Part I

Differentiation

Partial Derivative

- $\frac{\partial f}{\partial x} = f_x = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$
- $\frac{\partial f}{\partial y} = f_y = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$
- sum rule $(u+v)' = u' + v'$
- product rule $(uv)' = u'v + uv'$
- quotient rule $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$
- chain rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
- inverse function rule $\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}$
- logarithmatic rule $(a^b)' = a^b\left(b'\ln a + \frac{b}{a}a'\right)$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\tan x)' = \sec^2 x$
- $(\operatorname{cosec} x)' = \left(\frac{1}{\sin x}\right)' = -\operatorname{cosec} x \cot x$
- $(\sec x)' = \left(\frac{1}{\cos x}\right)' = \sec x \tan x$
- $(\cot x)' = \left(\frac{1}{\tan x}\right)' = -\operatorname{cosec}^2 x$
- $(\arcsin x)' \xrightarrow{x=\sin y} \frac{dx}{dy} = \cos y = \sqrt{1-\sin^2 y} \longrightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$
- $(\arccos x)' \xrightarrow{x=\cos y} \frac{dx}{dy} = -\sin y = -\sqrt{1-\cos^2 y} \longrightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$
- $(\arctan x)' \xrightarrow{x=\tan y} \frac{dx}{dy} = \sec^2 y = \frac{1}{\cos^2 y} = \frac{\cos^2 y + \sin^2 y}{\cos^2 y} = 1 + \tan^2 y \longrightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$
- $[f(x)^n]' = nf'(x)f(x)^{n-1}$
- $(e^{nx})' = ne^{nx}$
- $(\ln x)' = \frac{1}{x}$
- $(\log_a x)' = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{x\ln a}$
- $(a^x)' \xrightarrow{y=a^x} (\ln y)' = (\ln a^x)' = (x\ln a)' \xrightarrow{(\ln y)' = \frac{1}{y}\frac{dy}{dx}} \frac{dy}{dx} = y\ln a = a^x \ln a$

Total Deriavative

$u(t) = f(x(t), y(t)) \longrightarrow \frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt}$

High Order Deriavative

- $f_{xy}(a,b) = f_{yx}(a,b)$
- Leibniz's rule $(fg)^{(n)} = \sum_{r=0}^{n} {}^nC_r f^{(n-r)} g^{(r)}$ $\quad$ $*\,{}^nC_r = \frac{n!}{(n-r)!\,r!}$

Estimating Small Changes

$\delta f(x,y) \approx \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$

Taylor's Expension

- about $x=a$ $\quad f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \longrightarrow f(x) = f(a) + f'(a)(x-a) + o(1)$

$\longrightarrow f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$

* Peano Reminder $R_n(x) = o((x-a)^n)$

Lagrange Reminder $R_n(x) = \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-a)^{n+1}$ where $x_0 \in (a, x)$

· about (a,b) $f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$+ \frac{1}{2!}[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2] + \dots$

Stationary Point

stationary point (a,b) occurs when $f_x = f_y = 0$

· maxima if $\Delta(a,b) = [f_{xy}(a,b)]^2 - f_{xx}(a,b)f_{yy}(a,b) < 0$ and either $f_{xx}(a,b) < 0$ or $f_{yy}(a,b) < 0$

· minima if $\Delta(a,b) = [f_{xy}(a,b)]^2 - f_{xx}(a,b)f_{yy}(a,b) < 0$ and either $f_{xx}(a,b) > 0$ or $f_{yy}(a,b) > 0$

· saddle point $\Delta(a,b) = [f_{xy}(a,b)]^2 - f_{xx}(a,b)f_{yy}(a,b) > 0$

Newton Raphson Method

if a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n = 0, 1, 2, \dots$

Integration

Basic techniques

· $\int \sin(x)\,dx = -\cos(x) + C$

· $\int \cos(x)\,dx = \sin(x) + C$

· $\int \tan(x)\,dx = \ln|\sec(x)| + C$

· $\int \operatorname{cosec}(x)\,dx = \ln|\operatorname{cosec}(x) - \cot(x)| + C$

· $\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$

· $\int \cot(x)\,dx = \ln|\sin(x)| + C$

· $\int \sin^2(x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$

· $\int \cos^2(x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$

· $\int \tan^2(x)\,dx = \tan x - x + C$

· $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$

· $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

· $\int \ln x\,dx = x\ln x - x + C$

· $\int \log_a x\,dx = \int \frac{\ln x}{\ln a}\,dx = \frac{1}{\ln a}(x\ln x - x) + C$

