Calculus of Variations

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The Euler-Lagrange equation

find function $y(x)$ that extremises $I = \int_{x_1}^{x_2} f(x, y, y')\, dx$

$\longrightarrow$ let minimiser be $y = u(x)$

$\longrightarrow$ apply small perturbation $y = u(x) + \varepsilon \eta(x)$ where $\eta(x_1) = \eta(x_2) = 0$

$\longrightarrow \int_{x_1}^{x_2} f(x, u+\varepsilon\eta, u'+\varepsilon\eta')\, dx > \int_{x_1}^{x_2} f(x, u, u')\, dx = I|_{\varepsilon=0}$

$\longrightarrow$ for all suitable choice of $\eta(x)$, $\frac{dI}{d\varepsilon}\Big|_{\varepsilon=0} = 0$

$\longrightarrow \int_{x_1}^{x_2} \left(\frac{df}{dx}\frac{dx}{d\varepsilon} + \frac{df}{dy}\frac{dy}{d\varepsilon} + \frac{df}{dy'}\frac{dy'}{d\varepsilon} \right) dx = \int_{x_1}^{x_2} \left(\frac{df}{dy}\eta + \frac{df}{dy'}\eta' \right) dx = 0$

$\xrightarrow{\text{integration by parts}} \int_{x_1}^{x_2} \frac{df}{dy}\eta\, dx + \left[\frac{df}{dy'}\eta \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{df}{dy'}\right)\eta\, dx = 0$

$\longrightarrow \int_{x_1}^{x_2} \eta \left(\frac{df}{dy} - \frac{d}{dx}\frac{df}{dy'} \right) dx = 0$

$\longrightarrow \frac{df}{dy} - \frac{d}{dx}\frac{df}{dy'} = 0$

Special cases

find function $y(x)$ that extremises $I = \int_{x_1}^{x_2} f(x, y)\, dx \longrightarrow \frac{df}{dy} = 0$

find function $y(x)$ that extremises $I = \int_{x_1}^{x_2} f(x, y')\, dx \longrightarrow \frac{df}{dy'} = \text{constant}$

Beltrami identity

find function $y(x)$ that extremises $I = \int_{x_1}^{x_2} f(y, y')\, dx$

$\longrightarrow \frac{\partial f}{\partial x} = 0$ while $\frac{df}{dx} = \frac{df}{dy}\frac{dy}{dx} + \frac{df}{dy'}\frac{dy'}{dx} = \frac{df}{dy}y' + \frac{df}{dy'}y''$

$\xrightarrow{y'\left(\frac{df}{dy} - \frac{d}{dx}\frac{df}{dy'}\right) = 0} y'\frac{d}{dx}\left(\frac{df}{dy'}\right) + \frac{df}{dy'}y'' = y'\frac{d}{dx}\left(\frac{df}{dy'}\right) + \frac{df}{dy'}\frac{d}{dx}(y') = 0$

$\longrightarrow \frac{df}{dx} = \frac{d}{dx}\left(y'\frac{df}{dy'}\right)$

$\longrightarrow \frac{d}{dx}\left(f - y'\frac{df}{dy'}\right) = 0$

$\longrightarrow f - y'\frac{df}{dy'} = \text{constant}$

Shortest route

find shortest route between $A(x_1, y_1)$ and $B(x_2, y_2)$

$\longrightarrow$ find route $y(x)$ that minimises $\int_{x_1}^{x_2} L\,dx$

$\longrightarrow dL = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\,dx$

$\longrightarrow \int_{x_1}^{x_2} L\,dx = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\,dx$

$\longrightarrow \frac{dL}{dy'} = \text{constant}$

$\longrightarrow y'(1 + y'^2)^{-\frac{1}{2}} = A \longrightarrow y'^2 = A^2(1 + y'^2) \longrightarrow y' = \pm\sqrt{\frac{A^2}{1 - A^2}} = m$

$\longrightarrow y = mx + c \longrightarrow$ apply boundary conditions

Brachistochrone

find shape of slide that particle released from A will reach B in the shortest possible time

$\longrightarrow$ find function $y(x)$ that minimises $\int_{x_1}^{x_2} t\,dx$

$\longrightarrow dt = \frac{dL}{v} = \frac{1}{v}\sqrt{1 + y'^2}\,dx \xrightarrow{\frac{1}{2}mv^2 = mgy} \sqrt{\frac{1 + y'^2}{2gy}}\,dx$

$\longrightarrow \int_{x_1}^{x_2} t\,dx = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1 + y'^2}{y}}\,dx$

$\longrightarrow \sqrt{\frac{1 + y'^2}{y}} - y' \frac{d}{dy'}\left(\sqrt{\frac{1 + y'^2}{y}}\right) = \text{constant}$

