2018 Question 3:

3. A two-dimensional, parallel-sided channel of height $2h$ has its upper wall moving with velocity $2U$ and its lower wall moving with velocity $-0.5U$. A flow in the channel is caused solely by the movement of the walls and no pressure gradient is applied.

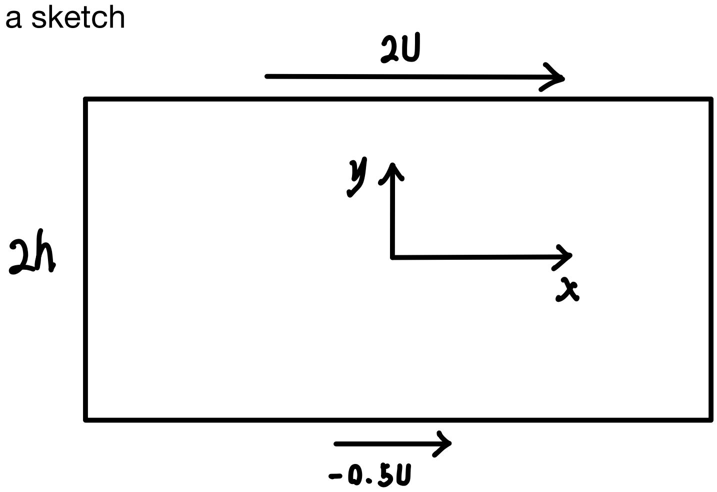
   (a) What assumptions are made in a fully-developed, incompressible, steady laminar channel flow about the behaviour of the velocity?

   (b) Starting from the equations of motion for steady viscous flow, derive an expression for the velocity profile across the duct.

   (c) Calculate the vorticity profile.

   (d) The lower wall is now brought to rest. Determine the change in shear stress at the wall after steady flow conditions have been re-established.

Start with a sketch



(a) Fully developed flow: $\dfrac{\partial u}{\partial x} = 0$

In 2D, $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \quad \therefore \dfrac{\partial v}{\partial y} = 0 \quad \therefore$ velocity v constant

But $v(y=0) = 0 \longrightarrow v = 0$ everywhere

$\longrightarrow \dfrac{\partial u}{\partial x} = 0$ everywhere

(b) Equations of motion for steady flow (u-component):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\therefore \quad 0 = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\because \quad \frac{dp}{dx} = 0 \qquad \therefore \frac{\partial^2 u}{\partial y^2} = 0$$

Intergrate twice $\therefore \frac{\partial u}{\partial y} = a \quad \therefore \partial u = a\,\partial y \quad \therefore u = ay + b$

Apply no-slip boundary condition

$u(h) = 2U \longrightarrow 2U = ah + b$

$u(-h) = -0.5U \longrightarrow -0.5U = a(-h) + b$

$\therefore a = \frac{5U}{4h},\ b = \frac{3U}{4}$

$\therefore u(y) = \frac{U}{4}\left(5\frac{y}{h} + 3\right)$

(c) Recalling the definition of Vorticity:

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\because \ v = 0$$

$$\therefore \ \frac{\partial v}{\partial x} = 0$$

$$\therefore \ \omega_z = -\frac{\partial u}{\partial y} = -\frac{5U}{4h}$$

(d) Shear stress at lower wall when u=-0.5*U*

$$\tau_w = \mu \frac{du}{dy}\Big|_{y=-h} = \mu \frac{5U}{4h}$$

Lower wall is now u = 0.

$$u = ay + b$$

$$\therefore \begin{cases} 0 = -ah + b \\ 2U = ah + b \end{cases} \qquad \therefore \begin{cases} a = U \\ b = \frac{U}{h} \end{cases}$$

$$\therefore \ u(y) = \frac{U}{h} y + U$$

$$\therefore \ \tau_{w(new)} = \mu \frac{\partial u}{\partial y} = \mu \frac{U}{h}$$

$$\therefore \ \Delta \tau_w = \tau_w - \tau_{w(new)}$$

$$= \frac{1}{4} \mu \frac{U}{h}$$

2019 Question 3:

(a) Sketch the velocity profile, in the form y/δ versus u/U_e, for a boundary layer subject to a zero pressure gradient. Also sketch the corresponding profile for a boundary layer in adverse pressure gradient.

