Equation of Motion

Governing Flow Equations

Conservation of mass

$$\frac{\partial m}{\partial t} = \frac{\partial(\rho V)}{\partial t} = 0 \longrightarrow \frac{\partial \rho}{\partial t} + \frac{\partial(\rho U_i)}{\partial x_i} = \frac{\partial \rho}{\partial t} + \frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} + \frac{\partial(\rho W)}{\partial z} = 0$$

Conservation of momentum

$$\frac{\partial(mU_j)}{\partial t} = \frac{\partial(\rho V U_j)}{\partial t} = F = F_{pressure} + F_{viscous} = -pA_j + \Sigma \sigma_{ij} A_i \longrightarrow \frac{\partial(\rho U_j)}{\partial t} + \frac{\partial(\rho U_i U_j)}{\partial x_i} = -\frac{pA_j}{V} + \frac{\partial \sigma_{ij}}{\partial x_i}$$

Conservation of energy

$\frac{\partial(\rho E)}{\partial t} + \nabla \cdot J = 0$ where E energy per unit mass $= e_{th}$ specific thermal energy $+ \frac{1}{2}(U^2 + V^2 + W^2) = H$ enthalpy $- \frac{p}{\rho}$

and J energy flux vector $= J_{convective} + J_{viscous} + J_{heat} = \rho U_i H - U_i \cdot \sigma_{ij} + q_i$

Equation of states

p and $e = f(\rho, T) \xrightarrow{\text{calorifically perfect}} p = \rho RT$ and $pv = RT$ and $e = C_v T$ and $h = e + pv = C_p T$

The 3-D Incompressible Navier-Stokes Equations

Conservation of mass

$$\frac{\partial U_i}{\partial x_i} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0$$

Conservation of momentum

$$\frac{DU_i}{Dt} = \frac{\partial U_i}{\partial t} + \underbrace{U_j \frac{\partial U_i}{\partial x_j}}_{\text{non-linear}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x_i}}_{\text{non-local}} + \nu \nabla^2 U_i \xrightarrow{i=1} \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$$

non-local atomospheric pressure change with altitude

Vorticity

$$\omega = \nabla \times u = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ U & V & W \end{vmatrix}$$

Bernoulli equation

for a steady $\frac{\partial U_i}{\partial t} = 0$, inviscid $\mu = \nu = 0$, irrotational $\omega = 0$, incompressible $\rho =$ constant flow field.

$$U_i \frac{\partial U_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{F_{body\ force}}{\rho V}$$

$$\xrightarrow{i=1} U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad ①$$

$$\xrightarrow{i=2} U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g \quad ②$$

$$\xrightarrow{i=3} U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \quad ③$$

and $\omega = \nabla \times u = 0$

$$\xrightarrow{i} \frac{\partial W}{\partial y} - \frac{\partial V}{\partial z} = 0 \quad ④$$

$$\xrightarrow{j} \frac{\partial U}{\partial z} - \frac{\partial W}{\partial x} = 0 \quad ⑤$$

$$\xrightarrow{k} \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = 0 \quad ⑥$$

substitude ④, ⑤, ⑥ into ①, ②, ③

$$\left(U\frac{\partial U}{\partial x} + V\frac{\partial V}{\partial x} + W\frac{\partial W}{\partial x}\right) + \left(U\frac{\partial U}{\partial y} + V\frac{\partial V}{\partial y} + W\frac{\partial W}{\partial y}\right) + \left(U\frac{\partial U}{\partial z} + V\frac{\partial V}{\partial z} + W\frac{\partial W}{\partial z}\right) = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} + \frac{\partial p}{\partial z}\right) - g$$

$$\longrightarrow u\nabla u = \frac{1}{2}\nabla u^2 = -\frac{1}{\rho}\nabla p - g$$

integrating on both side, get $\frac{1}{2}\rho u^2 + p + \rho g h = constant$ where $h = \sqrt{x^2+y^2+z^2}$

The 2-D Reynold-averaged Navier-Stokes (RANs)

Time averaging

where $S = \frac{1}{T}\int_0^T \hat{S}\, dt$

and time averaging $\bar{\hat{s}} = S$ and $\bar{s} = 0$

let $\hat{S} = S + s$ and $\hat{T} = T + t$ $\longrightarrow$ $\overline{St} = \overline{sT} = 0$ and $\overline{ST} = ST$ and $\overline{st}$ $\begin{cases} > 0 & \text{corelated} \\ = 0 & \text{uncorelated} \\ < 0 & \text{anti-corelated} \end{cases}$

Conservation of momentum

let $\hat{U} = U + u$, $\hat{V} = V + v$ and $\hat{p} = p + p'$

$$\xrightarrow{x} (U+u)\frac{\partial(U+u)}{\partial x} + (V+v)\frac{\partial(U+u)}{\partial y} = -\frac{1}{\rho}\frac{\partial(p+p')}{\partial x} + \nu\nabla^2(U+u)$$

$$\xrightarrow{\text{time averaging}} U\frac{\partial U}{\partial x} + \overline{u\frac{\partial u}{\partial x}} + V\frac{\partial U}{\partial x} + \overline{v\frac{\partial u}{\partial x}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 U$$

$$\xrightarrow{\text{add } \overline{u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} \ (= \bar{u}\cdot 0 = 0) \text{ to LHS}} U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\partial}{\partial x}(\overline{u^2}) - \frac{\partial}{\partial y}(\overline{uv}) + \nu\nabla^2 U$$

Thin shear-layer approximation

$\frac{d\delta}{dx} \ll 1$ i.e. $\frac{\delta}{x} \ll 1$ where $\ll$ means $O(\frac{1}{100})$, $\frac{\partial p}{\partial y} \approx 0$ and $\frac{\partial}{\partial x}(\overline{u^2}) \approx 0$

for continuity, $V \approx U\frac{\delta}{x} \longrightarrow U\frac{\partial U}{\partial x} \sim O(\frac{U^2}{x})$ and $V\frac{\partial U}{\partial y} \sim O(U\frac{\delta}{x}\frac{U}{\delta}) = O(\frac{U^2}{x})$ have same order

$\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x}$

$\xrightarrow{\text{laminar}} U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 U}{\partial y^2}$

$\xrightarrow{\text{turbulent}} U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 U}{\partial y^2} - \frac{\partial}{\partial y}(\overline{uv})$ $\longleftarrow$ turbulent moment flux grow much more rapid

Turbulence models

Reynold stresses $\rho\overline{u_i u_j} \xrightarrow{\text{incompressible}} \overline{u_i u_j}$

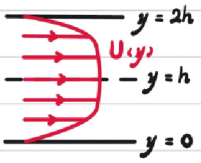
eddy viscosity assumption $-\overline{uv} = \nu_t\frac{\partial U}{\partial y}$ where $\nu_t = l_0^2$ mixing length $\frac{\partial U}{\partial y}$

Pressure gradient

now defined by free-stream condition $\xrightarrow{U = U_e} U_e\frac{\partial U_e}{\partial x} + \cancel{V_e\frac{\partial U_e}{\partial y}}^{0} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\cancel{\frac{\partial^2 U_e}{\partial y^2}}^{0} - \cancel{\frac{\partial}{\partial y}(\overline{uv})_e}^{0}$

$\longrightarrow \frac{dp}{dx} = -\rho U_e\frac{\partial U_e}{\partial x} = \mu\frac{\partial^2 U}{\partial y^2}\Big|_{y=0}$ $\longrightarrow$ boundary layer equation is parabolic

Parallel plates



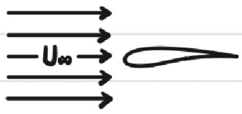
$\xrightarrow{\text{laminar}} U\cancel{\frac{\partial U}{\partial x}}^{0} + \cancel{V}^{0}\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 U}{\partial y^2} \longrightarrow \int_0^h \mu\frac{\partial^2 U}{\partial y^2}dy = \int_0^h \frac{dp}{dx}dy \xrightarrow{\frac{\partial U}{\partial y}|_{y=h} = 0} -\mu\frac{\partial U}{\partial y}\Big|_{y=0} = -\tau_w = \frac{dp}{dx}h$

$\xrightarrow{\text{turbulent}} U\cancel{\frac{\partial U}{\partial x}}^{0} + \cancel{V}^{0}\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 U}{\partial y^2} - \frac{\partial}{\partial y}(\overline{uv}) \longrightarrow \int_0^h \xrightarrow{\frac{\partial U}{\partial y}|_{y=h} = 0 \text{ and } [\overline{uv}]_0^h = 0} -\mu\frac{\partial U}{\partial y}\Big|_{y=0} = -\tau_w = \frac{dp}{dx}h$

Transonic Small Perturbation (TSP) Equation

Small perturbation

for wing with thin aerofoil section $\tau = \frac{t}{c} < 10\%$

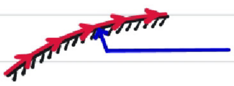


$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} U_\infty \\ 0 \\ 0 \end{bmatrix}$$

$\xrightarrow{\text{induces small perturbations}} \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} U_\infty + \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix}$ where $\begin{bmatrix} \left|\frac{\tilde{u}}{U_\infty}\right| \ll 1 \\ \left|\frac{\tilde{v}}{U_\infty}\right| \ll 1 \\ \left|\frac{\tilde{w}}{U_\infty}\right| \ll 1 \end{bmatrix}$ and $\begin{bmatrix} \tilde{u} = 0 \text{ at } x = \pm\infty \\ \tilde{v} = 0 \text{ at } y = \pm\infty \\ \tilde{w} = 0 \text{ at } z = \pm\infty \end{bmatrix}$

Boundary condition

velocity should be tangent to surface



surface $F_s(x, y, z) = 0$ e.g. for sphere, $F_s = x^2 + y^2 + z^2 - R^2$

$\longrightarrow u \cdot \nabla F_s = U\frac{\partial F_s}{\partial x} + V\frac{\partial F_s}{\partial y} + W\frac{\partial F_s}{\partial z} = 0$ where normal to the surface $n_s = \nabla F_s$

Full potential equation

for flow of perfect gas that assumed to be inviscid $\mu = \nu = 0$, steady $\frac{\partial U_i}{\partial t} = 0$ and isentropic $p = f(\rho)$

conservation of momentum $\rho U_i \frac{\partial U_i}{\partial x_i} = -\frac{\partial p}{\partial x_i} = -\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x_i} = -a^2 \frac{\partial \rho}{\partial x_i}$ ⑦

conservation of energy $H = e + \frac{1}{2}u^2 = \frac{p}{(\gamma-1)\rho} + \frac{1}{2}u^2 = \frac{a_0^2}{\gamma-1} = \frac{a^2}{\gamma-1} + \frac{1}{2}(U^2+V^2+W^2)$

$\xrightarrow{\text{free-stream}} H = \frac{a_\infty^2}{\gamma-1} + \frac{1}{2}U_\infty^2 = \frac{a^2}{\gamma-1} + \frac{1}{2}[(U_\infty+\tilde{u})^2 + \tilde{v}^2 + \tilde{w}^2]$ ⑧

Transonic Small Perturbation (TSP) equation

dividing ⑦ by a^2, using ⑧ and neglecting terms containing $\tilde{u}^2$, $\tilde{v}^2$ and $\tilde{w}^2$

$$(1-M_\infty)^2\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} = \frac{M_\infty^2(\gamma+1)}{U_\infty}\tilde{u}\frac{\partial \tilde{u}}{\partial x} + \frac{M_\infty^2(\gamma-1)}{U_\infty}\tilde{u}\left(\frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z}\right) + \frac{M_\infty^2}{U_\infty}\tilde{v}\left(\frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{v}}{\partial x}\right) + \frac{M_\infty^2}{U_\infty}\tilde{w}\left(\frac{\partial \tilde{u}}{\partial z} + \frac{\partial \tilde{w}}{\partial x}\right)$$

$\xrightarrow{\text{neglect non-linear RHS}}$ Prandtl-Glauert equation $(1-M_\infty^2)\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} = 0$

$\xrightarrow{\text{at } M_\infty \to 1} (1-M_\infty^2)\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} = \frac{M_\infty^2(\gamma+1)}{U_\infty}\tilde{u}\frac{\partial \tilde{u}}{\partial x}$

introducing velocity potential ϕ for small perturbation $(\tilde{u}, \tilde{v}, \tilde{w}) = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$ in irrotational flow $\omega = 0$

$\longrightarrow (1-M_\infty^2)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{M_\infty^2(\gamma+1)}{U_\infty}\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2}$

Prandtl-Glauert equation

$(1-M_\infty^2)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$ with soild boundary condition $U_\infty \frac{\partial F_s}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial F_s}{\partial y} + \frac{\partial \phi}{\partial z}\frac{\partial F_s}{\partial z} = 0$

$\xrightarrow{\text{for } M_\infty \ll 1}$ Laplace's equation $\nabla^2 \phi = 0$

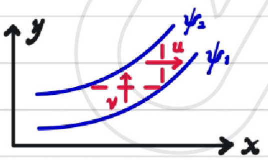
Numerical Solution of Incompressible, Potential Flow

Governing Equations

Potential flow

irrotational $\omega = 0$ and inviscid $\mu = \nu = 0$

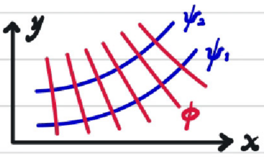
Stream function



flow cannot cross streamline

$u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$ and $V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$, $V_\theta = -\frac{\partial \psi}{\partial r}$

Velocity potential

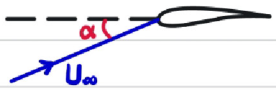


$$\vec{\nabla\psi}\cdot\vec{\nabla\phi} = 0 \longrightarrow \psi \perp \phi$$

$$u = \frac{\partial\phi}{\partial x},\ v = \frac{\partial\phi}{\partial y} \quad \text{and} \quad V_r = \frac{\partial\phi}{\partial r},\ V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

Effects of Thickness and Camber

Linearized pressure coefficient



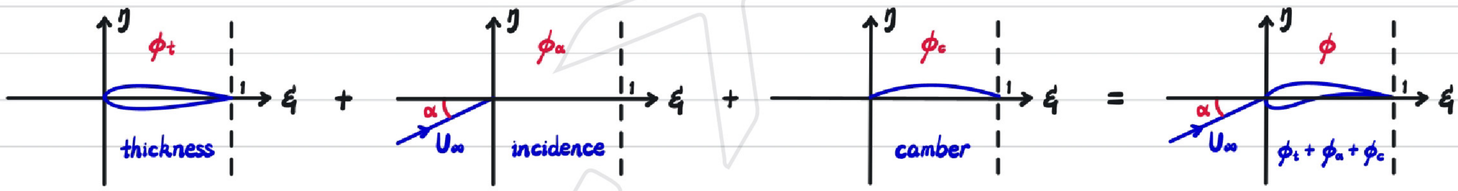
$$\xrightarrow{\text{induces small perturbations}} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U_\infty \cos\alpha + \tilde{u} \\ U_\infty \sin\alpha + \tilde{v} \end{bmatrix}$$

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2} \xrightarrow{\text{Bernoulli equation}} \frac{\frac{1}{2}\rho U_\infty^2 - \frac{1}{2}\rho u^2}{\frac{1}{2}\rho U_\infty^2} = 1 - \frac{u^2}{U_\infty^2}$$

$$\xrightarrow{\text{induces small perturbations}} 1 - \frac{(U_\infty\cos\alpha + \tilde{u})^2 + (U_\infty \sin\alpha + \tilde{v})^2}{U_\infty^2} = 1 - \frac{(U_\infty^2 \cancelto{1}{\cos^2\alpha} + 2U_\infty \cancelto{1}{\cos\alpha}\,\tilde{u} + \cancelto{0}{\tilde{u}^2}) + (U_\infty^2 \cancelto{0}{\sin^2\alpha} + \cancelto{0}{2U_\infty \sin\alpha\,\tilde{v}} + \cancelto{0}{\tilde{v}^2})}{U_\infty^2}$$

$$\longrightarrow C_p = -\frac{2\tilde{u}}{U_\infty} = -\frac{2}{U_\infty}\frac{\partial\phi}{\partial x} \quad \text{e.g. linearity} \quad \tilde{u}_1 + \tilde{u}_2 = \frac{\partial\phi_1}{\partial x} + \frac{\partial\phi_2}{\partial x} = \frac{\partial}{\partial x}(\phi_1 + \phi_2) = -\frac{U_\infty}{2}(C_{p1} + C_{p2})$$