$\longrightarrow y^{-\frac{1}{2}}(1 + y'^2)^{\frac{1}{2}} - y' y^{-\frac{1}{2}}[y'(1 + y'^2)^{-\frac{1}{2}}] = y^{-\frac{1}{2}}(1 + y'^2 - y'^2)(1 + y'^2)^{-\frac{1}{2}} = y^{-\frac{1}{2}}(1 + y'^2)^{-\frac{1}{2}} = \text{constant}$

$\longrightarrow y(1 + y'^2) = D^2 \longrightarrow \frac{dy}{dx} = \pm\sqrt{\frac{D^2}{y} - 1} \longrightarrow x = \int \pm\left(\frac{D^2}{y} - 1\right)^{-\frac{1}{2}} dy = \int \pm\sqrt{\frac{y}{D^2 - y}}\,dy$

$\longrightarrow$ let $y = D^2 \sin^2\theta$, $\int \pm\left(\frac{D^2}{y} - 1\right)^{-\frac{1}{2}} dy = \int \pm\frac{\sin\theta}{\cos\theta}(2D^2 \sin\theta\cos\theta)\,d\theta = \pm 2D^2 \int \sin^2\theta\,d\theta$

$\longrightarrow x = \pm D^2 \int 1 - \cos(2\theta)\,d\theta = \pm D^2 [\theta - \frac{1}{2}\sin(2\theta) + \cancel{c}]$ 0 as curve pass through origin

$$\xrightarrow{\phi = 2\theta,\ a = \frac{1}{2}D^2} \begin{cases} x = a(\phi - \sin\phi) \\ y = a(1 - \cos\phi) \end{cases}$$

Constraint problems

the smallest possible k is the one for $f(x, y) = k$ and constraint $g(x, y) = g_0$ contour just touch

$\longrightarrow \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix} \parallel \begin{pmatrix} \partial g/\partial x \\ \partial g/\partial y \end{pmatrix} \longrightarrow \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix} + \lambda \begin{pmatrix} \partial g/\partial x \\ \partial g/\partial y \end{pmatrix} = 0$

$\longrightarrow$ define $L(x, y) = f(x, y) + \lambda g(x, y)$ where $\frac{dL}{dx} = \frac{dL}{dy} = 0$

find function $y(x)$ that extremises $I = \int_{x_1}^{x_2} f(x, y, y')\, dx$

subject to constraint $J = \int_{x_1}^{x_2} g(x, y, y')\, dx = J_0$

$\longrightarrow$ find function $y(x)$ that extremises $L = I + \lambda J = \int_{x_1}^{x_2} f(x, y, y') + \lambda g(x, y, y')\, dx$

$\longrightarrow \frac{\partial}{\partial y}(f + \lambda g) - \frac{d}{dx}\left[\frac{\partial}{\partial y'}(f + \lambda g)\right] = 0$

Systems of Euler-Lagrange equation

find function $y(x)$ that extremises $I = \int_{x_1}^{x_2} f(x, y_1, y_2, \dots, y_N, y_1', y_2', \dots, y_N')\, dx$

$\longrightarrow \frac{df}{dy_i} - \frac{d}{dx}\frac{df}{dy_i'} = 0, \quad i = 1, 2, \dots, N$

Complex Analysis

Functions of a Complex Variable

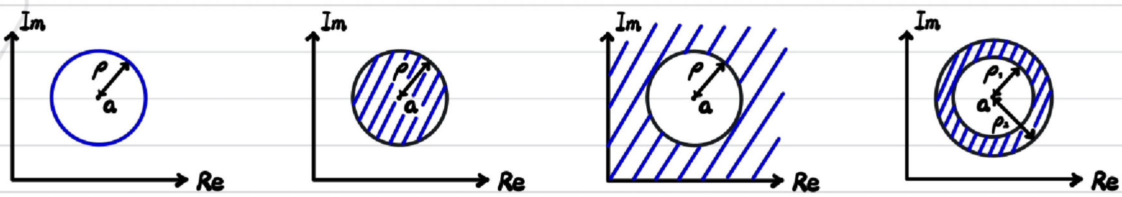
Complex numbers

$x + iy$

- $\longrightarrow (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$
- $\longrightarrow (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + 2ixy$
- $\longrightarrow \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1 x_2 + y_1 y_2}{x_2 + y_2} + i\frac{x_2 y_1 - x_1 y_2}{x_2 + y_2}$
- $\longrightarrow \overline{x + iy} = x - iy \longrightarrow \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ and $|z| = z\overline{z}$
- $\longrightarrow$ complex plane $z = x + iy = re^{i\theta}$ where $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \arg z = \tan^{-1}(\frac{y}{x})$
- $\longrightarrow$ polar form $x = r\cos\theta$, $y = r\sin\theta \longrightarrow z = r(\cos\theta + i\sin\theta) = re^{i\theta}$
- $\longrightarrow$ triangle inequality $|z_1 + z_2| \le |z_1| + |z_2| \longrightarrow |z_1 - z_2| \ge |z_1| - |z_2|$
- $\longrightarrow$ De Moivre's formula $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

nth roots $w^n = z \longrightarrow w = z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\frac{\theta}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right], \ k \in \mathbb{Z}$

curves and regions



$|z - a| = \rho \qquad |z - a| < \rho \qquad |z - a| > \rho \qquad \rho_1 < |z - a| < \rho_2$