· $\int a^x\,dx = \frac{a^x}{\ln a} + C$

· $\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln|1+x^2| + C$

· $\int \frac{1}{1+x^2} = \arctan(x) + C$

· By parts $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$

Integration by partial fraction

- $\int \frac{px+q}{(x-a)(x-b)}\,dx = \int \frac{A}{x-a} + \frac{B}{x-b}\,dx$
- $\int \frac{px+q}{(x-a)^2}\,dx = \int \frac{A}{(x-a)^2} + \frac{B}{x-a}\,dx$
- $\int \frac{px^2+qx+r}{(x-a)(x^2-bx+c)}\,dx = \int \frac{A}{x-a} + \frac{Bx+C}{x^2-bx+c}\,dx$

Trigonometric functions

- Binomial theorem

$\int \sin^m(x)\cos^n(x)\,dx \xrightarrow{m \text{ odd}} \int \sin^m(x)\cos^{2k+1}(x)\,dx = \int \sin^m(x)\,[1-\sin^2(x)]^k \cos(x)\,dx$ * use $u = \sin(x)$

$\xrightarrow{n \text{ odd}} \int \sin^{2k+1}(x)\cos^n(x)\,dx = \int [1-\cos^2(x)]^k \cos^n(x)\sin(x)\,dx$ * use $u = \cos(x)$

$\xrightarrow{m,n \text{ even}} \ldots$ * $\sin^2(x) = \frac{1-\cos(2x)}{2}$, $\cos^2(x) = \frac{1+\cos(2x)}{2}$, $\sin x\cos x = \frac{\cos(2x)}{2}$

- Half-angle substitution

let $t = \tan(\frac{\theta}{2})$, $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $d\theta = \frac{2}{1+t^2}\,dt$

Areas

- $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (f(x)-g(x))\,dx$ * $f \geq g$ on $[a,b]$
- Solid of revolution $V = \int_a^b S(x)\,dx = \pi \int_a^b (f^2(x) - g^2(x))\,dx$

Complex numbers

- $\sqrt{-1} = i$
- $(a+ib)+(c+id) = (a+c)+i(b+d)$
- $(a+ib)(c+id) = ac+iad+ibc-bd = ac-bd+i(ad+bc)$
- $\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{ac+bd+i(bc-ad)}{c^2-d^2} = \frac{ac+bd}{c^2-d^2} + i\frac{bc-ad}{c^2-d^2}$
- Conjugate 共軛的 $z = a+ib \longrightarrow \bar{z} = a-ib \longrightarrow Re(z) = \frac{z+\bar{z}}{2}$, $Im(z) = \frac{z-\bar{z}}{2}$

$z = a+ib \longrightarrow \theta = \arg(z) = \arctan(\frac{b}{a})$ $(0 \leq \arg(z) \leq 2\pi)$, $|z| = \sqrt{a^2+b^2} \longrightarrow z = |z|e^{i\theta}$

- $\arg(zw) = \arg(z) + \arg(w)$
- $\arg(\frac{z}{w}) = \arg(z) - \arg(w)$
- $|zw| = |z||w|$

Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta) \longrightarrow e^{i\pi}+1 = 0$

De Moivre's theorem $\cos(n\theta) + i\sin(n\theta) = e^{i(n\theta)} = (e^{i\theta})^n = [\cos(\theta)+i\sin(\theta)]^n$

Fundemental theorem of algebra for any n^{th} order polynomial, there are n complex solutions

Geometry of the complex plane $\{ z \in C : "z \text{ satisfies some condition} \}$

Series

Test of convergence

- **Comparision test** $0 \le x_n \le y_n \longrightarrow$ if $\sum_{n\ge0} y_n$ converges, $\sum_{n\ge0} x_n$ converges

  $\longrightarrow$ if $\sum_{n\ge0} x_n$ diverges, $\sum_{n\ge0} y_n$ diverges

- **the ratio test** $\cdot \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} > 1 \longrightarrow$ diverges $\cdot \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} < 1 \longrightarrow$ converges $\cdot \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = 1 \longrightarrow ?$

- **the integral test** $\sum_{n=1} f(n) \longrightarrow \lim_{n\to\infty} \int_1^n f(x)\,dx \longrightarrow \infty$, diverges

  $\longrightarrow < \infty$, converges

- **alternating series** $\cdot$ for $x \ge 0$, $\sum_{n\ge1} (-1)^n x_n$ $\cdot$ for $x \le 0$, $\sum_{n\ge1} (-1)^{n+1} x_n$

  $\longrightarrow$ if $\lim_{n\to\infty} x_n = +R < \infty$, alternating series converges

Taylor's Theorem

reminder $= R_n = \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-a)^{n+1} \longrightarrow$ error $[a,b] = \max[R_n(a), R_n(b)]$