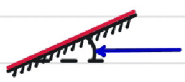
Superposition of solutions



where $\eta = \frac{y}{c}$, $\xi = \frac{x}{c}$ and $0 \le \xi \le 1 \longrightarrow \eta_t = \frac{1}{2}(\eta_u + \eta_l)$ and $\eta_c = \frac{1}{2}(\eta_u - \eta_l)$

Linearized boundary condition

streamlines to be tangential to the surface



local slope $\eta' = \frac{d\eta}{d\xi}$

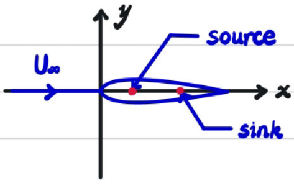
$$\longrightarrow \eta' = \frac{V}{U} = \frac{U_\infty \sin\alpha + \tilde{v}}{U_\infty\cos\alpha + \tilde{u}} \longrightarrow \frac{U_\infty \alpha + \tilde{v}}{U_\infty} = \alpha + \frac{\tilde{v}}{U_\infty} = \alpha + \frac{1}{U_\infty}\frac{\partial\phi}{\partial y}$$

therefore $\pm\eta_t' = \frac{1}{U_\infty}\frac{\partial\phi_t}{\partial y}$, $\eta_\alpha' = 0 = \alpha + \frac{1}{U_\infty}\frac{\partial\phi_\alpha}{\partial y}$ and $\eta_c' = \frac{1}{U_\infty}\frac{\partial\phi_c}{\partial y}$

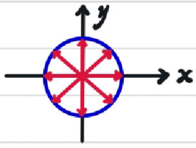
Surface Singularity Methods

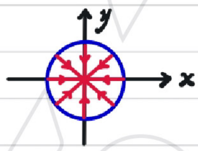
Source distribution

a symmetrical aerofoil generate no lift ⟶ no circulation ⟶ use sources and sinks generate streamline

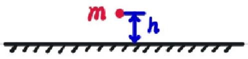


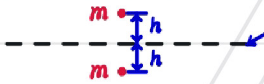
with boundary condition $\pm \eta_t'(\frac{x}{c}) = \frac{\tilde{v}}{U_\infty} = \frac{1}{U_\infty}\frac{\partial \phi_t}{\partial y}$

isolated source 

$V_r = \frac{m}{2\pi r}$ $\longleftrightarrow$ isolated sink 

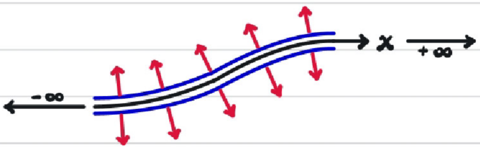
$V_r = -\frac{m}{2\pi r}$

source above rigid wall 

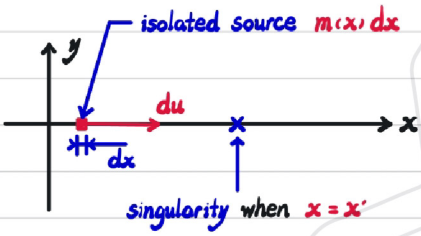
$=$ 

no flow through the wall

source along chord line with strength $m(x)$ unit length



$2v\,dx = m\,dx \longrightarrow m(x) = 2U_\infty \eta_t'(\frac{x}{c}) \longrightarrow m(x)$ determined by $\eta_t'(\frac{x}{c})$ alone

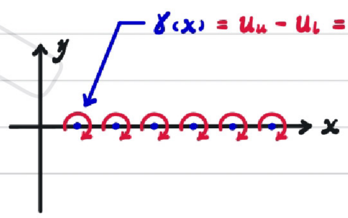
for thin section 

boundary condition is enforced along x-axis

at x', $du = \frac{m(x)\,dx}{2\pi(x'-x)} \longrightarrow u = \int_0^c \frac{m(x)}{2\pi(x'-x)}\,dx$

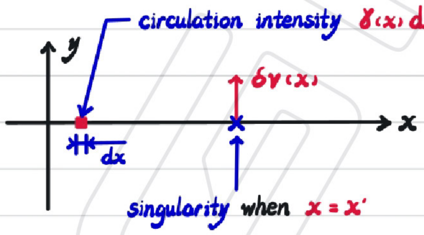
Vorticity distributions

with boundary condition $\eta_c' = \alpha + \frac{\tilde{v}}{U_\infty} = \alpha + \frac{1}{U_\infty}\frac{\partial}{\partial y}(\phi_c + \phi_\alpha)$

for small camber and incidence 

boundary condition is enforced along x-axis

where circulation $\Gamma = \oint u \cdot dl = \int_0^c \gamma(x)\,dx$



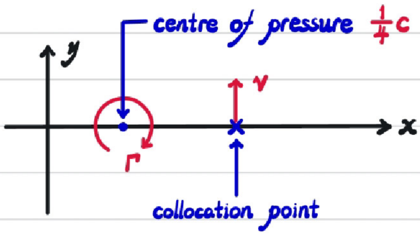
at x', $\delta v(x') = -\frac{\gamma(x)\,dx}{2\pi(x'-x)} \longrightarrow v(x') = -\int_0^c \frac{\gamma(x)}{2\pi(x'-x)}\,dx$

as $x \to c$, $u_u = u_l$ to satisfied Kutta condition

Lumped vortex model

vortex above rigid wall

for a flat plate $\eta' = \frac{d\eta}{d\xi} = 0$



with boundary condition $0 = \alpha + \frac{\tilde{v}}{U_\infty} = \alpha + \frac{1}{U_\infty}\frac{\partial \phi_a}{\partial y}$

at x', $v(x') = -\frac{\Gamma}{2\pi(x'-x)} = -\frac{\Gamma}{2\pi(x'-\frac{1}{4}c)}$

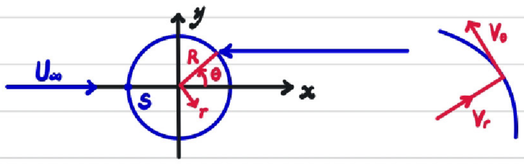
at small incidence, $C_L = \frac{L}{\frac{1}{2}\rho U_\infty^2 c} = \frac{\rho U_\infty \Gamma}{\frac{1}{2}\rho U_\infty^2 c} = \frac{2\Gamma}{U_\infty c}$

for a flat plate, $C_L = 2\pi\alpha \longrightarrow \Gamma = U_\infty c\pi\alpha$

$\longrightarrow v(x') = -\frac{U_\infty c\pi\alpha}{2\pi(x'-\frac{1}{4}c)} \xrightarrow{\text{boundary condition}} 0 = \alpha - \frac{1}{U_\infty}\frac{U_\infty c\pi\alpha}{2\pi(x'-\frac{1}{4}c)} \longrightarrow$ collocation point $x' = \frac{3}{4}c$

for $C_L = 0 \longrightarrow$ collocation point $x' = \frac{1}{2}c$

Flow around circular cylinder



for uniform flow, $V_\theta = -U_\infty \sin\theta \longrightarrow \psi_{flow} = U_\infty r \sin\theta$

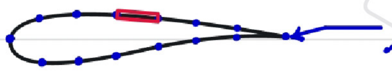
for doublet, $\psi_{doublet} = -\frac{k}{2\pi r}\sin\theta$

$\longrightarrow \psi = \psi_{flow} + \psi_{doublet} = U_\infty r\sin\theta - \frac{k}{2\pi r}\sin\theta \longrightarrow V_r = U_\infty r\cos\theta - \frac{k}{2\pi r}\cos\theta \xrightarrow{V_r = 0 \text{ at } r=R} k = 2\pi R^2 U_\infty$

at stagnation point S, $V_\theta = -U_\infty \sin\pi + \frac{k}{2\pi R^2}\sin\pi = 0$

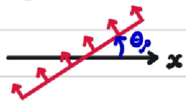
Panel code method

divide surface with varies points connected by 'panel' with source density m

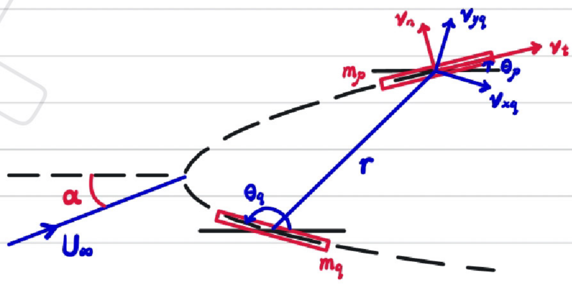


for aerofoil one of point must at trailing edge

angle of element to x-axis θ_p



body coordinate system



find velocity components at the control point collocation point of the pth element due to qth element

$v_{xq} = f(m_q)$ and $v_{yq} = f(m_q) \longrightarrow v_n(p,q) = A_{pq} m_q$ and $v_t(p,q) = B_{pq} m_q$

for the effect of U_∞ and $\alpha \longrightarrow U_n(p) = U_\infty \sin(\alpha - \theta_p)$ and $U_t(p) = U_\infty \cos(\alpha - \theta_p)$

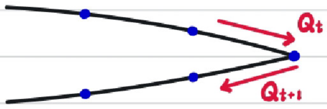
with linearized boundary condition $\sum_{q=1}^{N} v_n(p,q) + U_n(p) = \sum_{q=1}^{N} A_{pq} m_q + U_\infty \sin(\alpha - \theta_p) = 0 \longrightarrow$ find m_q

therefore find tangential velocity $Q_p = \sum_{q=1}^{N} v_t(p,q) + U_t(p) = \sum_{q=1}^{N} B_{pq} m_q + U_\infty \cos(\alpha - \theta_p)$

apply unit circulation $\Gamma = U_\infty c \xrightarrow{\text{linearized boundary condition}}$ find $m_{q,r} \longrightarrow$ find $Q_{p,r}$

$\xrightarrow{\text{superposition}} Q_{p,total} = Q_p + kQ_{p,r} \longrightarrow C_p(p) = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 - \frac{u^2}{U_\infty^2} = 1 - \frac{Q_{p,total}}{U_\infty^2}$

to fulfill Kutta condition



flow leaves trailing edge in smooth manner $Q_t + Q_{t+1} = 0$

$\longrightarrow (Q_t + kQ_{t,r}) + (Q_{t+1} + kQ_{t+1}) = \Delta Q_t + k\Delta Q_t = 0 \longrightarrow$ find k

$\longrightarrow \Gamma_{total} = kU_\infty c \longrightarrow L = 2k$

Boundary Layers

Introduction

Reynolds number

$Re = \frac{\text{internal force}}{\text{viscous force}} = \frac{\frac{1}{2}\rho u^2}{\frac{1}{2}\mu\frac{u}{D}} = \frac{\rho u D}{\mu} = \frac{uD}{\nu}$ i.e. $Re_x = \frac{\rho u x}{\mu} \longrightarrow$ viscous effect $\propto \frac{1}{Re}$

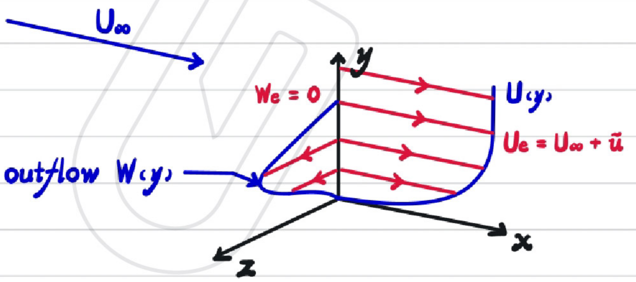
Boundary conditions

impermeability boundary condition $u \cdot \nabla F_s = 0 \longrightarrow V = 0$

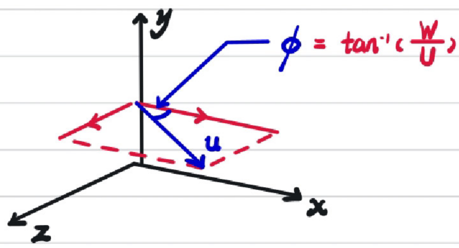
no-slip boundary condition $u \times \nabla F_s = 0 \longrightarrow U = W = 0$

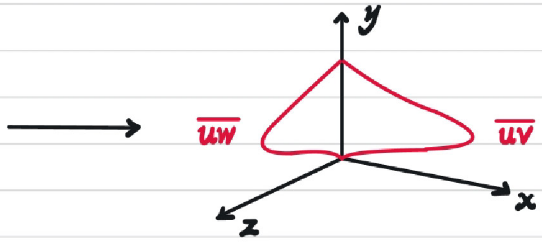
Boundary layer theory

a class of flow dominated by large mean shear which is predominantly in one direction



for total velocity

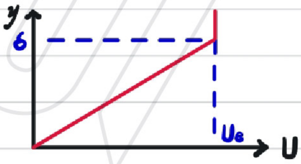




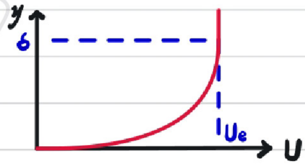
2-D boundary layers $W = 0$, $\frac{\partial}{\partial z} = 0$, $\frac{\partial}{\partial t} = 0$ and $\overline{uw} = \overline{vw} = 0$ even $\overline{u^2}, \overline{v^2}, \overline{w^2} \neq 0$

Pressure gradient

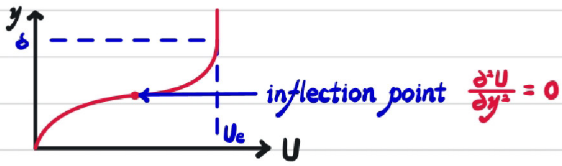
zero pressure gradient $\frac{dp}{dx} = -\rho U_e \frac{\partial U_e}{\partial x} = \mu \frac{\partial^2 U}{\partial y^2}\Big|_{y=0} = 0$



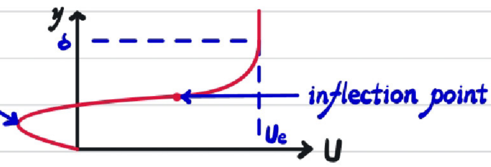
favourable pressure gradient $\frac{dp}{dx} = -\rho U_e \frac{\partial U_e}{\partial x} = \mu \frac{\partial^2 U}{\partial y^2}\Big|_{y=0} < 0$



adverse pressure gradient $\frac{dp}{dx} = -\rho U_e \frac{\partial U_e}{\partial x} = \mu \frac{\partial^2 U}{\partial y^2}\Big|_{y=0} > 0$



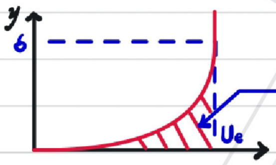
$\xrightarrow{\text{destabilizing}}$ seperation occurs reversal flow



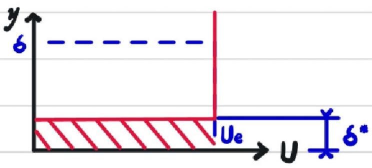
Momentum Integral Equation

Displacement thickness

viscid



viscous blockage = inviscid



mass flow deficit $= \int_0^\infty \rho (U_e - U)\, dy = \rho U_e \delta^* \longrightarrow \frac{\delta^*}{\delta} = \int_0^1 (1 - \frac{U}{U_e})\, d(\frac{y}{\delta})$

Momentum thickness

momentum flux deficit $= \int_0^\infty \rho U (U_e - U)\, dy = \rho U_e^2 \theta \longrightarrow \frac{\theta}{\delta} = \int_0^1 \frac{U}{U_e}(1 - \frac{U}{U_e})\, d(\frac{y}{\delta})$

