Principal of Fluid Flow

Flow properties and variables

Flow variables · u, v, w velocity · P pressure

Flow properties · ρ density · μ dynamic viscosity · $\nu = \frac{\mu}{\rho}$ kinetic viscosity

Shear stress in flow

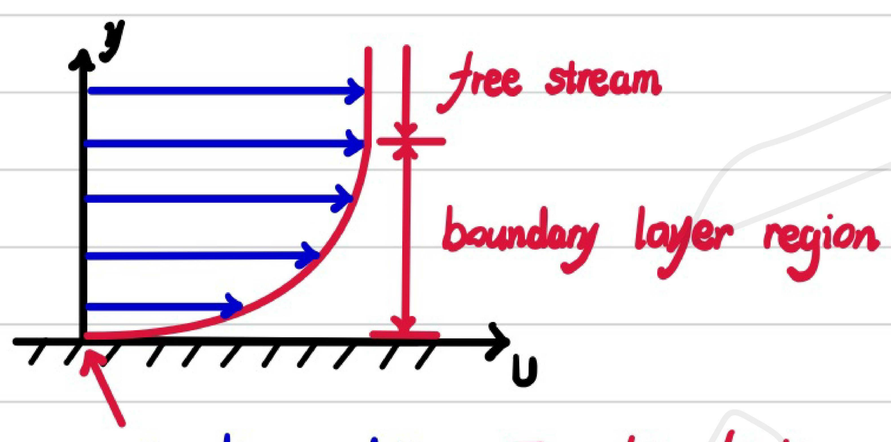
Newtonian fluids stress $\propto$ time rate of deformation

* fluid in motion must resist a shear stress 剪切应力

Viscosity $\mu \propto T$ Sutherland's law

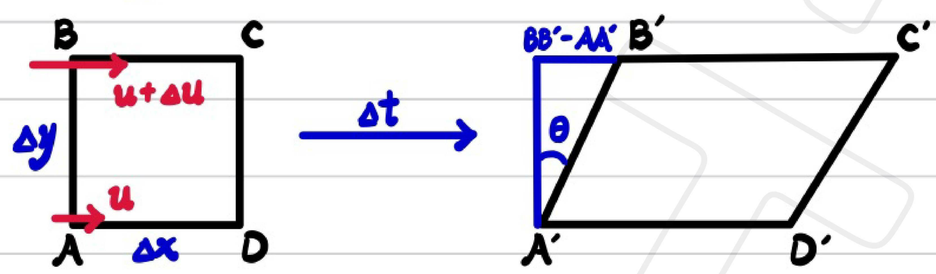
· $T\uparrow \longrightarrow \mu\downarrow$ for liquid · $T\uparrow \longrightarrow \mu\uparrow$ for gas

* **vicous stress** τ always significant near a soild surface



no slip condition τ = skin friction leads to drag force at the surface

Rate of shear strain consider a square element in the boundary layer at time t



$\Delta\theta = \arctan\left(\frac{BB' - AA'}{\Delta y}\right) \xlongequal{\text{small angle approximation}} \frac{\Delta u \Delta t}{\Delta y}$

$\xrightarrow{\Delta t \to 0} \frac{\partial \theta}{\partial t} = \frac{\partial u}{\partial y}$

$AA' = u\Delta t$ & $BB' = (u + \Delta u)\Delta t$ $\xrightarrow{\tau \propto \frac{\partial \theta}{\partial t}}$ **shear stress** $\tau = \mu \frac{\partial u}{\partial y}$ * $\tau_w = \mu \frac{\partial u}{\partial y}\Big|_w$

Velocity and pressure

Pressure $P = \frac{F}{A} = \rho R T$

· normally consider pressure at a point in the fluid or on a surface $A \to 0$

· pressure at point is equal in all directions and is a scalar value

Velocity u, v, w and free-stream velocity V_∞ (U_∞)

· velocity is a vector value

Incompressible and compressible flow

Incompressible flow

· low speed $M < 0.3$ ($u < 100 m/s$) · constant temperature T const $\longrightarrow$ ρ and μ const

Compressible flow

Relative Motion

frame of reference

Eulerian moving fluid past a station object = Lagrangain moving object through station fluid

Dimensional Analysis

All terms in an equation must have same equations

Dimensions

Quantities · mass [M] · length [L] · time [T]

· v velocity $[\frac{L}{T}]$ · a acceleration $[\frac{L}{T^2}]$ · F force $[\frac{ML}{T^2}]$

· P pressure $[\frac{M}{LT^2}]$ · ρ density $[\frac{M}{L^3}]$ · μ viscosity $[\frac{M}{LT}]$ · ν dynamic viscosity $[\frac{L^2}{T}]$

Dimensionless Numbers or Coefficient

Dynamic pressure $q = \frac{1}{2}\rho V_\infty^2$

Pressure coefficient $C_p = \frac{P - P_\infty}{\frac{1}{2}\rho V_\infty^2}$

Lift coefficient $C_L = \frac{L}{\frac{1}{2}\rho V_\infty^2 A}$

Drag coefficient $C_D = \frac{D}{\frac{1}{2}\rho V_\infty^2 A}$

Pitching moment coefficient $C_M = \frac{M}{\frac{1}{2}\rho V_\infty^2 A c}$

Reynolds number $Re = \frac{\text{internal force}}{\text{viscous force}} = \frac{\frac{1}{2}\rho u^2}{\frac{1}{2}\mu \frac{u}{D}} = \frac{\rho u D}{\mu} = \frac{uD}{\nu}$

Mach number $M = \frac{\text{speed of flow}}{\text{speed of sound}} = \frac{u}{a}$ $\quad * \; a = \sqrt{\gamma R T}$

Buckingham's Rule

Example $L = f(\rho, u, A, a, \mu)$

$\longrightarrow C_L = \frac{L}{\frac{1}{2}\rho u^2 A} = f(\rho, u, A, a, \mu) = \ldots + c\rho^\alpha u^\beta A^\gamma a^\delta \mu^\theta + \ldots$

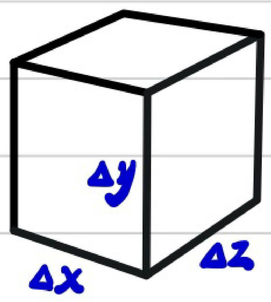
$= \left[\frac{M}{L^3}\right]^\alpha \left[\frac{L}{T}\right]^\beta \left[L^2\right]^\gamma \left[\frac{L}{T}\right]^\delta \left[\frac{M}{LT}\right]^\theta = [M^0 L^0 T^0]$

$\longrightarrow M: \alpha + \theta = 0, \; L: -3\alpha + \beta + 2\gamma + \delta - \theta = 0, \; T: -\beta - \delta - \theta = 0$

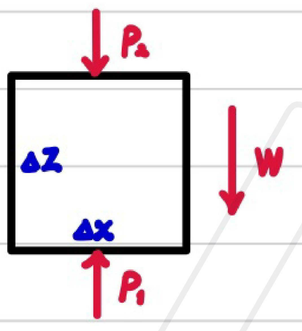
$\longrightarrow \alpha = -\theta = 2\gamma, \; \beta = -\delta + \alpha \longrightarrow C_L = \left[\frac{\rho u A^{\frac{1}{2}}}{\mu}\right]^\alpha \left[\frac{a}{u}\right]^\delta = f(M, Re)$

Similarity

Control Volume Analysis



Hydrostatics

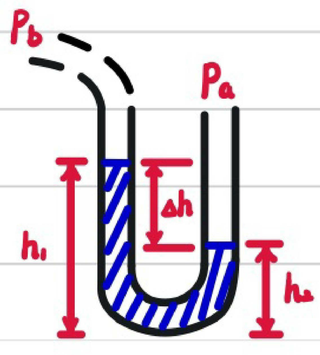


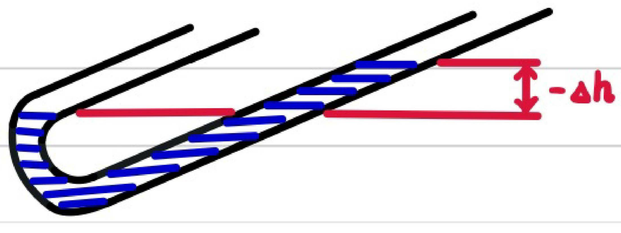
$W = mg = \rho V g = \rho \Delta x \Delta y \Delta z g = (P_1 - P_2) A = -\Delta P \Delta x \Delta y$

$\longrightarrow \frac{\Delta P}{\Delta z} = -\rho g \longrightarrow \frac{dP}{dz} = -\rho g$

Buoyancy force $P_1 - P_2 = -\rho g (h_1 - h_2)$

Manometers $P_b = P_a + \Delta P = P_a - \rho g \Delta h$





The atomsphere

In troposphere $T(z) = T_0 + \lambda z \longrightarrow \rho = \frac{P}{RT} = \frac{P}{R(T_0+\lambda z)}$

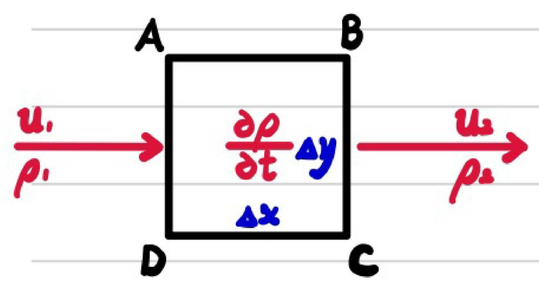
$$\xrightarrow{\frac{dP}{dz} = -\rho g} \frac{dP}{dz} = -\frac{Pg}{R(T_0+\lambda z)} \longrightarrow \ln\left(\frac{P_2}{P_1}\right) = -\frac{g}{R\lambda}\ln\left(\frac{T_0+\lambda z_2}{T_1+\lambda z_1}\right) = \ln\left(\frac{T_0+\lambda z_2}{T_1+\lambda z_1}\right)^{-\frac{g}{R\lambda}} \longrightarrow \frac{P_2}{P_1} = \left(\frac{T_0+\lambda z_2}{T_0+\lambda z_1}\right)^{-\frac{g}{R\lambda}}$$

Conservation 守恒 of mass

The rate of change (in time) of mass within a control volume plus the net flux of mass out of the control volume equal to zero

Mass flow rate $Q = \dot{m} = \rho\dot{V} = \rho u A$ = constant (for steady flow)

