Elastic Continua

Four Key Concepts

Equilibrium $\Sigma F = 0$

Compatibility* $\varepsilon_{xx} = \frac{du}{dx}$ etc.

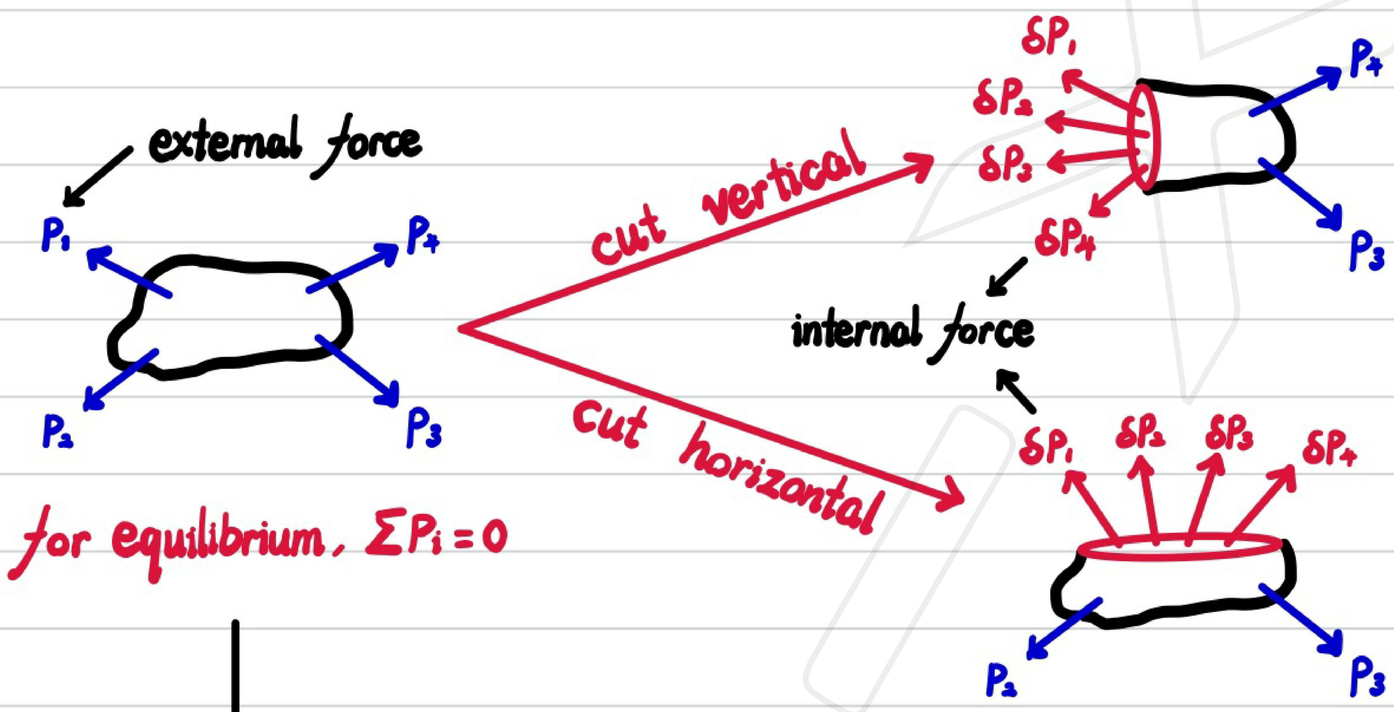
Constitutive Law (Hooke's Law)* $\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}) + \alpha\Delta T$ etc. * poisson's ratio ν

$\gamma_{xy} = \frac{\tau_{xy}}{G}$ etc. * shear modulus $G = \frac{E}{2(1+\nu)}$

Failure model
- material yield 屈服
- material break
- structural instability (buckling)
- dynamic deformation
- dynamic instability
- excessive 过度的 deformation

