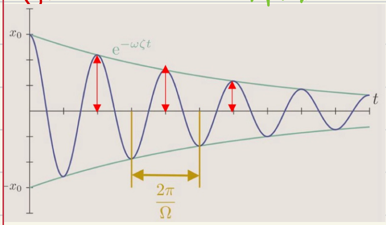
Under-damped vibration

$\zeta < 1$

$x(t) = e^{-\zeta\omega_n t}(C_1\cos(\Omega t) + C_2\sin(\Omega t))$

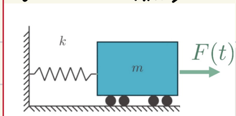


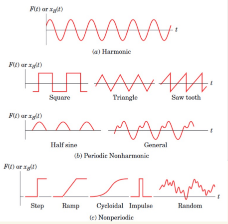
Logarithmic Decrement 用来算 next Amplitude



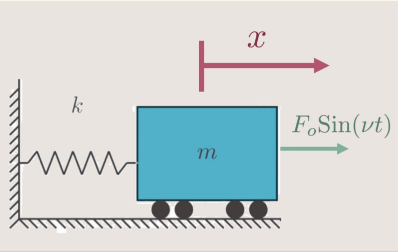
$$\ln\left(\frac{x(t)}{x(t+T)}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

Force vibrations





Forced dependents Loads



Force undamped vibration

Equation of motion:

$m\ddot{x} + kx = F_o \sin(\nu t)$

ν: forcing frequency

Solution $x(t) = \underbrace{x_c(t)}_{\text{solution of } m\ddot{x}+kx=0 \text{ complementary solution}} + \underbrace{x_p(t)}_{\text{particular solution}}$

设 $x_p(t) = A\sin(vt)$

$\ddot{x}_p = -Av^2\sin(vt)$

$m(-Av^2\cancel{\sin vt}) + k(A\cancel{\sin vt}) = F_0\cancel{\sin vt}$

for particular solution $x_p(t)$

$$x_p(t) = A\sin(vt)$$
$$A = \frac{F_0/k}{1-(v/\omega_n)^2} \longleftarrow A(k - mv^2) = F_0$$

$\omega_n = \sqrt{\frac{k}{m}} \longrightarrow m = \frac{k}{\omega_n^2}$ 代

Amplification factor

$$M = \frac{A}{F_0/k} = \frac{1}{1-(v/\omega_n)^2}$$

力慢慢施加时的位移

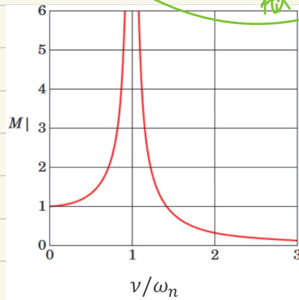
决定分母符号

What happens when $\frac{v}{\omega_n} < 1$?

A is + (in phase)

What happens when $\frac{v}{\omega_n} > 1$?

A is − (+π)



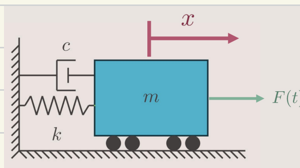
phase = 0 means 每次推，幅度↑

phase = π means 每次推，幅度↓

if
- $M \approx 1$ 动态效果≈静态效果 resonance
- $M \to \infty$ 共振，动态力会毁了system
- $M \to 0$ 动态几乎不影响

Force damped

Equation of motion



$$m\ddot{x} + c\dot{x} + kx = F(t)$$
$$x(t) = x_c(t) + x_p(t)$$

Inertial force + Damping force + Stiffness force = External force

forcing function $F(t) = F_0\cos(vt) \longrightarrow x(t) = A\cos(vt - \phi)$

Amplitude unknown

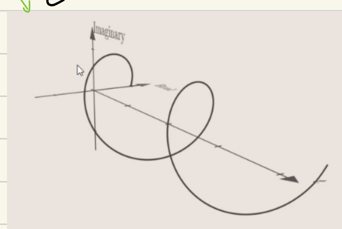
frequency of force, known

phase difference unknown

$F(t) = F_0 e^{\omega t} \longrightarrow x(t) = Ae^{i(vt-\phi)}$ 为什么不用丢Im( )? - 等到最后再丢