Elementary functions

exponential $e^{z} \equiv e^{x}(\cos y + i\sin y) \longrightarrow e^{z_1+z_2} = e^{z_1}e^{z_2}$ and $(e^{z})^{n} = e^{nz}$

trigonometrical $\begin{cases} \cos z = \frac{1}{2}(e^{iz}+e^{-iz}) \longrightarrow \cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \\ \sin z = \frac{1}{2i}(e^{iz}-e^{-iz}) \longrightarrow \sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \end{cases}$

hyperbolic $\begin{cases} \cosh z = \frac{1}{2}(e^{z}+e^{-z}) \longrightarrow \cosh(z_1+z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 \\ \sinh z = \frac{1}{2}(e^{z}-e^{-z}) \longrightarrow \sinh(z_1+z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2 \end{cases}$

$\longrightarrow$ $\cosh iz = \cos z$, $\cos iz = \cosh iz$ and $\sinh iz = i\sin z$, $\sin iz = i\sinh z$

Complex functions

$f : D \to \mathbb{C}$ and $f : z \mapsto w$

neighbourhood $|z - z_0| < \varepsilon \longrightarrow$ punctured neighbourhood $0 < |z - z_0| < \varepsilon$

limit $w_0 = \lim_{z\to z_0} f(z) \longrightarrow |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon$

continuity $f(z_0) = \lim_{z\to z_0} f(z)$

differentiable if and only if $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exist $\xrightarrow{\delta z = z - z_0} f'(z_0) = \lim_{\delta z\to 0} \frac{f(z_0+\delta z)-f(z_0)}{\delta z}$

analytic function in a region $R \in \mathbb{C}$ is differentiable for all $z \in R$

entire analytic everywhere in $\mathbb{C}$

The Cauchy - Riemann equations

$$f'(z) = \lim_{\delta z\to 0} \frac{f(z_0+\delta z)-f(z_0)}{\delta z} = \lim_{\delta x,\delta y\to 0} \frac{u(x_0+\delta x, y_0+\delta y) - u(x_0,y_0) + i[v(x_0+\delta x, y_0+\delta y) - v(x_0,y_0)]}{\delta x + i\delta y}$$

$$\xrightarrow{\delta y = 0} f'(z) = \lim_{\delta x\to 0} \frac{u(x_0+\delta x, y_0) - u(x_0,y_0) + i[v(x_0+\delta x, y_0) - v(x_0,y_0)]}{\delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\xrightarrow{\delta x = 0} f'(z) = \lim_{\delta y\to 0} \frac{u(x_0, y_0+\delta y) - u(x_0,y_0) + i[v(x_0, y_0+\delta y) - v(x_0,y_0)]}{i\delta y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$\longrightarrow$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

polar form $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

plotting $n_u \cdot n_v = \begin{pmatrix} \partial u/\partial x \\ \partial u/\partial y \end{pmatrix} \cdot \begin{pmatrix} \partial v/\partial x \\ \partial v/\partial y \end{pmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0 \longrightarrow$ curves u and v intersect at $90°$

Laplace's equation $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

Multivalued functions

$\theta = \arg z$ is multivalued

principle value $\mathrm{Arg}z$ which satisfied $-\pi < \theta \le \pi$ $\longrightarrow$ branch cut along negative real axis

$\ln z = \ln re^{i\theta} = \ln r + i\theta = \ln|z| + i\arg z$ $\longrightarrow$ branch point at $z = 0$ $\longrightarrow$ $\mathrm{Ln}z = \ln|z| + i\mathrm{Arg}z$

$z^{\frac{1}{n}} = r^{\frac{1}{n}}[\cos(\frac{\theta + 2k\pi}{n}) + i\sin(\frac{\theta + 2k\pi}{n})],\ k \in \mathbb{Z}$ $\longrightarrow$ repeat with period of n $\longrightarrow$ has n branches

$z^a = e^{\ln z^a} = e^{\ln r^a}e^{i(a\theta)} = r^a[\cos(a\theta + 2ka\pi) + i\sin(a\theta + 2ka\pi)],\ k \in \mathbb{Z}$

Complex Line Integration

The complex line integral

$\int_c f(z)dz = \int_c f[g(t)]g'(t)dt$ for $C(t, g(t); a, b)$ if $g(t)$ is differentiable function of t

integrate z^n with respect to z, anticlockwise around unit circle

$\longrightarrow \oint_c f(z)dz = \int_0^{2\pi} (e^{it})^n(ie^{it})dt = \int_0^{2\pi} ie^{i(n+1)t}dt$