1-D

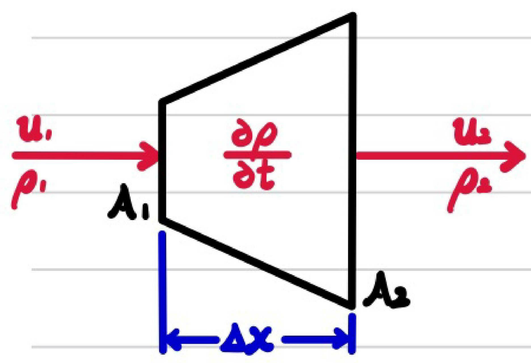


$$\frac{\partial m}{\partial t} = \frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z$$

$$+ (\dot{m}_{out} - \dot{m}_{in}) = \rho_2\left(\frac{\Delta x}{\Delta t}\right)_2 \Delta y \Delta z - \rho_1\left(\frac{\Delta x}{\Delta t}\right)_1 \Delta y \Delta z = (\rho u)_2 \Delta y \Delta z - (\rho u)_1 \Delta y \Delta z = 0$$

$$\xrightarrow{\frac{1}{\Delta x \Delta y \Delta z}} \frac{\partial \rho}{\partial t} + \frac{(\rho u)_2 - (\rho u)_1}{\Delta x} = \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0$$

1-D duct



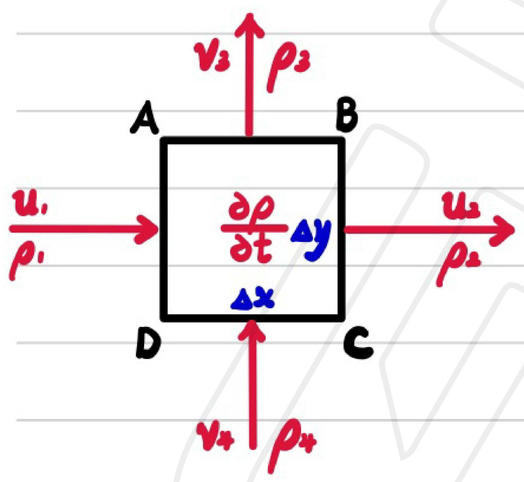
$$\frac{\partial m}{\partial t} = \frac{\partial \rho}{\partial t}\bar{A}\Delta x = \frac{\partial \rho}{\partial t}\frac{A_1 + A_2}{2}\Delta x$$

$$+ \dot{m}_{out} - \dot{m}_{in} = (\rho u A)_2 - (\rho u A)_1 = 0$$

$$\xrightarrow{\frac{1}{\Delta x}} \frac{\partial \rho}{\partial t}\frac{A_1 + A_2}{2} = \frac{\partial(\rho u A)}{\partial x} \xrightarrow{\Delta x \to 0} A_1 \to A_2 \longrightarrow A\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u A)}{\partial x} = 0$$

* steady flow $\xrightarrow{\rho u A = \text{const}} \frac{\partial(\rho u A)}{\partial x} = 0 \longrightarrow \frac{\partial \rho}{\partial t} = 0$

2-D



$$\frac{\partial m}{\partial t} = \frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z$$

$$+ (\dot{m}_{out} - \dot{m}_{in}) = (\rho u)_2 \Delta y \Delta z - (\rho u)_1 \Delta y \Delta z + (\rho v)_4 \Delta x \Delta z - (\rho v)_3 \Delta x \Delta z = 0$$

$$\xrightarrow{\frac{1}{\Delta x \Delta y \Delta z}} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$$\longrightarrow \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

* steady flow $\xrightarrow{\rho = \text{const}} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nabla \cdot u = 0$

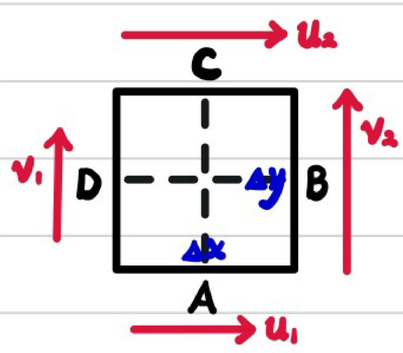
3-D $\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$

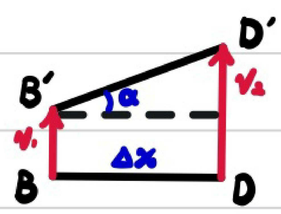
$\longrightarrow \frac{D\rho}{Dt}$ total derivative $+ \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \frac{D\rho}{Dt} + \rho \cdot \nabla \cdot u = 0$

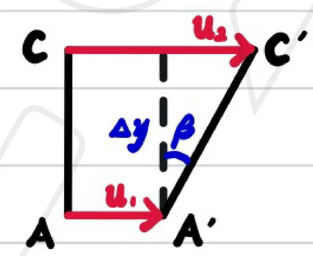
* steady flow $\xrightarrow{\rho = \text{const}} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot u = 0$

Rotation of a Fluid Element in 2D

Vorticity 涡度







$\Delta\alpha = \frac{\Delta v \Delta t}{\Delta x} \xrightarrow{\omega_\alpha} \frac{\partial \alpha}{\partial t} = \frac{\partial v}{\partial x}$ (G)

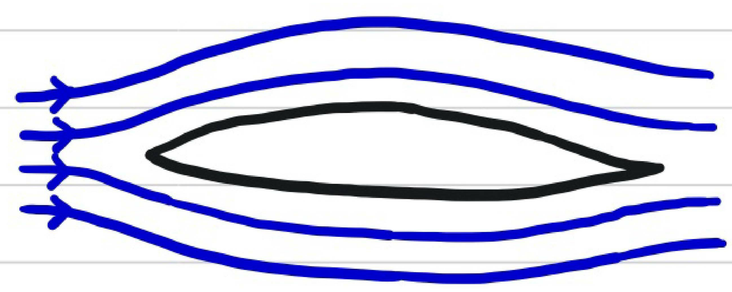
$\Delta\beta = \frac{\Delta u \Delta t}{\Delta y} \xrightarrow{\omega_\beta} \frac{\partial \beta}{\partial t} = \frac{\partial u}{\partial y}$ (Ω)

$\xrightarrow{\bar{\omega}(G) = \frac{1}{2}\Sigma\omega(G)} \omega_z = \frac{1}{2}\left(\frac{\partial \alpha}{\partial t} - \frac{\partial \beta}{\partial t}\right) \xrightarrow{\text{ignore } \frac{1}{2}} \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

* due to the effects of viscosity near the soild surface

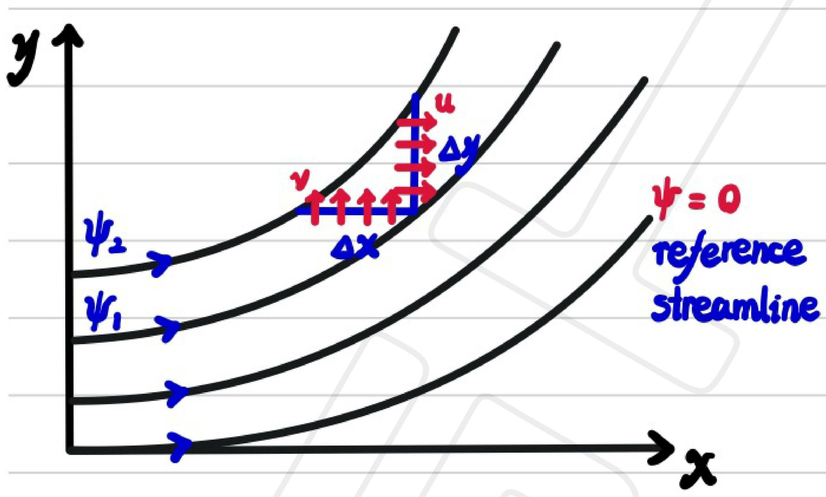
Streamlines

A streamline body is one where flow follows the body surface without seperating



Stream function

ψ * constant along the streamline satisfies 2D mass conservation



$(\uparrow)\ \psi_2 - \psi_1 = \int u\,dy = u\Delta y \longrightarrow u = \frac{\partial \psi}{\partial y}$

$(\rightarrow)\ \psi_1 - \psi_2 = \int v\,dx = v\Delta x \longrightarrow v = -\frac{\partial \psi}{\partial x}$

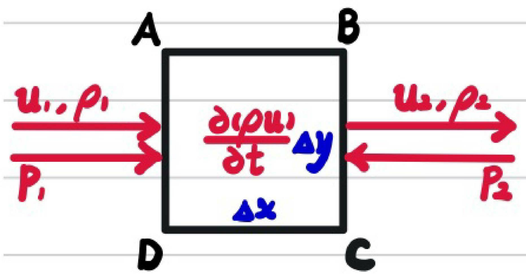
* steady flow $\xrightarrow{\rho = \text{const}} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$

* irrotational flow $\xrightarrow{\omega_z = 0} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0$

Conservation of momentum

The rate of change (in time) of momentum within a control volume plus the net flux of momentum out of the control volume equal to applied force acting on the control volume

1-D



$$\frac{\partial(mu)}{\partial t} = \frac{\partial(\rho u)}{\partial t}\Delta x\Delta y\Delta z + ((\dot{m}u)_2 - (\dot{m}u)_1) = (\rho u^2)_1 \Delta y\Delta z - (\rho u^2)_2 \Delta y\Delta z$$

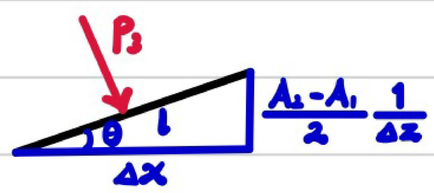
$$= F_1 - F_2 = P_1 \Delta y\Delta z - P_2 \Delta y\Delta z$$

$$\xrightarrow{\frac{1}{\Delta x\Delta y\Delta z}} \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} = -\frac{\partial P}{\partial x}$$

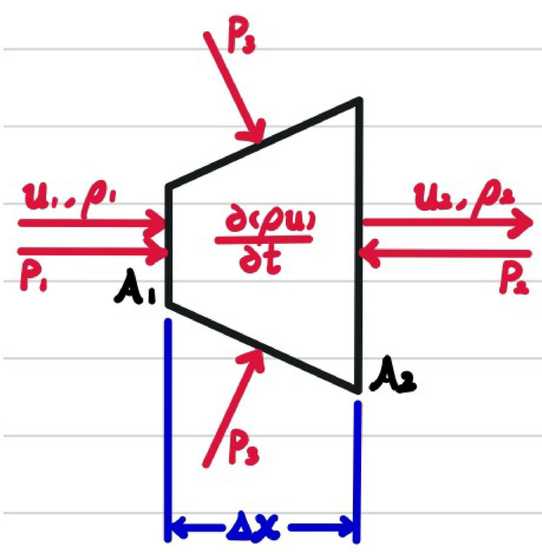
$$\rightarrow \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u\left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x}\right) = -\frac{\partial P}{\partial x}$$

$$\xrightarrow{\text{1-D mass}} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