(b) Explain why the shape of the velocity profile close to the wall depends on the pressure gradient.

(c) A boundary layer profile is given by:

$$\frac{u}{U_e} = 3\left(\frac{y}{\delta}\right)^2 - 2\left(\frac{y}{\delta}\right)^3.$$

Show that the boundary layer is on the verge of separation.

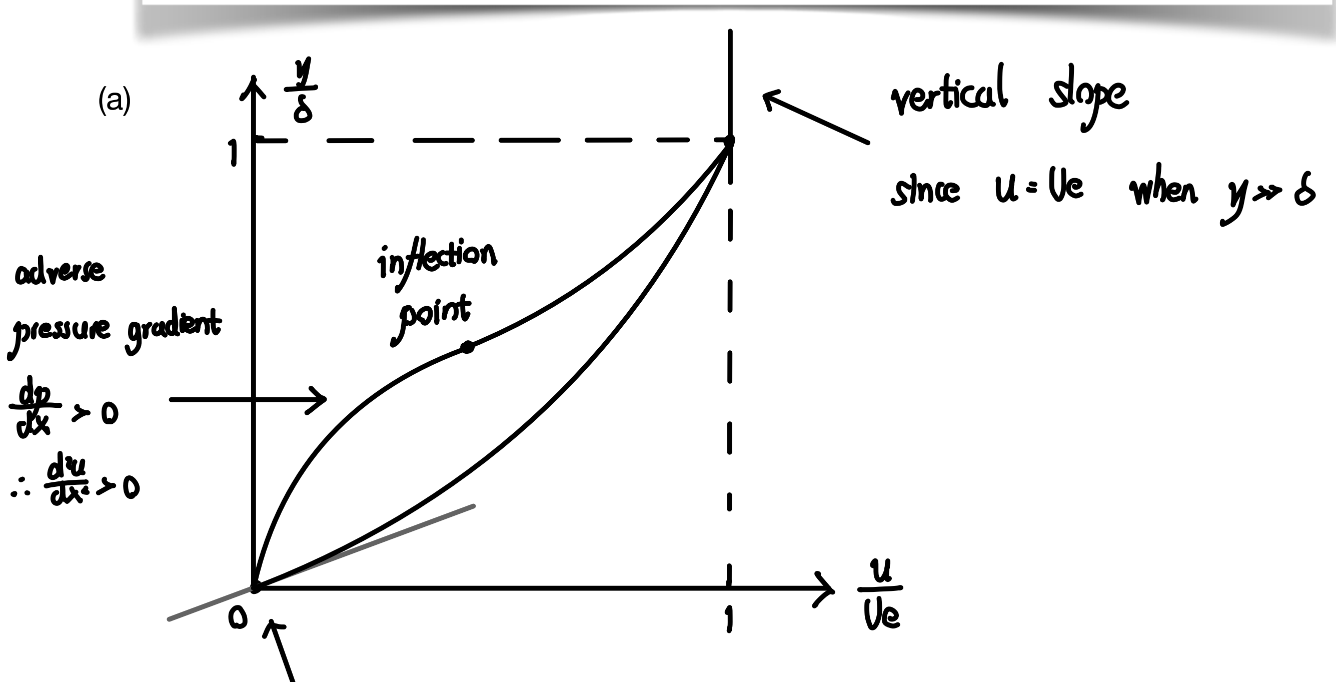
(d) The mass flow deficit of a boundary layer (the "displacement thickness", δ^*) is defined as:

$$\frac{\delta^*}{\delta} = \int_0^1 \left(1 - \frac{u}{U_e}\right) d(y/\delta).$$

Determine its value for the given profile.