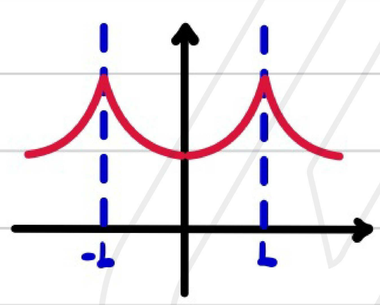
l'Hôpital's rule

$$\lim_{x\to0} \text{ or } \lim_{x\to\infty} \frac{a(x)}{b(x)} = \lim_{x\to0} \text{ or } \lim_{x\to\infty} \frac{a'(x)}{b'(x)} = \lim_{x\to0} \text{ or } \lim_{x\to\infty} \frac{a''(x)}{b''(x)} = \dots$$

Fourier series

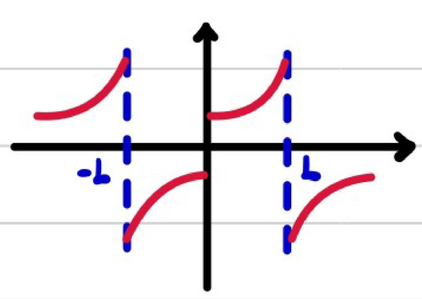
Creating periodic function

- **even extension**



$f(-x) = f(x)$

- **odd extension**



$f(-x) = -f(x)$

- **even functions** $\int_{-L}^{L} f(x)\,dx = \int_{L}^{-L} f(x)\,dx$

- **odd functions** $\int_{-L}^{L} f(x)\,dx = 0$

General formulae

if $f(x)$ is periodic with period $2L$, $f(x+2L) = f(x)$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

where $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \qquad n = 0, 1, 2, \ldots$

and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \qquad n = 1, 2, 3, \ldots$

- if $f(x)$ is an even function of $x \longrightarrow a_n = \frac{2}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$ and $b_n = 0$
- if $f(x)$ is an odd function of $x \longrightarrow a_n = 0$ and $b_n = \frac{2}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$

Parseval's Theorem

$$\frac{1}{L}\int_{-L}^{L} f(x)^2\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Part II

Linear ODEs

All terms multiplying y or derivates of y have to be a $f(x)$ or a constant

1st Order ODEs

Direct integration $\frac{dy}{dx} = f(x) \longrightarrow y = \int f(x)\,dx$

Seperation of variables $\frac{dy}{dx} = f(x)g(y) \longrightarrow \int \frac{1}{g(y)}dy = \int f(x)\,dx$

Homogenous $\frac{dy}{dx} = f(\frac{y}{x}) \longrightarrow u = \frac{y}{x} \longrightarrow \frac{dy}{dx} = (ux)' = u + x\frac{du}{dx} = f(u)$

General linear $\frac{dy}{dx} + f(x)y = g(x) \longrightarrow I(x) = e^{\int f(x)dx} \longrightarrow I(x)\frac{dy}{dx} + I(x)f(x)y = I(x)g(x)$

$\longrightarrow I\frac{dy}{dx} + \frac{dI}{dx}y = \frac{dIy}{dx} = I(x)g(x)$

Exact $\frac{dy}{dx} = -\frac{f(x,y)}{g(x,y)}$ when $\frac{df(x,y)}{dy} = \frac{dg(x,y)}{dx}$

$\longrightarrow \int f(x,y)\,dx + \int g(x,y)\,dy \overset{u(x,y)}{=\!=} \frac{du}{dx}dx + \frac{du}{dy}dy = 0$

$\longrightarrow \frac{du}{dx} = f(x,y),\ \frac{du}{dy} = g(x,y) \longrightarrow u(x,y) = \int f(x,y)\,dx = \int g(x,y)\,dy$

Inexact $\frac{dy}{dx} = -\frac{f(x,y)}{g(x,y)}$ when $\frac{df(x,y)}{dy} \neq \frac{dg(x,y)}{dx} \xrightarrow{I(x,y)} \frac{dx}{dy} = \frac{If(x,y)}{Ig(x,y)} \longrightarrow \frac{dIf(x,y)}{dy} = \frac{dIg(x,y)}{dx}$

$\longrightarrow If(x,y)\,dx + Ig(x,y)\,dy \overset{u(x,y)}{=\!=} \frac{du}{dx}dx + \frac{du}{dy}dy = 0$

$\longrightarrow \frac{du}{dx} = If(x,y),\ \frac{du}{dy} = Ig(x,y) \longrightarrow u(x,y) = \int If(x,y)\,dx = \int Ig(x,y)\,dy$

Bernouli $\frac{dy}{dx} + f(x)y = g(x)y^k \longrightarrow y = u^a,\ a = \frac{1}{1-k} \longrightarrow \frac{dy}{dx} = au^{a-1}\frac{du}{dx}$

2nd Order ODEs

Homogenous $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \xrightarrow{y = Ae^{\alpha x}} a\alpha^2 + b\alpha + c = 0 \longrightarrow \alpha_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