1-D duct

* 

* $(\rightarrow)$ $P_3 \sin\theta A = P_3 \frac{\left(\frac{A_2 - A_1}{2\Delta z}\right)}{l}(l\Delta z) = \frac{1}{2}P_3(A_2 - A_1)$



$$\frac{\partial(mu)}{\partial t} = \frac{\partial(\rho u)}{\partial t}\Delta x\frac{A_1 + A_2}{2} + ((\dot{m}u)_2 - (\dot{m}u)_1) = (\rho u^2 A)_2 - (\rho u^2 A)_1$$

$$= F_1 - F_2 = (PA)_1 - (PA)_2 + 2\left[\frac{1}{2}P_3(A_2 - A_1)\right]$$

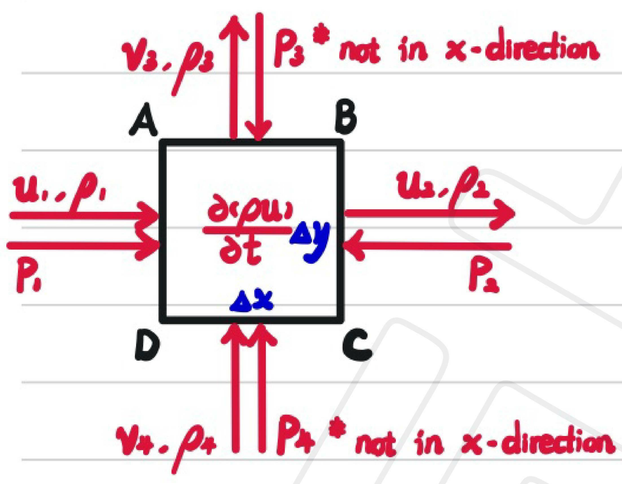
$$\xrightarrow[A_1 \to A_2]{\frac{1}{\Delta x}} A\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 A)}{\partial x} = -\frac{\partial(PA)}{\partial x} + P\frac{\partial A}{\partial x} = -A\frac{\partial P}{\partial x}$$

$$\rightarrow A\left(\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x}\right) + u\left(A\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u A)}{\partial x}\right) = A\left(\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x}\right) = -A\frac{\partial P}{\partial x}$$

$$\xrightarrow{\text{1-D duct mass}} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

2-D

$$\frac{\partial(mu)}{\partial t} = \frac{\partial(\rho u)}{\partial t}\Delta x\Delta y\Delta z$$



$$+((\dot{m}u)_2 - (\dot{m}u)_1 + (\dot{m}v)_4 - (\dot{m}v)_3) = (\rho u^2)_2 \Delta y\Delta z - (\rho u^2)_1 \Delta y\Delta z + (\rho u v)_4 \Delta x\Delta z - (\rho u v)_3 \Delta x\Delta z$$

$$= F_1 - F_2 + F_3 - F_4 = P_1 \Delta y\Delta z - P_2 \Delta y\Delta z$$

$$\xrightarrow{\frac{1}{\Delta x\Delta y\Delta z}} \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho u v)}{\partial y} = -\frac{\partial P}{\partial x}$$

$$\rightarrow u\left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y}\right) + \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x}$$

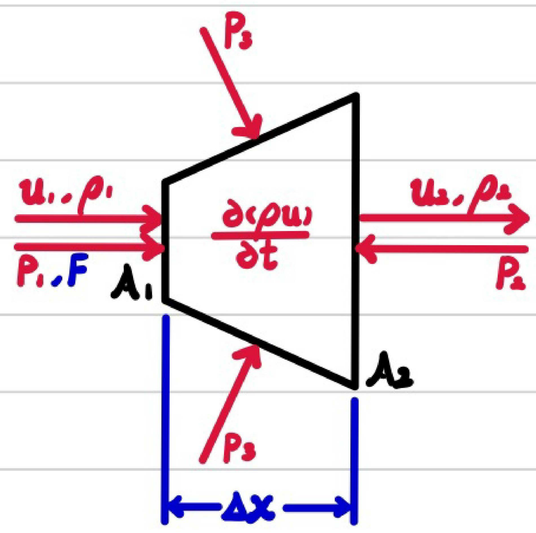
$$\xrightarrow{\text{2-D mass}} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

Euler equations

$$\frac{Du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \longrightarrow \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y},\ \frac{Dw}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial w}$$

Force on a Control Volume

e.g.

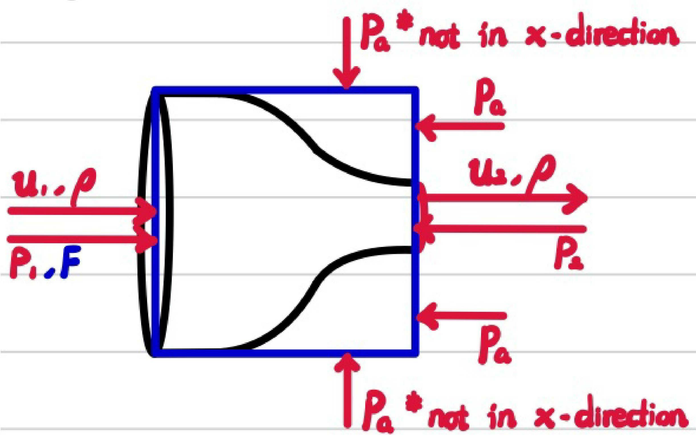


$$\frac{\partial(\rho u)}{\partial t} A\Delta x + \partial(\rho u^2 A) = F - \partial(PA) + P\partial A$$

$$\xrightarrow{\text{steady flow}} (\rho u^2 A)_2 - (\rho u^2 A)_1 = F + P_1A_1 - P_2A_1 + P_3(A_2 - A_1)$$

$$\longrightarrow F = -P_1A_1 + P_2A_1 - P_3(A_2 - A_1) + (\rho u^2 A)_2 - (\rho u^2 A)_1$$

e.g.



$$\xrightarrow{\rho = \text{const}} \rho(uA)_1 = \rho(uA)_2$$

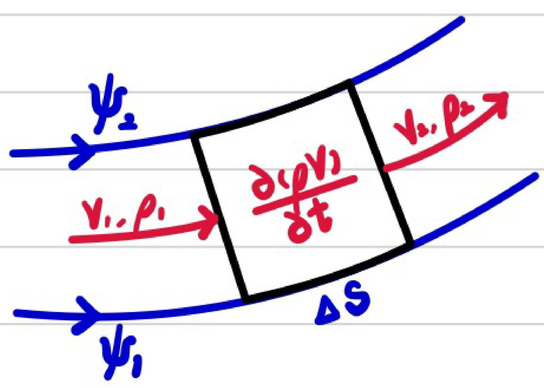
$$\xrightarrow{\text{assumptions}} P_1 + \tfrac{1}{2}\rho u_1^2 = P_2 + \tfrac{1}{2}\rho u_2^2 = P_a + \tfrac{1}{2}\rho u_2^2 \longrightarrow P_2 = P_a$$

$$\xrightarrow{\text{steady flow}} (\rho u^2 A)_2 - (\rho u^2 A)_1 = F + P_1A_1 - P_2A_2 - P_a(A_1 - A_2)$$

$$\longrightarrow F = (P_a - P_1)A_1 + (\rho u^2 A)_1 - (\rho u^2 A)_2$$

Bernoulli's Equation

Euler equation along a streamline



$$\frac{DV}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial s} \xrightarrow{\text{streamline}} \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s} = -\frac{1}{\rho}\frac{\partial P}{\partial s}$$

$$* \text{ steady flow } \xrightarrow{\frac{\partial V}{\partial t} = 0} V\frac{\partial V}{\partial s} = -\frac{1}{\rho}\frac{\partial P}{\partial s}$$

$$\xrightarrow{\frac{\partial V^2}{\partial s} = 2V\frac{\partial V}{\partial s}} \frac{1}{2}\frac{\partial V^2}{\partial s} = -\frac{1}{\rho}\frac{\partial P}{\partial s}$$

Incompressible Bernoulli's Equation

$$\frac{1}{2}\frac{\partial V^2}{\partial s} = -\frac{1}{\rho}\frac{\partial P}{\partial s} \longrightarrow \frac{1}{2}\frac{\partial V^2}{\partial s} + \frac{1}{\rho}\frac{\partial P}{\partial s} = \frac{\partial}{\partial s}\left(\frac{1}{2}V^2 + \frac{P}{\rho}\right) = 0 \longrightarrow \frac{1}{2}V^2 + \frac{P}{\rho} = \text{constant}$$

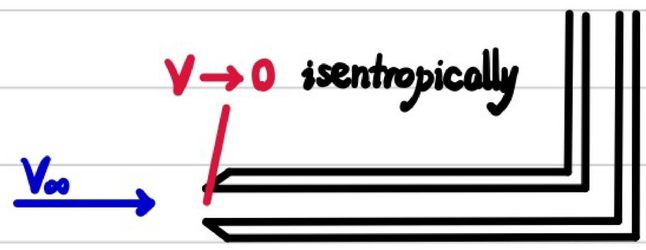
$$\xrightarrow{P = P_0 \text{ when } V = 0} P_0 = P + \tfrac{1}{2}\rho V^2$$

$$* \; C_p = \frac{P - P_\infty}{\frac{1}{2}\rho V_\infty^2} \xrightarrow{P_\infty + \frac{1}{2}\rho V_\infty^2 = P + \frac{1}{2}\rho V^2} \frac{P_\infty + \frac{1}{2}\rho(V_\infty^2 - V^2) - P_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - \left(\frac{V}{V_\infty}\right)^2$$

- steady flow $\frac{\partial V}{\partial t} = 0$
- incompressible flow $\frac{\partial \rho}{\partial t} = 0$
- inviscid flow $\mu = 0 \longrightarrow Re = \infty$
- irrotational $\omega_z = 0$ * cannot be used inside boundary layers and wakes

