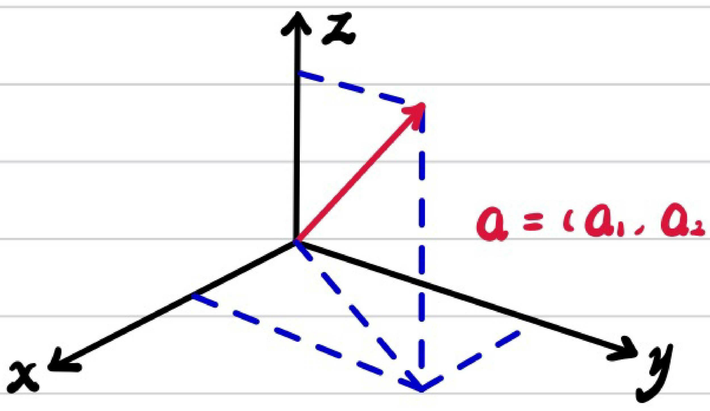
Vector Space

Vector

Cartersian Coordinate

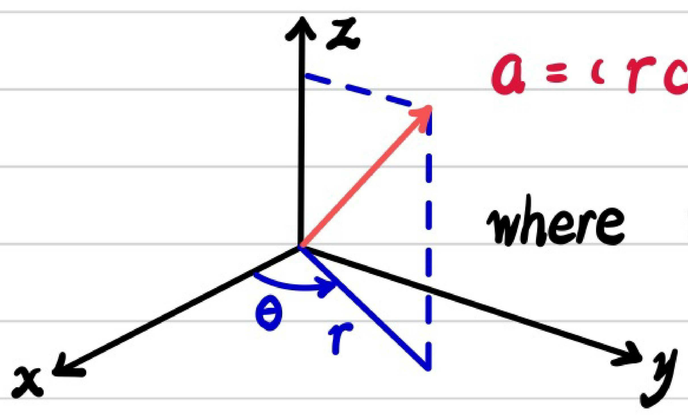


$a = (a_1, a_2, a_3) = a_1 i + a_2 j + a_3 k$

Vector Addition & Substraction

$a \pm b = (a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3)$

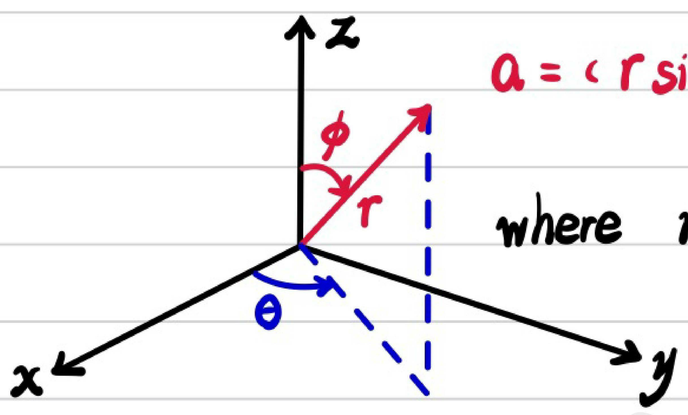
Cylindrical Coordinate



$a = (r\cos\theta, r\sin\theta, z)$

where $r \geq 0$ & $0 \leq \theta \leq 2\pi$

Spherical Coordinate



$a = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)$

where $r \geq 0$ & $0 \leq \phi \leq \pi$ & $0 \leq \theta \leq 2\pi$

Inner Product

Inner Product

$a \cdot b = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1 b_1 + a_2 b_2 + a_3 b_3$

Properties

① $a \cdot a \geq 0$ ($a \cdot a = 0$ if and only if $a = 0$)

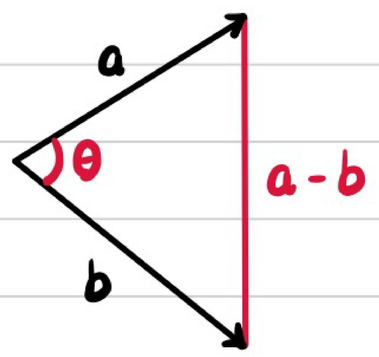
② $(\alpha a) \cdot b = \alpha (a \cdot b)$

③ $a \cdot (b + c) = a \cdot b + a \cdot c$

④ $a \cdot b = b \cdot a$

Norm length of the vector $\|a\| = \sqrt{a \cdot a} = \sqrt{a_1^2 + a_2^2 + b_2^2}$

Theorem $a \cdot b = \|a\|\|b\| \cos\theta$ where $a, b \in \mathbb{R}^3$ & $0 \leq \theta \leq \pi$



- the law of cosines $\|a-b\|^2 = \|a\|^2 + \|b\|^2 - 2\|a\|\|b\| \cos\theta$
- norm $\|a-b\|^2 = (a-b)\cdot(a-b) = a\cdot a - 2\cdot a\cdot b + b\cdot b$

$$= \|a\|^2 - 2a\cdot b + \|b\|^2$$

Cauchy-Schwarz Inequality $|a\cdot b| \leq \|a\|\|b\|$

$$a\cdot b = \|a\|\|b\| \cos\theta \xrightarrow{-1 \leq \cos\theta \leq 1} |a\cdot b| \leq \|a\|\|b\|$$