So, $F(t) = F_0 e^{ivt}$ $\quad e^{ivt}$

$F(t) = F_0\cos(vt) + iF_0\sin(vt)$



$$-mr^2Ae^{\cancel{i(vt-\phi)}} + CirAe^{\cancel{i(vt-\phi)}} + kAe^{\cancel{i(vt-\phi)}} = F_0 e^{\cancel{ivt}}$$

$$-mr^2A + CivA + kA = F_0$$

$$A(\underbrace{-mv^2 + k}_{Re} + \underbrace{Cvi}_{Im}) = F_0$$

$= Re^{i\phi}$

$$A = \frac{F_0}{\sqrt{(k-mr^2)^2 + (Cv)^2}} \longleftarrow \text{always positive 取绝对值}$$

$$\tan\phi = \frac{Cv}{k - mv^2}$$

Magnification Factor 与1比较，跟前同理

$$M = \frac{A}{F_0/k} = \frac{\frac{\cancel{F_0}}{\sqrt{(k-mv^2)^2+(cv)^2}}}{\frac{\cancel{F_0}}{k}} = \frac{k}{\sqrt{(k-mv^2)^2+(cv)^2}} = \frac{1}{\sqrt{(1-(\frac{v}{\omega_n})^2)^2 + 4\zeta^2(\frac{v}{\omega_n})^2}}$$

let $\omega_n = \sqrt{\frac{k}{m}}$ in, $m = \frac{k}{\omega_n^2}$

Resonance

$V_c = \omega_n\sqrt{1-2\zeta^2}$ 时，Amplitude 最大

= (resonant frequency) not the same as natural frequency

Transmitted force

Define F_{T_0} : $F_T(t) = \underbrace{kx(t)}_{\text{弹簧力}} + \underbrace{c\dot{x}(t)}_{\text{阻尼力}}$

Solution $x(t) = A\sin(\nu t - \phi)$

$\dot{x} = A\nu\cos(\nu t - \phi)$

代入，$F_T(t) = kA\sin(\nu t-\phi) + cA\nu\cos(\nu t - \phi)$

$F_T(t) = F_{T_0}\sin(\nu t - \phi + \theta)$ 合并

$F_{T_0} = \sqrt{(kA)^2 + (c\nu A)^2}$

Force Transmissibility 传给地基的力

$T = \frac{F_{T_0}}{F_0}$

代入 $A = \frac{F_0}{\sqrt{(k-m\nu^2)^2+(c\nu)^2}}$

$T>1$ 传给地基的力 > 外力

$T=1$ 传给地基的力 = 外力

$T<1$ 传给地基的力 < 外力 Vibration Isolation

$$T = \sqrt{\frac{k^2+c^2\nu^2}{(k-m\nu^2)^2+(c\nu)^2}} = \sqrt{\frac{1+(2\zeta\frac{\nu}{\omega_n})^2}{1+(\frac{\nu}{\omega_n})^2((\frac{\nu}{\omega_n})^2+4\zeta^2-2)}}$$

同理可推得 $\frac{X}{Y} = T = \sqrt{\frac{k^2+c^2\nu^2}{(k-m\nu^2)^2+(c\nu)^2}} = \sqrt{\frac{1+(2\zeta\frac{\nu}{\omega_n})^2}{1+(\frac{\nu}{\omega_n})^2((\frac{\nu}{\omega_n})^2+4\zeta^2-2)}}$

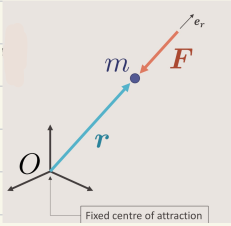
Orbital Mechanics

Central force motion

universal Gravitational constant $6.67\times10^{-11} m^3 kg^{-1} s^{-2}$

$F = -\frac{GMm}{|r|^2} e_r$

attractive 单位向量



Two-body problems

- if $M \gg m$, m is negligible.