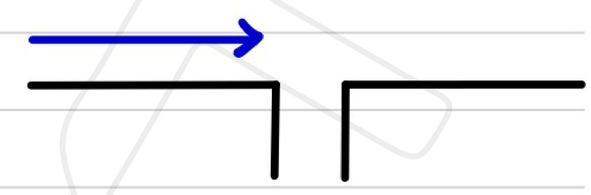
Measurement of Pressures

Pitot tube



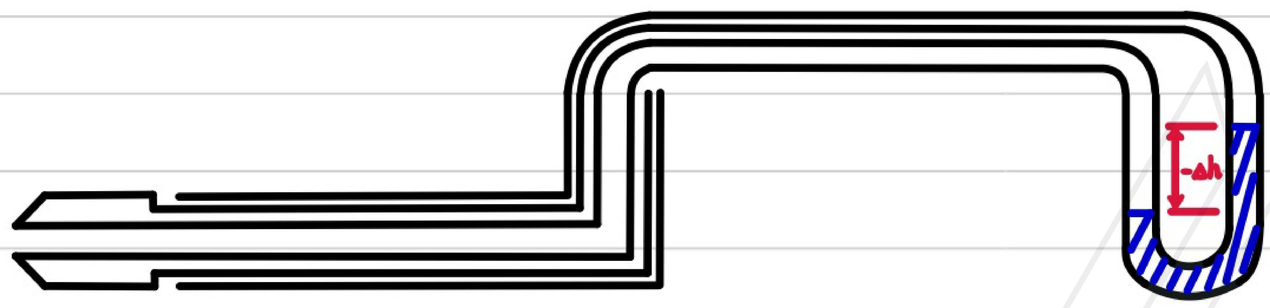
→ P_o

Static pressure hole



→ P_s

Pitot-static tube



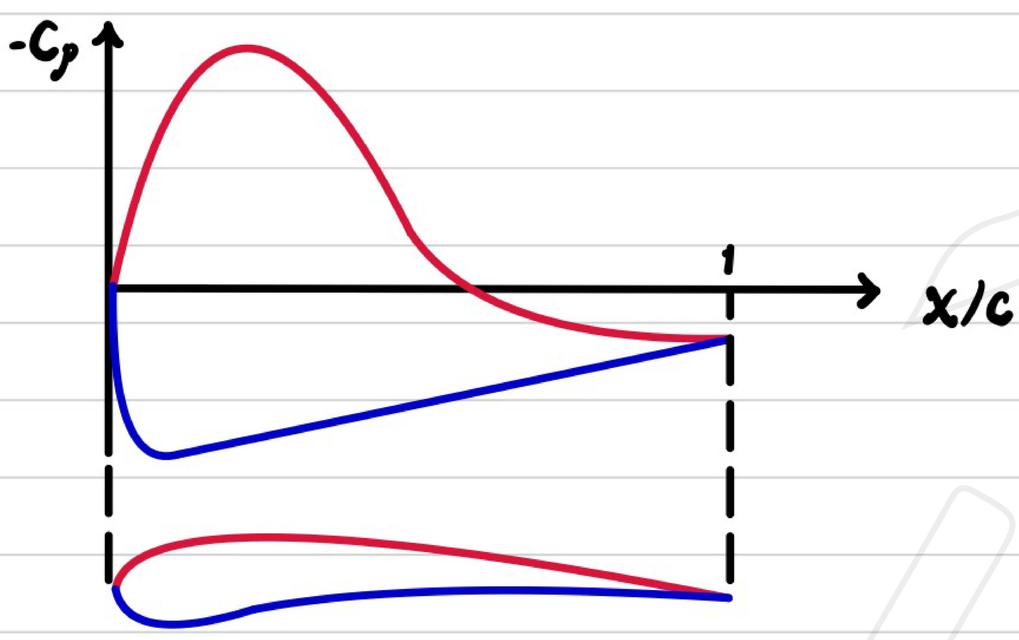
→ P_o, P_s

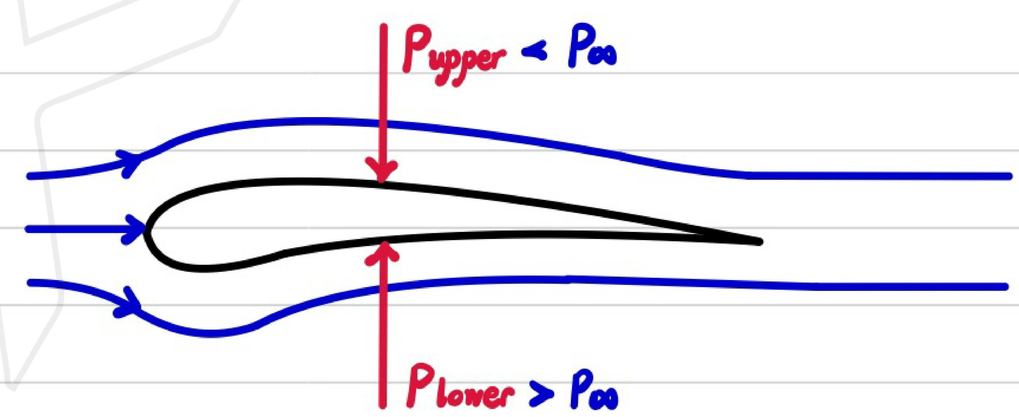
P_o total pressure = P_s local static pressure + $\frac{1}{2}\rho V^2$ dynamic pressure

Lift

Lift Generation

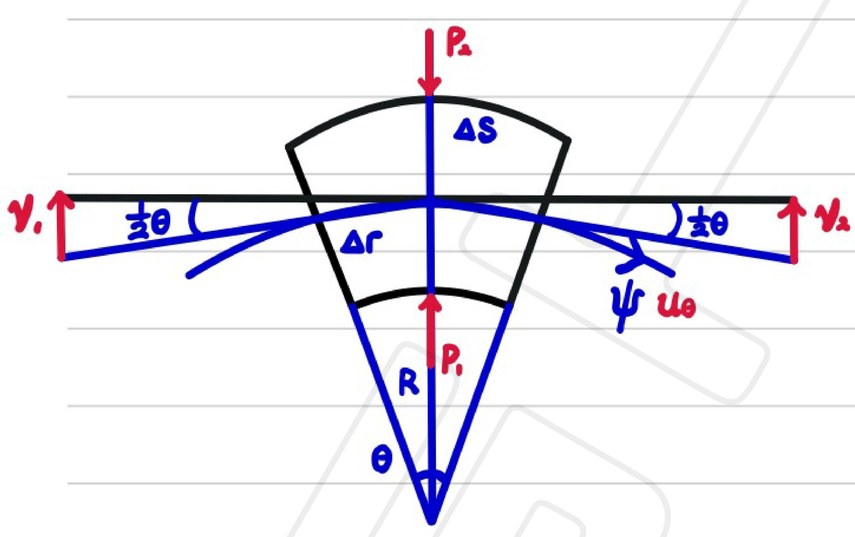
Pressure difference $F(\uparrow) = P_{lower} - P_{upper}$ net force on body produce lift





Momentum balance

Pressure gradient normal to the streamlines



Net momentum flux = pressure forces

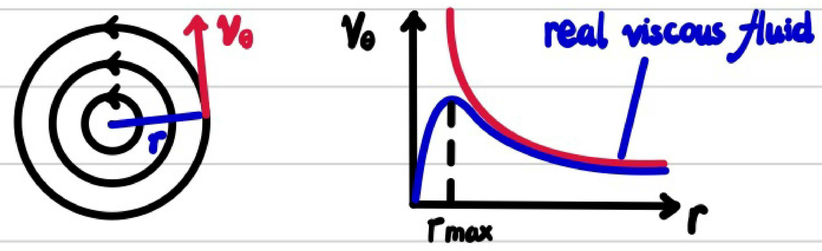
$$\rho u_\theta V_2 \Delta r \Delta z - \rho u_\theta V_1 \Delta r \Delta z = (P_1 - P_2)\Delta S \Delta z = \Delta P R\theta \Delta z$$

$$\xrightarrow{V_1 = -V_2 = u_\theta \sin(\frac{\theta}{2}) \approx u_\theta(\frac{\theta}{2})} -\rho u_\theta^2 \theta \Delta r \Delta z = -\Delta P R\theta \Delta z$$

$$\xrightarrow{\frac{1}{R\theta\Delta r\Delta z}} \frac{\rho u_\theta^2}{R} = \frac{\Delta P}{\Delta r} \longrightarrow F_{centri} = \frac{\rho u_\theta^2}{R} = \frac{\partial P}{\partial r} = F_p$$

Circulation 环量

Ideal or point vortex



V_θ circumferenal velocity $\propto \frac{1}{r}$

- $r < r_{max}$ forced vortex
- $r > r_{max}$ ideal or point vortex

Circulation or strength of vortex

$\Gamma = \oint v \, dS$ in clockwise

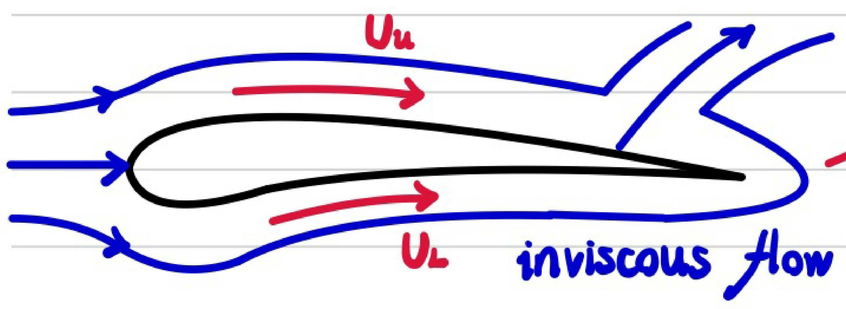
$$\Gamma = -2\pi r V_\theta \longrightarrow V_\theta = -\frac{\Gamma}{2\pi r} \xrightarrow{v = -\frac{\partial \psi}{\partial x}} V_\theta = -\frac{\partial \psi}{\partial r} \longrightarrow \frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r} \longrightarrow \psi = \frac{\Gamma}{2\pi} \ln(r)$$

$$* \ \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = -\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0$$

flow is still irrotational as $r \uparrow$

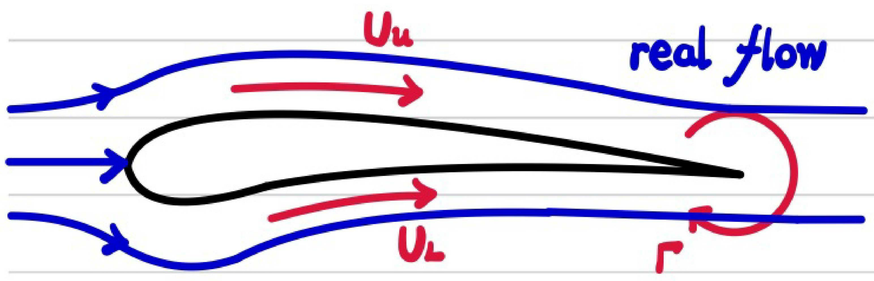
Kutta-Joukowiski theorem

L' lift per unit span acting on airfoil $= \rho u \Gamma$



infinite streamline culvature $\xrightarrow{\Gamma = 0} L = 0$

$P_u < P_L \xrightarrow{P_0 = P_s + \frac{1}{2}\rho u^2} U_u > U_L \longrightarrow$ positive circulation