Triangle Inequality $\|a+b\| \leq \|a\| + \|b\|$

- $\|a+b\|^2 = (a+b)\cdot(a+b) = \|a\|^2 + 2a\cdot b + \|b\|^2$
- $(\|a\| + \|b\|)^2 = \|a\|^2 + 2\|a\|\|b\| + \|b\|^2$

$|a\cdot b| \leq \|a\|\|b\|$

Cross Product

Cross Product

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$$

Properties

① $a \times b = -b \times a$

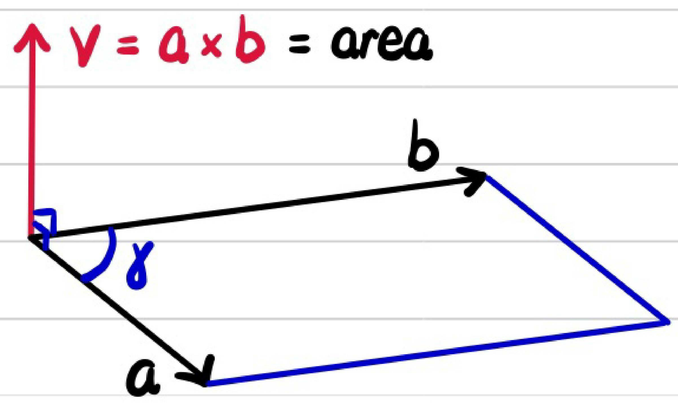
② distribution law $a \times (\alpha b + \beta c) = \alpha(a \times b) + \beta(a \times c)$

③ cyclic permutation 循环置换 $i \times j = k$ & $i \times k = j$ & $k \times j = i$

④ $\|a \times b\| = \|a\|\|b\| \sin\theta$

* $\|a \times b\|^2 = \|a\|^2\|b\|^2(1 - \cos^2\theta) = \|a\|^2\|b\|^2 - (a\cdot b)^2$

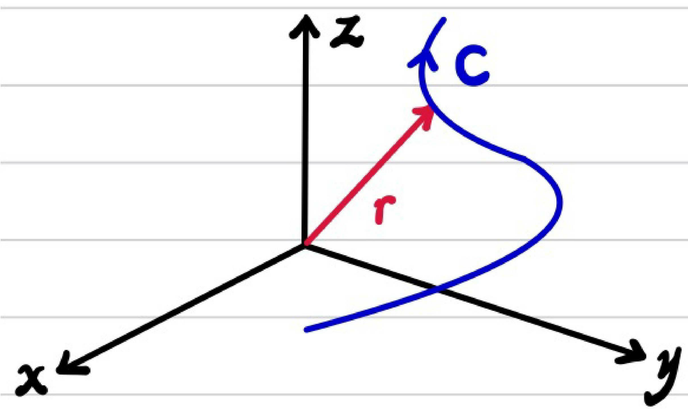
⑤ Right-hand Rule



Curve (path)

Parametric Representation of a Curve (path)

let $r : \underset{\text{domain}}{\underline{R}} \longrightarrow \underset{\text{range}}{\underline{R^3}}$ and $t \in R$, a curve C is then defined such that

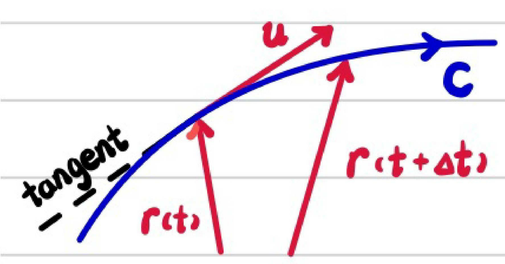


$r(t) = (x(t), y(t), z(t))$

where t is a parameter of the curve

Tangent Vector to a Curve (velocity)

let a curve C defined such that $r : R \longrightarrow R^3$ and $t \in R$



$$\frac{dr(t)}{dt} = \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt}\right)$$

* acceleration $= \dfrac{d^2 r(t)}{dt^2}$

Differentiation of Scalar-valued Function

Level Set (contour) 水平集

Scalar-valued Function $f : \underset{\text{domain}}{\underline{R^n}} \longrightarrow \underset{\text{range}}{\underline{R}}$ for $n \geq 1$

Level Set $f : R^n \longrightarrow R$ and $c \in R$, at all the points $x \in R^n$ $f(x) = c$

- level curve $n = 2$
- level surface $n = 3$ draw x-y & x-z / y-z plane first

Gradient

Gradient (vector form) $f : R^n \longrightarrow R$ $\nabla f = \text{grad} f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)$

Properties

① $\nabla(\alpha f(x)) = \alpha \nabla f(x)$

② $\nabla(f(x) + g(x)) = \nabla f(x) + \nabla g(x)$

③ $\nabla(f(x) g(x)) = g(x) \nabla f(x) + f(x) \nabla g(x)$

Tangent Plane

let $z = f(x, y)$, tangent plane on $(x_0, y_0, f(x_0, y_0))$ by Taylor's expansion

$$z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}\right]_{(x_0, y_0)}(x - x_0) + \left[\frac{\partial f}{\partial y}\right]_{(x_0, y_0)}(y - y_0)$$