Stress and Strain in 3D

Stress Tensor



for equilibrium, $\Sigma P_i = 0$

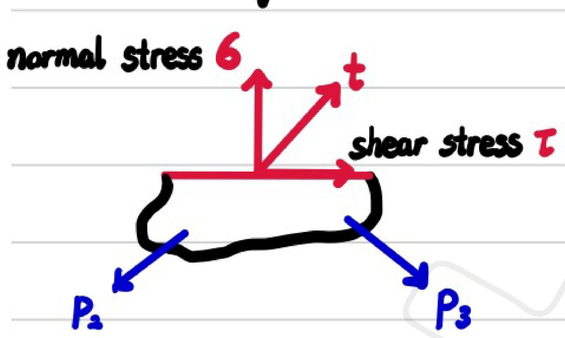
for equilibrium, $P_3 + P_4 + \Sigma \delta P_i = 0$

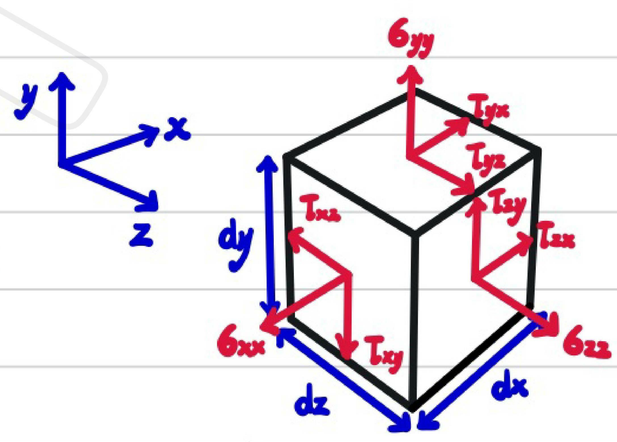
$\longrightarrow \lim_{\delta \to 0} \frac{\delta P}{\delta A} = \frac{dP}{dA} = t$ traction

$\longrightarrow P_3 + P_4 + \int_{A_h} t_i \, dA = 0$

$\longrightarrow P_2 + P_3 + \int_{A_v} t_i \, dA = 0$

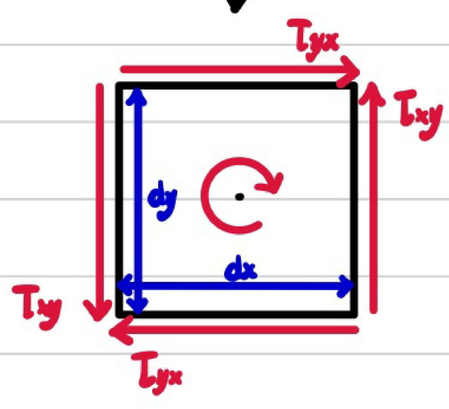
* traction is a tensor 张量





$3 \times 6 = 18$ (pieces of information)