If $b^2 - 4ac > 0$, $2 \times R$ roots $\longrightarrow y = Ae^{\alpha_1 x} + Be^{\alpha_2 x}$

If $b^2 - 4ac = 0$, $1 \times R$ roots $\longrightarrow \alpha = \lambda_1,\ \lambda_1 = -\frac{b}{2a},\ y = (A + Bx)e^{\lambda_1 x}$

If $b^2 - 4ac < 0$, $2 \times C$ roots $\longrightarrow \alpha = \lambda_1 \pm i\lambda_2,\ \lambda_1 = -\frac{b}{2a},\ \lambda_2 = \frac{\sqrt{4ac-b^2}}{2a},\ y = [A\cos(\lambda_2 x) + B\sin(\lambda_2 x)]e^{\lambda_1 x}$

Non-Homogenous $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \longrightarrow y = y_h + y_p$

$y_h \longrightarrow a\frac{d^2y_h}{dx^2} + b\frac{dy_h}{dx} + cy_h = 0$

$y_p \longrightarrow a\frac{d^2y_p}{dx^2} + b\frac{dy_p}{dx} + cy_p = f(x)$

- $f(x) = x^2 \longrightarrow y_p = Ax^2 + Bx + C$
- $f(x) = e^{nx} \longrightarrow y_p = Ae^{nx}$ or Axe^{nx}
- $f(x) = \sin(nx)$ or $\cos(nx) \longrightarrow y_p = A\sin(nx) + B\cos(nx)$ or $Ax\sin(nx) + Bx\cos(nx)$

Euler ODEs

$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0 \xrightarrow{y = Ax^\alpha} (a-1)\alpha + \alpha^2 + b = 0 \longrightarrow \alpha_{1,2} = \frac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2}$

If $(a-1)^2 - 4b > 0$, $2\times$ R roots $\longrightarrow y = Ax^{\alpha_1} + Bx^{\alpha_2}$

If $(a-1)^2 - 4b = 0$, $1\times$ R roots $\longrightarrow \alpha = \lambda_1$, $\lambda_1 = -\frac{(a-1)}{2}$, $y = (A + B\ln x)x^{\lambda_1}$

If $(a-1)^2 - 4b < 0$, $2\times$ C roots $\longrightarrow \alpha = \lambda_1 + i\lambda_2$, $\lambda_1 = -\frac{(a-1)}{2}$, $\lambda_2 = \frac{\sqrt{4b - (a-1)^2}}{2}$, $y = [A\cos(\lambda_2 \ln x) + B\sin(\lambda_2 \ln x)]x^{\lambda_1}$

Laplace Transform

- $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

Linearity

- $L\{f(t) + g(t)\} = L\{f(t)\} + L\{g(t)\}$
- $L\{kf(t)\} = kL\{f(t)\}$

Derivatives

- $L\{\frac{df}{dt}\} = sL\{f(t)\} - f(0)$
- $L\{\frac{d^2f}{dt^2}\} = s^2L\{f(t)\} - sf(0) - \frac{df}{dt}(0)$

Inverse

$f(t) = L^{-1}L\{f(t)\} = L^{-1}\{F(s)\}$

- $L^{-1}\{F(s) + G(s)\} = L^{-1}\{F(s)\} + L^{-1}\{G(s)\}$
- $L^{-1}\{kF(s)\} = kL^{-1}\{F(s)\}$

$$\begin{tikzcd}
\text{ODE} \arrow[r, "L"] \arrow[d, dashed] & \text{Algebraic problem 代数问题} \arrow[d] \\
\text{solution} & \text{Solution of algebraic problem} \arrow[l, "-L"']
\end{tikzcd}$$

Periodic functions

$$f(t) = f(t+T) \longrightarrow L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_0^\infty e^{-s(t+T)} f(t+T)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT} \underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{L\{f(t)\}}$$