Chain Rule

let $f = f(r(t))$ where $r(t) = (x(t), y(t), z(t))$ $\quad \frac{\partial f(r(t))}{\partial t} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial t} = \nabla f \cdot \frac{\partial r}{\partial t}$

Directional Derivative

the derivative of $f(x)$ along unit direction vector $b \in R^n$

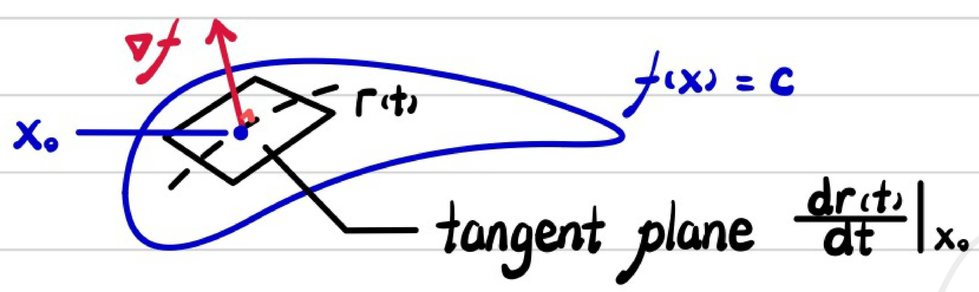
$$\left.\frac{\partial f(x_0 + sb)}{\partial s}\right|_{s=0} = \left(\frac{\partial f}{\partial (x_0 + sb)} \cdot \frac{\partial (x_0 + sb)}{\partial s}\right)\Big|_{s=0} = \nabla f(x_0) \cdot b$$

Direction of Maximum Increase $\nabla f(x_0)$

$$* \ \nabla f(x_0) \cdot b = \|\nabla f(x_0)\| \underbrace{\|b\|}_{=1 \text{ for unit vector}} \cos\theta \xrightarrow{\theta = 0} \nabla f(x_0) \cdot b = \|\nabla f(x_0)\| \longrightarrow b = \frac{\nabla f(x_0)}{\|\nabla f(x_0)\|}$$

Gradient Normal to Level Set

for $f(x) = c$ ∇f is a vector normal to level set



$$\left.\frac{\partial f(r(t))}{\partial t}\right|_{x_0} = 0 = \nabla f(x_0) \cdot \left.\frac{\partial r}{\partial t}\right|_{x_0}$$

* unit normal vector $n = \frac{\nabla f(x_0)}{\|\nabla f(x_0)\|}$

Laplacian

Laplacian (Laplace Operator) $\quad \nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \dots + \frac{\partial^2 f}{\partial x_n^2}$

Laplace Equation $\quad \nabla^2 f = 0$

Jacobian

Vector-valued Function $\quad F = R^m \longrightarrow R^n$ for m & $n \geq 1$

Jacobian

$$\frac{\partial F(x)}{\partial x} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \dots & \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \dots & \frac{\partial F_2}{\partial x_m} \\ \dots & \dots & \dots & \dots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \dots & \frac{\partial F_n}{\partial x_m} \end{bmatrix}$$

Divergence

Divergence let $F : R^3 \longrightarrow R^3$ $\quad \nabla \cdot F = \operatorname{div} F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$

* $\nabla \cdot (\nabla f) = \nabla^2 f$ where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$

Physical Interpretation

$\nabla \cdot u = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$ amount of outgoing fluid flux per unit volume

Curl

Curl

let $F : R^3 \longrightarrow R^3$ $\quad \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

Physical Interpretation

$\nabla \times F$ rotation of small particle embedded to the vector field

Vector Differential Identities

Theorem let $F, G : R^3 \longrightarrow R^3$ and $f, g : R^3 \longrightarrow R$

① $\nabla \times (\nabla f) = 0$

② $\nabla \cdot (\nabla \times F) = 0$

③ $\nabla \cdot (fF) = f(\nabla \cdot F) + \nabla f \cdot F$

④ $\nabla \cdot (F \times G) = (\nabla \times F) \cdot G - F \cdot (\nabla \times G)$

⑤ $\nabla \times (fF) = f(\nabla \times F) + \nabla f \times F$

⑥ $\nabla^2 (fg) = f \nabla^2 g + g \nabla^2 f + 2(\nabla f \cdot \nabla g)$

⑦ $\nabla \cdot (\nabla f \times \nabla g) = 0$

⑧ $\nabla \cdot (f \nabla g - g \nabla f) = f \nabla^2 g - g \nabla^2 f$

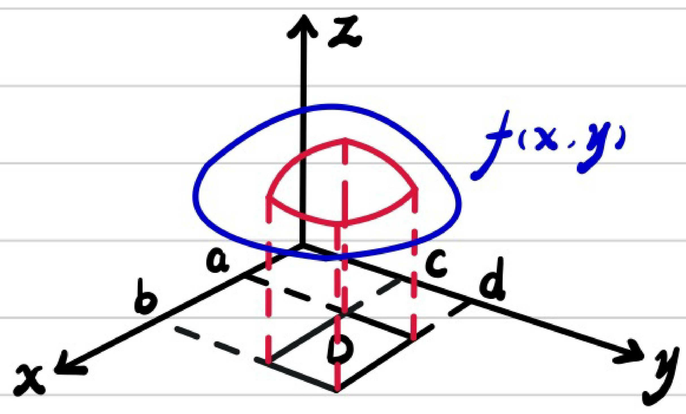
⑨ $\nabla \times (\nabla \times F) = \nabla (\nabla \cdot F) - \nabla^2 F$