$18 \rightarrow 9$

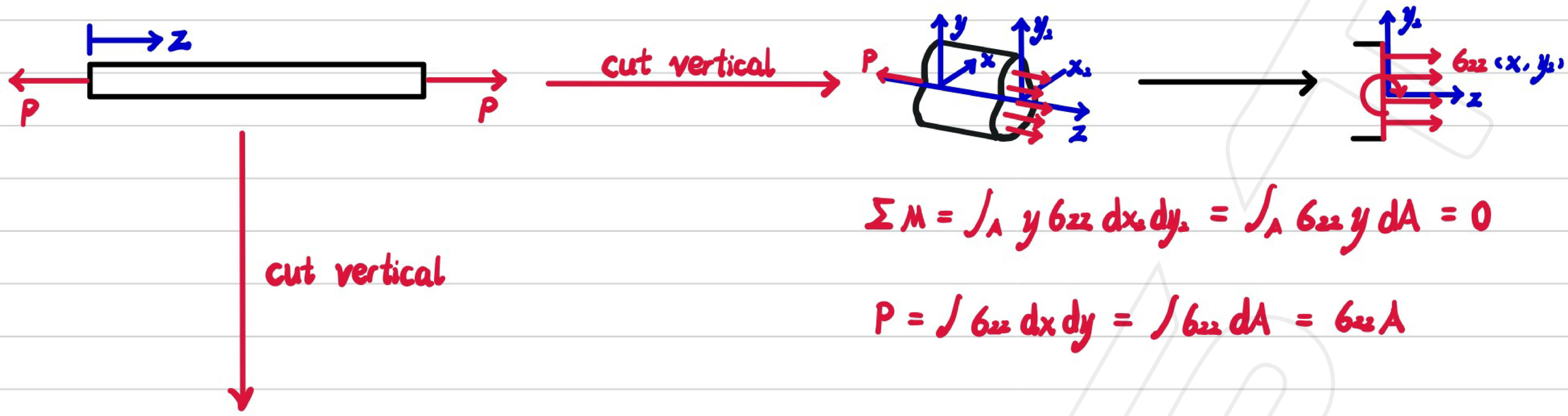


$\Sigma M = 0 = 2(\tau_{yx}\, dx\, dz \frac{dy}{2}) - 2(\tau_{xy}\, dy\, dz \frac{dx}{2})$

$\longrightarrow \tau_{xy} = \tau_{yx} \longrightarrow \tau_{ij} = \tau_{ji}$

$9 \rightarrow 6$ (independent components)

1-D rod structure

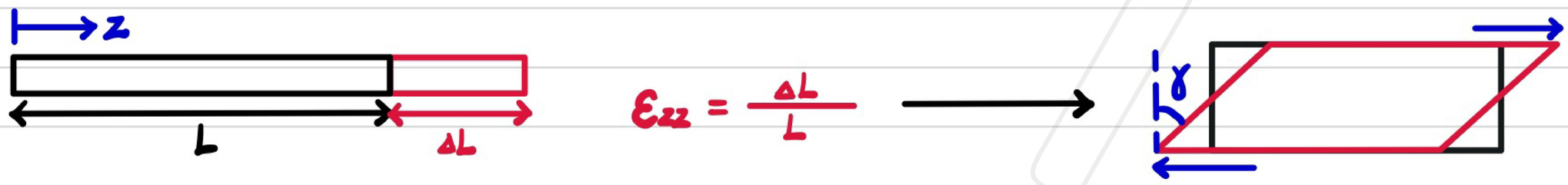


$$\Sigma M = \int_A y\,\sigma_{zz}\,dx_2\,dy_2 = \int_A \sigma_{zz}\,y\,dA = 0$$

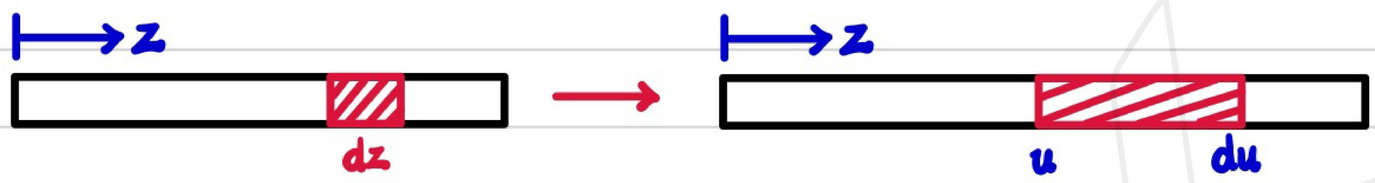
$$P = \int \sigma_{zz}\,dx\,dy = \int \sigma_{zz}\,dA = \sigma_{zz} A$$

$$\sigma_{zz} = \frac{P}{A}$$

Strain



$$\varepsilon_{zz} = \frac{\Delta L}{L}$$



$$\varepsilon_{zz} = \frac{\Delta L}{L} = \frac{(u + du) - u}{dz} = \frac{du}{dz}$$