$$\longrightarrow L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

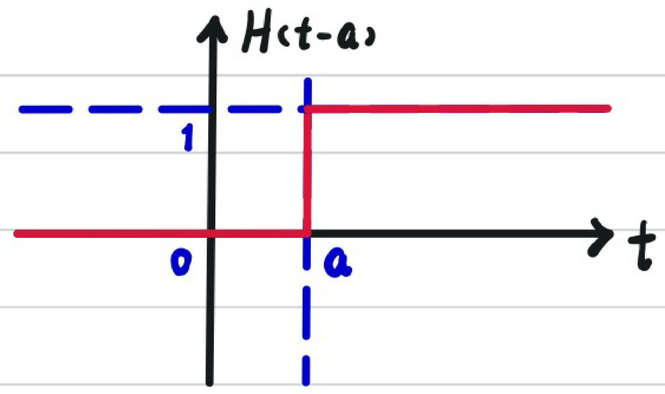
Exponential multiplication

$$f(t) = e^{at} g(t) \longrightarrow L\{f(t)\} = G(s-a)$$

Heaviside function

$$f(t) = H(t-a)\, g(t-a) \longrightarrow L\{f(t)\} = e^{-sa} G(s)$$

*

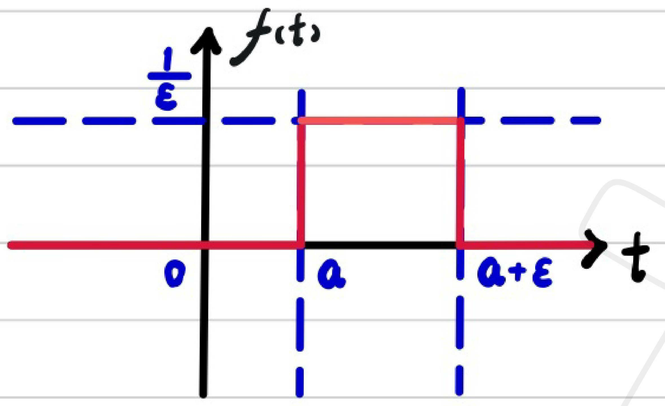


$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \longrightarrow H(t-a) = \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$$

Top-hat function

$$f(t) = \delta_a(t) \longrightarrow L\{f(t)\} = \frac{e^{-sa}(1-e^{\varepsilon s})}{\varepsilon s}$$

*



$$\delta_a(t) \begin{cases} \frac{1}{\varepsilon} & a \leq t \leq a+\varepsilon \\ 0 & t < a,\ t > a+\varepsilon \end{cases} = \frac{1}{\varepsilon}\left[H(t-a) - H(t-(a+\varepsilon))\right]$$

Direc & Delta function

$$f(t) = \delta(t-a) \longrightarrow L\{f(t)\} = e^{-sa}$$

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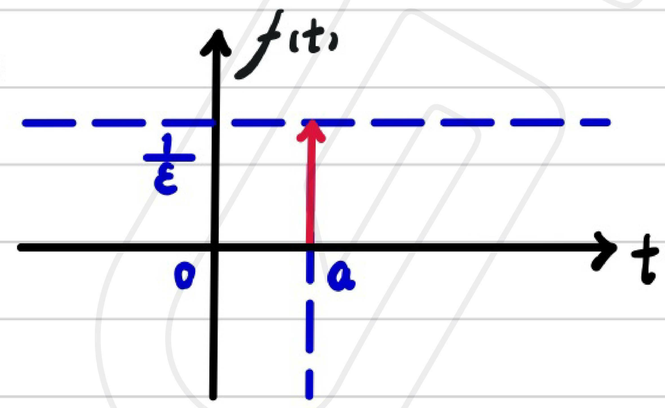


$$\delta(t) = \lim_{\varepsilon \to 0} \delta_a(t)$$

Part III

Matrices

Identity matrix

$$I = \begin{bmatrix} 1 & 0 & \cdots & \\ 0 & 1 & & \\ \cdots & & & \\ & & & 1 \end{bmatrix}$$

Addition only valid when $A^{N\times N} + B^{N\times N}$

- $A+B = B+A$
- $A+(B+C) = (A+B)+C$

Multiplication only valid when $A^{N\times M} \times B^{K\times N}$

- $AB \neq BA$
- $A(BC) = (AB)C$
- $c(A+B) = cA + cB$
- $C(A+B) = CA + CB$
- $(A+B)C = AC + BC$

$$\underset{A}{\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}} \underset{x}{\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}} = \underset{B}{\begin{bmatrix} m_{11}b_{11} + m_{12}b_{21} + m_{13}b_{31} \\ m_{21}b_{11} + m_{22}b_{21} + m_{23}b_{31} \\ m_{31}b_{11} + m_{32}b_{21} + m_{33}b_{31} \end{bmatrix}}$$

Inverse $AA^{-1} = A^{-1}A = I$

When $A^{(N\times N)}$ is invertible $\longrightarrow$ A^{-1} exists

$Ax = B \longrightarrow A^{-1}Ax = A^{-1}B$

$\longrightarrow Ix = A^{-1}B \longrightarrow x = A^{-1}B$

Transformation

$$A = \begin{bmatrix} a+bi \\ d \\ c-ei \end{bmatrix}$$

$\longrightarrow$ transpose $A^T = \begin{bmatrix} a+bi & d & c-ei \end{bmatrix}$

- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(A^{-1})^T = (A^T)^{-1}$

$\longrightarrow$ conjugate 共軛 transpose $A^H = (A^*)^T = (\bar{A})^T = \begin{bmatrix} a-bi & d & c+ei \end{bmatrix}$