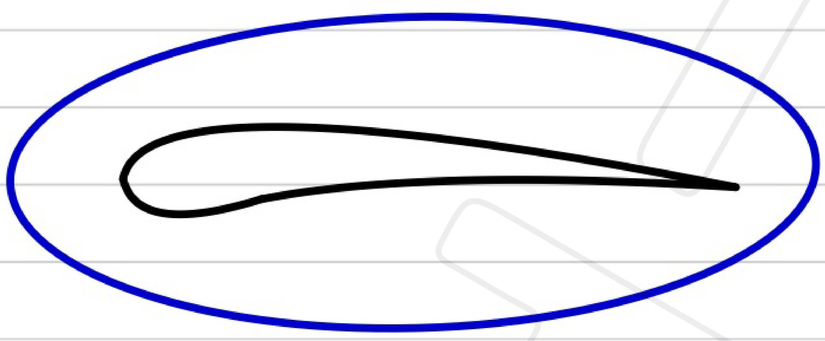


* Kutta condition flow leaves "smoothly" from trailing edge

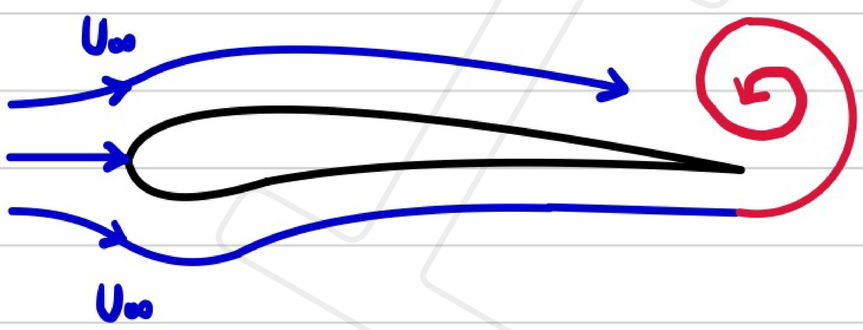
$\longrightarrow$ Lift causes a $+\Gamma$ around an airfoil

Kelvin's circulation theorem

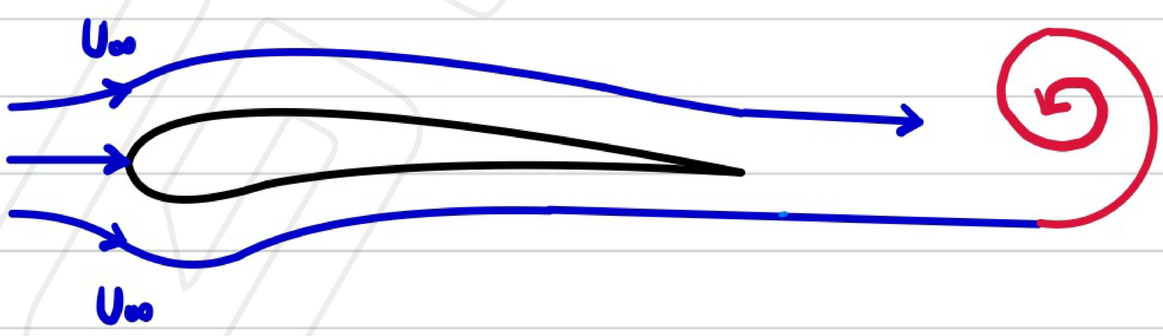
The rate of change (in time) of circulation around a closed curve consisting of the same fluid element is zero $\longrightarrow \frac{D\Gamma}{Dt} = 0$

① 

$\Gamma = 0$

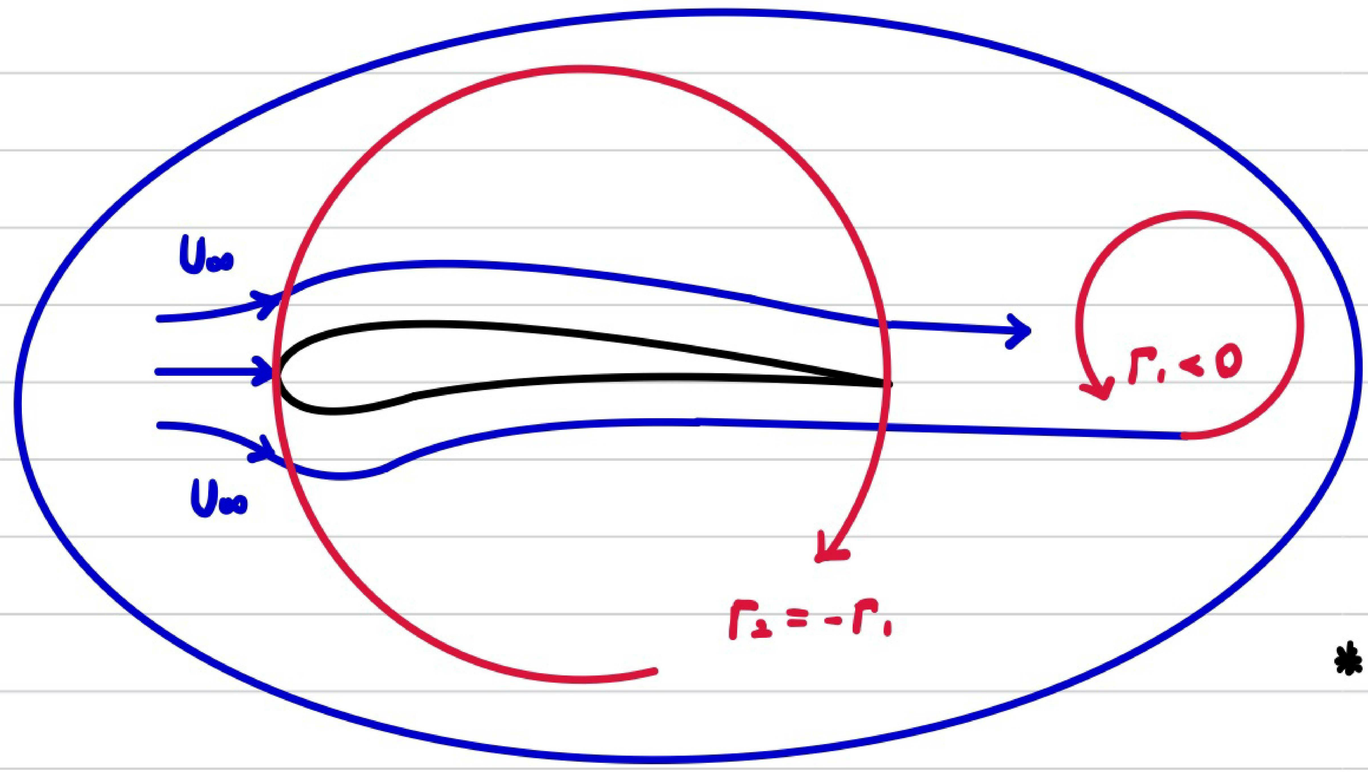
② 

vorticity is formed

③ 

vorticity has been swept away

④



$\frac{D\Gamma}{Dt} = 0 \longrightarrow \Gamma = 0$

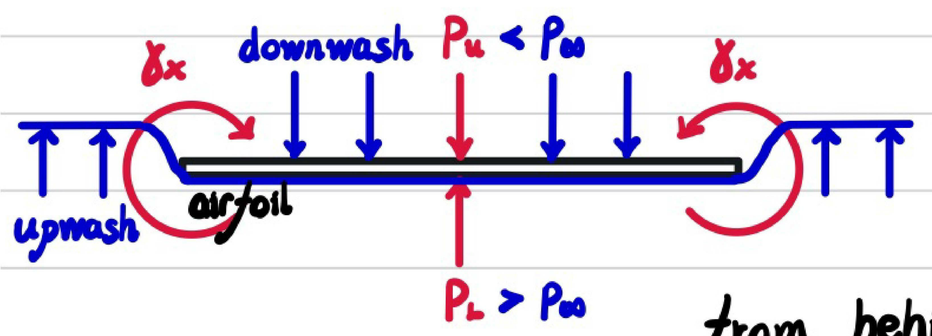
$\longrightarrow \Gamma_1 + \Gamma_2 = 0$

$\longrightarrow \Gamma_2 = -\Gamma_1$

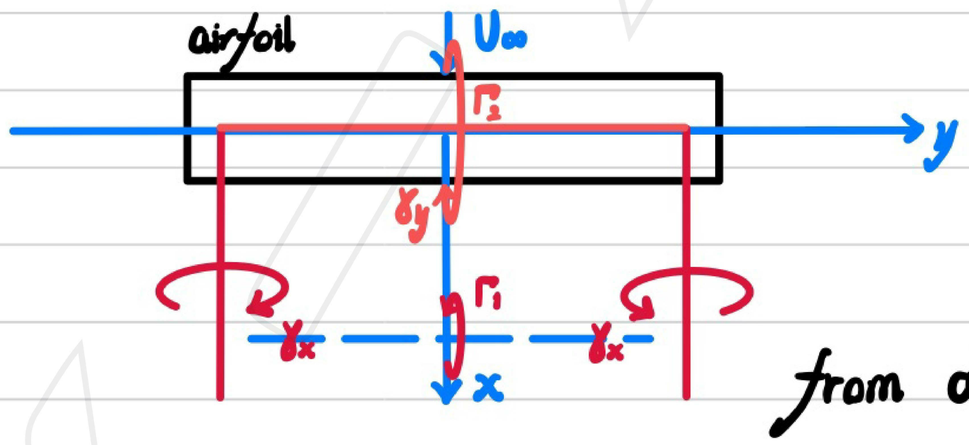
* $U_U = U_\infty + \Gamma_2 > U_L = U_\infty - \Gamma_2$

$\longrightarrow$ $+\Gamma$ around an airfoil causes Lift

Helmholtz's theorem In an inviscid flow, vortex line cannot end in space



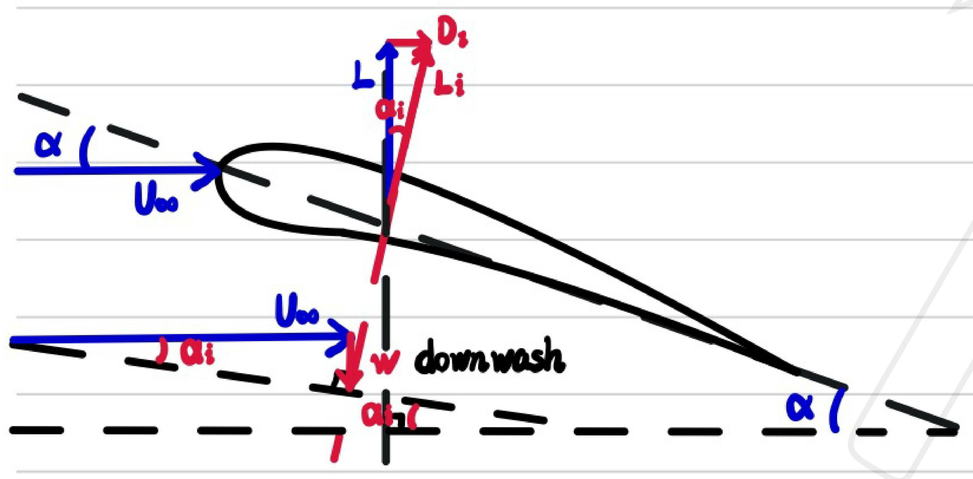
from behind



from above

γ circulation intensity $\longrightarrow$ $\Gamma = \int \gamma_x \, dx = \int \gamma_y \, dy$