Pin-jointed Framework

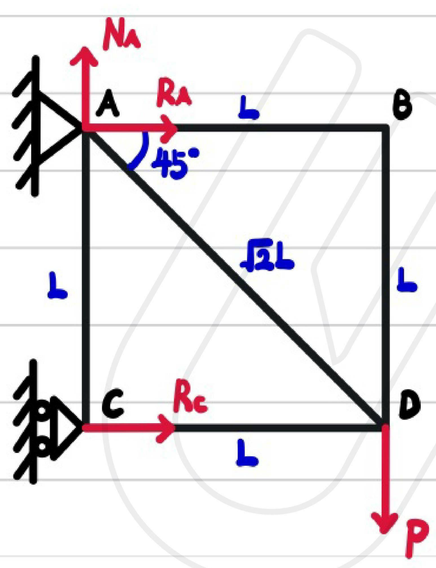
Classification

Zero moment condition $\Sigma M = 0 \longrightarrow$ only axial force exist

Small deformation assumption Deformation are always assumed to be small

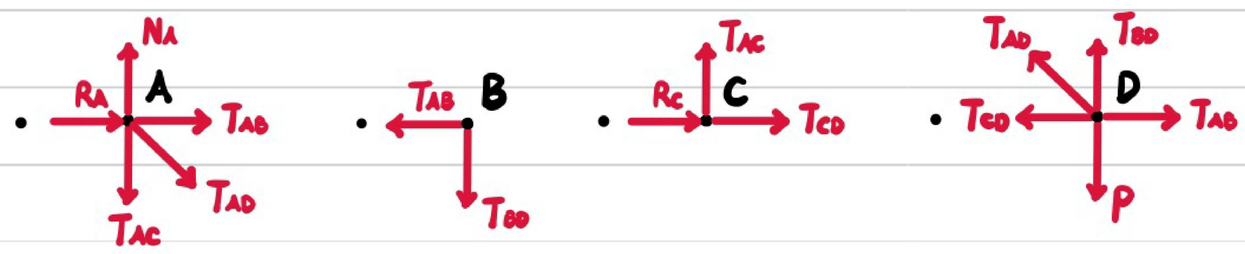
Bar Forces

Simple pin-jointed framework



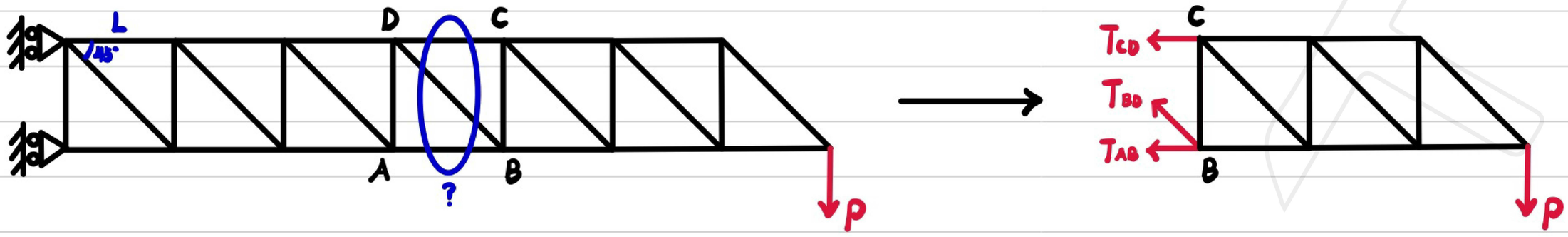
Global equilibrium

- $\Sigma F(\uparrow) = N_A - P = 0$
- $\Sigma F(\downarrow) = R_A + R_C = 0$
- $\Sigma M_A = R_C \cdot L - P \cdot L = 0$