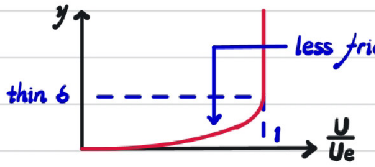
to maximising $\frac{\theta}{\delta}$, $\frac{\partial}{\partial y}[\frac{U}{U_e}(1 - \frac{U}{U_e})] = \frac{1}{U_e}\frac{\partial U}{\partial y} - \frac{1}{U_e^2}\frac{\partial U^2}{\partial y} = \frac{1}{U_e}(1 - \frac{2U}{U_e})\frac{\partial U}{\partial y} = 0 \longrightarrow 1 - \frac{2U}{U_e} = 0 \longrightarrow U = \frac{1}{2}U_e$

$\longrightarrow (\frac{\theta}{\delta})_{max} = \int_0^1 \frac{1}{4}\, d(\frac{y}{\delta})$

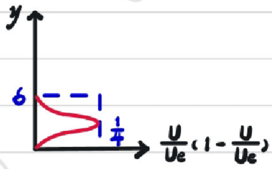
Shape factor

$H = \frac{\delta^*}{\theta}$ e.g. for $\frac{dp}{dx} = 0$, $H_{laminar} \approx 2.6$ and $H_{turbulent} \approx 1.4$ $\xrightarrow{\text{more momentum}}$ more resistant to seperation

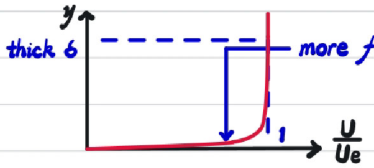
laminar low Re_x



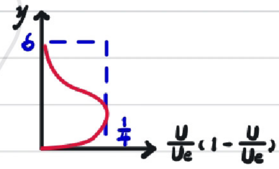
less friction $\frac{\partial U}{\partial y}\Big|_{wall}$ smaller



turbulent high Re_x



more friction more resistant to seperation



$H \to 1$ as $Re \to \infty$

Momentum integral equation

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 U}{\partial y^2} = U_e\frac{\partial U_e}{\partial x} + \frac{1}{\rho}\frac{d\tau}{dy} \longrightarrow \int_0^\delta \left(U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} - U_e\frac{\partial U_e}{\partial x}\right)dy = \left[\frac{1}{\rho}\tau\right]_0^\delta = -\frac{\tau_w}{\rho}$$

$$\xrightarrow{\text{continuity}} \int_0^\delta \left(U\frac{\partial U}{\partial x} - \int_0^y \frac{\partial U}{\partial x}dy\,\frac{\partial U}{\partial y} - U_e\frac{\partial U_e}{\partial x}\right)dy = \int_0^\delta \left(U\frac{\partial U}{\partial x} - U_e\frac{\partial U_e}{\partial x}\right)dy - \left(\left[U\int_0^y \frac{\partial U}{\partial x}dy\right]_0^\delta - \int_0^\delta U\frac{\partial U}{\partial x}dy\right) = -\frac{\tau_w}{\rho}$$

$$\longrightarrow \int_0^\delta \left(U\frac{\partial U}{\partial x} - U_e\frac{\partial U_e}{\partial x}\right)dy - \left(U_e\int_0^\delta \frac{\partial U}{\partial x}dy - \int_0^\delta U\frac{\partial U}{\partial x}dy\right) = \int_0^\delta \left(2U\frac{\partial U}{\partial x} - U_e\frac{\partial U}{\partial x} - U_e\frac{\partial U_e}{\partial x}\right)dy = -\frac{\tau_w}{\rho}$$

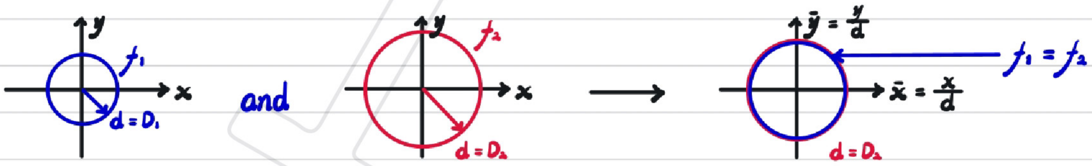
$$\xrightarrow{\frac{\partial (U_e U)}{\partial x} = U_e\frac{\partial U}{\partial x} + U\frac{\partial U_e}{\partial x}} \int_0^\delta \frac{\partial}{\partial x}\left[U(U_e - U)\right]dy + \frac{\partial U_e}{\partial x}\int_0^\delta (U_e - U)\,dy = \frac{\tau_w}{\rho}$$

$$\longrightarrow \frac{d}{dx}(U_e^2\theta) + \delta^* U_e\frac{dU_e}{dx} = \frac{\tau_w}{\rho} \xrightarrow{\frac{d}{dx}(U_e^2\theta) = U_e^2\frac{d\theta}{dx} + 2U_e\theta\frac{dU_e}{dx}} \frac{d\theta}{dx} + (H+2)\frac{dU_e}{dx}\frac{\theta}{U_e} = \frac{\tau_w}{\rho U_e^2} = \frac{C_f}{2}$$

for turbulent flow $\int_0^\delta -\frac{\partial}{\partial y}(\overline{uv})\,dy = [\overline{uv}]_0^\delta = 0$

Exact Laminar Boundary Layer Solution

Self-similar solutions



for $Re \gg 1$ $\xrightarrow{\text{internal force dominate}}$ θ small and no preferred length $\xrightarrow{\text{similarity solutions}}$ $\frac{U}{U_e} = f\left(\frac{y}{\delta}\right)$

$\longrightarrow$ viscous length $\delta \sim \sqrt{\nu t}$ and viscous time $t \sim \frac{x}{U_e}$ $\longrightarrow$ $\delta \sim \sqrt{\frac{\nu x}{U_e}}$

Blasius equation

for incompressible ρ = constant 2-D flow over a flat plate $\frac{\partial U_e}{\partial x} = 0$ $\longrightarrow$ $\frac{dp}{dx} = -\rho U_e\frac{\partial U_e}{\partial x} = 0$

$\longrightarrow \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = 0$ and $U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = \nu \frac{\partial^2 U}{\partial y^2}$

$\longrightarrow FF'' + 2F''' = 0$ with $F(\eta) = \int f(\eta)\, d\eta$, $f(\eta) = \frac{U}{U_e}$, $\eta \propto \frac{y}{\delta} \longrightarrow \eta = y\sqrt{\frac{U_e}{\nu x}}$

the solution $U = U(\eta)$ is self-similar

boundary condition
- $\xrightarrow{\text{wall}} \eta = 0 \quad F(0) = 0$ and $F'(0) = f(0) = 0$
- $\xrightarrow{\text{free-stream}} \eta = \infty \quad F'(\infty) = f(\infty) = 1$
- $\xrightarrow{\text{assumption}} F''(0) = A \longrightarrow$ iterate to find A for $F'(\infty) = 1$

for flate plate, $\delta \approx 5.0\sqrt{\frac{\nu x}{U_e}}$ where $\eta_0 \approx 5.0$

$$\delta^* = \delta\int_0^1 (1 - \frac{U}{U_e})\, d(\frac{y}{\delta}) = \sqrt{\frac{\nu x}{U_e}}\int_0^{\eta_0}(1 - F')\, d\eta = \sqrt{\frac{\nu x}{U_e}}[\eta - F]_0^{\eta_0} \xrightarrow{\text{Howarth's table}} \delta^* = 1.721\sqrt{\frac{\nu x}{U_e}} = 1.721 \times Re_x^{-\frac{1}{2}}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} = \frac{\mu\frac{\partial U}{\partial y}\big|_{y=0}}{\frac{1}{2}\rho U_e^2} = \frac{2\nu}{U_e^2}U_e\frac{\partial F'}{\partial y}\Big|_{y=0} = \frac{2\nu}{U_e^2}U_e\frac{\partial F'}{\partial \eta}\Big|_{\eta=0}\sqrt{\frac{U_e}{\nu x}} = 2F''\big|_{\eta=0}\sqrt{\frac{\nu}{U_e x}}$$

$$\xrightarrow{\text{Howarth's table}} C_f = 0.664\sqrt{\frac{\nu}{U_e x}} = 0.664\, Re_x^{-\frac{1}{2}}$$

$$\xrightarrow{\text{all the momentum losses due to } \tau_w} \theta = 0.664\sqrt{\frac{\nu x}{U_e}} = 0.664 \times Re_x^{-\frac{1}{2}}$$

Falkner-Skan equation

for $\frac{dp}{dx} \neq 0$, if $U_e \propto x^m \xrightarrow{\text{self-similar velocity profile}} F''' + \frac{1}{2}(m+1)FF'' + m(1 - F'^2) = 0$

Analytical solution

$$\frac{U}{U_e} = 2\eta - \eta^2$$

Thwaites' Approximate Method

Pressure gradient parameter

at wall, $\frac{dp}{dx} = -\rho U_e\frac{\partial U_e}{\partial x} = \nu\frac{\partial^2 U}{\partial y^2}\Big|_{y=0} \longrightarrow m = \frac{\theta^2}{U_e}(\frac{\partial^2 U}{\partial y^2})_w = -\frac{\theta^2}{\nu}\frac{dU_e}{dx} = -\lambda \longrightarrow m \propto \frac{dp}{dx}$

Shear stress parameter

$\tau_w = \mu\frac{\partial U}{\partial y}\Big|_{y=0} \longrightarrow l = \frac{\theta}{U_e}(\frac{\partial U}{\partial y})_w \longrightarrow l \propto \tau_w$

Thwaites' approximation method

for a laminar boundary layer, assume profiles at distance x_0 from the stagnation point $\longrightarrow \frac{U}{U_e} = f(\frac{y}{\delta}, m)$

$\frac{d\theta}{dx} + (H+2)\frac{dU_e}{dx}\frac{\theta}{U_e} = \frac{\tau_w}{\rho U_e^2} = \frac{C_f}{2} \xrightarrow{\text{multiply } 2U_e\theta} U_e\frac{d\theta^2}{dx} = 2\nu[m(H+2)+l] = \nu L(m)$

$\xrightarrow{\text{Thwaites shows } L(m) = 0.45+6m} U_e\frac{d\theta^2}{dx} = 0.45\nu - 6\theta^2\frac{dU_e}{dx} \longrightarrow \frac{d}{dx}(U_e^6\theta^2) = 0.45\nu U_e^5$

$\longrightarrow \theta^2(x_0) = 0.45\, U_e^{-6}(x_0)\,\nu\int_0^{x_0} U_e^5\, dx \longleftarrow$ x followed the surface of boundary layer ≠ streamwise

for $0 \le \lambda \le 0.1$, $l = 0.22 + 1.57\lambda - 1.80\lambda^2$ and $H = 2.61 - 3.75\lambda + 5.24\lambda^2 \longrightarrow \frac{dp}{dx} < 0$

for $-0.1 \le \lambda \le 0$, $l = 0.22 + 1.402\lambda + \frac{0.018\lambda}{0.107+\lambda}$ and $H = 2.088 + \frac{0.0731}{0.14+\lambda} \longrightarrow \frac{dp}{dx} > 0$

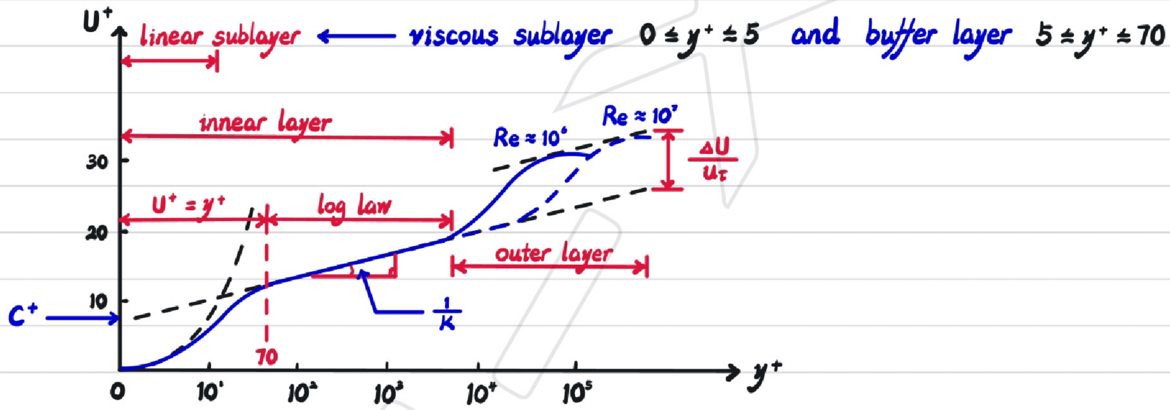
seperation occurs at $\left.\frac{\partial U}{\partial y}\right|_{y=0} = 0$ while $\frac{dp}{dx} > 0 \longrightarrow l = 0 \longrightarrow m = 0.082$

The Turbulent Boundary Layer

The turbulent boundary layer

wall friction velocity $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$

non-dimensional 'wall-variables' $y^+ = \frac{y}{\delta_\nu} = \frac{yu_\tau}{\nu}$ and $U^+ = \frac{U}{u_\tau}$



Log law

for the near-wall region $y < \delta \longrightarrow \frac{\partial}{\partial y} > \frac{\partial}{\partial x}$ or $\frac{\partial}{\partial z}$, $\frac{dp}{dx} \ll \frac{\tau_w}{y} \xrightarrow{\tau_w \text{ dominant}} U = f(y, \tau_w, \rho, \nu)$

$\xrightarrow{\text{dimensional analysis}} U^+ = f(y^+)$, $\frac{y}{u_\tau}\frac{\partial U}{\partial y} = f(y^+)$ and $\frac{-\overline{uv}}{u_\tau^2} = f(y^+)$

$\xrightarrow{\text{high Re, } y^+ \to \infty}$ assume $\frac{y}{u_\tau}\frac{\partial U}{\partial y} = \frac{1}{\kappa}$ and $\frac{-\overline{uv}}{u_\tau^2} = \frac{\text{turbulent stress}}{\text{wall shear stress}} = 1$

$\longrightarrow \frac{U}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{yu_\tau}{\nu}\right) + C$ where von Kármán constant $\kappa \approx 0.41$

for zero pressure gradient $\frac{dp}{dx} = 0$, smooth-wall boundary layer $C \approx 5.2$

Law of the wake

outside viscous sublayer $\longrightarrow$ $\frac{\Delta U}{u_\tau} = \frac{U_e - U}{u_\tau} = f(\frac{y}{\delta})$ and $\frac{-\overline{uv}}{u_\tau^2} = f(\frac{y}{\delta})$

for flat plate $\frac{\partial U_e}{dx} = 0$, zero pressure gradient $\frac{dp}{dx} = 0$ boundary layer, define empirically Coles' profile

$\frac{U}{u_\tau} = \frac{1}{\kappa} \ln(\frac{y u_\tau}{\nu}) + C + \frac{\Pi(x)}{\kappa} \omega(\eta)$ where $\omega(\eta) = 1 - \cos(\pi \eta)$ and $\eta = \frac{y}{\delta}$

the wake function

for $Re_\theta \geq 5000$, $\Pi(x) \longrightarrow$ constant

at edge of boundary layer $\xrightarrow{\omega(1) = 1 - (-1) = 2}$ $\frac{U_e}{u_\tau} = \frac{1}{\kappa} \ln(\frac{y u_\tau}{\nu}) + C + \frac{2\Pi(x)}{\kappa}$ $\longrightarrow$ $\frac{\Delta U}{u_\tau} = \frac{U_e - U}{u_\tau} = \frac{2\Pi(x)}{\kappa}$