  then, 小质量绕固定中心运动

Kepler's laws of Planetary motion

$r = \frac{h^2}{GM(1+e\cos\phi)}$

1st. The orbits of planets are ellipses with the sun at one focus

椭圆轨道，太阳焦点，

2nd. The line joining a planet to the sun seeps out area at a constant rate

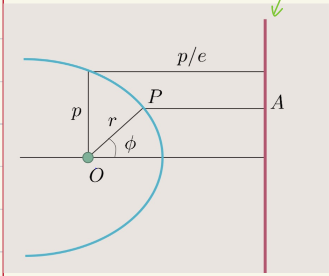
行星与太阳连线扫过面积速率恒定 $\frac{dA}{dt} = \frac{h}{2}$

3rd. The square of the period of a planet's orbit is proportional to the cube of the major axis of this orbit

$T^2 \propto r^3$

Conic section

directrix

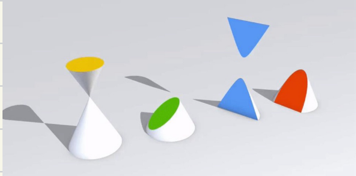


由一个 forcus 和一条 line 组成

$e = \frac{PO}{PA}$ 焦点 准线 离心率

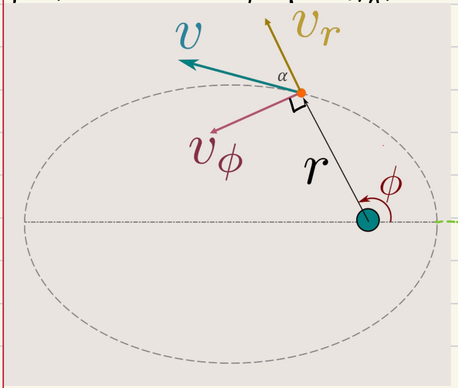
$PA = \frac{p}{e} - r\cos(\phi)$

$$\boxed{r = \frac{p}{1+e\cos\phi}}$$



eccentricity
- $e=0$ circle
- $e<1$ ellipse 椭圆
- $e=1$ parabola
- $e>1$ hyperbola 双曲

Polar coord. and specific angular momentum



$\phi = 0$

Define Polar coord:

Angular momentum is conserved:

$h = r \times v$ cross product

$h = r \cdot v\sin(\alpha)$ (取与 r 垂直的)

$h = \underbrace{r \cdot v_\phi}_{\text{constant}}$

Derivation of kepler's 1st and 2nd Law

$\dot{r}$ polar coordinate, radius direction, transverse direction

$$F = \frac{-GMm}{r^2}\hat{e}_r = m[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + \underbrace{(r\ddot{\theta} + 2\dot{r}\dot{\theta})}_{0}\hat{e}_\theta]$$

$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$

$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0$ 同乘 r

$\frac{d}{dt}(r^2\dot{\theta}) = 0 \rightarrow r^2\dot{\theta}$ is constant 替换

prove of kepler's 1st

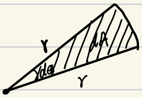
for radius direction

$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}$

$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{GM}{r^2}$

$\frac{1}{r} = \frac{GM}{h^2}(1+e\cos(\phi+\psi))$

$r = \underbrace{\frac{h^2}{GM(1+e\cos(\phi+\psi))}}_{\text{conic}}$



$dA = \frac{1}{2}r^2 d\theta$

$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta}$

$\dot{A} = \frac{1}{2}\underbrace{r^2\dot{\theta}}_{\text{constant}}$

prove of kepler's 2nd

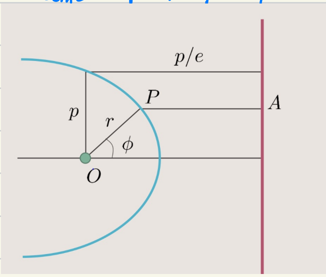
$\dot{A} = \frac{1}{2}h$

Prove of kepler's third law

$T = \frac{A}{\dot{A}} = \frac{\pi a b}{\frac{h}{2}}$

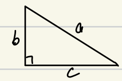
from kepler's 1st law, $\frac{1}{r} = \frac{GM}{h^2} + \frac{GM}{h^2} e\cos\phi$ $\longrightarrow p = \frac{h^2}{GM} \longrightarrow h = \sqrt{pGM}$