- $\Sigma F(\uparrow) = 0$
- $\Sigma F(\rightarrow) = 0$

Method of sections



Global equilibrium

- $\Sigma M_B = T_{CD} \cdot L + P \cdot 3L = 0$
- $\Sigma M_D = T_{AB} \cdot L + P \cdot 4L = 0$
- $\Sigma M_C = T_{AB} \cdot L + T_{BD} \cdot \frac{\sqrt{2}}{2} L + P \cdot 3L = 0$

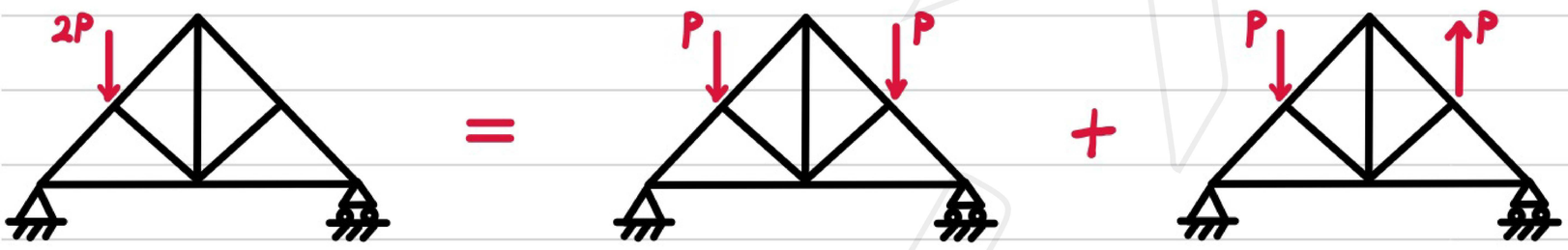
Nodal Displacement 节点位移

Mechanism system that can undergo large geometry change without addition of strain

Maxwell equation M = no. of bars b + no. of reaction forces r − 2 × no. of pin joints $2j$

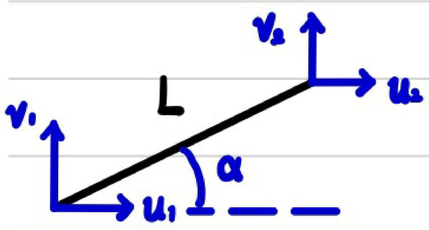
- $M < 0$ mechanism
- $M = 0$ statically determine / mechanism + redundancy 冗余 / infinitesimal mechanism
- $M > 0$ statically indetermine 超静定

Symmetry and superposition



Direct approach

bar extension $e = \varepsilon L = \frac{\sigma}{E} L = \frac{TL}{AE}$



$e = \Delta L = (u_2 - u_1)\cos\alpha + (v_2 - v_1)\sin\alpha$

Virtual work $1^* \cdot \delta = \Sigma T^*_{ij} \cdot e_{ij}$

Statically-indeterminate Framework

direct approach $e = (u_2 - u_1)\cos\alpha + (v_2 - v_1)\sin\alpha = \frac{TL}{AE} \longrightarrow T = \frac{AE}{L}(\ldots) \longrightarrow \Sigma T(\uparrow) \text{ and } (\rightarrow) = 0$

Beams

Applied Loads

Distributed force

$F = \int w(x)\, dx$

$M = \int w(x)\, x\, dx$

Concentrated force

Moments

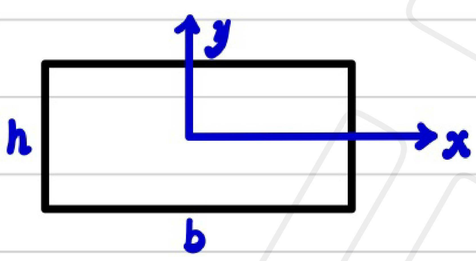
Support type

Geometric Properties of An Area

1st moment of area $\cdot\ Q_x = \int_A y\, dA \qquad \cdot\ Q_y = \int_A x\, dA$