Multiple Integral

Double Integral

Double Integral

let $f : D \longrightarrow R$ and $D \subset R^2$ $\quad \iint_D f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$ = volume below $f(x,y)$

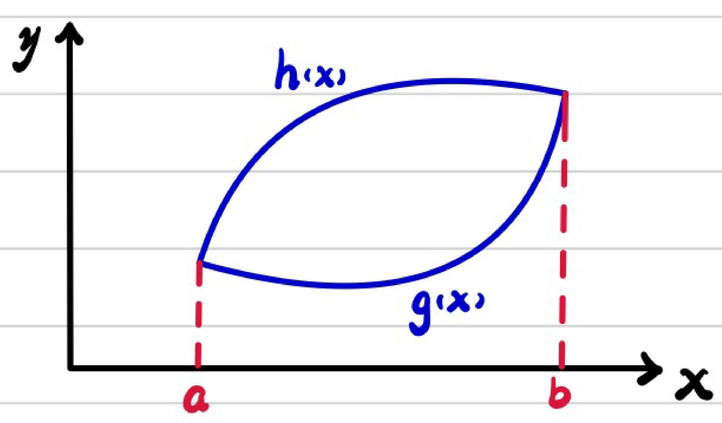


* if $f(x,y) = 1$, $\iint_D f(x,y)\,dA$ = area of D

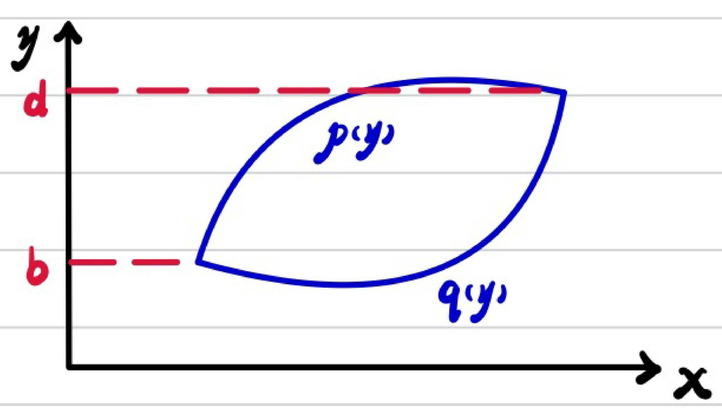
Fubini's Theorem $\quad \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$

Over a General Region

· type I $\quad \iint_D f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$



· type II $\quad \iint_D f(x,y)\,dA = \int_c^d \int_{q(y)}^{p(y)} f(x,y)\,dy\,dx$



Triple Integral

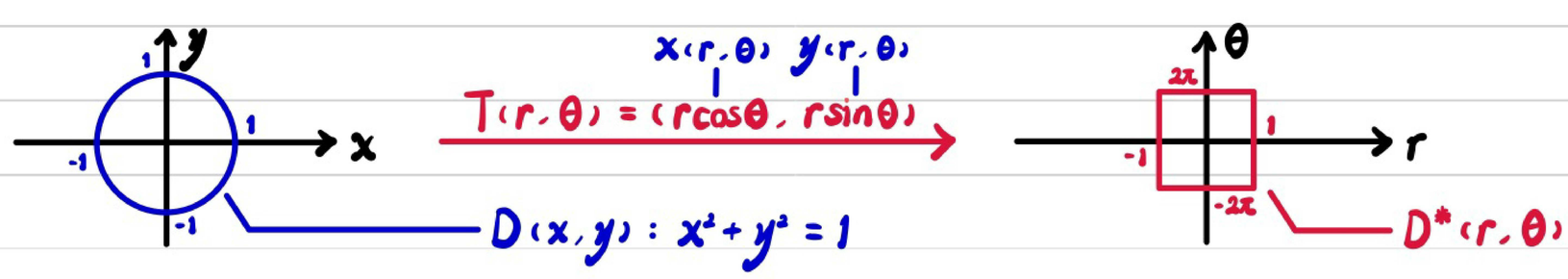
Triple Integral

let $f : B \longrightarrow R$ where $D = [a,b] \times [c,d] \times [e,f]$ $\quad \iiint_D f(x,y,z)\,dV = \int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$

* if $f(x,y,z) = 1$, $\iiint_D f(x,y,z)\,dV$ = volume of D

The Change of Variables

Change of Variables



· double integral $\iint_{D(x,y)} f(x,y)\,dx\,dy = \iint_{D^*(u,v)} f(x(u,v), y(u,v)) \left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right| du\,dv$

· triple integral $\iiint_{D(x,y,z)} f(x,y,z)\,dx\,dy\,dz = \iiint_{D^*(u,v,w)} f(u,v,w) \left|\det\left(\frac{\partial(x,y,z)}{\partial(u,v,w)}\right)\right| du\,dv\,dw$

Circular Coordinate $\iint_{D(x,y)} f(x,y)\,dx\,dy = \iint_{D^*(r,\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$