conic : $\frac{1}{r} = \frac{1}{p} + \frac{e}{p}\cos\phi$



For the ellipse in Cartesian coordinates

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$



$c^2 = a^2 - b^2$

So $\left(\frac{c}{a}\right)^2 + \left(\frac{p}{b}\right)^2 = 1$

$\frac{a^2 - b^2}{a^2} + \frac{p^2}{b^2} = 1$

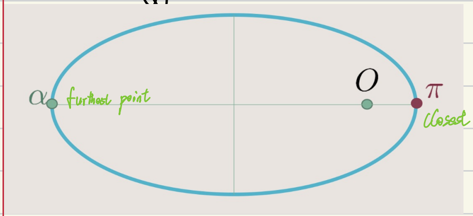
$\frac{b^2}{a^2} = \frac{p^2}{b^2} \longrightarrow p = \frac{b^2}{a}$

So $h = \frac{b}{\sqrt{a}}\sqrt{GM}$

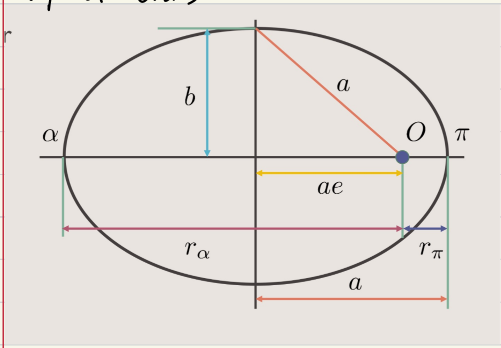
$T = \frac{2\pi a b}{\frac{b}{\sqrt{a}}\sqrt{GM}} = \frac{2\pi a^{3/2}}{\sqrt{GM}}$

$T^2 \propto a^3$

Orbit Terminology



Elliptical Orbits



$a = (r_\pi + r_\alpha)/2$

$b = \sqrt{r_\pi r_\alpha} = a\sqrt{1-e^2}$

Speed in A and B ?

$r = \frac{h^2}{GM(1+e\cos\phi)}$

for π $\phi = 0$ for α $\phi = \pi$

$r(\phi=0) = \frac{h^2}{GM(1+e)}$ $r(\pi) = \frac{h^2}{GM(1-e)}$

$\frac{r_\alpha}{r_\pi} = \frac{1+e}{1-e} = \frac{18.3}{4.6}$ $e = 0.4131$

$r_\pi = \frac{h^2}{GM(1+e)}$ $h = r_\pi \cdot V_A$

$V_A = \sqrt{\frac{GM \times 1.4131}{r_\pi}}$, $GM = 3.9867 \times 10^{14}\, m^2/s^2$

$V_A = 5547\, m/s$

Specific energy

- Gravitational PE per unit mass.

$$-\frac{GM}{r}$$ Potential

- Specific energy of orbital

$$E = KE + GPE$$

$$r_\pi = \frac{h^2}{GM(1+e)}$$

$$h^2 = r_\pi GM(1+e) \qquad h = r_\pi V_\pi$$

$$V_\pi^2 = \frac{GM(1+e)}{r_\pi}$$

$$E = \frac{1}{2}\frac{GM(1+e)}{r_\pi} - \frac{GM}{r_\pi}$$

$$E = \frac{GM(e-1)}{r_\pi} = \frac{GM(e-1)}{2\cdot\left(\frac{h^2}{GM(1+e)}\right)} = \frac{(GM)^2(e^2-1)}{2h^2}$$

for Parabolic orbit

$e=1 \quad E=0$

for Elliptical orbit

$0<e<1 \longrightarrow E<0$

$$E = \frac{GM(e-1)}{2r_\pi}$$

$$E = -\frac{GM}{2a}$$

$r_\pi = a - a_E$

for circular orbit

$e=0 \quad E = -\frac{GM}{2\cdot r_c}$

for circular Orbits

1st law $\quad r = \frac{h^2}{GM}$

2nd law $\quad GM = gR^2$

Apply Kepler's first law in circular orbit

$$r_c = \frac{h^2}{GM(1+e\cos\theta)} \ ; \ e=0$$

$$r_c = \frac{h^2}{GM} \ ; \ h = r_c \cdot V_c$$

$$r_c = \frac{r_c^2 \cdot V_c^2}{GM} \qquad V_c = \sqrt{\frac{GM}{r_c}}$$

Velocity in orbit

Synchronous orbit 同步轨道 (same period)