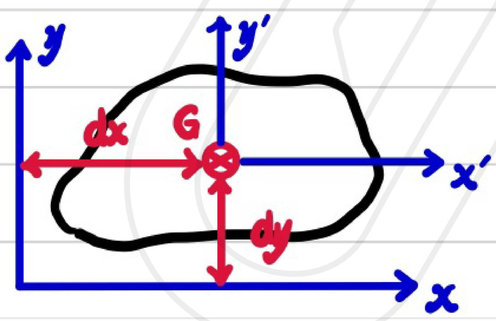
centroid of area $\cdot\ \bar{x} = \frac{1}{A}\int_A x\, dA = \frac{\sum_{i=1}^{N} \bar{x}_i A_i}{\sum_{i=1}^{N} A_i} \qquad \cdot\ \bar{y} = \frac{1}{A}\int_A y\, dA = \frac{\sum_{i=1}^{N} \bar{y}_i A_i}{\sum_{i=1}^{N} A_i}$

2nd moment of area (moment of inertia) $\cdot\ I_x = \int_A y^2\, dA \qquad \cdot\ I_y = \int_A x^2\, dA$



$$I_x = \int_A y^2\, dA = \int_A y^2 b\, dy = b\int_{-\frac{h}{2}}^{\frac{h}{2}} y^2\, dy = \frac{bh^3}{12}$$

$$I_y = \int_A x^2\, dA = \int_A x^2 h\, dx = h\int_{-\frac{b}{2}}^{\frac{b}{2}} x^2\, dx = \frac{hb^3}{12}$$



$I_x = I_{x'} + A\, dy^2 \qquad * \ dy = |\bar{y}' - \bar{y}|$

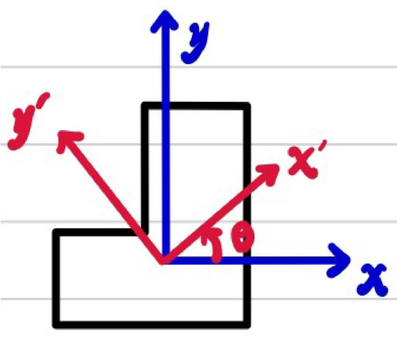
$I_y = I_{y'} + A\, dx^2 \qquad * \ dx = |\bar{x}' - \bar{x}|$

Cross product inertia of area

- $I_{xy} = \int_A xy\, dA$ only exist for unsymmetrical cross section

$$= I_{x'y'} + A\,dx\,dy$$

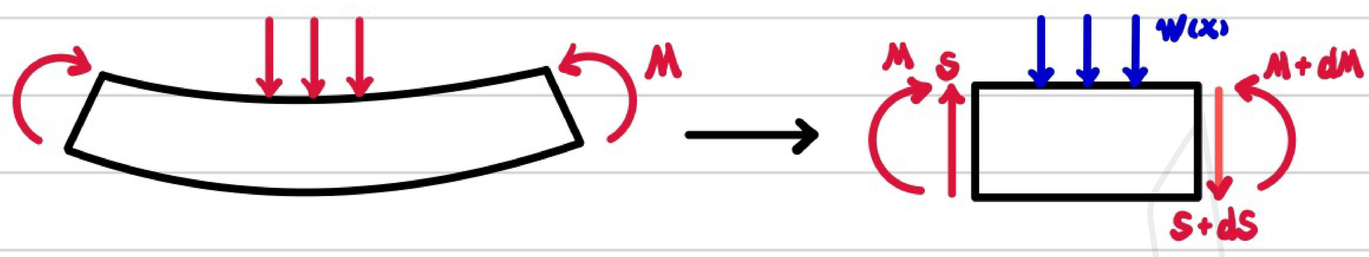
Transmation of second and product moments of area



