Partial Differential Equation

Linear ODEs Recap

Direct Integration $\frac{dy}{dx} = f(x) \longrightarrow y = \int f(x)\,dx$

Seperation of Variables $\frac{dy}{dx} = f(x)g(y) \longrightarrow \int \frac{1}{g(y)}\,dy = \int f(x)\,dx$

Homogenous $\frac{dy}{dx} = f(\frac{y}{x}) \longrightarrow$ let $u = \frac{y}{x}$, $\frac{dy}{dx} = u + \frac{du}{dx}x = f(u)$

General Linear $\frac{dy}{dx} + f(x)y = g(x) \longrightarrow$ let $I = e^{\int f(x)dx}$, $\frac{d(Iy)}{dx} = I(x)g(x)$

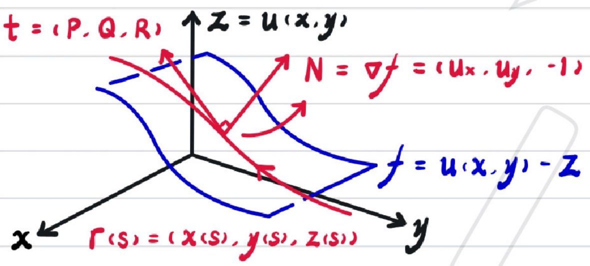
Exact $\frac{dy}{dx} = -\frac{f(x,y)}{g(x,y)}$ where $\frac{df}{dy} = \frac{dg}{dx} \longrightarrow u(x,y) = \int f\,dx = \int g\,dy$

Inexact $\frac{dy}{dx} = -\frac{f(x,y)}{g(x,y)}$ where $\frac{df}{dy} \neq \frac{dg}{dx} \longrightarrow \frac{d(If)}{dy} = \frac{d(Ig)}{dx} \longrightarrow u = \int If\,dx = \int Ig\,dy$

Bernouli $\frac{dy}{dx} + f(x)y = g(x)y^k \longrightarrow$ let $y = u^{\frac{1}{1-k}}$, $\frac{dy}{dx} = \frac{1}{1-k}u^{\frac{1}{1-k}-1}\frac{du}{dx} = gu^{\frac{k}{1-k}} - fu^{\frac{1}{1-k}}$

First Order PDEs

General Solution Method



$t \cdot N = 0$

$\longrightarrow (P, Q, R) \cdot (u_x, u_y, -1)$

$\longrightarrow P(x,y)\frac{\partial u}{\partial x} + Q(x,y)\frac{\partial u}{\partial y} = R(x,y,u)$

let $\frac{dr(s)}{ds} = t \longrightarrow (\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}) = (P, Q, R) \longrightarrow \frac{dy}{dx} = \frac{Q}{P}$ & $\frac{dz}{dx} = \frac{du}{dx} = \frac{R}{P}$

$\longrightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R}$ for $R \neq 0$

$\frac{dx}{P} = \frac{dy}{Q}$ & $du = 0$ for $R = 0$

e.g. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \longrightarrow \frac{dx}{1} = \frac{dy}{1}$ & $du = 0 \longrightarrow \frac{dy}{dx} = 1$ & $du = 0$

$\longrightarrow \underbrace{y = x + C_1}_{\text{characteristic curve}}$ & $u = C_2 \longrightarrow u = f(C_1) = f(y - x)$

Boundary Conditions

e.g. $y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = x^2 + y^2$, $u(x,y) = 1 + x^2$ when $y = 0$, $u(x,y) = 1 + y^2$ when $x = 0$

$$\longrightarrow \frac{dx}{y} = \frac{dy}{x} = \frac{du}{x^2+y^2} \longrightarrow \frac{dy}{dx} = \frac{x}{y} \ \& \ \frac{du}{dx} = \frac{x^2}{y} + y = x\frac{dy}{dx} + y = \frac{d}{dx}(xy)$$

$$\longrightarrow y^2 = x^2 + C_1 \ \& \ u = xy + C_2 \longrightarrow u = xy + f(y^2 - x^2)$$

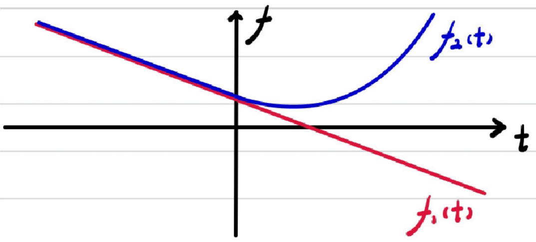
$$\longrightarrow 1 + x^2 = f(-x^2) \ \& \ 1 + y^2 = f(y^2) \longrightarrow f(t) = 1 + |t| \longrightarrow u = xy + 1 + |y^2 - x^2|$$