(e) On your sketch from part (a), shade the area corresponding to the displacement thickness. Explain what it means and why your estimate of δ^* is large in this case.

(a)



$\frac{\partial p}{\partial x}\Big|_{y=0} = \nu \frac{\partial^2 u}{\partial y^2}\Big|_{y=0} \quad \therefore \frac{\partial^2 u}{\partial y^2}\Big|_{y=0} = 0 \quad \therefore \frac{du}{dy} = a$

$\therefore u = ay + b$ $\therefore$ profile must be a straight line at 0

(b) At wall:

$$\cancel{u\frac{\partial u}{\partial x}}\bigg|_w + \cancel{v\frac{\partial u}{\partial y}}\bigg|_w = -\frac{1}{\rho}\frac{\partial p}{\partial x}\bigg|_w + \cancel{\nu\frac{\partial^2 u}{\partial x^2}}\bigg|_w + \nu\frac{\partial^2 u}{\partial y^2}\bigg|_w$$

$$\therefore \frac{\partial^2 u}{\partial y^2}\bigg|_{wall} \propto \frac{\partial p}{\partial x}\bigg|_{wall}$$

(c) $$\frac{u}{U_e} = 3\left(\frac{y}{\delta}\right)^2 - 2\left(\frac{y}{\delta}\right)^3$$

At the point of seperation, $\tau_w = 0$

$$\therefore \frac{\partial\left(\frac{u}{U_e}\right)}{\partial y} = \frac{6}{\delta}\left(\frac{y}{\delta}\right) - \frac{6}{\delta}\left(\frac{y}{\delta}\right)^2$$

so at $y=0$, $\frac{\partial\left(\frac{u}{U_e}\right)}{\partial y} = 0$

$$\therefore \frac{\partial u}{\partial y} = 0$$

$$\therefore \tau_w = 0$$

(d) $\frac{\delta^*}{\delta} = \int_0^1 (1 - \frac{u}{U_e}) \, d\frac{y}{\delta}$

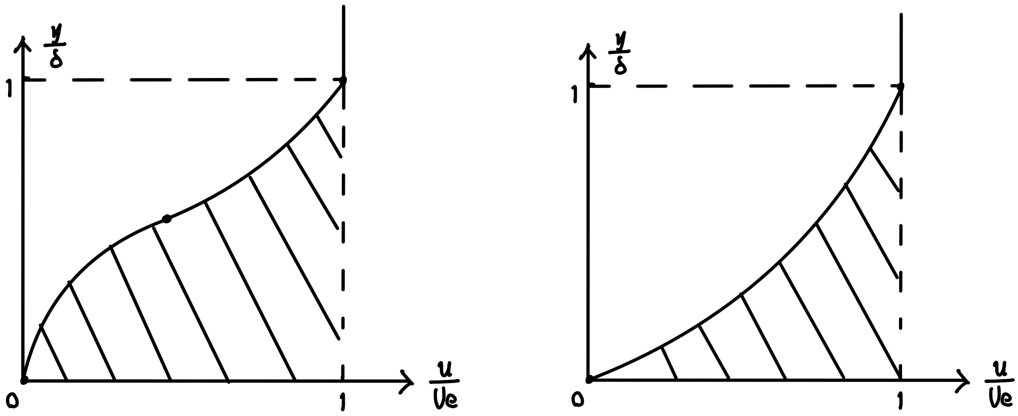
$= \int_0^1 (1 - 3(\frac{y}{\delta})^2 + 2(\frac{y}{\delta})^3 \, d\frac{y}{\delta}$

$= [\frac{y}{\delta} - (\frac{y}{\delta})^3 + \frac{1}{2}(\frac{y}{\delta})^4]_0^1$

$= 1 - 1 + \frac{1}{2}$

$= \frac{1}{2}$

(e)



The thickness that we must displace the wall vertically to generate an inviscid flow (slip condition) without boundary layer