- $(A+B)^H = A^H + B^H$
- $(AB)^H = B^H A^H$
- $(A^{-1})^H = (A^H)^{-1}$

Gaussian Elimination

① $\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 2 & 4 & -2 \\ \begin{matrix} 4-2\times2 \\ 0 \end{matrix} & \begin{matrix} 9-4\times2 \\ 1 \end{matrix} & \begin{matrix} -3-(-2)\times2 \\ 1 \end{matrix} \\ \begin{matrix} -2-2\times(-1) \\ 0 \end{matrix} & \begin{matrix} -3-4\times(-1) \\ 1 \end{matrix} & \begin{matrix} 7-(-2)\times(-1) \\ 5 \end{matrix} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ \begin{matrix} 2-2\times2 \\ 4 \end{matrix} \\ \begin{matrix} 10-2\times(-1) \\ 12 \end{matrix} \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & \begin{matrix} 1-1\times1 \\ 0 \end{matrix} & \begin{matrix} 5-1\times1 \\ 4 \end{matrix} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ \begin{matrix} 12-4\times1 \\ 8 \end{matrix} \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

② $\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$* \det\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} = 0 \longrightarrow$ linear dependent

$\downarrow$

$\begin{bmatrix} 1 & -2 \\ \begin{matrix} 3-1\times3 \\ 0 \end{matrix} & \begin{matrix} -6-(-2)\times3 \\ 0 \end{matrix} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \begin{matrix} 3-1\times3 \\ 0 \end{matrix} \end{bmatrix}$

$\therefore$ cannot be solved

③ $\begin{bmatrix} 0 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & -2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

LU Decomposition

$Ax = b \longrightarrow LUx = b \longrightarrow$ solve $L(Ux) = b$, find $Ux \longrightarrow$ solve Ux, find x

① $\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} & & \\ 2 & & \\ -1 & & \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ \begin{matrix} 4-2\times2 \\ 0 \end{matrix} & \begin{matrix} 9-4\times2 \\ 1 \end{matrix} & \begin{matrix} -3-(-2)\times2 \\ 1 \end{matrix} \\ \begin{matrix} -2-2\times(-1) \\ 0 \end{matrix} & \begin{matrix} -3-4\times(-1) \\ 1 \end{matrix} & \begin{matrix} 7-(-2)\times(-1) \\ 5 \end{matrix} \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & \begin{matrix} 1-1\times1 \\ 0 \end{matrix} & \begin{matrix} 5-1\times1 \\ 4 \end{matrix} \end{bmatrix}$

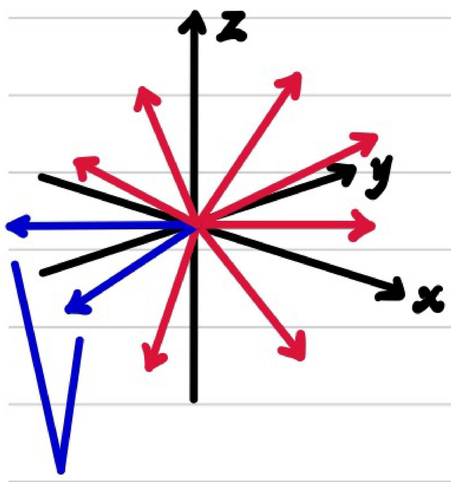
$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} \longrightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

Space

Vector space — If u and v in the vector space



① $u+v$ in the vector space

② cu in the vector space * c is a real scalar

③ $u+0=u$ * pass the origin

Subspace — subset of a vector space that itself is a vector space

Column space — span of the columns of the matrix $\longrightarrow$ $C(A)$

Rank — number of dimensions in the column space

Row space — $R(A) = C(A^T)$

Null space — the set of all solution to $Ax = 0$ $\longrightarrow$ $N(A)$

① $$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ 1 \\ x_3 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* nullity = 3

Determinants

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \longrightarrow \det(A) = |A| = a_{11}a_{22} - a_{12}a_{21}$$

$$\longrightarrow A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \longrightarrow M = \begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

$$\longrightarrow \text{Cofactor} \quad C_{ij} = (-1)^{(i+j)} M_{ij} \longrightarrow C = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

$$\xrightarrow{\text{Laplace expansion}} \det(A) = \sum_{i=1}^{N} a_{ip} C_{ip} \quad \text{or} \quad \sum_{j=1}^{N} a_{qj} C_{qj}$$

① $\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} \left(-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}\right) + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

② $\det(A) = a_{12} \left(-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}\right) + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{32} \left(-\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}\right)$

If $A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ or $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \longrightarrow \det(A) = a_{11} a_{12} a_{13}$