Domain of Definition

specific boundary condition only allow us determine f for a certain range of values

Uniqueness

for $f(t) = 1 - 2t$ when $t \le 0$



$$f_1(t) = 1 - 2t \quad \text{for } -\infty < t < \infty$$

$$f_2(t) = \begin{cases} 1 - 2t & \text{for } t \le 0 \\ (t-1)^2 & \text{for } t > 0 \end{cases}$$

Second Order PDEs

Characteristic Equation

- first order

$$\begin{bmatrix} P & Q \\ dx & dy \end{bmatrix}\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} R \\ du \end{bmatrix} \longrightarrow \begin{vmatrix} P & Q \\ dx & dy \end{vmatrix} = P\,dy - Q\,dx = 0 \longrightarrow \frac{dy}{dx} = \frac{Q}{P}$$

- second order

$$\begin{bmatrix} R & 2S & T \\ dx & dy & 0 \\ 0 & dx & dy \end{bmatrix}\begin{bmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{bmatrix} = \begin{bmatrix} f(x, y, u_x, u_y) \\ d(u_x) \\ d(u_y) \end{bmatrix}$$

$$\longrightarrow \begin{vmatrix} R & 2S & T \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = R(dy)^2 - 2S\,dx\,dy + T(dx)^2 = R\left(\frac{dy}{dx}\right)^2 - 2S\frac{dy}{dx} + T = 0$$

$$\longrightarrow \frac{dy}{dx} = \frac{S \pm \sqrt{S^2 - RT}}{R}$$

Standard (canonical) Form

$$\frac{dy}{dx} = a \ \& \ b \longrightarrow y = \begin{cases} \int a\,dx = A + C_1 \\ \int b\,dx = B + C_2 \end{cases} \quad \text{where} \begin{cases} C_1 = \xi(x, y) \\ C_2 = \eta(x, y) \end{cases}$$

Change of Variable

- $u_x = u_\xi \xi_x + u_\eta \eta_x$ - $u_y = u_\xi \xi_y + u_\eta \eta_y$
- $u_{xx} = \frac{\partial}{\partial x} u_x$ - $u_{yy} = \frac{\partial}{\partial y} u_y$ - $u_{xy} = \frac{\partial}{\partial y} u_x = \frac{\partial}{\partial x} u_y$

Hypobolic PDEs

$S^2 - RT > 0 \longrightarrow$ 2 real solution $\longrightarrow f(u_{\xi\eta}) = \ldots$

* wave equation $u_{xx} - u_{yy} = 0$

e.g. $2u_{xx} + 6u_{xy} + 4u_{yy} + 2 = 0$, $u = 2y^2$ and $u_x = -3y$ when $x = 0$

$$\longrightarrow \frac{dy}{dx} = \frac{3 \pm \sqrt{9-8}}{2} = 1.5 \pm 0.5 = 2 \text{ or } 1 > 0$$

$$\longrightarrow \begin{cases} y = 2x + C_1 \\ y = x + C_2 \end{cases} \longrightarrow \begin{cases} \xi = y - 2x \\ \eta = y - x \end{cases} \longrightarrow \begin{cases} u_x = -2u_\xi - u_\eta \\ u_y = u_\xi + u_\eta \end{cases}$$

$$\longrightarrow \begin{cases} u_{xx} = 4u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \\ u_{xy} = -2u_{\xi\xi} - 3u_{\xi\eta} - u_{\eta\eta} \\ u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \end{cases} \longrightarrow 2u_{xx} + 6u_{xy} + 4u_{yy} = -2u_{\xi\eta} = -2$$

$$\longrightarrow u_{\xi\eta} = 1 \longrightarrow u_\xi = \int u_{\xi\eta}\, d\eta = \eta + F(\xi) \longrightarrow u = \int u_\xi\, d\xi = \xi\eta + f(\eta) + g(\xi)$$

$$\longrightarrow u = (y-2x)(y-x) + f(y-2x) + g(y-x) \longrightarrow u_x = (4x-3y) - 2f'(y-2x) - g'(y-x)$$

$$\longrightarrow 2y^2 = y^2 + f(y) + g(y) \;\;\&\;\; -3y = -3y - 2f'(y) - g'(y)$$

$$\longrightarrow f(y) + g(y) = y^2 \qquad \begin{cases} f'(y) + g'(y) = 2y \\ -2f'(y) - g'(y) = 0 \end{cases}$$

$$\longrightarrow \begin{cases} f'(t) = -2t \\ g'(t) = 4t \end{cases} \longrightarrow \begin{cases} f(t) = -t^2 \\ g(t) = 2t^2 \end{cases} \longrightarrow u = (y-2x)(y-x) - (y-2x)^2 + 2(y-x)^2$$