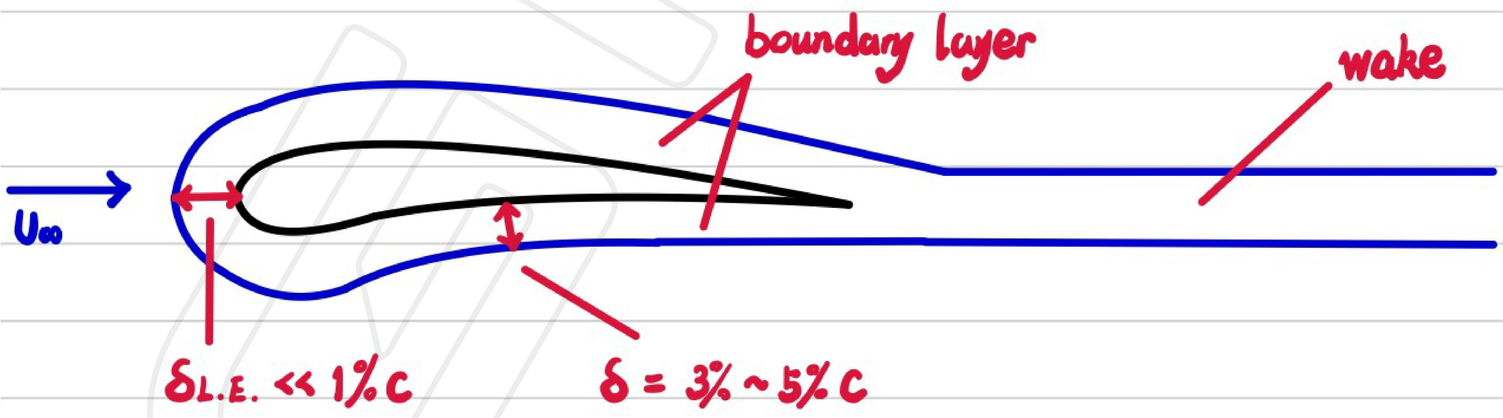
Induced drag



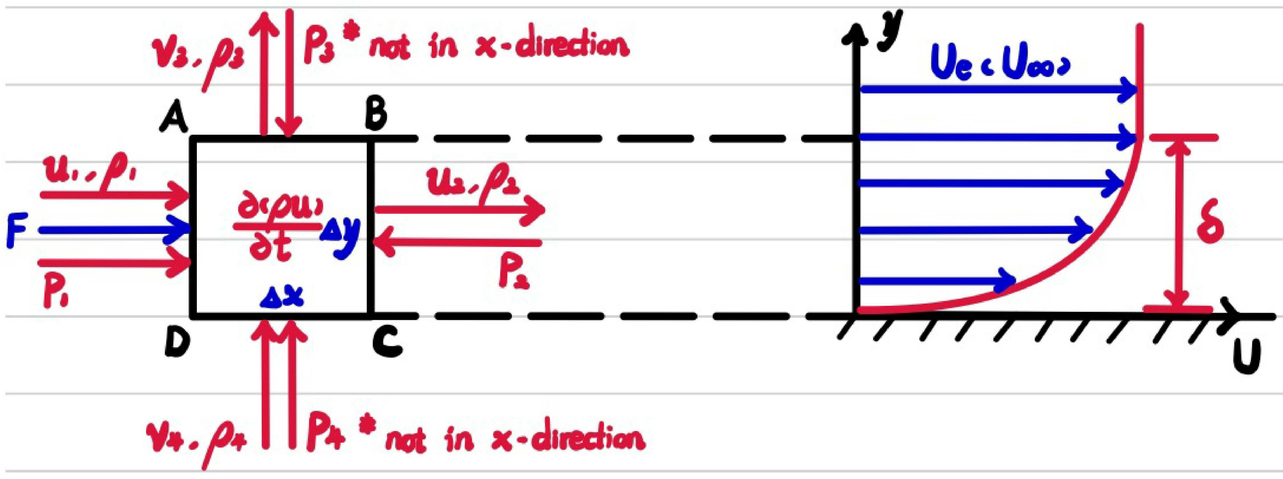
$D_i = L \tan(\alpha_i) \approx L\alpha_i \longrightarrow D \propto L$

Boundary Layer

Viscous Momentum Conservation



Viscous forces on the fluid in control volume



$F = \Delta\tau\Delta x\Delta z$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{1}{\rho}\frac{\partial \tau}{\partial y} \xlongequal{\tau = \mu\frac{du}{dy}} -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

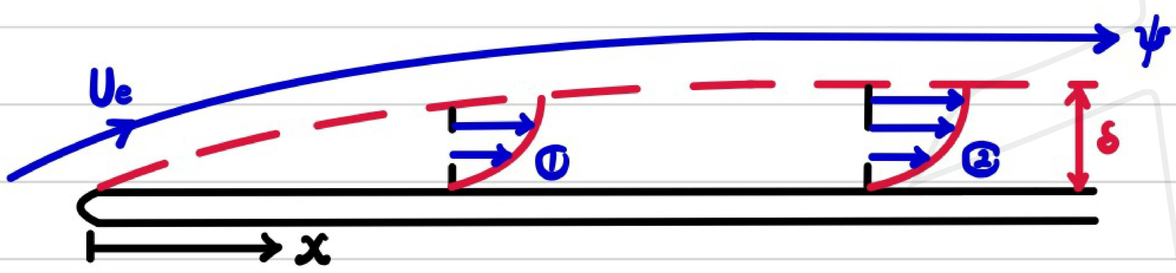
$$\xrightarrow{\nu = \frac{\mu}{\rho}} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

* Ignored $\nu\frac{\partial^2 u}{\partial x^2}$ since $\nu\frac{\partial^2 u}{\partial y^2} \gg \nu\frac{\partial^2 u}{\partial x^2}$

Laminar Boundary Layer

Viscous forces for steady flow

* steady flow $\xrightarrow{u = const}$ $u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$



· $\tau_{w①} > \tau_{w②}$

· $C_{f①} > C_{f②}$

Laminar boundary layer thickness

· convection 对流 $x[L] = f(U_e[\frac{L}{T}], t[T])$ · diffusion 扩散 $\delta[L] = f(\nu[\frac{L^2}{T}], t[T], \ldots)$

$\longrightarrow$ At time $t = T$ · $x = U_eT$ · $\delta \propto \sqrt{\nu T}$

$\longrightarrow \frac{\delta}{x} \propto \sqrt{\frac{\nu}{U_e^2 T}} = \sqrt{\frac{\mu}{\rho U_e (U_e T)}} = \sqrt{\frac{\mu}{\rho U_e x}} \longrightarrow \frac{\delta}{x} \propto Re^{-\frac{1}{2}}$ for $Re_x < 3 \sim 5 \times 10^5$

$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$ and $\omega = \frac{\partial v}{\partial x} - \frac{\partial z}{\partial y} = 0$ * $\frac{\partial P}{\partial y} = 0$ in boundary layer $\longrightarrow \frac{\partial P}{\partial x} \rightarrow \frac{dP}{dx}$

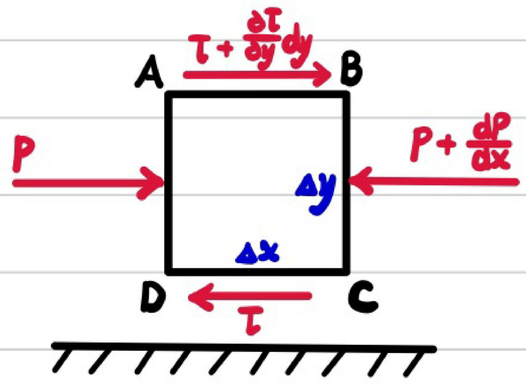
* For flat plate, $\delta = 5.3\sqrt{\frac{\nu x}{U_e}} \longrightarrow \frac{\delta}{x} = 5.3 Re^{-\frac{1}{2}}$

Pressure gradient

U_e varies with x for airfoil $\xrightarrow[\text{outside of boundary layer}]{\frac{\partial}{\partial x}P_0 = 0 = \frac{\partial}{\partial x}(P + \frac{1}{2}\rho U_e^2)}$ $0 = \frac{\partial P}{\partial x} + \rho U_e\frac{\partial U_e}{\partial x}$ * $\frac{\partial(\rho U_e)}{\partial x} = 0$ for steady flow

$\longrightarrow \frac{\partial P}{\partial x} = -\rho U_e\frac{\partial U_e}{\partial x} \longrightarrow \frac{\partial P}{\partial x} \propto -\frac{\partial U_e}{\partial x}$

Seperation when $\tau_w = 0 \longrightarrow \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0$



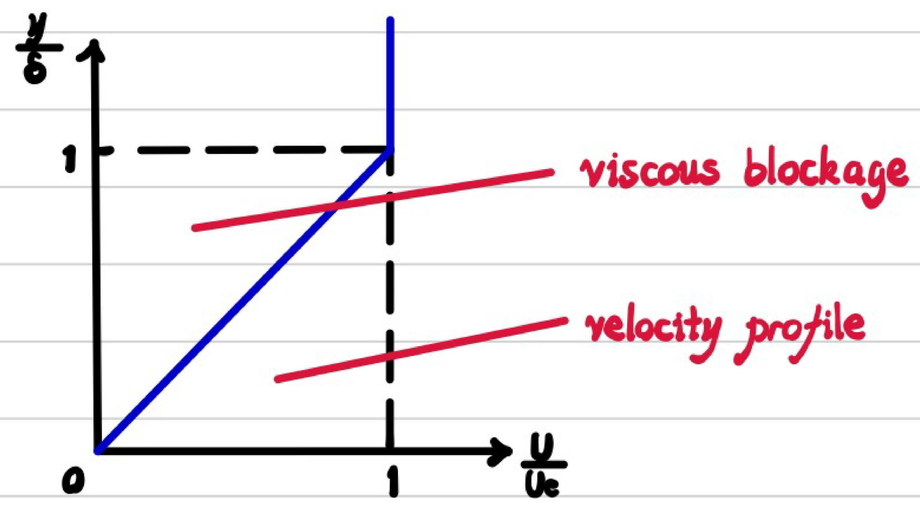
$\longrightarrow$ flow could brought to rest and reversed

* At seperation point $\tau_w = 0 \longrightarrow \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0$ * $y = u = v = 0$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = 0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} \longrightarrow \frac{\partial P}{\partial x} = \mu \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} \longrightarrow \frac{\partial P}{\partial x} \propto \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0}$$