- Nullity > 0 $\longrightarrow$ Rank < Dimension $\longrightarrow$ Singular 奇异矩阵 $\longrightarrow$ Determinant = 0
- $\det(A) = \det(LU) = \det(L)\det(U)$
- $\det(A^{-1}) = \dfrac{1}{\det(A)}$

Inverse $A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)} = \dfrac{C^T}{\det(A)}$

Cramer's Rule $Ax = B \longrightarrow x_i = \dfrac{\det(B_i)}{\det(A)}$

Eigenvectors & Eigenvalues 特征向量 & 特征值

$Au = \lambda u = \lambda I u \longrightarrow (A - \lambda I)u = 0$ * $u \in N(A - \lambda I)$ as nutility of $(A - \lambda I) > 0$

eigenvector eigenvalue $\longrightarrow \det(A - \lambda I) = 0$

$\longrightarrow$ find $\lambda \longrightarrow$ find u

- $\det(A) = \lambda_1 \lambda_2 \dots \lambda_N$
- $\operatorname{trace}(A) = \lambda_1 + \lambda_2 + \dots + \lambda_N$

Eigenvalue decomposition

$$Au_i = \lambda u_i \longrightarrow AU = U\Lambda = [u_1 | u_2 | \dots | u_N] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \dots & \\ & & & \lambda_N \end{bmatrix}$$

$$\longrightarrow A = U\Lambda U^{-1} \text{ or } \Lambda = U^{-1}AU \longrightarrow A^N = U\Lambda^N U^{-1} = U \begin{bmatrix} \lambda_1^N & & & \\ & \lambda_2^N & & \\ & & \dots & \\ & & & \lambda_N^N \end{bmatrix} U^{-1}$$

Matrix exponentials

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \longrightarrow e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

$$\xrightarrow{A = U\Lambda U^{-1}} e^A = UU^{-1} + U\Lambda U^{-1} + \frac{U\Lambda^2 U^{-1}}{2!} + \frac{U\Lambda^3 U^{-1}}{3!} + \dots = Ue^{\Lambda}U^{-1}$$

System of ODEs

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \dots a_{1N}x_N$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \dots a_{2N}x_N$$

$$\dots\dots$$

$$\frac{dx_N}{dt} = a_{N1}x_1 + a_{N2}x_2 + \dots a_{NN}x_N$$

$$\longrightarrow \frac{dx}{dt} = Ax = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & & \\ \dots & & \dots & \\ a_{N1} & & & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_N \end{bmatrix}$$

$$\xrightarrow{A = U\Lambda U^{-1}} \frac{dx}{dt} = U\Lambda U^{-1}x \longrightarrow \frac{dU^{-1}x}{dt} = \Lambda U^{-1}x$$

$$\xrightarrow{z = U^{-1}x} \frac{dU^{-1}x}{dt} = \frac{dz}{dt} = \Lambda z \longrightarrow \text{find } x$$

Diagonalisation 对角化

If $A \in C^{N\times N}$ has N distinct λ_i then the associated u_i will be linearly independent

$\longrightarrow U^{-1}$ exist $\longrightarrow$ A can be diagonalised * $AU = \Lambda U \rightarrow A = U\Lambda U^{-1} \longrightarrow U$ diagonalised A

- when nullity$(A - \lambda I) = k$ (times that λ repeated) $\longrightarrow$ diagonalisable

Similar Matrices

$B = P^{-1}AP \longrightarrow$ matrix $A \in C^{n\times n}$ is similar to matrix $B \in C^{n\times n}$ * P is non-singular matrix

- $\lambda_{Bi} = \lambda_{Ai}$
- $u_{Bi} = P^{-1}u_{Ai}$
- $\det(A) = \det(B)$
- $\text{trace}(A) = \text{trace}(B)$

Cayley - Hamilton Theorem

charateristic polynomial $P(\lambda) = \det(A - \lambda I) \longrightarrow P(A) = 0$

① Express A^3 when

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$P(\lambda) = \det(A - \lambda I) = (1-\lambda)^2 - (-4) = \lambda^2 - 2\lambda + 5$

$\therefore P(A) = A^2 - 2A + 5I = 0 \qquad \therefore A^2 = 2A - 5I$

$\therefore A^3 = A^2 A = 2A^2 - 5A = 4A - 10I - 5A = -A - 10I$

Inner Products 内积

- $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
- $\langle cx, y\rangle = c\langle x, y\rangle$
- $\langle x, y\rangle^* = \langle y, x\rangle$
- $\langle x, x\rangle > 0$ if $x \neq 0$

① $x, y \in R^3 \longrightarrow \langle x, y\rangle = y^T x = x_1 y_1 + x_2 y_2 + x_3 y_3$ $\quad$ * $x = [x_1\ x_2\ x_3]^T$, $y = [y_1\ y_2\ y_3]^T$