$$a_{geo} = \left(\frac{T^2 g R^2}{4\pi^2}\right)^{\frac{1}{3}}$$

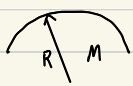
apply Kepler's 3rd law

$$GM = \left(\frac{2\pi}{T}\right)^2 a^3$$

$$a = \left(\frac{gR^2T^2}{4\pi^2}\right)^{\frac{1}{3}}$$

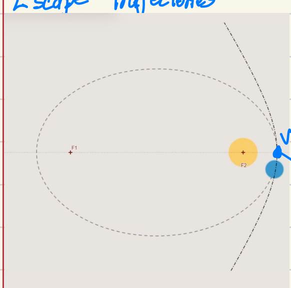
equilibrium

$$mg = \frac{GMm}{r^2}$$



$GM = gR^2$

Escape Trajectories

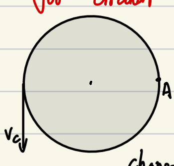


when $\begin{cases} e \geq 1 \longrightarrow \text{hyperbolic} \\ e = 1 \longrightarrow \text{parabolic trajectory, minimum energy} \end{cases}$

Escape velocity

$$\Delta V = (\sqrt{2}-1)V_c$$

for circular orbit



Circular orbit $e=0$ $\quad V_c = \sqrt{\frac{GM}{r}}$

$$E = \frac{V_c^2}{2} - \frac{GM}{r}$$

So $\Delta V = (\sqrt{2}-1)V_c$

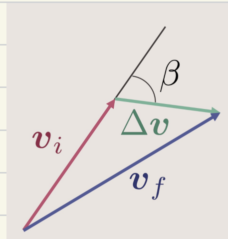
At the point A

change to parabolic $e=1$

$$E = 0 = \frac{V_E^2}{2} - \frac{GM}{r} \longrightarrow V_E = \sqrt{\frac{2GM}{r}}$$

Orbits transfers

-We want to transfers $\begin{cases} \text{efficiently to minimize } \Delta V, \\ \text{fast in shortest time.} \end{cases}$



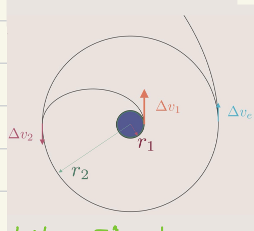
Velocity Impulse (instantaneous velocity impulse ΔV)

$$V_f = V_i + \Delta V$$

$$V_f^2 = V_i^2 + \Delta V^2 + 2 V_i \Delta V \cos\beta$$

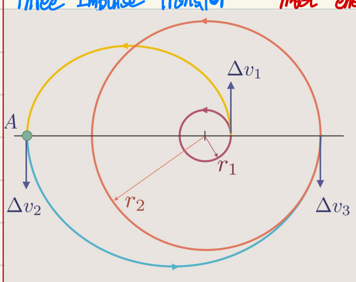
$$\longrightarrow \Delta E = \frac{1}{2}\Delta V^2 + V_i \Delta V \cos(\beta)$$

The Hohmann Transfer (circular to circular)



大转小ΔE为negative

Three Impulse Transfer most energy efficient way



Orbitals

Kepler's three laws $\qquad h = r \cdot v_\phi$

① $r = \dfrac{h^2}{GM(1+e\cos(\phi))}$ $\quad \cos\phi$ 用来算 $r_\alpha \quad r_\pi \longrightarrow \dfrac{r_\pi}{r_\alpha} = \dfrac{1-e}{1+e}$

② $\dfrac{dA}{dt} = \dfrac{h}{2} \longrightarrow \dfrac{A}{T} = \dfrac{h}{2}$

$A = \pi ab$

T — 一个period

③ $GM = \left(\dfrac{2\pi}{T}\right)^2 a^3$

$a = \dfrac{r_\alpha + r_\pi}{2} = \sqrt{r_\alpha r_\pi}$, $b = a\sqrt{1-e^2}$

Orbits transfer

$\Delta V = (\sqrt{2}-1)V_c$

- Hohmann $\longrightarrow$ 时间最少
- three impulse $\longrightarrow$ 能量最少，时间更长

  condition : $r_2 > 12 r_1$