Lifting-Line Theory

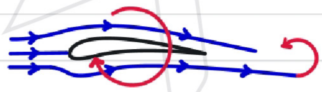
Wing of Finite Span

Basic theories

Kutta-Joukowiski theorem L' per unit span $= \rho U_\infty \Gamma$

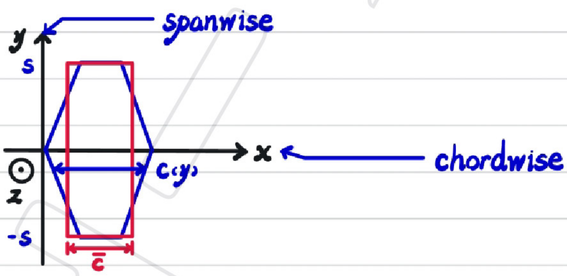
starting vortex

Kelvin's circulation theorem $\longrightarrow$



$\frac{D\Gamma}{Dt} = 0$ $\longrightarrow$ $\Gamma_{bound} + \Gamma_{shed} = 0$

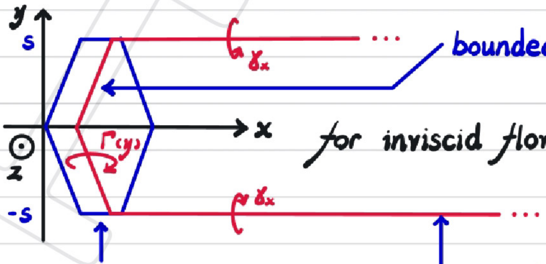
wing of finite span



$AR = \frac{b^2}{S_{ref}} = \frac{(2s)^2}{2s\bar{c}} = \frac{2s}{\bar{c}}$

where large $AR \longrightarrow AR > 4$

Helmholtz theorem



bounded vortex line // leading edge

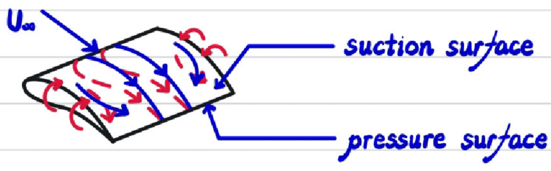
for inviscid flow $\mu = \nu = 0$, vortex line cannot end in space

at tips $p_u = p_l \longrightarrow \Gamma(\pm s) = 0$

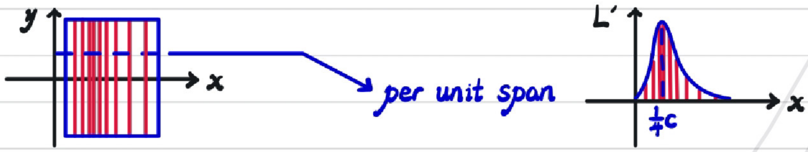
trailing vortex line // x-axis

Circulation intensity

velocity vectors have spanwise components



bounded vortex line



circulation intensity $\gamma_y(x,y) = u_u - u_l = 2u = 2\frac{d\phi}{dx} = \mp\frac{d\Gamma(y)}{dx} \longrightarrow \Gamma(y) = \int_0^c \gamma_y(x,y)\,dx$

streamwise vorticity $\gamma_x(x,y) = 2v = 2\frac{d\phi}{dy} = \pm\frac{d\Gamma(y)}{dy}$

Helmholtz theorem $\frac{\partial\gamma_x}{\partial y} = -\frac{\partial\gamma_y}{\partial x} \longrightarrow \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = \nabla^2\phi = \frac{1}{2}\left(\frac{\partial\gamma_y}{\partial x} + \frac{\partial\gamma_x}{\partial y}\right) = 0$

Downwash and Induced Drag

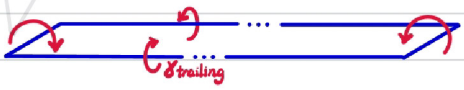
Vortex sheet

bound vorticity ⟶ generate lift

trailing vorticity $\xrightarrow{\text{over wing and in wake}}$ induce downwash

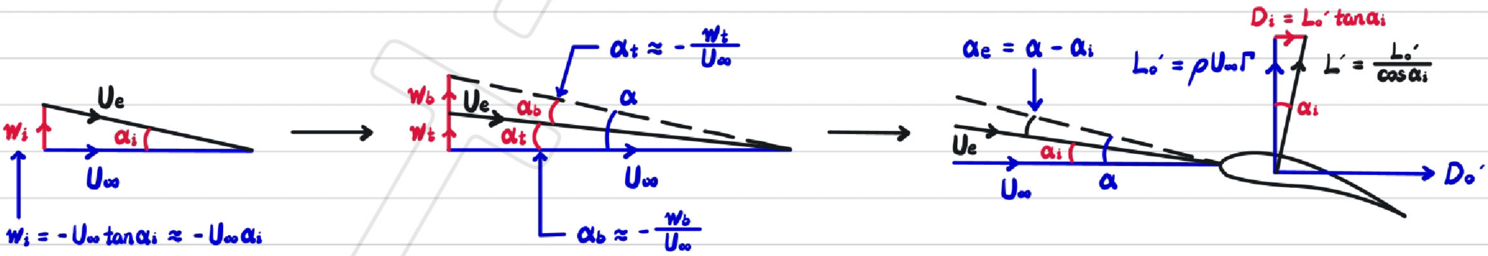
trailing vorticity $\xrightarrow{\text{far downstream}}$ only contributor to downwash

vortex sheet in the plane $z = 0$ $\quad \Gamma_{bound} = \int_0^c \gamma_y\,dx$



$\Gamma_{shed} = -\Gamma_{bound}$

Downwash



Induced Drag

an inviscid phenomenon ⟶ typically $C_{D_i} \approx 3\%\, C_L$

Lifting-line equation

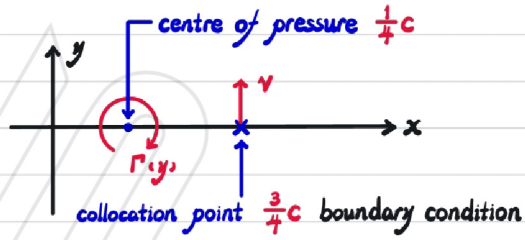
linearised boundary condition $0 = \alpha + \frac{w}{U_\infty} \xrightarrow{\text{small angle approximation}} w_b + w_t + U_\infty\alpha = 0$

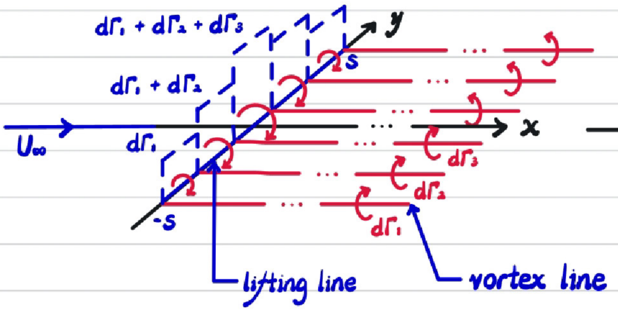
Prandtl's Theory for Large AR Wings

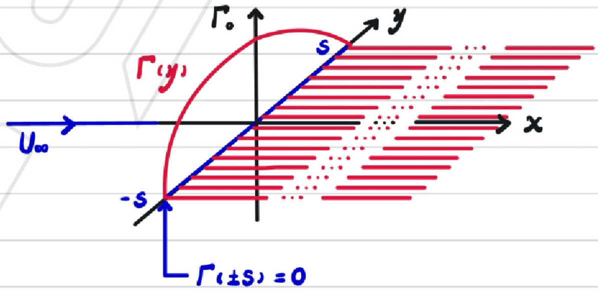
Lifting line

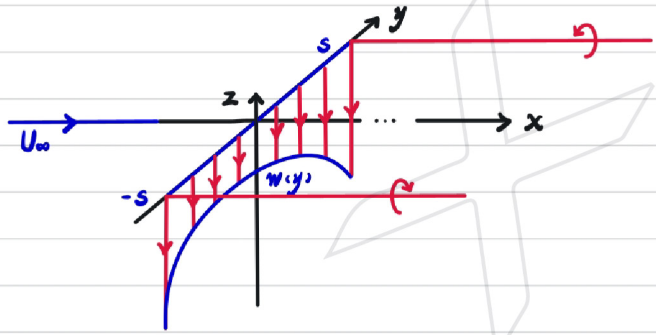
for thin plate in the plane $z = 0$ $\longrightarrow$ small disturbance

bound vorticity 

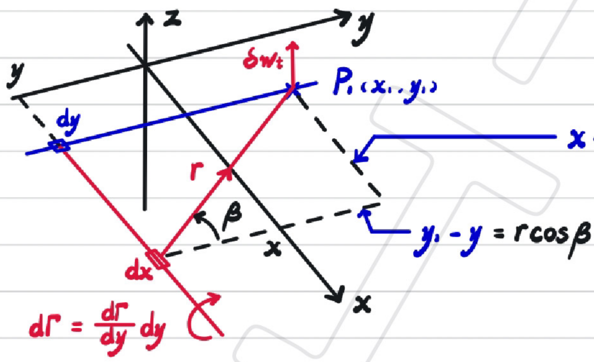
= single lumped vortex 

$\longrightarrow$ 

$\xrightarrow{\int dx}$ 

$\xrightarrow{\int dy}$ 

Prandtl's lifting-line equation



$x - x_1 = (y_1 - y)\tan\beta \longrightarrow dx = (y_1 - y)\sec^2\beta \, d\beta$

$y_1 - y = r\cos\beta$

downwash at P_1 due to trailing vorticity $\xrightarrow{\text{for a vortex line}}$ Biot-Savart law $\delta w_t = -\frac{d\Gamma}{4\pi r^2}\cos\beta\,\delta x$

$$\longrightarrow \delta w_t = -\frac{d\Gamma}{4\pi}\int_0^\infty \frac{\cos\beta}{r^2}dx = -\frac{d\Gamma}{4\pi}\int_0^{\frac{\pi}{2}} \frac{\cos^2\beta}{(y-y_1)^2}\cos\beta\,(y-y_1)\sec^2\beta\,d\beta = -\frac{d\Gamma}{4\pi}\int_0^{\frac{\pi}{2}}\frac{\cos\beta}{(y-y_1)}d\beta = -\frac{d\Gamma}{4\pi(y-y_1)}$$

$$\xrightarrow{\text{for all vortex line}} w_t(x_1, y_1) = -\frac{1}{4\pi}\int_{-s}^{s}\frac{\frac{d\Gamma}{dy}dy}{y_1 - y} = -\frac{1}{4\pi}\int_{-s}^{s}\frac{\gamma_x(x, y)}{y_1 - y}dy$$

downwash at P_1 due to bound vorticity $\longrightarrow$ $w_b(x_1, y_1) = 2\left(-\frac{1}{4\pi}\int_{x_{L.E.}}^{x_{T.E.}}\frac{\frac{d\Gamma}{dx}dx}{x_1 - x}\right) = -\frac{1}{2\pi}\int_{x_{L.E.}}^{x_{T.E.}}\frac{\gamma_y(x, y_1)}{x_1 - x}dx$

downwash at P_1 $\quad w(x_1,y_1) = w_b(x_1,y_1) + w_t(x_1,y_1) = -\frac{1}{2\pi}\int_{x_{L.E.}}^{x_{T.E.}}\frac{\gamma_y(x,y_1)}{x_1-x}dx - \frac{1}{4\pi}\int_{-s}^{s}\frac{\frac{d\Gamma}{dy}dy}{y_1-y}$

$\xrightarrow{\text{lifting-line equation}} \alpha(y_1) = -\frac{1}{U_\infty}(w_b+w_t) = \frac{1}{2\pi U_\infty}\int_{x_{L.E.}}^{x_{T.E.}}\frac{\gamma_y(x,y_1)}{x_1-x}dx + \frac{1}{4\pi U_\infty}\int_{-s}^{s}\frac{\frac{d\Gamma}{dy}dy}{y_1-y}$

Effective angle of attack

for lifting-line equation, $\alpha(y_1)$ must be independent of x $\longrightarrow F(y_1) = \int_{x_{L.E.}}^{x_{T.E.}}\frac{\gamma_y(x,y_1)}{x_1-x}dx$

assume solution is $\gamma_y(x,y_1) = \frac{F(y_1)}{\pi}\sqrt{\frac{x_{T.E.}(y)-x}{x-x_{L.E.}(y)}} \xrightarrow{C_L'(y)=\frac{\rho U_\infty\Gamma}{\frac{1}{2}\rho U_\infty^2 c}} C_L'(y_1) = \frac{2}{U_\infty c}\int_{x_{L.E.}}^{x_{T.E.}}\frac{F(y_1)}{\pi}\sqrt{\frac{x_{T.E.}(y)-x}{x-x_{L.E.}(y)}}dx$

$\xrightarrow{x=x_{L.E}+\frac{c}{2}(1-\cos\theta)} C_L'(y_1) = \frac{2}{U_\infty c}\int_{x_{L.E.}}^{x_{T.E.}}\frac{F(y_1)}{\pi}\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}dx \xrightarrow{\times\frac{1+\cos\theta}{1+\cos\theta}} \frac{2}{U_\infty c}\int_0^\pi\frac{F(y_1)}{\pi}\frac{1+\cos\theta}{\sin\theta}(\frac{c}{2}\sin\theta\, d\theta)$

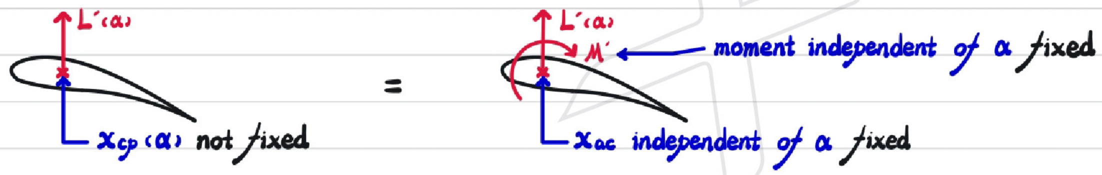
$\longrightarrow C_L'(y_1) = \frac{F(y_1)}{U_\infty} \longrightarrow \frac{1}{2\pi U_\infty}\int_{x_{L.E.}}^{x_{T.E.}}\frac{\gamma_y(x,y_1)}{x_1-x}dx = \frac{C_L'(y_1)}{2\pi}$

for a thin 2-D section $\quad C_L'(y_1) = \frac{2\Gamma(y_1)}{U_\infty c} = a_0\alpha_e(y) \quad$ where $\quad a_0 = 2\pi$

$\longrightarrow \frac{1}{2\pi U_\infty}\int_{x_{L.E.}}^{x_{T.E.}}\frac{\gamma_y(x,y_1)}{x_1-x}dx = \alpha_e(y_1)$

$\longrightarrow \alpha(y_1) = \alpha_e(y_1) + \frac{1}{4\pi U_\infty}\int_{-s}^{s}\frac{\frac{d\Gamma}{dy}dy}{y_1-y} = \alpha_e(y_1) + \alpha_i(y_1)$ and $\gamma_y(x,y_1) = 2\alpha_e U_\infty\sqrt{\frac{c-x}{x}}$

Centre of pressure and aerodynamic centre



$\longrightarrow x_{ac}L' = M'$

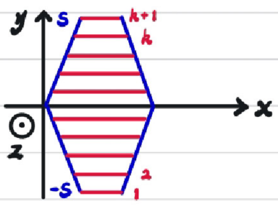
$C_M' = \frac{M'}{\frac{1}{2}\rho U_\infty^2c^2} = \frac{1}{\frac{1}{2}\rho U_\infty^2c^2}\int_0^c \Delta p\, x\, dx = \frac{1}{\frac{1}{2}\rho U_\infty^2c^2}\int_0^c\frac{dL}{dx}x\,dx = \frac{1}{\frac{1}{2}\rho U_\infty^2c^2}\int_0^c\rho U_\infty\gamma_y x\,dx$

$\longrightarrow C_M' = \frac{4\alpha_e}{c^2}\int_0^c\sqrt{\frac{c-x}{x}}\,x\,dx \xrightarrow{x=\frac{c}{2}(1-\cos\theta)} \frac{4\alpha_e}{c^2}\int_0^\pi\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}[\frac{c}{2}(1-\cos\theta)]\frac{c}{2}\sin\theta\, d\theta = \frac{1}{2}\pi\alpha_e$

for a thin 2-D section $\quad C_L' = a_0\alpha_e = 2\pi\alpha_e \longrightarrow \frac{x_{ac}}{c} = \frac{C_M'}{C_L'} = \frac{1}{4} \longrightarrow x_{ac} = x_{cp} = \frac{1}{4}c$