- $\langle x+y, z\rangle = z^T\langle x+y\rangle = z^T x + z^T y = \langle x, z\rangle + \langle y, z\rangle$
- $\langle cx, y\rangle = y^T\langle cx\rangle = c y^T x = c\langle x, y\rangle$
- $\langle x, y\rangle^* = (y^T x)^* \overset{\in R^3}{=\!=} y^T x = x^T y = \langle y, x\rangle$
- $\langle x, x\rangle = x^T x = x_1^2 + x_2^2 + x_3^2 > 0$ if $x \neq 0$

② $x, y \in C^N \longrightarrow \langle x, y\rangle = y^H x = x_1 y_1^* + \ldots + x_n y_n^*$ $\quad$ * $x = [x_1\ x_2 \ldots x_N]^H$, $y = [y_1\ y_2 \ldots y_N]^H$

- $\langle x+y, z\rangle = z^H\langle x+y\rangle = z^H x + z^H y = \langle x+z\rangle + \langle y+z\rangle$
- $\langle cx, y\rangle = y^H\langle cx\rangle = c y^H x = c\langle x, y\rangle$
- $\langle x, y\rangle^* = (y^H x)^* = ((y^H x)^*)^T = (y^H x)^H = x^H y = \langle y, x\rangle$
- $\langle x, x\rangle = x^H x = x_1^* x_1 + x_2^* x_2 + \ldots + x_N^* x_N = |x_1|^2 + |x_2|^2 + \ldots + |x_N|^2 > 0$ if $x \neq 0$

Orthogonality 正交性

$\langle x, y\rangle = 0$ when $x, y \in V$

Norms 范数

two-norm $\|x\|_2 = (\langle x, x\rangle)^{\frac{1}{2}}$

① $x \in R^n$ $\|x\|_2 = (x^T x)^{\frac{1}{2}} = (x_1^2 + x_2^2 + \dots x_N^2)^{\frac{1}{2}}$

② $x \in C^n$ $\|x\|_2 = (x^H x)^{\frac{1}{2}} = (|x_1|^2 + |x_2|^2 + \dots + |x_N|^2)^{\frac{1}{2}}$

$\langle x, y \rangle = 0 + \|x\|_2 = \|y\|_2 = 1 \longrightarrow x, y \in V$ are orthonormal

Hermitian Matrix

$A^H = (\bar{A})^T = A$ when $A \in C^{n \times n}$

- $(x^H A x)^H = x^H A^H x = x^H A x \longrightarrow x^H A x$ is real for any $x \in C^{n \times 1}$
- $x^H A x = x^H \lambda x = \lambda x^H x \longrightarrow \lambda = \dfrac{x^H A x}{x^H x} \longrightarrow$ every λ of Hermitian matrix is real
- Let $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$ $(\lambda_1 \neq \lambda_2)$

$$x_2^H A x_1 = \lambda_1 x_2^H x_1$$

$$(A^H x_2)^H x_1$$

$$A = A^H$$

$$(\lambda_2 x_2)^H x_1$$

$$\lambda_2 x_2^H x_1 = \lambda_1 x_2^H x_1$$

$$\rightarrow (\lambda_2 - \lambda_1) x_2^H x_1 = 0 \xrightarrow{\lambda_1 \neq \lambda_2} x_2^H x_1 = \langle x_1, x_2 \rangle = 0 \longrightarrow x_i \text{ and } x_j \text{ are orthogonal if } \lambda_i \neq \lambda_j$$

$$Q = [x_1 | x_2 | x_3 | \dots | x_n] \longrightarrow Q^H Q = Q Q^H = I \xrightarrow{I = QQ^{-1}} Q^H = Q^{-1}$$

Unitary Matrix

$U^H U = U U^H = I$ when $U \in C^{n \times n}$

- $\|Ux\|_2 = (\langle Ux, Ux \rangle)^{\frac{1}{2}} = (Ux)^H (Ux) = x^H U^H U x \overset{U^H U = I}{=\!=} x^H x = \|x\|_2 \longrightarrow \|Ux\|_2 = \|x\|_2$
- $Ux = \lambda x \longrightarrow \|Ux\|_2 = \|\lambda x\|_2 = |\lambda| \|x\|_2 \longrightarrow |\lambda| = \dfrac{\|Ux\|_2}{\|x\|_2} = 1 \longrightarrow$ Every λ has $|\lambda| = 1$
- $\dots \longrightarrow x_i$ and x_j are orthogonal if $\lambda_i \neq \lambda_j$

Normal Matrix

$N^H N = N N^H$ when $N \in C^{n \times n}$

- $\dots \longrightarrow x_i$ and x_j are orthogonal if $\lambda_i \neq \lambda_j$