Cylinderical Coordinate $\iiint_{D(x,y,z)} f(x,y,z)\,dx\,dy\,dz = \iiint_{D^*(r,\theta,z)} f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$

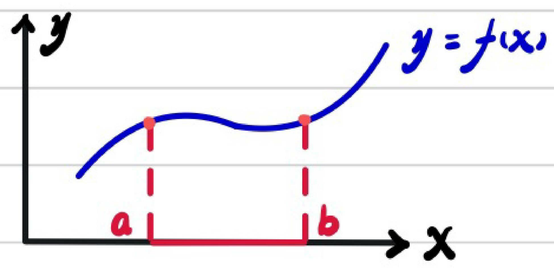
Spherical Coordinate

$\iiint_{D(x,y,z)} f(x,y,z)\,dx\,dy\,dz = \iiint_{D^*(r,\phi,\theta)} f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)\, r^2\sin\phi\, dr\,d\phi\,d\theta$

Line Integral

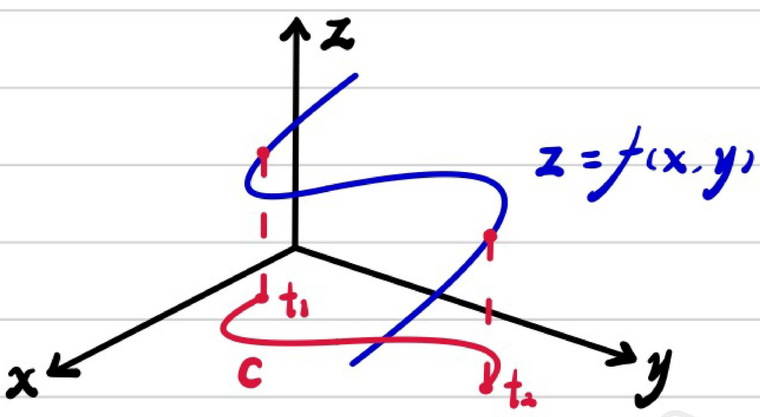
Introduction

Integral integration over an axis



$\int_a^b f(x)\,dx$

Line Integral integration over a curve



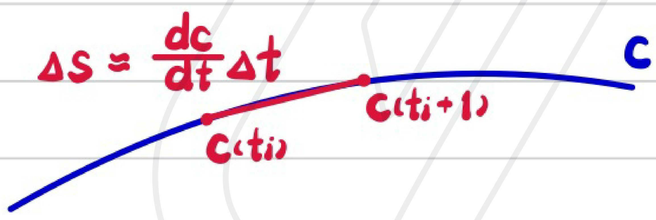
$\int_c f(x,y)\,ds$ where $c : [t_1, t_2] \longrightarrow R^2$

* $L = \int_c -[p(x,y) - p_\infty](n \cdot j)\,ds$ where c : curve of airfoil surface

Line Integral

Scalar-valued Function

$\int_c f(x,y,z)\,ds = \lim_{N\to\infty} \sum_{i=1}^{N} f(x(t_i), y(t_i), z(t_i)) \|\Delta s\| = \int_a^b f(c(t)) \left\|\frac{dc}{dt}\right\| dt$ for $c(t) = (x(t), y(t), z(t))$



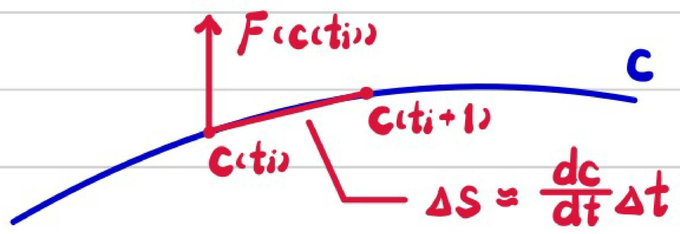
* for $f(x,y,z) = 1$, $\int_c f(x,y,z)\,ds$ = length of c

for $c[a,b] \longrightarrow R^2$ and $f : R^2 \longrightarrow R$, $\int_c f\,ds$ = area of fence

Vector-valued Function

$\int_c F(x,y,z)\cdot ds = \lim_{N\to\infty}\sum_{i=1}^{N} F(x(t_i), y(t_i), z(t_i))\cdot \Delta s = \int_a^b F(c(t))\cdot \frac{dc}{dt}\,dt$ for $c(t) = (x(t), y(t), z(t))$

$= \int_c (F_1, F_2, F_3)\cdot\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt = \int_c F_1\,dx + F_2\,dy + F_3\,dz$



* $\int_c F\,ds = \int_a^b F(c(t))\cdot\frac{dc}{dt}\,dt = \int_a^b \left(F(c(t))\cdot\frac{dc/dt}{\|dc/dt\|}\right)\left\|\frac{dc}{dt}\right\| dt$

$W = \int_c F\,ds$ for F = force field & c = object trajectory

Reparametrisation

Reparametrisation

let $h : [a,b] \longrightarrow [a_1, b_1]$ be an one-to one map and $c : [a_1, b_1] \longrightarrow R^3$