Numerical solution

divide wing into $k+1$ spanwise stations



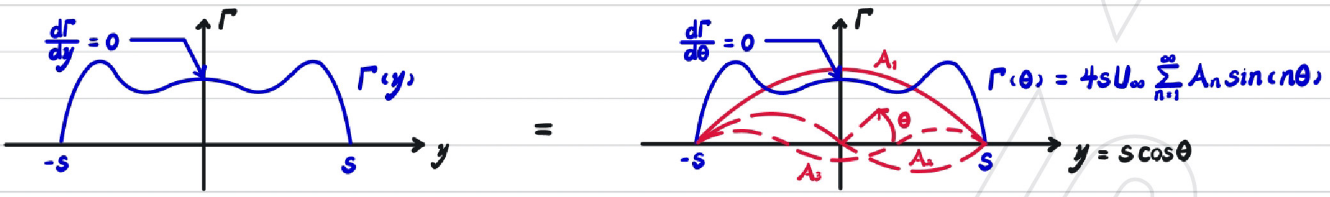
guess $\Gamma(y_1), \Gamma(y_2), \dots, \Gamma(y_{k+1})$ $\longleftarrow$ $+D\cdot$error

$\longrightarrow$ compute $\alpha_i(y_1), \alpha_i(y_2), \dots, \alpha_i(y_k) \longrightarrow \alpha_e(y) = \alpha(y) - \alpha_i(y)$ $\quad$ $|\text{error}| > \varepsilon$

$\longrightarrow \Gamma(y) = \frac{1}{2}a_0\alpha_e(y)U_\infty c(y) \longrightarrow$ error $= \Gamma_{guess}(y) - \Gamma_{result}(y)$ $\xrightarrow{|\text{error}| \le \varepsilon} \Gamma(y_1), \Gamma(y_2), \dots, \Gamma(y_{k+1})$

Collocation Solution Method

Fourier series



$$w_i(y_1) = -\frac{1}{4\pi}\int_{-s}^{s}\frac{\frac{d\Gamma}{dy}dy}{y_1 - y} \xrightarrow{dy = -s\sin\theta d\theta} -\frac{1}{4\pi}\int_{-s}^{s}\frac{\frac{d\Gamma}{d\theta}\left(\frac{dy}{d\theta}\right)^{-1}}{y_1 - y}dy = -\frac{1}{4\pi}\int_{\pi}^{0}\frac{-4U_\infty\sum_{n=1}^{\infty}\frac{A_n n\cos(n\theta)}{\sin\theta}}{s\cos\theta_1 - s\cos\theta}(-s\sin\theta\, d\theta)$$

$$\longrightarrow w_i(\theta_1) = -\frac{U_\infty}{\pi}\int_0^{\pi}\frac{\sum_{n=1}^{\infty}A_n n\cos(n\theta)}{\cos\theta - \cos\theta_1}d\theta = -U_\infty\sum_{n=1}^{\infty} nA_n\frac{\sin(n\theta_1)}{\sin\theta_1}$$

Symmetric loading

$$\sin[n(\pi-\theta)] = \cancel{\sin(n\pi)}^{\,0}\cos(n\theta) - \cos(n\pi)\sin(n\theta) = (-1)^{n+1}\sin(n\theta)$$

$\xrightarrow{n \text{ odd}}$ $\sin(n\theta) = \sin[n(\pi-\theta)]$ $\longrightarrow$ symmetric loading $\Gamma(\theta) = \Gamma(-\theta)$

$\xrightarrow{n \text{ even}}$ $\sin(n\theta) = -\sin[n(\pi-\theta)]$ $\longrightarrow$ antisymmetric loading $\Gamma(\theta) = \Gamma(-\theta)$ $\longleftarrow$ discard

Fourier series solution

$$\alpha(\theta_1) = \alpha_e(\theta_1) + \frac{1}{4\pi U_\infty}\int_{-s}^{s}\frac{\frac{d\Gamma}{dy}dy}{y_1 - y} = \frac{2\Gamma(\theta_1)}{a_0 U_\infty c} + \frac{1}{4\pi U_\infty}\int_{-s}^{s}\frac{\frac{d\Gamma}{dy}dy}{y_1 - y} = \frac{8s\sum_{n=1}^{\infty}A_n\sin(n\theta_1)}{a_0 c} + \sum_{n=1}^{\infty} nA_n\frac{\sin(n\theta_1)}{\sin\theta_1}$$

$$\longrightarrow \sum_{n=1}^{\infty} A_n \sin(n\theta)(n\mu + \sin\theta) = \mu\alpha\sin\theta \quad \text{where} \quad \mu = \frac{a_0 c}{8s}$$

Total lift

$$L = \int_{-s}^{s} L_0' dy = \int_{-s}^{s}\rho U_\infty \Gamma\, dy = \rho U_\infty\int_{\pi}^{0} 4sU_\infty\sum_{n=1}^{\infty}A_n\sin(n\theta)(-s\sin\theta\, d\theta) \xrightarrow{\text{orthogonality}} 4\rho U_\infty^2 s^2 A_1\left(\tfrac{1}{2}\pi\right)$$

$$\longrightarrow C_L = \frac{L}{\frac{1}{2}\rho U_\infty^2 S} = \frac{4\pi s^2 A_1}{S} = \pi \mathit{AR}\, A_1$$

Total induced drag

$$D_i = \int_{-s}^{s} L_0' \tan\alpha_i\, dy = \int_{-s}^{s}\rho U_\infty\Gamma\left(-\frac{w_i}{U_\infty}\right)dy = \rho U_\infty\int_{\pi}^{0} 4sU_\infty\sum_{m=1}^{\infty}A_m\sin(m\theta)\sum_{n=1}^{\infty}nA_n\frac{\sin(n\theta)}{\sin\theta}(-s\sin\theta\, d\theta)$$

$$= 4\rho U_\infty^2 s^2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} nA_mA_n\int_0^{\pi}\sin(n\theta)\sin(m\theta)\,d\theta \xrightarrow{\text{orthogonality}} 4\rho U_\infty^2 s^2\sum_{n=1}^{\infty}nA_n^2\left(\tfrac{1}{2}\pi\right)$$

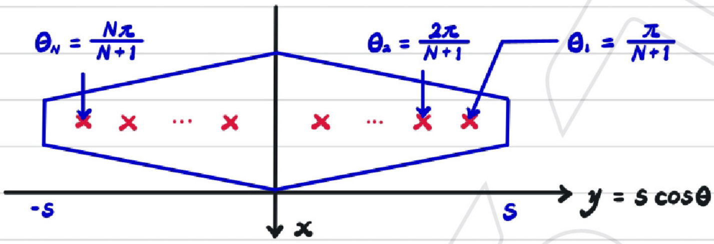
$$\longrightarrow C_{D_i} = \frac{D_i}{\frac{1}{2}\rho U_\infty^2 S} = \frac{4\pi s^2\sum_{n=1}^{\infty}nA_n^2}{S} = \pi\mathit{AR}\sum_{n=1}^{\infty}nA_n^2$$

$$\xrightarrow{\text{symmetric loading}} C_{D_i} = \pi\mathit{AR}\,(A_1^2 + 3A_3^2 + 5A_5^2 + \dots)$$

Solution for general platform

assume that all coefficients A_n are negiligible for $n > N$

$\longrightarrow$ choose N collocation points



symmetric loading

at θ_1 = at θ_N $\longrightarrow$ $\mu_1\alpha_1\sin\theta_1 = A_1\sin\theta_1(\mu_1 + \sin\theta_1) + A_3\sin3\theta_1(3\mu_1 + \sin\theta_1) + \ldots$

at θ_2 = at θ_{N-1} $\longrightarrow$ $\mu_2\alpha_2\sin\theta_2 = A_1\sin\theta_2(\mu_2 + \sin\theta_2) + A_3\sin3\theta_2(3\mu_2 + \sin\theta_2) + \ldots$

...

$\longrightarrow$ find $A_1, A_3, A_5, \ldots$

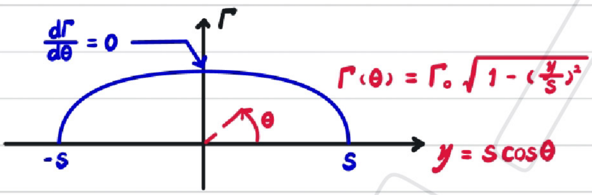
at flap edge, $N+1$ is a multiple of 3 $\longrightarrow$ collocation point invalid

Elliptic Loading

Minimum induced drag

$$C_{Di} = \pi A\!\!R \sum_{n=1}^{\infty} nA_n^2 \xrightarrow{\text{let } \delta = \sum_{n=2}^{\infty} nA_n^2} \pi A\!\!R A_1^2(1+\delta) \longrightarrow C_{Di} = \frac{C_L^2}{\pi A\!\!R}(1+\delta)$$

minimised induced drag $\delta = 0$ $\longrightarrow$ $\Gamma(\theta) = 4sU_\infty A_1\sin\theta = \Gamma_0\sin\theta = \Gamma_0\sqrt{1-(\frac{y}{s})^2}$



elliptical loading

Downwash velocity

$$w_t = -\frac{1}{4\pi}\int_{-s}^{s}\frac{\frac{d\Gamma}{dy}dy}{y_1 - y} = -U_\infty\sum_{n=1}^{\infty} nA_n\frac{\sin(n\theta_1)}{\sin\theta_1}$$

$\longrightarrow$ $w_t = -U_\infty A_1 = -U_\infty\frac{\Gamma_0}{4sU_\infty} = -U_\infty\frac{C_L}{\pi A\!\!R}$ constant along the span

Lift curve slope

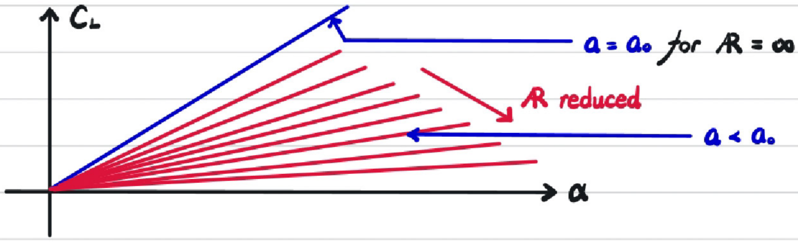
for a wing with no twist $\alpha(y) = \text{constant}$ $\longrightarrow$ $\alpha_e = \alpha - \alpha_i = \alpha + \frac{w_t}{U_\infty} = \alpha - \frac{C_L}{\pi A\!\!R} = \text{constant}$

$\longrightarrow$ $C_L'(y) = a_0\alpha_e = \frac{2\Gamma(y)}{U_\infty c} = \text{constant}$ $\longrightarrow$ $\Gamma(\theta) \propto c(\theta)$

$$\frac{1}{a_0} = \left(\frac{\partial C_L}{\partial\alpha_e}\right)^{-1} = \frac{\partial\alpha_e}{\partial\alpha}\left(\frac{\partial C_L}{\partial\alpha}\right)^{-1} = \frac{\partial}{\partial\alpha}\left(\alpha - \frac{C_L}{\pi A\!\!R}\right)\frac{1}{a} = \left(1 - \frac{a}{\pi A\!\!R}\right)\frac{1}{a}$$

$\longrightarrow \frac{1}{a_0} = \frac{1}{a} - \frac{1}{\pi AR}$

2-D equivalent lift curve slope decreases as AR reduced



Compressible Flow

Isentropic Flow

Change in entropy

for calorically perfect gas $\Delta S = S_2 - S_1 = C_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} = C_v \ln \frac{T_2}{T_1} + R \ln \frac{V_2}{V_1}$

for isentropic process i.e. adiabatic and reversible $\longrightarrow S_2 = S_1 \longrightarrow \frac{p_2}{p_1} = (\frac{\rho_2}{\rho_1})^{\gamma} = (\frac{T_2}{T_1})^{\frac{\gamma}{\gamma-1}} = (\frac{a_2}{a_1})^{\frac{2\gamma}{\gamma-1}}$

Streamline

for a streamline of the steady flow ρ = constant connects the stagnation point $U_0 = 0$

H = constant $\longrightarrow C_p T_0 = C_p T + \frac{U^2}{2}$ where $C_p = \frac{\gamma R}{\gamma - 1} = C_v + R$

$\xrightarrow{a^2 = \gamma RT} \frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2$

$\xrightarrow{\text{isentropic}} \frac{p_0}{p} = (1 + \frac{\gamma-1}{2} M^2)^{\frac{\gamma}{\gamma-1}}$ and $\frac{\rho_0}{\rho} = (1 + \frac{\gamma-1}{2} M^2)^{\frac{1}{\gamma-1}}$

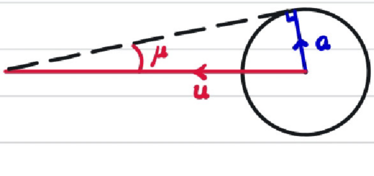
Prantl-Meyer Expansion

Compressibility

$\tau = -\frac{1}{V}\frac{dV}{dp} = \frac{1}{\rho}\frac{d\rho}{dp}$

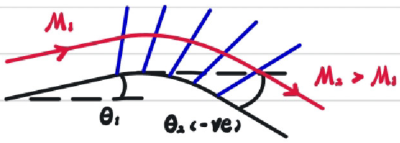
Mach wave

isentropic expansion



Mach angle $\mu = \arcsin(\frac{a}{u}) = \arcsin(\frac{1}{M})$

Expansion fans



for right-running wave $\theta - \nu(M) = \text{constant}$

for left-running wave $\theta + \nu(M) = \text{constant}$

$\xrightarrow{\text{isentropic}}$ $\frac{T_2}{T_1} = (\frac{T_{02}}{T_2})^{-1}\frac{T_{01}}{T_1}$, $\frac{\rho_2}{\rho_1} = (\frac{\rho_{02}}{\rho_2})^{-1}\frac{\rho_{01}}{\rho_1}$ and $\frac{p_2}{p_1} = (\frac{p_{02}}{p_2})^{-1}\frac{p_{01}}{p_1}$

1D Isentropic Duct Flow

The convergent-divergent nozzle

$(M^2 - 1)\frac{du}{u} = \frac{dA}{A}$

- $\xrightarrow{M<1}$ subsonic $du \propto -dA$
- $\xrightarrow{M>1}$ supersonic $du \propto dA$

Chocking

at throat of area A^* $M = 1$ $\longrightarrow$ $\frac{A}{A^*} = f(M)$ where $A \geq A^*$

$\xrightarrow{\text{isentropic}}$ $\frac{T_2}{T_1} = (\frac{T_{02}}{T_2})^{-1}\frac{T_{01}}{T_1}$, $\frac{\rho_2}{\rho_1} = (\frac{\rho_{02}}{\rho_2})^{-1}\frac{\rho_{01}}{\rho_1}$ and $\frac{p_2}{p_1} = (\frac{p_{02}}{p_2})^{-1}\frac{p_{01}}{p_1}$

Rankine-Hugoniot Discontinouity Relation

General form

for discontinous adiabatic process $u_s(U_2 - U_1) = F_2^n - F_1^n$ where $F^n = F_x n_x + F_y n_y + F_z n_z$

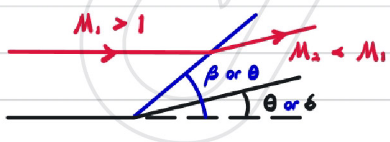
Steady normal shock

$p_1 < p_2$, $T_1 < T_2$, $\rho_1 < \rho_2$

$\dot{m}_1 = \dot{m}_2$, $T_{01} = T_{02}$, $p_{01} > p_{02}$, $S_1 > S_2$

$\xrightarrow{\text{adiabatic}}$ $\frac{\rho_{02}}{\rho_{01}} = \frac{\rho_{02}}{\rho_2}\frac{\rho_2}{\rho_1}(\frac{\rho_{01}}{\rho_1})^{-1}$ and $\frac{p_{02}}{p_{01}} = \frac{p_{02}}{p_2}\frac{p_2}{p_1}(\frac{p_{01}}{p_1})^{-1}$

Steady oblique shock



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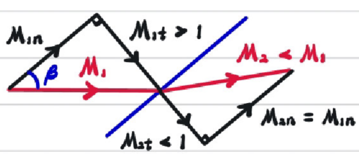