Parabolic PDEs

$S^2 - RT = 0 \longrightarrow$ 1 real solution $\longrightarrow$ chose η to satisfy $\xi_x \eta_y - \xi_y \eta_x \neq 0$

* heat equation $u_{xx} = u_t$

Elliptic PDEs

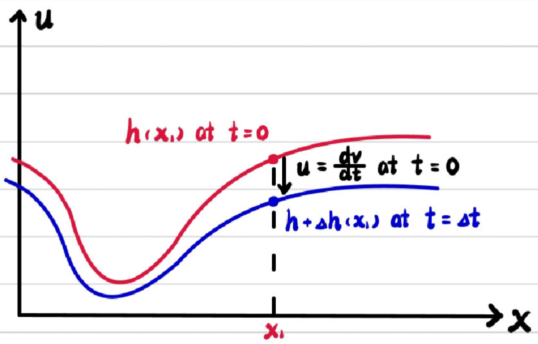
$S^2 - RT < 0 \longrightarrow C_{1,2} = a \pm ib \longrightarrow \xi = a,\ \eta = b \longrightarrow f(u_{\xi\xi}, u_{\eta\eta}) = \dots$

* Laplace's equation $u_{xx} + u_{yy} = 0$

The Wave Equation

Wave Equation

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow u_{tt} - c^2 u_{xx} = 0 \longrightarrow \sqrt{S^2 - RT} = c > 0$ hypobolic



$\longrightarrow \frac{dx}{dt} = \frac{0 \pm \sqrt{-(-c^2)}}{1} = \pm c$

$\longrightarrow x = \begin{cases} ct + C_1 \\ -ct + C_2 \end{cases} \longrightarrow \begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$

$\longrightarrow u_{\xi\eta} = 0 \longrightarrow u = f(x-ct) + g(x+ct)$

D'Alembert's Solution

initial condition $\longrightarrow$ at $t=0$ $h(x) = u(x,0)$ & $v(x) = u_t(x,0)$

$$\longrightarrow \begin{cases} h(x) = f(x-ct) + g(x+ct)\big|_{t=0} = f(x) + g(x) \\ v(x) = -cf'(x-ct) + cg'(x-ct)\big|_{t=0} = -cf'(x) + cg'(x) \end{cases}$$

$$\longrightarrow \int_a^x v(y)\,dy = \int_a^x -cf'(y) + cg'(y)\,dy$$

$$\longrightarrow \frac{1}{c}\int_a^x v(y)\,dy = [-f(x) + g(x)] - [-f(a) + g(a)]$$

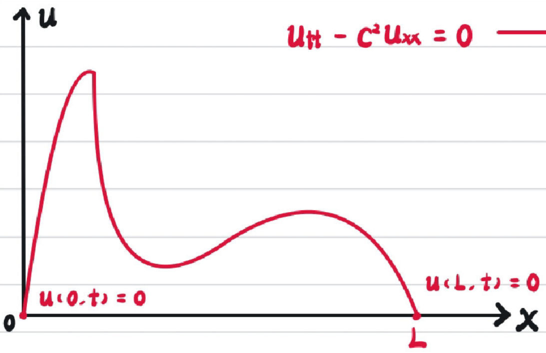
$$\longrightarrow \begin{cases} \frac{1}{c}\int_a^x v(y)\,dy + h(x) = 2g(x) - [-f(a) + g(a)] \\ \frac{1}{c}\int_a^x v(y)\,dy - h(x) = -2f(x) - [-f(a) + g(a)] \end{cases}$$

$$\longrightarrow \begin{cases} g(x) = \frac{1}{2c}\int_a^x v(y)\,dy + \frac{1}{2}h(x) + \frac{1}{2}[-f(a) + g(a)] \\ f(x) = -\frac{1}{2c}\int_a^x v(y)\,dy + \frac{1}{2}h(x) - \frac{1}{2}[-f(a) + g(a)] \end{cases}$$

$$\longrightarrow u(x,t) = f(x-ct) + g(x+ct) = \frac{1}{2}[h(x-ct) + h(x+ct)] + \frac{1}{2c}\left[\int_a^{x+ct} v(y)\,dy - \int_a^{x-ct} v(y)\,dy\right]$$

$$= \frac{1}{2}[h(x-ct) + h(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v(y)\,dy$$