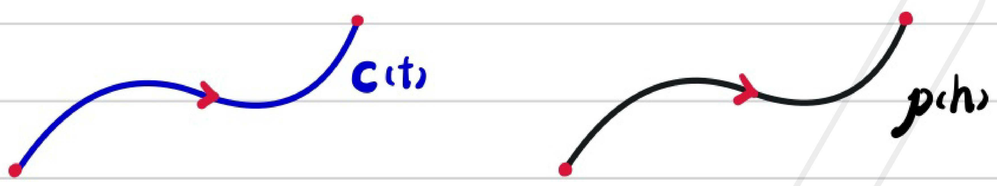
$p(h) = c(h(t))$ is a reparametrisation of $c(t)$

e.g. $c(t) = (t^2, 2-3t, 3t^3)$, $h(t) = -\sqrt{t}$, then $p(h) = c(h(t)) = (h^4, 2-3h^2, -3h^3)$

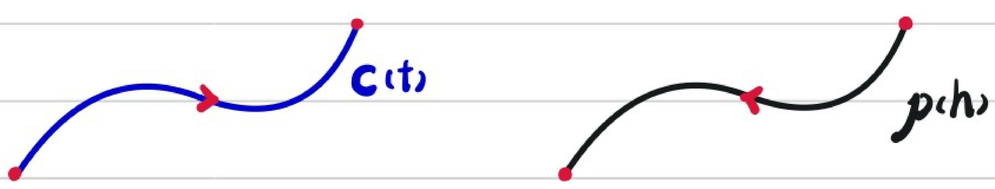
Scalar-valued Function $\int_p f\,ds = \int_c f\,ds$

Vector-valued Function

· orientation preserving $\int_p F\cdot ds = \int_c F\cdot ds$



· orientation reversing $\int_p F\cdot ds = -\int_c F\cdot ds$



* $\int_c F\cdot ds = \int_{c_1} F\cdot ds + \int_{c_2} F\cdot ds + \dots + \int_{c_n} F\cdot ds$

Oriented Close Curve

a curve with positive orientation $\longrightarrow$ ↺

Conservative Fields

Line Integral of a Gradient

let $f : R^3 \longrightarrow R$ and $c : [a,b] \longrightarrow R^3$

$\int_c \nabla f\cdot ds = \int_a^b \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\cdot\left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right) dt = \int_a^b \frac{\partial f}{\partial t}\,dt = f(c(b)) - f(c(a))$

* $F \longrightarrow \nabla f$ with constant C

Conservative Field

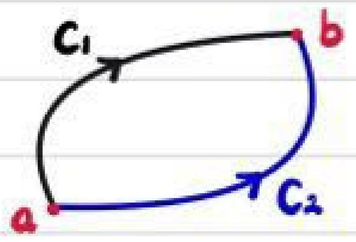
let $F: R^3 \longrightarrow R^3$ if $F = \nabla f$ F is a conservative field & f is a scalar potential

$\longrightarrow$ for any oriented closed curve c $\int_c F \cdot ds = \oint_c F \cdot ds = 0$

$\longrightarrow$ $\nabla \times F = \nabla \times (\nabla f) = 0 \longrightarrow$ irrotational

Path Independence if F is a conservative field

for any two simple curve c_1 and c_2 which have same end points

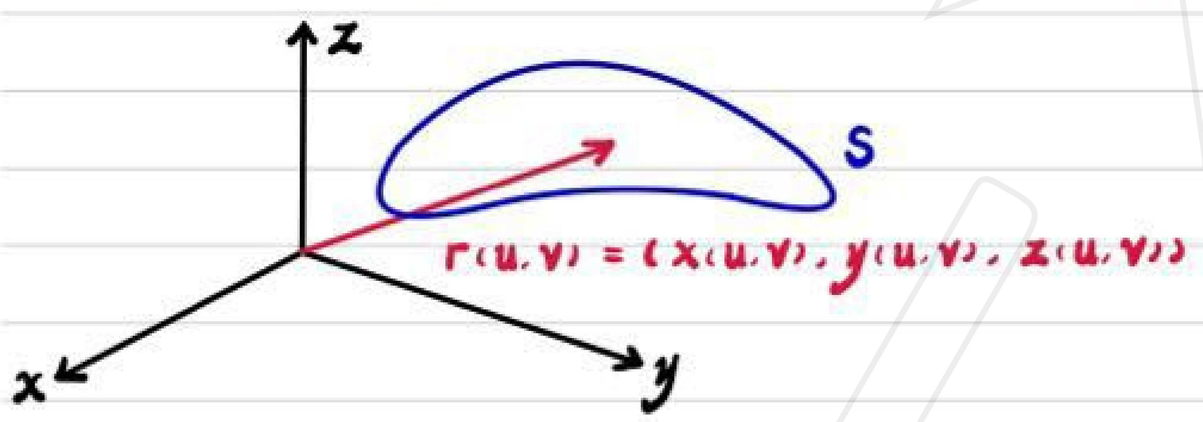


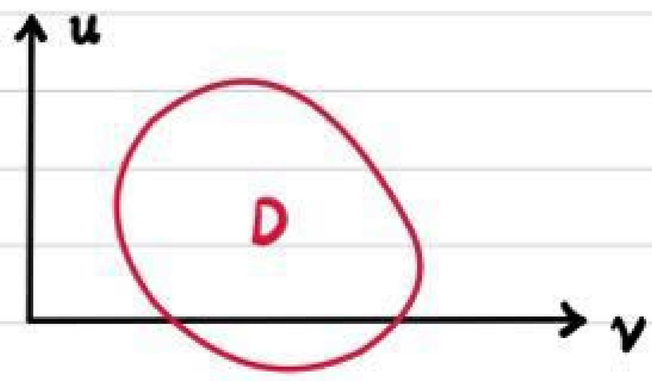
$$\int_{c_1} F \cdot ds = \int_{c_2} F \cdot ds = \int_a^b \nabla f \cdot ds = f(c(b)) - f(c(a))$$