$p_1 < p_2$, $T_1 < T_2$, $\rho_1 < \rho_2$

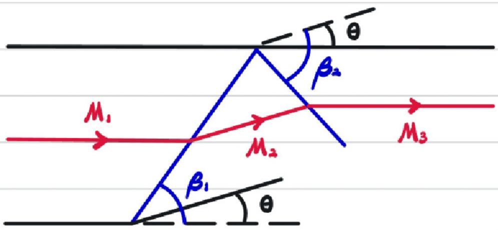
$\dot{m}_1 = \dot{m}_2$, $T_{01} = T_{02}$, $p_{01} > p_{02}$, $S_1 > S_2$

adiabatic $\longrightarrow \dfrac{\rho_{02}}{\rho_{01}} = \dfrac{\rho_{02}}{\rho_2}\dfrac{\rho_2}{\rho_1}\left(\dfrac{\rho_{01}}{\rho_1}\right)^{-1}$ and $\dfrac{p_{02}}{p_{01}} = \dfrac{p_{02}}{p_2}\dfrac{p_2}{p_1}\left(\dfrac{p_{01}}{p_1}\right)^{-1}$

Mach wave

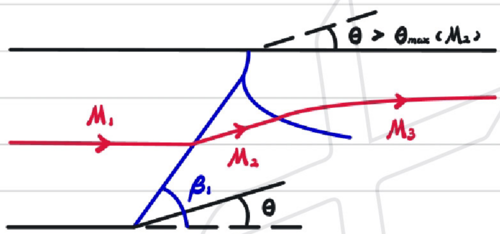
Regular reflection

from soild boundaries



Mach reflection

from soild boundaries

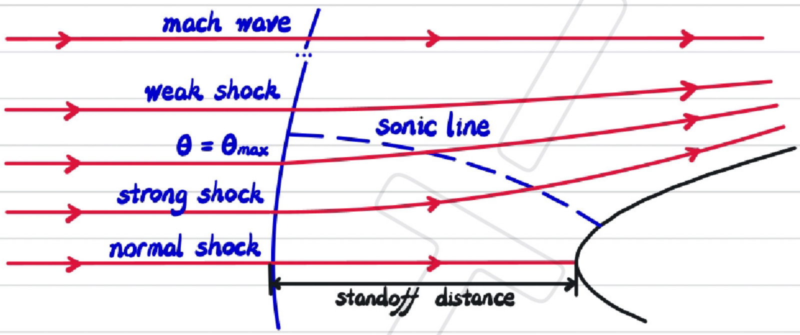


guess θ ← (|error| > ε)

→ find M_2

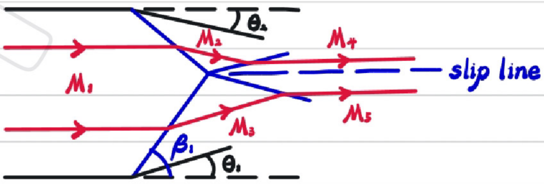
→ error = $\theta - \theta_{max}$ —(|error| ≤ ε)→ θ_{max}

Bow shock



Intersection of waves

from soild boundaries

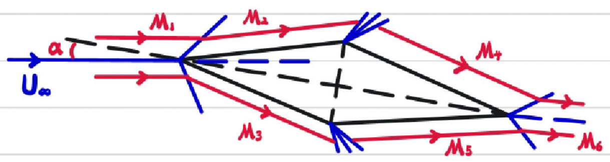


$M_4 \neq M_5$, $S_4 \neq S_5$

$p_4 = p_5$, $n_4 = n_5$

Supersonic expansion theory

combination of using oblique shock theory and Prandtl-Meyer expansion theory



find p_2, p_3 $\longrightarrow$ find p_4, p_5

$\longrightarrow F_2' = p_2 l_2,\ F_3' = p_3 l_3,\ F_4' = p_4 l_4,\ F_5' = p_5 l_5$

$\longrightarrow F_n' = F_{lower}' - F_{upper}' = (F_3'\cos\theta_3 + F_5'\cos\theta_5) - (F_2'\cos\theta_2 + F_4'\cos\theta_4)$

and $F_t' = F_{backward}' - F_{forward}' = (F_2'\sin\theta_2 + F_3'\sin\theta_3) - (F_4'\sin\theta_4 + F_5'\sin\theta_5)$

$\longrightarrow L' = F_n'\cos\alpha - F_t'\sin\alpha$ and $D' = F_n'\sin\alpha + F_t'\cos\alpha$

$\longrightarrow C_L' = \dfrac{L'}{\frac{1}{2}\rho U_\infty^2 c} = \dfrac{L'}{\rho M_1^2 \gamma R T_1 c} = \dfrac{L'}{p_1 M_1^2 \gamma c}$ and $C_D' = \dfrac{D'}{p_1 M_1^2 \gamma c}$

Higher Order Similarity Rules

Linear supersonic theory

for thin airfoil in supersonic flow with small deflection angle

for right-running wave $C_p = -\dfrac{2\theta}{\sqrt{M_\infty^2 - 1}} \longrightarrow C_{p_2} - C_{p_1} = -\dfrac{2(\theta_2 - \theta_1)}{\sqrt{M_\infty^2 - 1}}$

for left-running wave $C_p = \dfrac{2\theta}{\sqrt{M_\infty^2 - 1}} \longrightarrow C_{p_2} - C_{p_1} = \dfrac{2(\theta_2 - \theta_1)}{\sqrt{M_\infty^2 - 1}}$

$\longrightarrow C_L = \dfrac{4\alpha}{\sqrt{M_\infty^2 - 1}}$ and $C_D = \dfrac{4}{\sqrt{M_\infty^2 - 1}}\left[\alpha^2 + \int_0^1 (\theta_c^2 + \theta_t^2)\, d\left(\frac{x}{c}\right)\right]$ where $\theta_c = \dfrac{d\eta_c}{dx}$ and $\theta_t = \dfrac{d\eta_t}{dx}$

Rayleigh Flow

Application

combustor or afterburner of an aeroengine

Heat transfer

for compressible, inviscid, non-adiabatic, reversible 1-D calorically perfect ideal gas

$T_{02} = T_{01} + \dfrac{q}{C_p} \longrightarrow \dfrac{T_0}{T_0^*} = f(M)$ where $T_0 \doteq T_0^*$

Conservation of momentum

$\rho_2 U_2^2 - \rho_1 U_1^2 = p_1 - p_2 \xrightarrow{a^2 = \gamma RT} \rho_2 M_2^2 \gamma R T_2 - \rho_1 M_1^2 \gamma R T_1 = p_1 - p_2 \longrightarrow p_2 \gamma M_2^2 - p_1 \gamma M_1^2 = p_1 - p_2$

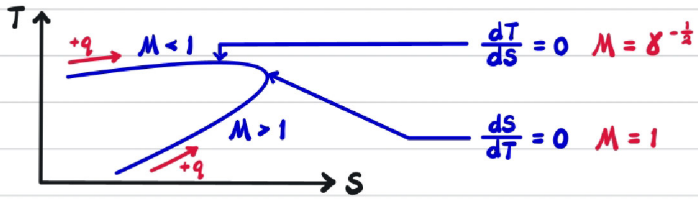
$\longrightarrow \dfrac{p_2}{p_1} = \dfrac{1 + \gamma M_1^2}{1 + \gamma M_2^2}$

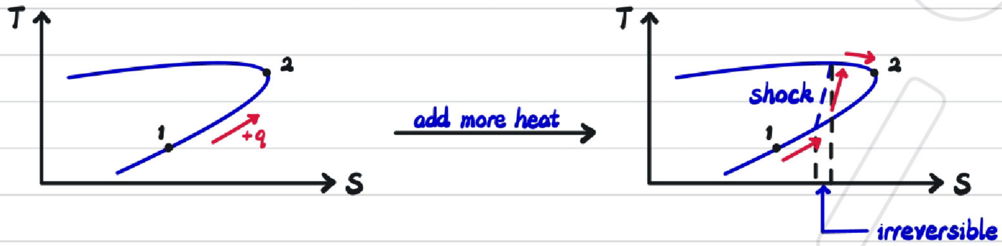
$$\longrightarrow \frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} \xrightarrow{\text{conservation of mass}} \frac{p_2}{p_1}\frac{U_2}{U_1} = \frac{p_2}{p_1}\frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}} \longrightarrow \frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\frac{M_2}{M_1}\right)^2 = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2\left(\frac{M_2}{M_1}\right)^2$$

$$\longrightarrow \frac{\rho_2}{\rho_1} = \frac{p_2}{p_1}\frac{T_1}{T_2} = \frac{p_1}{p_2}\left(\frac{M_1}{M_2}\right)^2 = \frac{1+\gamma M_2^2}{1+\gamma M_1^2}\left(\frac{M_1}{M_2}\right)^2$$

$$\longrightarrow \frac{T_{02}}{T_{01}} = \frac{T_{02}}{T_2}\frac{T_2}{T_1}\left(\frac{T_{01}}{T_1}\right)^{-1},\quad \frac{\rho_{02}}{\rho_{01}} = \frac{\rho_{02}}{\rho_2}\frac{\rho_2}{\rho_1}\left(\frac{\rho_{01}}{\rho_1}\right)^{-1} \text{ and } \frac{p_{02}}{p_{01}} = \frac{p_{02}}{p_2}\frac{p_2}{p_1}\left(\frac{p_{01}}{p_1}\right)^{-1}$$

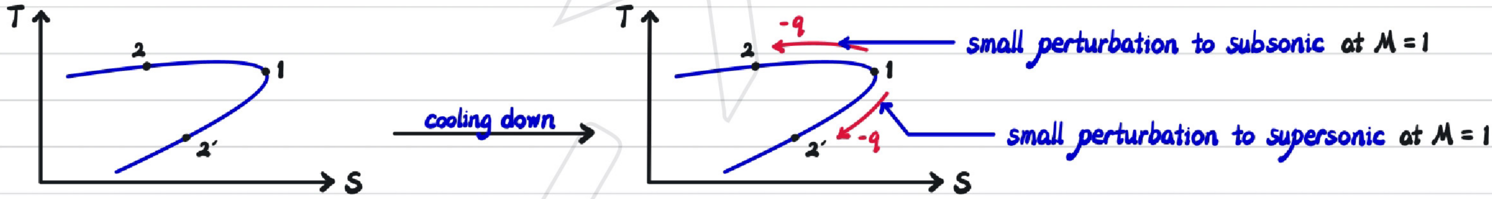
Mollier plots





$q(\Delta S^{subsonic}_{1\to 2}) > q(\Delta S^{supersonic}_{1\to 2})$

change inlet condition



Fanno Flow

Friction force

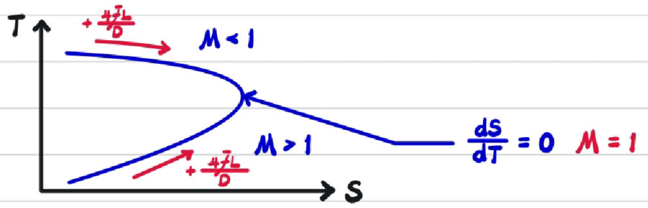
for compressible, adiabatic, 1-D calorically perfect ideal gas go through a cylindrical pipe

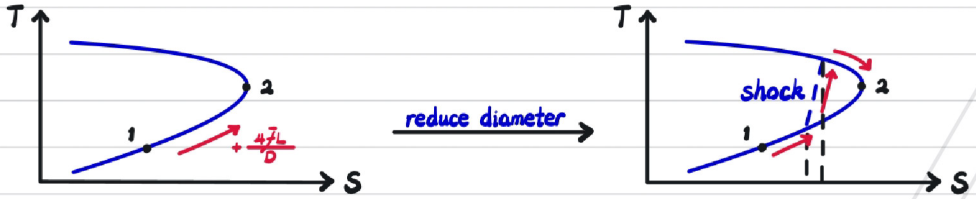
$$\left(\frac{4\bar{f}L}{D}\right)_2 = \left(\frac{4\bar{f}L}{D}\right)_1 - \frac{4\bar{f}L}{D} \longrightarrow \frac{4\bar{f}L}{D} = f(M) \text{ where } \frac{4\bar{f}L}{D} \geq 0$$

$$\xrightarrow{\text{adiabatic}} \frac{T_2}{T_1} = \frac{T_{02}}{T_2}\left(\frac{T_{01}}{T_1}\right)^{-1} \longrightarrow \frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{\gamma-1}} \text{ and } \frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

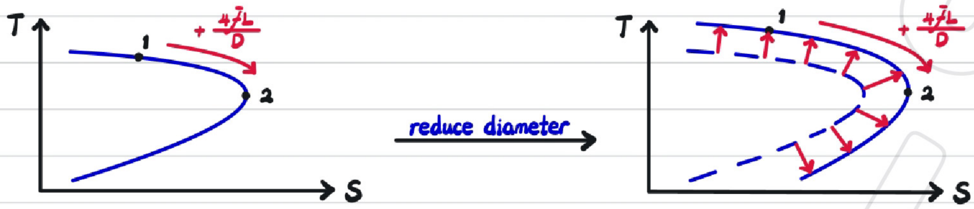
$$\longrightarrow \frac{\rho_{02}}{\rho_{01}} = \frac{\rho_{02}}{\rho_2}\frac{\rho_2}{\rho_1}\left(\frac{\rho_{01}}{\rho_1}\right)^{-1} \text{ and } \frac{p_{02}}{p_{01}} = \frac{p_{02}}{p_2}\frac{p_2}{p_1}\left(\frac{p_{01}}{p_1}\right)^{-1}$$

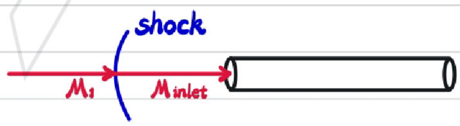
Mollier plots





$q(\Delta S_{1\to2}^{subsonic}) > q(\Delta S_{1\to2}^{supersonic})$





Fanno flow cannot transition from supersonic to subsonic because it is irreversible

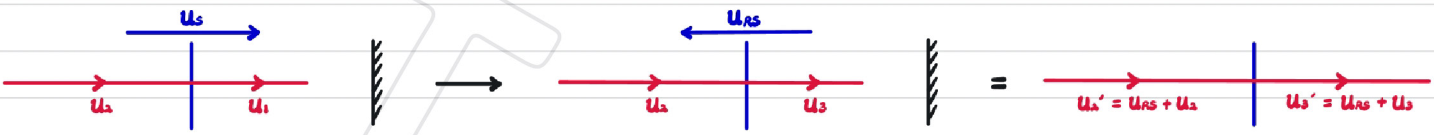
Unsteady Normal Shock

Unsteady normal shock

$\longrightarrow$ $M_s = M_1' = \frac{u_1'}{a_1} = \frac{u_1'}{\sqrt{\gamma R T_1}}$

$T = T'$, $p = p'$, $\rho = \rho'$ and $T_0 \neq T_0'$, $p_0 \neq p_0'$, $\rho_0 \neq \rho_0'$

Reflected normal shock



$$\frac{M_{RS}}{M_{RS}^2 - 1} = \frac{M_s}{M_s^2 - 1}\sqrt{1 + \frac{2(\gamma - 1)}{(\gamma + 1)^2}(M_s^2 - 1)\left(\gamma + \frac{1}{M_s^2}\right)} \longrightarrow M_{RS} = M_2' = \frac{u_2'}{a_2} = \frac{u_2'}{\sqrt{\gamma R T_2}}$$

$\longrightarrow$ high ΔT and Δp across reflected shock

$\longrightarrow$ shock tubes for obtaining high temperatures in gases $\longleftarrow$ very short period of time

Unsteady 1-D Isentrophic Flow

Acoustic theory

for infinitesmally small waves $\longrightarrow$ $\rho = \rho_\infty + \rho'$ and $U = u'$ $\longrightarrow$ $(u')^2 = (\rho')^2 = (\rho' u') = 0$

sound waves (for infinitesmally small waves)

stationary based flow $U_\infty = 0$ (for $U = u'$)

for unsteady, inviscid $\mu = \nu = 0$, isentropic flow

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho U)}{\partial x} = 0 \longrightarrow \cancelto{0}{\frac{\partial \rho_\infty}{\partial t}} + \frac{\partial \rho'}{\partial t} + \rho_\infty \frac{\partial u'}{\partial x} + \cancelto{0}{\rho' \frac{\partial u'}{\partial x}} + u' \cancelto{0}{\frac{\partial \rho_\infty}{\partial x}} + \cancelto{0}{u' \frac{\partial \rho'}{\partial x}} = 0$$