Seperation of Variables for wave equation on a finite domain



$u_{tt} - c^2 u_{xx} = 0 \longrightarrow$ for $u(x,t) = X(x)T(t)$, $X\frac{\partial^2 T}{\partial t^2} - c^2 \frac{\partial^2 X}{\partial x^2} T = 0$

$\longrightarrow \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda^2$

$$\longrightarrow \begin{cases} X'' + \lambda^2 X = 0 \\ T'' + \lambda^2 c^2 T = 0 \end{cases}$$

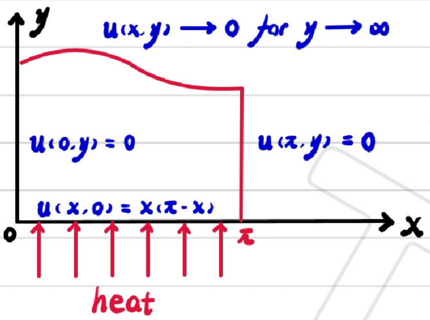
$$\longrightarrow \begin{cases} X = A\cos(\lambda x) + B\sin(\lambda x) \longrightarrow \begin{cases} x = 0,\ A = 0 \\ x = L,\ A\cos(\lambda L) + B\sin(\lambda L) = 0 \end{cases} \longrightarrow \sin(\lambda L) = 0 \longrightarrow \lambda = \frac{n\pi}{L} \\ T = C\cos(\lambda c t) + D\sin(\lambda c t) \longrightarrow C\cos(\frac{n\pi c}{L}t) + D\sin(\frac{n\pi c}{L}t) \end{cases}$$

$$\longrightarrow u = XT = \underbrace{\sum_{n=1}^{\infty}}_{\text{sum of all possible solutions}} B\sin(\frac{n\pi}{L}x)[C\cos(\frac{n\pi c}{L}t) + D\sin(\frac{n\pi c}{L}t)] = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L}x)[c_n\cos(\omega_n t) + d_n\sin(\omega_n t)]$$

for initial condition
$$\begin{cases} h(x) = u(x,0) = \sum_{n=1}^{\infty} c_n\sin(\frac{n\pi}{L}x) \xrightarrow{FS} c_n = \frac{2}{L}\int_0^L h(x)\sin(\frac{n\pi}{L}x)\,dx \\ v(x) = u_t(x,0) = \sum_{n=1}^{\infty} \omega_n d_n \sin(\frac{n\pi}{L}x) \xrightarrow{FS} \omega_n d_n = \frac{2}{L}\int_0^L u(x)\sin(\frac{n\pi}{L}x)\,dx \end{cases}$$

Laplace's Equation

Seperation of Variables heat diffusion in a plate



$u_{xx} + u_{yy} = 0 \longrightarrow$ for $u(x,y) = X(x)Y(t)$, $X''Y + XY'' = 0$

$\longrightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$

$$\longrightarrow \begin{cases} X'' + \lambda^2 X = 0 \longrightarrow X = A\cos(\lambda t) + B\sin(\lambda t) \\ Y'' - \lambda^2 Y = 0 \longrightarrow Y = Ce^{\lambda y} + De^{-\lambda y} \end{cases}$$

$$\longrightarrow \begin{cases} \text{for } x = 0,\ A = 0 \\ \text{for } x = \pi,\ A\cos(\lambda\pi) + B\sin(\lambda\pi) = 0 \end{cases} \longrightarrow \sin(\lambda\pi) = 0 \longrightarrow \lambda = n \text{ for } n = 1, 2, 3, \ldots$$

$$\longrightarrow u = XY = \sum_{n=1}^{\infty} B\sin(nx)(Ce^{\lambda y} + De^{-\lambda y}) = \sum_{n=0}^{\infty} \sin(nx)(c_n e^{n y} + d_n e^{-n y})$$

for initial condition
$$\begin{cases} u(x,\infty) = 0 = \sum_{n=0}^{\infty} c_n e^{n\infty} \longrightarrow c_n = 0 \\ u(x,0) = x(\pi - x) = \sum_{n=0}^{\infty} d_n \sin(nx) \longrightarrow d_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)\,dx \end{cases}$$

* orthogonal properties $\langle e^{jk\omega_0 t}, e^{jl\omega_0 t}\rangle = \int_T e^{j(k-l)\omega_0 t} = \begin{cases} T & k=l \\ 0 & k\neq l \end{cases}$