- $I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos(2\theta) - I_{xy}\sin(2\theta)$
- $I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos(2\theta) + I_{xy}\sin(2\theta)$
- $I_{xy'} = \frac{I_x - I_y}{2}\sin(2\theta) + I_{xy}\cos(2\theta) = 0$ for principal axis

Shear and Moment

Beam in bending



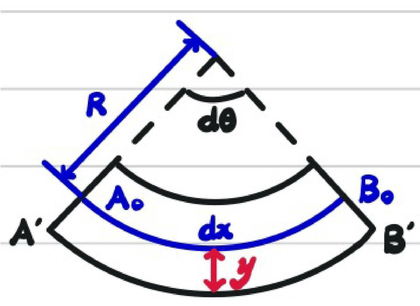
- $\frac{dS}{dx} = -w(z) \longrightarrow S = \int -w(z)\,dz$
- $\frac{dM}{dx} = S \longrightarrow M = \int S\,dz$

Singularity method 奇点法

$$f(x) = \langle x-a \rangle^n = \begin{cases} 0 & \text{when } x \le a \\ x-a & \text{when } x > a \end{cases}$$

- $\int f(x)\,dx = \frac{1}{n+1}\langle x-a\rangle^{n+1} + C$
- $\frac{df(x)}{dx} = n\langle x-a\rangle^{n-1}$

Bending stress



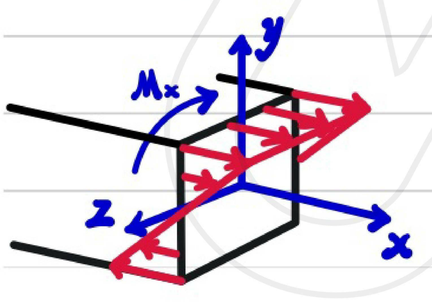
$$\varepsilon = \frac{\Delta L}{L} = \frac{A'B' - A_0B_0}{A_0B_0} = \frac{(R+y)\theta - R\theta}{R\theta} = \frac{y}{R}$$

$$\longrightarrow \sigma = E\varepsilon = \frac{y}{R}E \longrightarrow \frac{\sigma}{y} = \frac{E}{R_x}$$

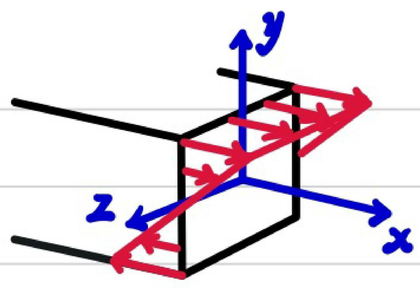
$\frac{dv}{dx} = \tan\theta \approx \theta$ & $\frac{1}{R} \approx -\frac{d^2v}{dx^2}$ from calculus

when $I_{xy} = 0$

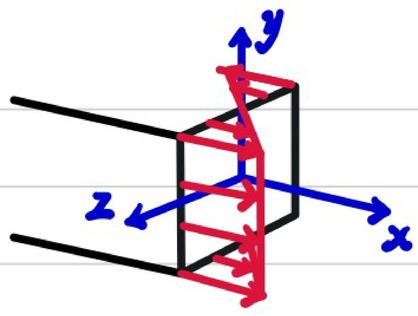
$$M_x = \int y\,dF = \int y(\sigma\,dA) = \int y\left[\frac{y}{R_x}E\right]dA = \frac{E}{R}\int y^2\,dA = \frac{E}{R}I_x$$



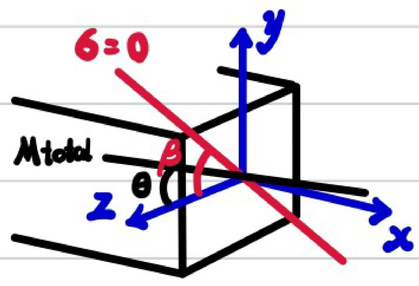
$\frac{\sigma}{y} = \frac{M_x}{I_x} = \frac{E}{R_x}$ where $\frac{1}{R_x} = -\frac{d^2v}{dz^2}$