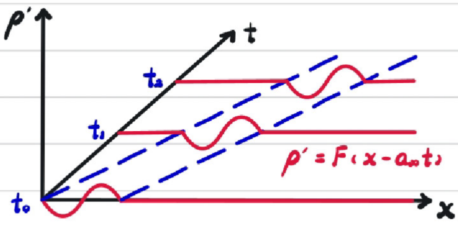
$$\longrightarrow \frac{\partial \rho'}{\partial t} + \rho_\infty \frac{\partial u'}{\partial x} = 0 \xrightarrow{\frac{\partial}{\partial t} \text{ and } \frac{\partial}{\partial x}} \frac{\partial^2 \rho'}{\partial t^2} + \rho_\infty \frac{\partial^2 u'}{\partial x \partial t} = 0 \; ① \text{ and } \frac{\partial^2 \rho'}{\partial x \partial t} + \rho_\infty \frac{\partial^2 u'}{\partial x^2} = 0 \; ②$$

also $$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{a^2}{\rho}\frac{\partial \rho}{\partial x} \longrightarrow \rho_\infty \frac{\partial u'}{\partial t} + \cancelto{0}{\rho' \frac{\partial u'}{\partial t}} + \cancelto{0}{\rho_\infty u' \frac{\partial u'}{\partial x}} + \cancelto{0}{\rho' u' \frac{\partial u'}{\partial x}} = -a_\infty^2 \cancelto{0}{\frac{\partial \rho_\infty}{\partial x}} - a_\infty^2 \frac{\partial \rho'}{\partial x}$$

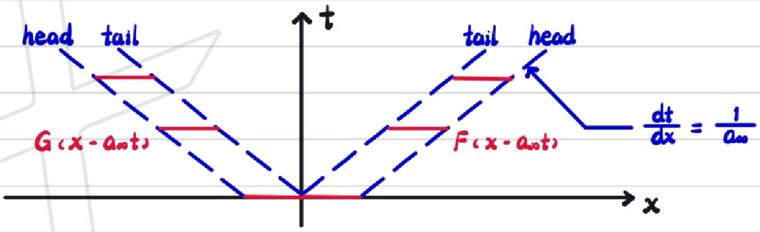
$$\longrightarrow \rho_\infty \frac{\partial u'}{\partial t} = -a_\infty^2 \frac{\partial \rho'}{\partial x} \xrightarrow{\frac{\partial}{\partial t} \text{ and } \frac{\partial}{\partial x}} \rho_\infty \frac{\partial^2 u'}{\partial t^2} = -a_\infty^2 \frac{\partial^2 \rho'}{\partial x \partial t} \; ③ \text{ and } \rho_\infty \frac{\partial^2 u'}{\partial x \partial t} = -a_\infty^2 \frac{\partial^2 \rho'}{\partial x^2} \; ④$$

substitude ④ into ① $\frac{\partial^2 \rho'}{\partial t^2} + a_\infty^2 \frac{\partial^2 \rho'}{\partial x^2} = 0 \longrightarrow \rho' = F(x - a_\infty t) + G(x + a_\infty t)$

right-running (for $F(x - a_\infty t)$) left-running (for $G(x + a_\infty t)$)



$\longrightarrow$

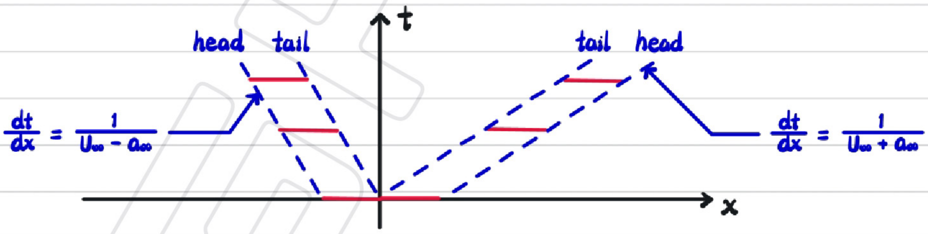


substitude ③ into ② $\frac{\partial^2 u'}{\partial t^2} + a_\infty^2 \frac{\partial^2 u'}{\partial x^2} = 0 \longrightarrow u' = P(x - a_\infty t) + Q(x + a_\infty t)$

- $\longrightarrow$ for right-running wave $u' = \frac{a_\infty}{\rho_\infty}\rho'$
- $\longrightarrow$ for left-running wave $u' = -\frac{a_\infty}{\rho_\infty}\rho'$

Method of characteristics

for finite strength isentropic waves with non-stationary based flow



for unsteady, inviscid $\mu = \nu = 0$, isentropic flow

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{2a}{\gamma - 1}\frac{\partial a}{\partial x} \; ① \text{ where } \frac{\partial p}{\partial a} = \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial a} = a^2 \frac{\partial}{\partial a}\left(\frac{\gamma p}{a^2}\right) = a^2\left(-2\frac{\gamma p}{a^3} + \frac{\gamma}{a^2}\frac{\partial p}{\partial a}\right) \longrightarrow \frac{\partial p}{\partial a} = \frac{2\gamma p}{(\gamma - 1)a} = \frac{1}{\rho}\frac{2a}{\gamma - 1}$$

$\longrightarrow \frac{2}{\gamma-1}\frac{\partial a}{\partial t} + a\frac{\partial U}{\partial x} = -\frac{2U}{\gamma-1}\frac{\partial a}{\partial x}$ ② where $\frac{\partial U}{\partial a} = \frac{2U}{(\gamma-1)a}$

for $\frac{dt}{dx} = \frac{1}{U+a}$, ①+② $\left[\frac{\partial U}{\partial t} + (U+a)\frac{\partial U}{\partial x}\right] + \frac{2}{\gamma-1}\left[\frac{\partial a}{\partial t} + (U+a)\frac{\partial a}{\partial x}\right] = 0 \longrightarrow 2\frac{\partial U}{\partial t} + \frac{4}{\gamma-1}\frac{\partial a}{\partial t} = 0$

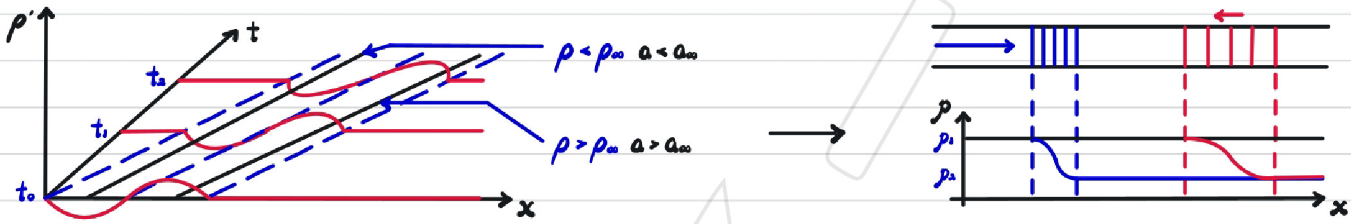
$\longrightarrow \Gamma^{+} = U + \frac{2a}{\gamma-1} = \text{constant}$ along C^{+} right-running wave

for $\frac{dt}{dx} = \frac{1}{U+a}$, ①−② $\left[\frac{\partial U}{\partial t} + (U-a)\frac{\partial U}{\partial x}\right] - \frac{2}{\gamma-1}\left[\frac{\partial a}{\partial t} + (U-a)\frac{\partial a}{\partial x}\right] = 0 \longrightarrow 2\frac{\partial U}{\partial t} - \frac{4}{\gamma-1}\frac{\partial a}{\partial t} = 0$

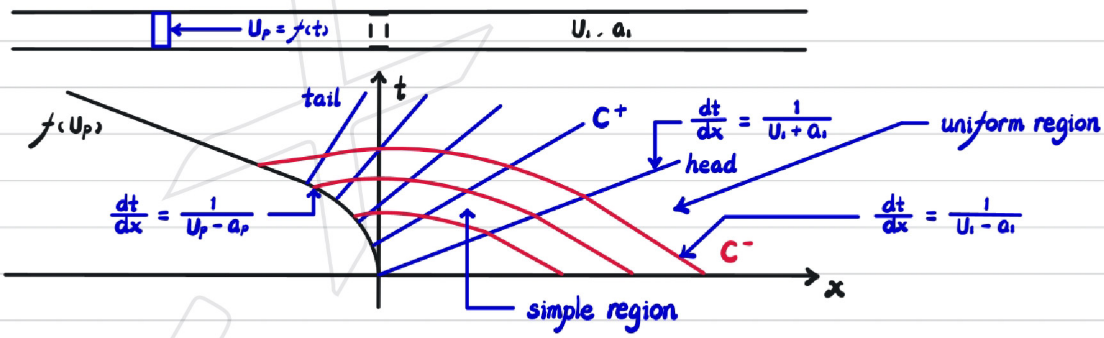
$\longrightarrow \Gamma^{-} = U - \frac{2a}{\gamma-1} = \text{constant}$ along C^{-} left-running wave

Unsteady expansion waves

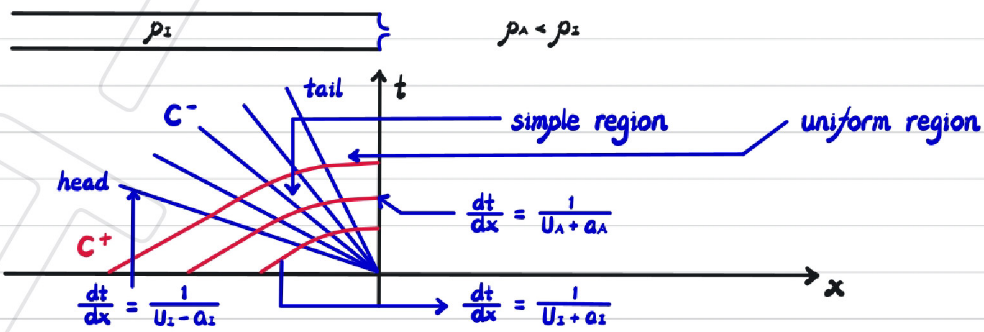
wave spreading $\longrightarrow$ higher wave speed at the head relative to the tail of the wave



accelerating pistons

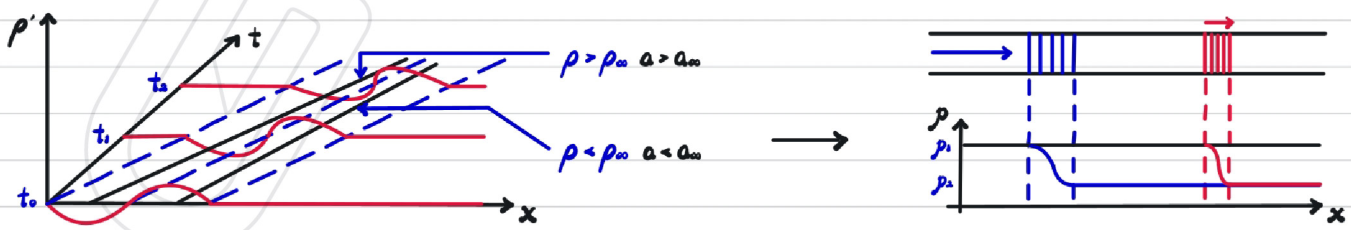


bursting diaphragms

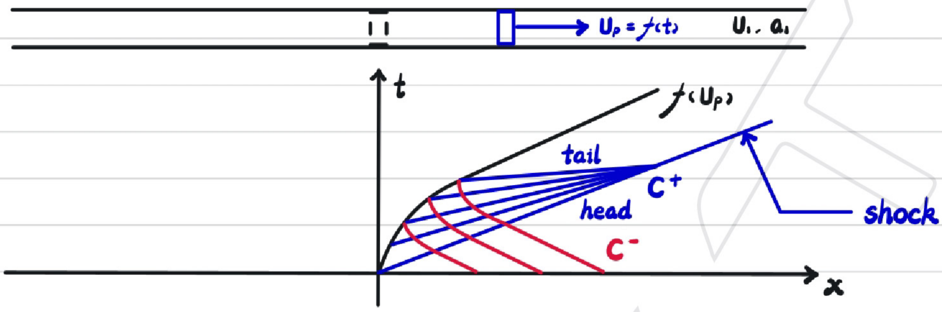


Unsteady compression waves

wave catching $\longrightarrow$ lower wave speed at the head relative to the tail of the wave



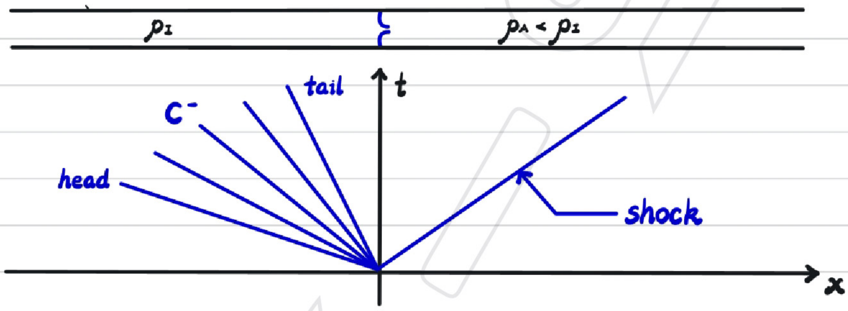
accelerating pistons



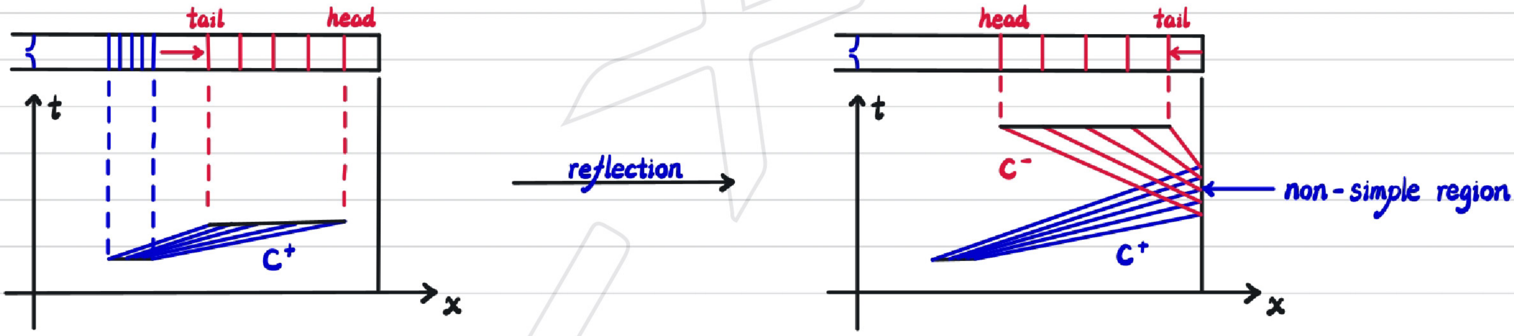
if $\frac{p_T}{p_H} < 2 \longrightarrow p_T \approx p_{TS}$ and $p_H \approx p_{HS} \xrightarrow{\text{shock table}}$ find M_s

Shock tubes

bursting diaphragms

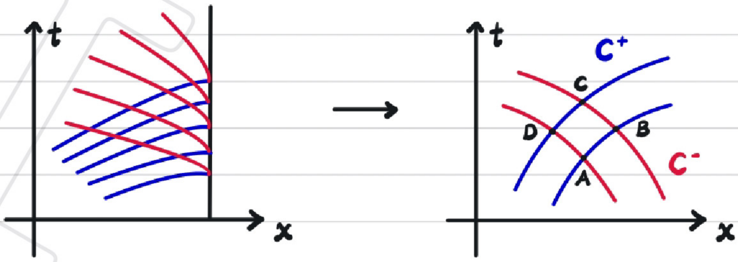


Wave reflection



where $\Gamma^+ = U + \frac{2a}{\gamma - 1} = \Gamma_W = \frac{2a_W}{\gamma - 1} = -\Gamma_R^+ = -(U_R + \frac{2a_R}{\gamma - 1})$

for non-simple region



$$\left(\frac{dt}{dx}\right)_{AB} \approx \frac{1}{2}\left(\frac{1}{U_A + a_A} + \frac{1}{U_B + a_B}\right)$$