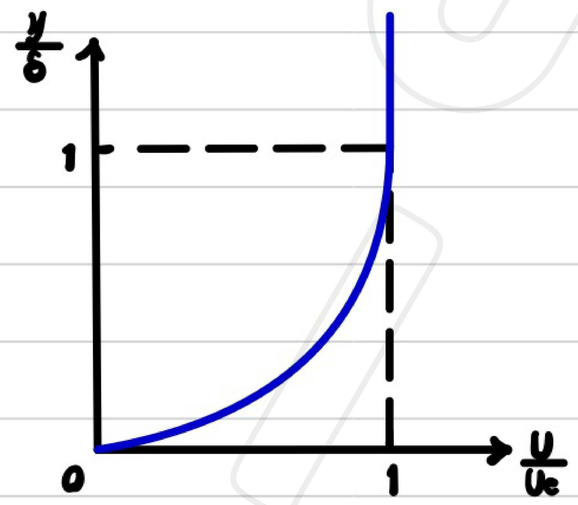
- zero pressure gradient $\xrightarrow{U_e}$

$$\frac{\partial U_e}{\partial x} = 0 \longrightarrow \frac{\partial P}{\partial x} = 0 \longrightarrow \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} = 0$$



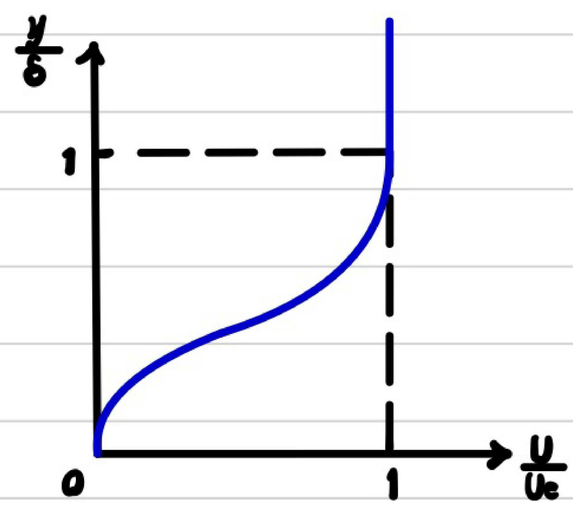
- favourable pressure gradient $\xrightarrow{U_e}$ * stabilizing

$$\frac{\partial U_e}{\partial x} > 0 \longrightarrow \frac{\partial P}{\partial x} < 0 \longrightarrow \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} < 0$$



- adverse pressure gradient $\xrightarrow{U_e}$ * destabilizing

$$\frac{\partial U_e}{\partial x} < 0 \longrightarrow \frac{\partial P}{\partial x} > 0 \longrightarrow \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} > 0$$



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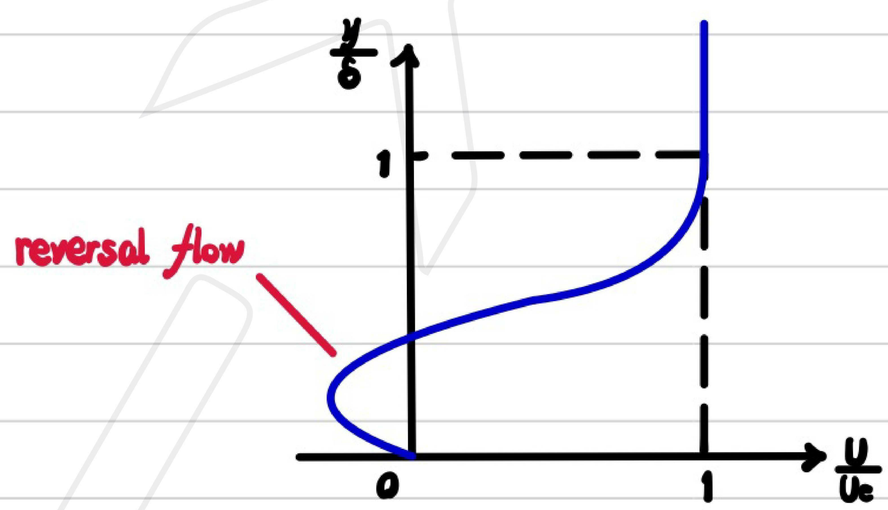
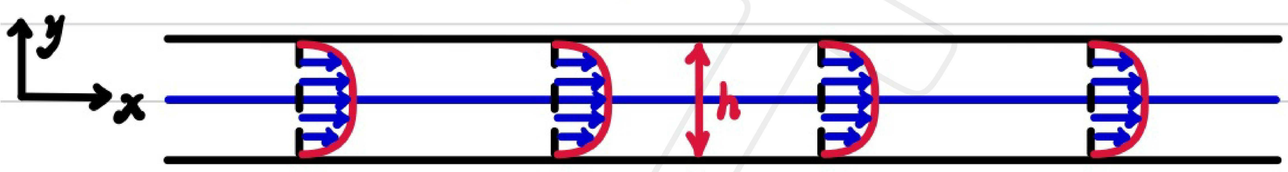


Steady, viscous, laminar flow in ducts



$\frac{\partial u}{\partial x} = 0$ for fully-developed flow

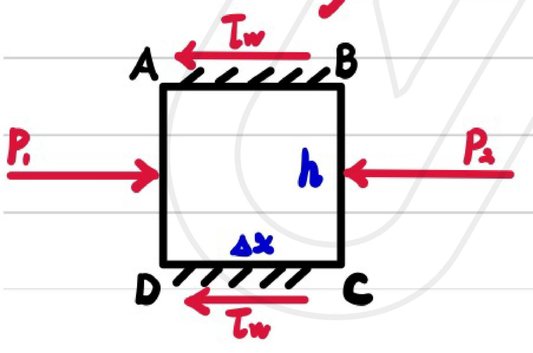
$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \omega = \frac{\partial v}{\partial x} - \frac{\partial z}{\partial y} = 0$$

* $\frac{\partial P}{\partial y} = 0$ in boundary layer $\longrightarrow \frac{\partial P}{\partial x} \rightarrow \frac{dP}{dx}$

$$\xrightarrow{v=0,\ \frac{\partial u}{\partial x}=0} \frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dP}{dx} \longrightarrow \frac{du}{dy} = \frac{1}{\mu}\frac{dP}{dx}y + C \longrightarrow u = \frac{1}{\mu}\frac{dP}{dx}\frac{y^2}{2} + Cy + D$$

* $P_2 < P_1 \longrightarrow \frac{dP}{dx} < 0$

* $u = 0$ at $y = \pm\frac{h}{2} \longrightarrow C = 0,\ D = \frac{1}{\mu}\frac{dP}{dx}\left(-\frac{h^2}{8}\right) \longrightarrow u = \frac{1}{\mu}\frac{dP}{dx}\left(\frac{y^2}{2} - \frac{h^2}{8}\right)$ parabolic



$$\tau_w = \mu\left(\frac{du}{dy}\right)\Big|_{y'=0} \text{ at } y' = y \mp \frac{h}{2} \longrightarrow \tau_{w,U} = \tau_{w,L} = \frac{dP}{dx}\left(-\frac{h}{2}\right)$$

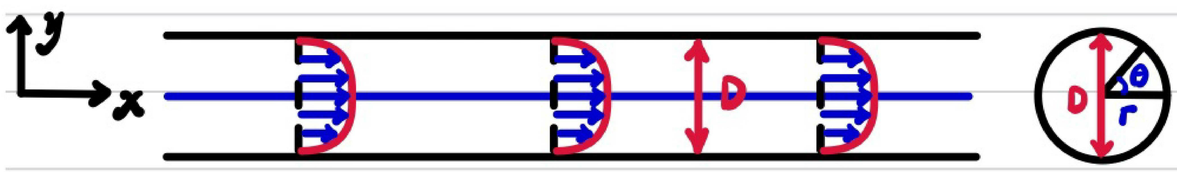
OR $(P_1 - P_2)h\Delta z = -\Delta P h \Delta z = (2\tau_w)\Delta x \Delta z \longrightarrow \tau_w = \frac{dP}{dx}\left(-\frac{h}{2}\right)$

Friction coefficient $C_f = \frac{\tau_w}{\frac{1}{2}\rho \bar{U}^2}$

$$\bar{U} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{\mu}\frac{dP}{dx}\left(\frac{y^2}{2}-\frac{h^2}{8}\right)dy \Big/ \int_{-\frac{h}{2}}^{\frac{h}{2}} dy = -\frac{dP}{dx}\frac{h^2}{12\mu} \longrightarrow \frac{dP}{dx} = \frac{-12\mu\bar{U}}{h} \longrightarrow \tau_w = \mp\frac{1}{\mu}\frac{dP}{dx}\left(\pm\frac{h}{2}\right) = \frac{6\bar{U}}{h}$$

$$\longrightarrow C_f = \frac{12\mu}{\rho\bar{U}h} = \frac{12}{Re}$$

Steady, viscous, laminar flow in circular pipes



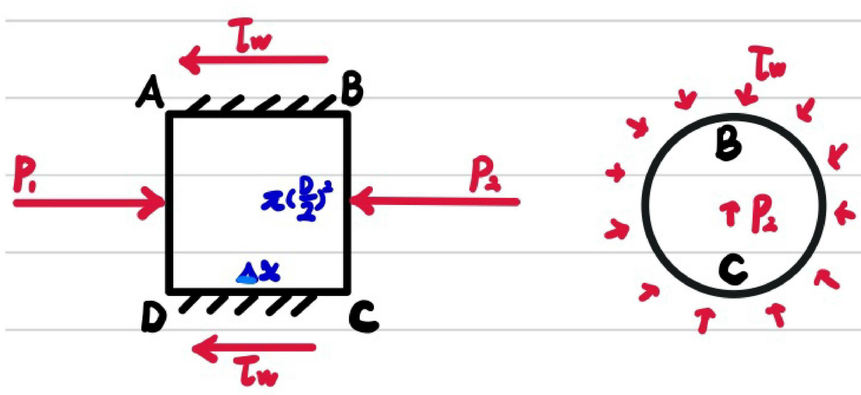
$\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial \theta} = 0$ for fully-developed flow

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

$$\xrightarrow{v=0,\ \frac{\partial u}{\partial x}=0} \frac{\mu}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{dP}{dx} \longrightarrow \mu r\frac{du}{dr} = \frac{dP}{dx}\left(\frac{r^2}{2}\right) + A$$

* $\frac{du}{dr} = 0$ at $r = 0 \longrightarrow A = 0 \longrightarrow \frac{du}{dr} = \frac{1}{\mu}\frac{dP}{dx}\frac{r}{2} \longrightarrow u = \frac{1}{\mu}\frac{dP}{dx}\frac{r^2}{4} + B$