Similarity Solution

heat diffusion on an infinite domain

in one-dimension $u_t = ku_{xx}\ (-\infty < x < \infty)$ $\longrightarrow$ try $u(x,t) = t^p g(\eta)$ where $\eta = \frac{x}{\sqrt{kt}}$

$\longrightarrow pt^{p-1}g(\eta) - t^p(\frac{1}{2}\frac{x}{\sqrt{k}}t^{-\frac{3}{2}})g'(\eta) = \frac{1}{t}t^p g''(\eta)$

$\longrightarrow g''(\eta) + \frac{1}{2}\eta g'(\eta) - pg(\eta) = 0$

when $p=0$, $g''(\eta) + \frac{1}{2}\eta g'(\eta) = 0 \longrightarrow f'(x) + \frac{1}{2}xf(x) = 0 \longrightarrow$ let $I(x) = e^{\int \frac{1}{2}x\,dx} = e^{\frac{1}{4}x^2}$

$\longrightarrow If'(x) + \frac{dI}{dx}f(x) = 0 \longrightarrow \frac{dIf(x)}{dx} = 0 \longrightarrow f(x) = Ce^{-\frac{1}{4}x^2} \longrightarrow g'(\eta) = Ce^{-\frac{1}{4}\eta^2}$

$\longrightarrow g(\eta) = C\int_{-\infty}^{\eta} e^{-\frac{1}{4}\eta^2}\,d\eta + D = \begin{cases} 2\sqrt{\pi},\ \eta \to \infty \longleftarrow t \to 0\ \&\ x>0 \\ 0,\ \eta \to \infty \longleftarrow t \to 0\ \&\ x<0 \end{cases}$

$\longrightarrow u(x,0) = \begin{cases} 2\sqrt{\pi}C + D & \text{for } x>0 \\ D & \text{for } x<0 \end{cases}$

Complex Variables

Analytic Functions

Differentiable

- $f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(h)}{h}$ limit taken from left or right is equal continuity to differentiable
- let $f(z) = u(x,y) + iv(x,y)$

$\longrightarrow f'(z) = \lim_{\delta z\to 0}\frac{f(z+\delta z) - f(z)}{\delta z}$ $\delta z \longrightarrow 0$ from any direction in complex plane

$\longrightarrow$ // Re-axis, $f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{u(x+h,y) + iv(x+h,y) - u(x,y) - v(x,y)}{h} = \frac{du}{dx} + i\frac{dv}{dx}$

$\longrightarrow$ // Im-axis, $f'(z) = \lim_{ik\to 0}\frac{f(z+ik)-f(z)}{ik} = \lim_{h\to 0}\frac{u(x,y+k) + iv(x,y+k) - u(x,y) - v(x,y)}{ik} = \frac{dv}{dy} - i\frac{du}{dy}$

Cauchey-Riemann Equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Cauchey-Riemann Equations hold + u_x, u_y, v_x, v_y all continuous $\longrightarrow f(z)$ differentiable

Analytic differentiable on a disc containing a differentiable point

Harmonic Functions

Harmonic Function function satisfy Laplace's equation on a domain D

for $f(z) = u(x,y) + iv(x,y)$, $\begin{cases} u_{xx} + u_{yy} = 0 \\ v_{xx} + v_{yy} = 0 \end{cases} \longrightarrow$ Harmonic Function

Harmonic Conjugate u & v above but also satisfies Cauchy - Riemann Equations

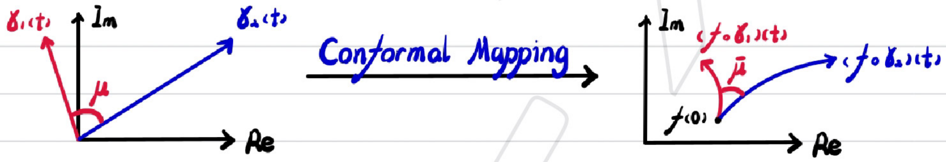
Conformal Mapping

Conformal Mapping

let G be an (open) subset of the complex plane and assume $f(z)$ is analytic on G

then f is called Conformal Mapping if $f'(z) \neq 0$ for any $z \in G$

Angles angle between lines are preserved under transformation



$$\mu = \arg[\gamma_1'(0)] - \arg[\gamma_2'(0)] = \frac{\arg[\gamma_1'(0)]}{\arg[\gamma_2'(0)]} = \bar{\mu} = \frac{\arg[f'(0)\gamma_1'(0)]}{\arg[f'(0)\gamma_2'(0)]} = \frac{\arg[\gamma_1'(0)]}{\arg[\gamma_2'(0)]}$$