+



$\longrightarrow \sigma = \frac{M_x}{I_x} y + \frac{M_y}{I_y} x = \frac{M_z}{I_z} y + \frac{M_y}{I_y} z$

* 

at neutral axis

$\sigma = 0 \longrightarrow \frac{M_z}{I_z} y + \frac{M_y}{I_y} z = 0 \longrightarrow y = -\frac{M_y I_z}{M_z I_y} z \longrightarrow y = -\frac{I_z}{I_y} \tan\theta \, z$

$\longrightarrow \tan\beta = -\frac{I_y}{I_z} \tan\theta$

when $I_{xy} \neq 0$

$$\sigma = \frac{\bar{\bar{M}}_x y}{I_x} + \frac{\bar{\bar{M}}_y x}{I_y} = \left(\frac{M_x I_y - M_y I_{xy}}{I_x I_y - I_{xy}^2} \right) y + \left(\frac{M_y I_x - M_x I_{xy}}{I_x I_y - I_{xy}^2} \right) x$$

Elastic curves

$$\frac{M_x}{I_x} = \frac{E}{R_x} = -\frac{E d^2 v}{dz^2} = -E \frac{d^2 v}{dx^2} \qquad * \; v(\uparrow)$$

$$\longrightarrow \frac{dv}{dx} = \int -\frac{M}{EI} dx \longrightarrow v = \iint -\frac{M}{EI} dx$$

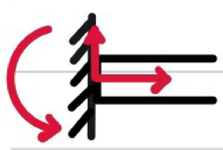
by virtual work $\quad \bar{1} \cdot \delta_y = \int \frac{M}{EI} \bar{M} \, dx$

Statically Indetermine Beams

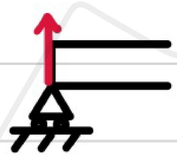
Definition when no. reactions > equilibrium equations

Elastic curves

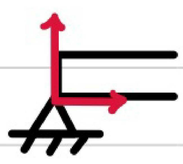
$\Sigma M = 0, \; \Sigma F = 0 \; + \; \frac{d^2 v}{dx^2}, \; \frac{dv}{dx}, \; v \longrightarrow$ solution



$v = 0, \; \frac{dv}{dx} = 0$

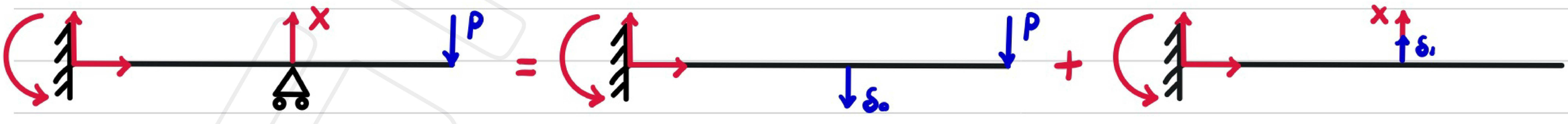


$v = 0$



$v = 0$

Unit load method



$$\delta_0 + X\delta_1 = \int \frac{\bar{M}_1}{EI} M_0 \, dx + \int \frac{\bar{M}_1}{EI} M_1 \, dx \; (M_1 = X\bar{M}_1) = \int \frac{\bar{M}_1}{EI} M_0 \, dx + X \int \frac{\bar{M}_1^2}{EI} dx = 0$$

Torsion of Circular Shaft

Relationship

$\frac{\tau}{r}$ shear stress $= \frac{T}{J}$ internal torque $= G\frac{d\theta}{dz}$ twist $\longrightarrow \theta = \int \frac{T}{GJ}\,dz$

* torsion constant $J = \frac{\pi}{2}(R_o^4 - R_i^4) \xrightarrow{\text{thin-walled tube}} 2\pi R^3 t$