Surface Integral

Introduction

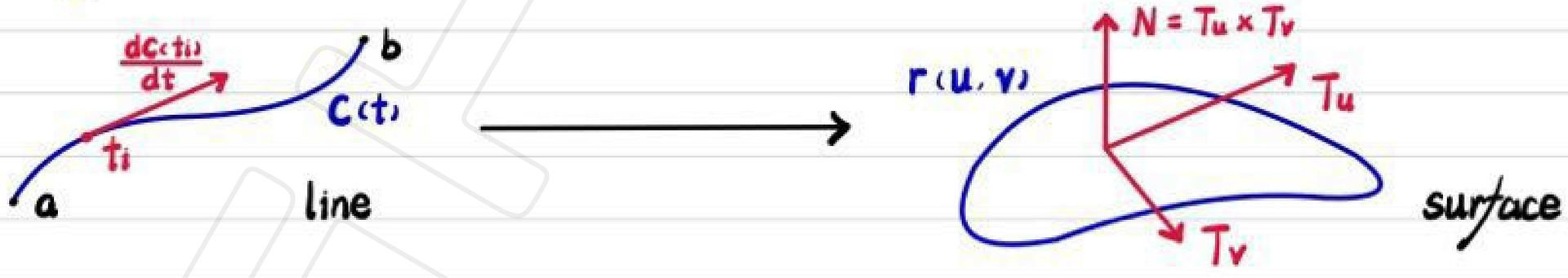
Surface Integral integration over a surface S in R^3 $\iint_S f(x,y,z)\, dA$





Parametric Representation of Surface

Tangent Vector

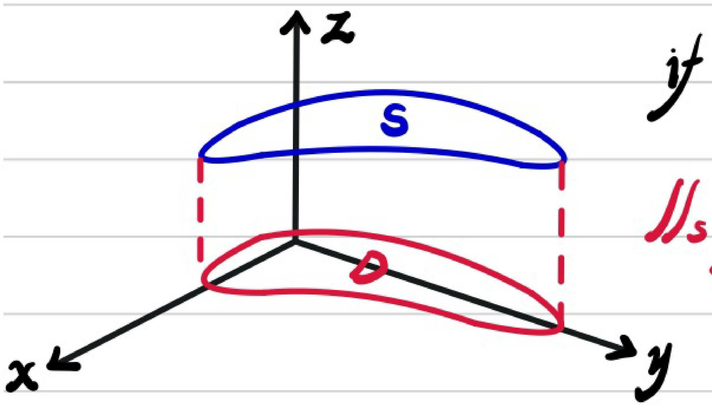


- $T_u = \frac{\partial r}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$
- $T_v = \frac{\partial r}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$
- $n = \frac{N}{\|N\|} = \frac{T_u \times T_v}{\|T_u \times T_v\|}$

Surface Integral

Scalar-valued Function $\iint_S f(x,y,z)\,dA = \iint_D f(x(u,v),y(u,v),z(u,v))\,\|T_u \times T_v\|\,du\,dv$

* for $f(x,y,z)=1$, $\iint_S f(x,y,z)\,dA$ = area of S



if $S: z = g(x,y)$ over region $(x,y) \in D$ in R^2 $\longrightarrow$ $S: z - g(x,y) = 0$

$\iint_S f(x,y,z)\,dA = \iint_D f(x,y,g(x,y))\,\|T_x \times T_y\|\,dx\,dy$

$= \iint_D f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy$

Vector-valued Function $\iint_S F(x,y,z)\cdot dS = \iint_D F(x(u,v),y(u,v),z(u,v))\cdot(T_u \times T_v)\,du\,dv$

* $\iint_S F\cdot dS = \iint_D F\cdot(T_u\times T_v)\,du\,dv = \iint_D F\cdot n\,\|T_u\times T_v\|\,du\,dv = \iint_S F\cdot n\,dA$

* $\iint_S F\cdot dS$ with $r(u,v)$ $=$ $-\iint_S F\cdot dS$ with $r(v,u)$ * $T_u \times T_v = -T_v \times T_u$

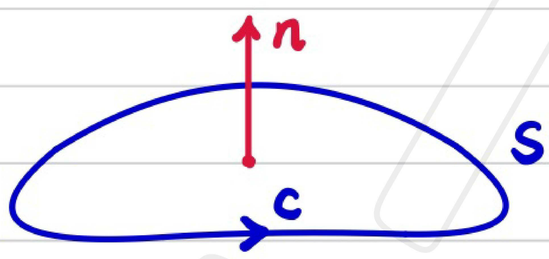
if $S: z = g(x,y)$ over region $\in D$ in R^2 $\longrightarrow$ $S: z - g(x,y) = 0$ $\longrightarrow$ $N = (-g_x, -g_y, 1)$