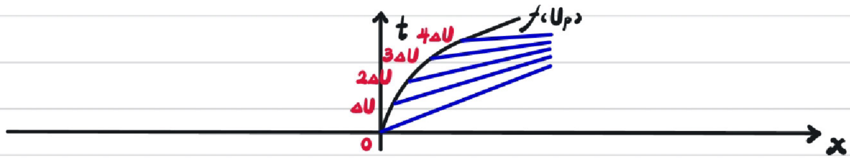
$$\left(\frac{dt}{dx}\right)_{DC} \approx \frac{1}{2}\left(\frac{1}{U_D + a_D} + \frac{1}{U_C + a_C}\right)$$

$$\left(\frac{dt}{dx}\right)_{AD} \approx \frac{1}{2}\left(\frac{1}{U_A - a_A} + \frac{1}{U_D - a_D}\right)$$

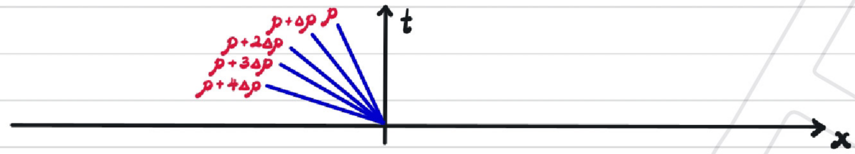
$$\left(\frac{dt}{dx}\right)_{BC} \approx \frac{1}{2}\left(\frac{1}{U_B - a_B} + \frac{1}{U_C - a_C}\right)$$

Discretisation

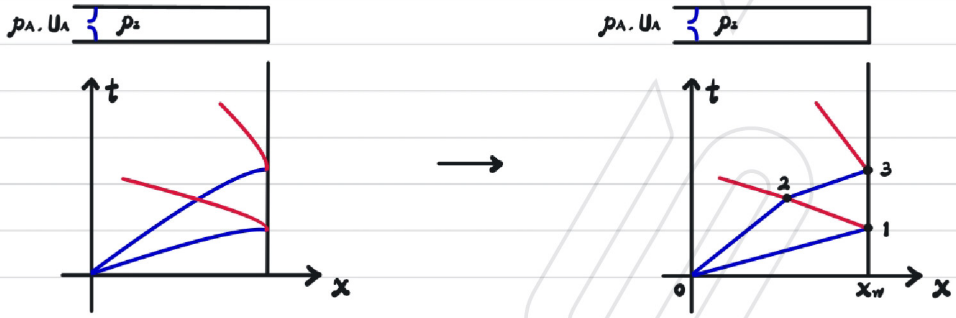
discretising velocity



discretising pressure



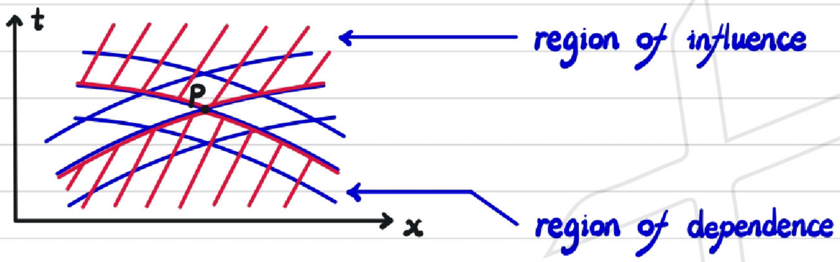
discretising reflection



given p_A, U_A, p_2, x_w and $U_w = U_1 = U_3 = 0 \longrightarrow$ find $a_1, a_2 \xrightarrow{(C^-)_{12}}$ find $U_2 \xrightarrow{(C^+)_{23}}$ find a_3

$\longrightarrow$ find $(\frac{dt}{dx})_{01}, (\frac{dt}{dx})_{12}, (\frac{dt}{dx})_{23} \longrightarrow$ find $t_{01} = f_{01}(x), \; t_{12} = f_{12}(x), \; t_{23} = f_{23}(x)$

Regions of influence and dependence



2-D Steady Inviscid Irrotational Flow

Method of characteristics

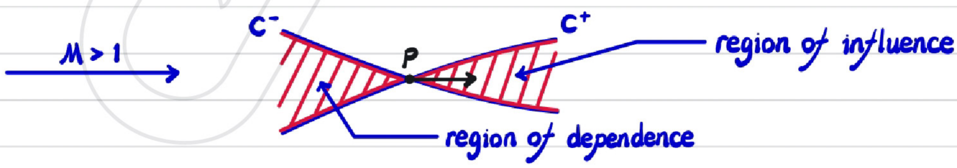
for steady $\rho = \text{constant}$, inviscid $\mu = \nu = 0$, adiabatic and irrotational $\omega = 0$ flow of perfect gas

$(U^2 - a^2)\frac{\partial U}{\partial x} + (V^2 - a^2)\frac{\partial V}{\partial y} + UV(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}) = 0$ where $H = c_pT + \frac{1}{2}u^2 = \text{constant} \longrightarrow \frac{a^2}{\gamma - 1} + \frac{U^2 + V^2}{2} = \frac{a_0^2}{\gamma - 1}$

assume flow is supersonic, flow angle $\theta = \tan^{-1}(\frac{V}{U}) \longrightarrow U = u\cos\theta, \; V = u\sin\theta$ and $a = u\sin\mu$

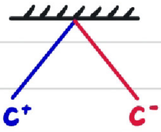
- for right-running wave $\frac{dy}{dx} = \tan(\theta + \mu) \longrightarrow \Gamma^+ = \theta - \nu(M) = \text{constant}$ along C^+
- for left-running wave $\frac{dy}{dx} = \tan(\theta - \mu) \longrightarrow \Gamma^- = \theta + \nu(M) = \text{constant}$ along C^-

Regions of influence and dependence

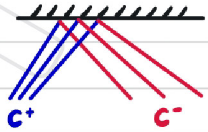


Boundary condition

soild wall $\longrightarrow$ C^+ reflects as C^-

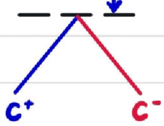


$\longrightarrow$ expansive reflects as expansive

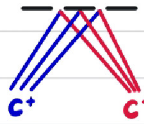


has constant pressure in isentropic compressible flow

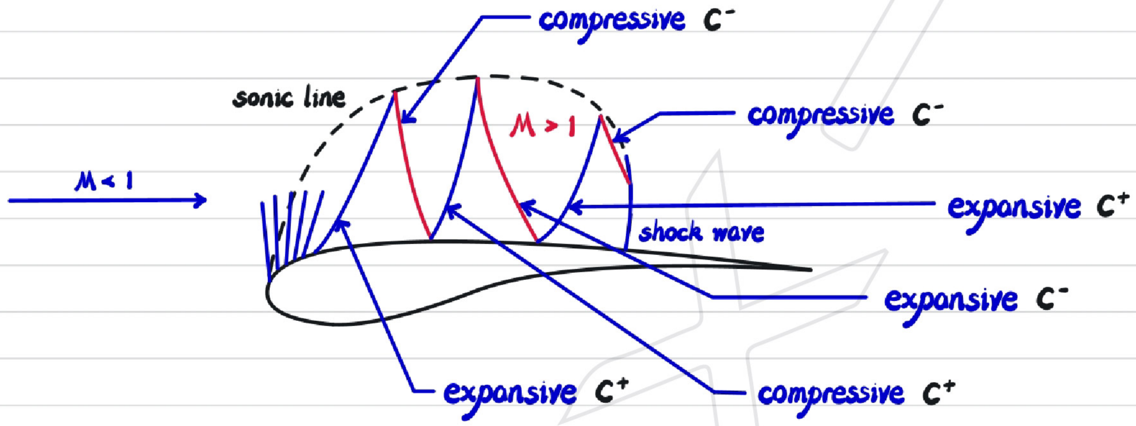
sonic line $\longrightarrow$ C^+ reflects as C^-



$\longrightarrow$ expansive reflects as compressive

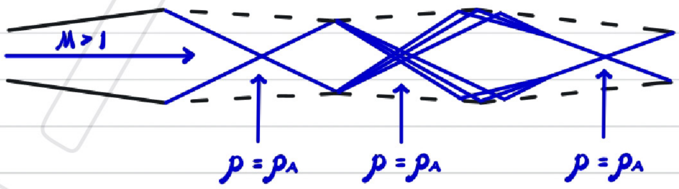


Transonic aerofoil

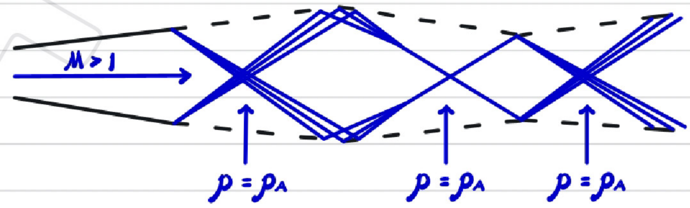


Supersonic jets

overexpanded jet $p_E < p_A$



underexpanded jet $p_E > p_A$

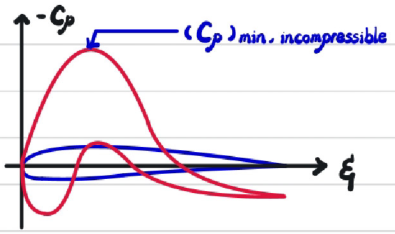


Supercritical aerofoil

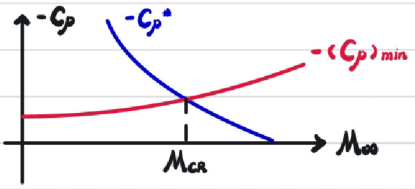
$$C_{pA} = \frac{p_A - p_\infty}{\frac{1}{2}\rho U_\infty^2} \xrightarrow{a=\sqrt{\gamma RT}} \frac{p_A - p_\infty}{\frac{1}{2}p_\infty \gamma M_\infty^2} = \frac{2}{\gamma M_\infty^2}\left(\frac{p_A}{p_\infty} - 1\right)$$

$$\xrightarrow{\text{isentropic}} C_{pA} = \frac{2}{\gamma M_\infty^2}\left[\left(\frac{p_{A0}}{p_A}\right)^{-1}\frac{p_{\infty 0}}{p_\infty} - 1\right] = \frac{2}{\gamma M_\infty^2}\left[\left(1+\frac{\gamma-1}{2}M_A^2\right)^{-\frac{\gamma}{\gamma-1}}\left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]$$

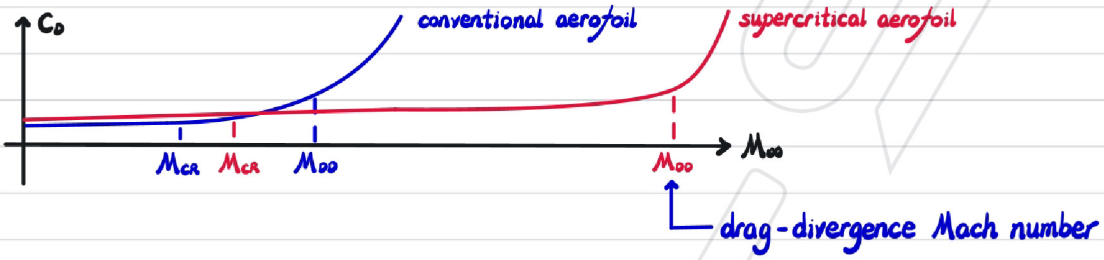
$$\xrightarrow{M_A = 1} C_p^* = \frac{2}{\gamma M_\infty^2}\left[\left(1+\frac{\gamma-1}{2}\right)^{-\frac{\gamma}{\gamma-1}}\left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]$$



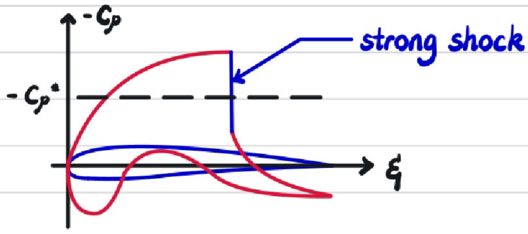
where Prandtl-Glauert correction $C_p = \dfrac{(C_p)_{incompressible}}{\sqrt{1-M_\infty^2}}$

$\longrightarrow$ 

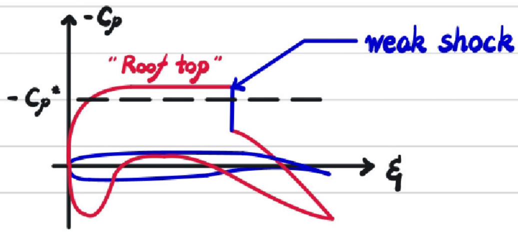
$$\frac{(C_p)_{min,\ incompressible}}{\sqrt{1-M_{CR}^2}} = \frac{2}{\gamma M_{CR}^2}\left[\left(1+\frac{\gamma-1}{2}\right)^{-\frac{\gamma}{\gamma-1}}\left(1+\frac{\gamma-1}{2}M_{CR}^2\right)^{\frac{\gamma}{\gamma-1}}-1\right]$$

$\longrightarrow$ 

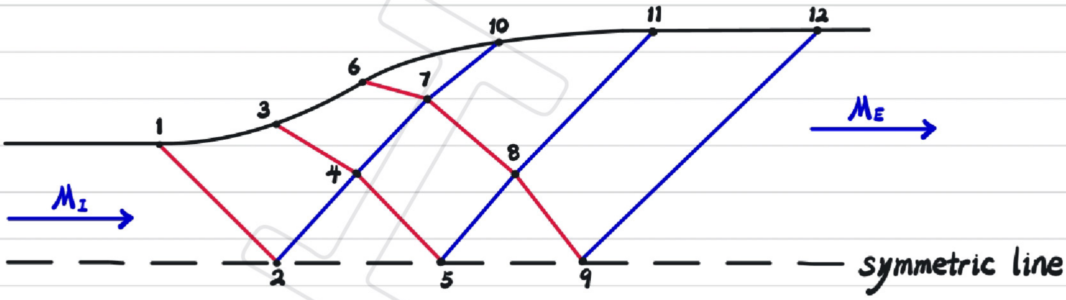
conventional aerofoil

$\longrightarrow$ 

supercritical aerofoil

$\longrightarrow$ 

Supersonic nozzle design



given M_1, x_1, y_1, $\theta_1 = 0°$, x_3, y_3, θ_3 and x_6, y_6 also M_E, $\theta_{12} = 0°$ $\longrightarrow$ $\nu_1 = \nu(M_1)$ and $\nu_{12} = \nu(M_E)$

$\xrightarrow{\text{symmetry}}$ $\theta_2 = \theta_5 = \theta_9 = 0°$ $\xrightarrow{(C^-)_{12} \text{ and } (C^+)_{912}}$ $\nu_1 = \nu_2$, $\nu_9 = \nu_{12}$ $\xrightarrow{(C^+)_{12\to3}}$ ν_3 $\xrightarrow{(C^-)_{34}}$ $\nu_4 = \nu_3$ where $\theta_4 = \theta_3$

$\xrightarrow{(C^-)_{45}}$ ν_5 $\xrightarrow{(C^+)_{12\to6} \text{ and } (C^-)_{912\to6}}$ ν_6, θ_6 $\xrightarrow{(C^-)_{67}}$ $\nu_7 = \nu_6$ where $\theta_7 = \theta_6$ $\xrightarrow{(C^+)_{58} \text{ and } (C^-)_{78}}$ ν_8, θ_8

$\xrightarrow{(C^+)_{710}}$ $\nu_{10} = \nu_7$ where $\theta_{10} = \theta_7$ $\xrightarrow{(C^+)_{811}}$ $\nu_{11} = \nu_8$ where $\theta_{11} = \theta_8$ $\longrightarrow$ find all x and y