* $u = 0$ at $r = \pm\frac{D}{2} \longrightarrow B = -\frac{1}{\mu}\frac{dP}{dx}\frac{D^2}{16} \longrightarrow u = \frac{1}{\mu}\frac{dP}{dx}\left(\frac{r^2}{4}-\frac{D^2}{16}\right)$



$$(P_1 - P_2)\pi\left(\frac{D}{2}\right)^2\Delta z = -\Delta P\frac{\pi D^2}{4}\Delta z = \tau_w \Delta x(\pi D)$$

$$\longrightarrow \tau_w = -\frac{D}{4}\frac{dP}{dx}$$

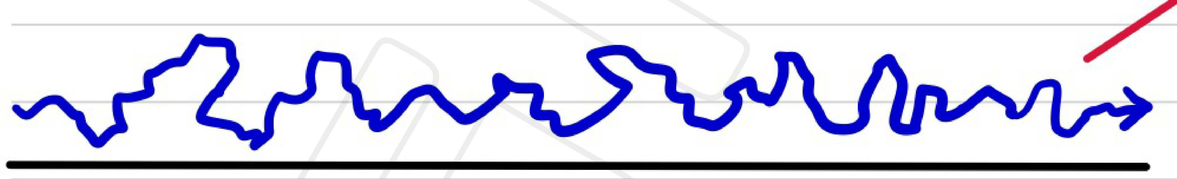
$$\bar{U} = \int_0^{2\pi}\int_0^{\frac{D}{2}} \frac{1}{\mu}\frac{dP}{dx}\left(\frac{r^2}{4}-\frac{D^2}{16}\right)dr\,(r\,d\theta) \Big/ \int_0^{2\pi}\int_0^{\frac{D}{2}} dr\,(r\,d\theta) = \int_0^{\frac{D}{2}} \frac{1}{\mu}\frac{dP}{dx}\left(\frac{r^2}{4}-\frac{D^2}{16}\right)2\pi r\,dr \Big/ \left(\frac{\pi D^2}{4}\right) = -\frac{D^2}{32\mu}\frac{dP}{dx}$$

OR $\bar{U} = \frac{1}{2}U_{max} = \frac{1}{2}\frac{1}{\mu}\frac{dP}{dx}\left(-\frac{D^2}{16}\right) = -\frac{D^2}{32\mu}\frac{dP}{dx}$

$$\longrightarrow C_f = \frac{\tau_w}{\frac{1}{2}\rho\bar{U}} = \frac{16}{Re}$$

Turbulence

small amplification of small disturbances



unsteady flow

Velocity profile

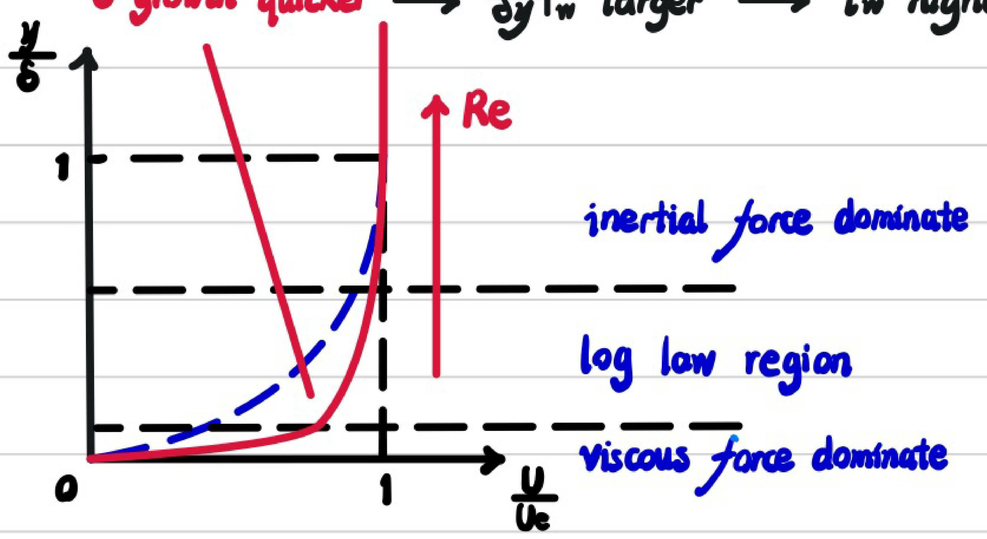
- internal timescale $t_i \propto \frac{\delta}{u}$
- viscous timescale $t_\nu \propto \frac{\delta^2}{\nu}$

$$Re = \frac{\text{internal force}}{\text{viscous force}} = \frac{\frac{1}{2}\rho u^2}{\frac{1}{2}\mu \frac{u}{\delta}} = \frac{\rho u D}{\mu}$$

$$= \frac{\frac{F_i}{m}}{\frac{F_v}{m}} = \frac{a_i}{a_v} = \frac{1/t_i}{1/t_v} = \frac{t_v}{t_i} \longrightarrow t_i \ll t_v$$

→ turbulent boundary layer growth much faster & thicker than laminar one

δ growth quicker → $\left.\frac{\partial u}{\partial y}\right|_w$ larger → τ_w higher



* log law $\frac{u}{u_\tau} = u\sqrt{\frac{\rho}{\tau_w}} = \frac{1}{k}\ln\left(\frac{y u_\tau}{\nu}\right) + C$

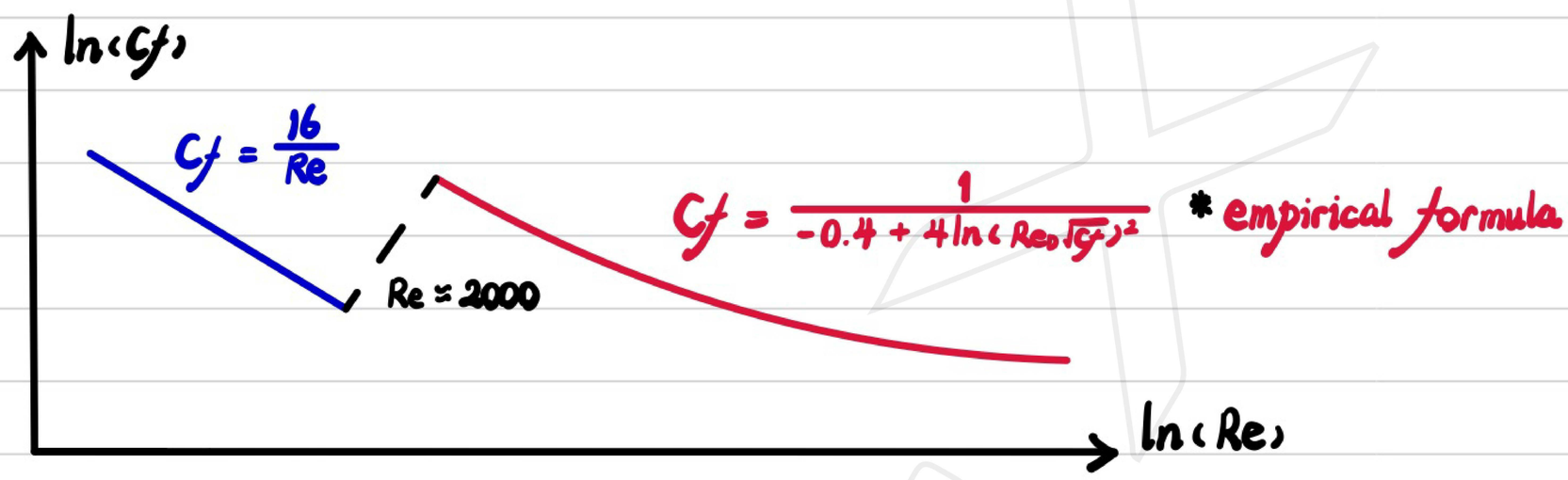
Transition flow

laminar → transition → turbulent

- surface roughness
- turbulence in freestream
- Increasing Re
- $\frac{\partial P}{\partial x} > 0$

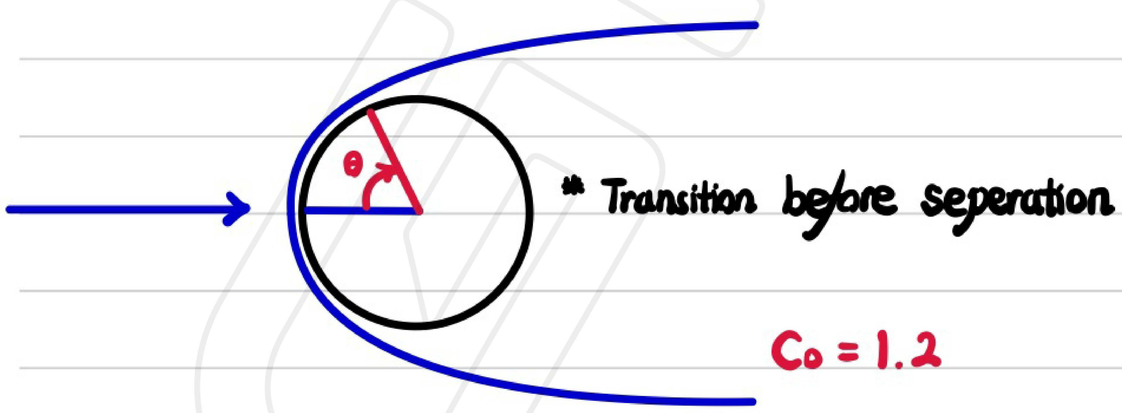
Pipe flow

$\frac{\delta}{x} = 0.37 Re_x^{-\frac{1}{5}}$ (Prandtl) $= 0.16 Re_x^{-\frac{1}{7}}$ (White)

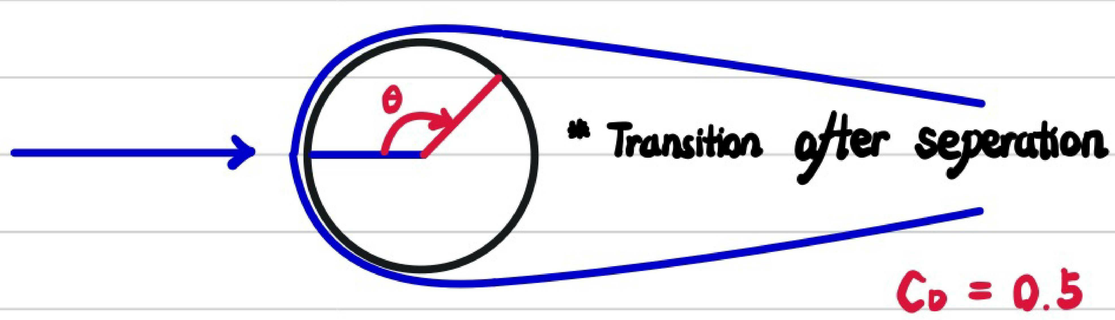


Benefits

- mixes flow effectively → disperses 驱散 pollutants
- seperate less readily 迟速地 → more momentum near the wall → higher $\frac{\partial P}{\partial x} > 0$
- C_D small for bluff 钝头 body → reduce drag * opposite with streamlined bodies
- Drag is not viscosity dominated → higher percent of pressure drag → good boundary layer



Subcritical $Re < 2\times10^5$



Supercritical $Re > 2\times10^5$