$\iint_S F\cdot dS = \iint_D F(x,y,g(x,y))\cdot(T_x\times T_y)\,dx\,dy = \iint_D F(x,y,g(x,y))\cdot\left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right)dx\,dy$

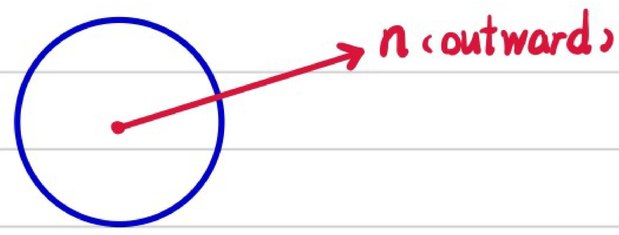
$= \iint_D -F(x)\frac{\partial g}{\partial x} - F(y)\frac{\partial g}{\partial y} + F(g(x,y))\,dx\,dy$

Oriented Surface surface with the following surface normal vector

· open surface



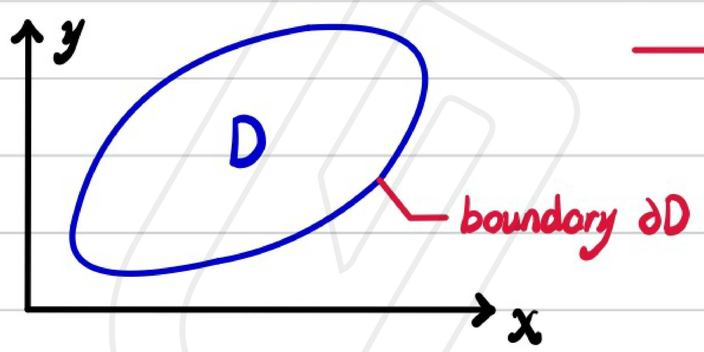
· closed surface



Integral Theorem

Green's Theorem

Green's Theorem $\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$



$\longrightarrow \int_{\partial D} F\cdot ds = \iint_D (\nabla\times F)\cdot k\,dx\,dy = \iint_D (\nabla\times F)\cdot k\,dA$

$(1,1)$

* only apply when D is a flat surface

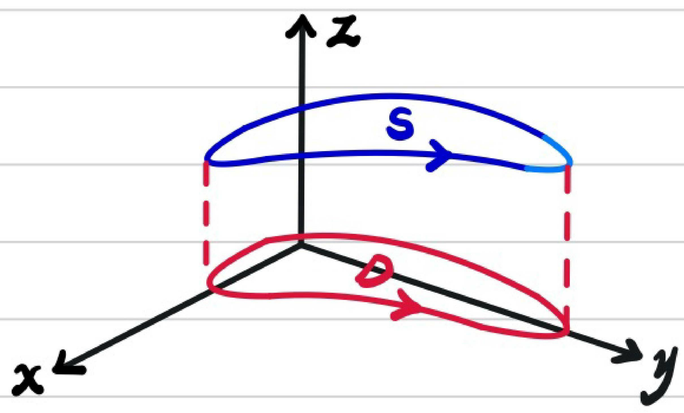
Area of a Region $A_D = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx = \frac{1}{2}\int_{\partial D} r^2\,d\theta$

· let $F = (-\frac{1}{2}y, \frac{1}{2}x)$, $A = \int_{\partial D} F \cdot ds = \iint_D (\nabla \times F) \cdot k\,dx\,dy = \iint_D dx\,dy$ = area of D

· let $F = (0, \frac{1}{2}r^2)$, $A = \int_{\partial D} F \cdot ds = \iint_D (\nabla \times F) \cdot k\,dr\,d\theta = \iint_D r\,dr\,d\theta = \int_D dx\,dy$ = area of D

Stoke's Theorem

Stoke's Theorem $\int_{\partial S} F \cdot ds = \iint_S (\nabla \times F) \cdot ds$



* on flat surface D,

$$\int_{\partial D} F \cdot ds = \iint_D (\nabla \times F) \cdot ds = \iint_D (\nabla \times F) \cdot n\,dA$$

$$= \iint_D (\nabla \times F) \cdot k\,dA \longrightarrow \text{Green's Theorem}$$

if S is infinitesimally small $\longrightarrow (\nabla \times F) \cdot n \approx \dfrac{\int_{\partial S} F \cdot ds}{\text{area of } S} = \dfrac{\iint_D (\nabla \times F) \cdot n\,dA}{\iint_D dA}$

Gauss' Divergence Theorem

Gauss' Divergence Theorem $\iiint_\Omega (\nabla \cdot F)\,dV = \iint_{\underbrace{\partial\Omega}_{\text{boundary surface}}} F \cdot ds = \iint_{\partial\Omega} F \cdot n\,dA$

if Ω is infinitesimally small $\longrightarrow \nabla \cdot F \approx \dfrac{\iint_{\partial\Omega} F \cdot n\,dA}{\text{volume of } \Omega} = \dfrac{\iint_{\partial\Omega} F \cdot ds}{\iint_{\partial\Omega} dV} = \dfrac{\iint_{\partial\Omega} \nabla \cdot F\,dV}{\iint_{\partial\Omega} dV}$