$\xrightarrow{n=-1} \oint_c f(z)dz = \int_0^{2\pi} i\,dt = 2\pi i$

$\xrightarrow{n\neq-1} \oint_c f(z)dz = [\frac{1}{n+1}e^{i(n+1)t}]_0^{2\pi} = 0$

sectionally smooth curve $\int_c f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz$

change in direction $\int_c f(z)dz = -\int_{-c} f(z)dz$

fundamental theorem of the calculus $\int_c f'(z)dz = f[g(b)] - f[g(a)] = f(z_2) - f(z_1)$

linearity $\int_c [k_1 f_1(z) + k_2 f_2(z)]dz = k_1\int_c f_1(z)dz + k_2\int_c f_2(z)dz$

upper bound $|\int_c f(z)dz| \le \int_c |f(z)||dz|$ $\longrightarrow$ $|\int_c f(z)dz| \le ML$ for $|f(z)| \le M$, $L = |\int_c dz|$

Cauchy's Integral Theorem

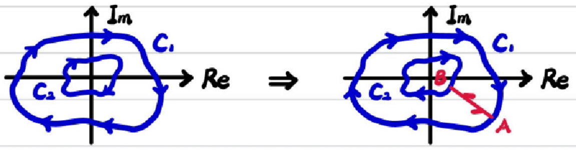
suppose $f(z)$ is analytic both on a simple contour C and in its interior. $\int_c f(z)dz = 0$

path-independence $C = C_1 - C_2$, since $\int_c f(z)dz = 0$ $\longrightarrow$ $\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$

The Deformation Theorem

suppose $f(z)$ is analytic both on two contours C_1, C_2 and in the region between them

$\longrightarrow$ create a sectionally smooth curve $\Gamma = C_1 + AB - C_2 + BA$

$\longrightarrow$ 

$\int_\Gamma f(z)\,dz = 0$

$\longrightarrow \int_{C_1} f(z)\,dz + \int_{AB} f(z)\,dz - \int_{C_2} f(z)\,dz + \int_{BA} f(z)\,dz = 0$

$\longrightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$

$\xrightarrow{\text{extended result}} \oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz + \dots$

Cauchy's Integral Formula

suppose $f(z)$ is analytic both on a simple contour C and in its interior

$\longrightarrow$ at any point z_0 in the interior of C, $f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz$

$\longrightarrow$ at any point z_0 in the interior of C, $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz$

Series and Singularites

Convergence

a series is convergent to $f(z)$ on some set $D \in \mathbb{C}$ if for all $z \in D$, $\lim_{n\to\infty} S_n(z) = f(z)$

pointwise convergent for any $z \in D$ and $\varepsilon > 0$, there exist some $N \in \mathbb{N}$ which may depend on z and ε

$\longrightarrow |S_n(z) - f(z)| < \varepsilon$ for all $n > N$

uniformly convergent for any $\varepsilon > 0$, there exist some $N \in \mathbb{N}$ that depend only on ε

$\longrightarrow |S_n(z) - f(z)| < \varepsilon$ for all $n > N$

$\longrightarrow$ for any sectionally smooth path $C \in D$, $\int_C f(z)\,dz = \sum_{j=1}^{\infty}\int_C f_j(z)\,dz$

$\longrightarrow$ for any $z \in D$, $f'(z) = \sum_{j=1}^{\infty} f_j'(z)$

Power series

$\sum_{n=0}^{\infty} C_n(z - z_0)^n = C_0 + C_1(z - z_0) + C_2(z - z_0)^2 + \dots$

ratio test the series convergence provided $\lim_{n\to\infty}\left|\frac{C_n(z-z_0)^n}{C_{n+1}(z-z_0)^{n+1}}\right| < 1$

$\longrightarrow \lim_{n\to\infty}|z-z_0|\left|\frac{C_n}{C_{n+1}}\right| < 1$

$\longrightarrow |z-z_0| < R$ where radius of convergence $R = \lim_{n\to\infty}\left|\frac{C_n}{C_{n+1}}\right|$

$\longrightarrow$ for any $R_1 < R$, the series uniformly convergent on the region $|z-z_0| \le R_1$

Taylor series

if $f(z)$ is analytic everywhere in $|z-z_0| \le R$

$\longrightarrow f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \dots$

Laurent series

if $f(z)$ is analytic on the annulus $R_1 < |z-z_0| < R_2$

$\longrightarrow f(z) = \sum_{n=0}^{\infty} C_n(z-z_0)^n = C_0 + C_1(z-z_0) + C_2(z-z_0)^2 + \dots$

$\longrightarrow$ for any simple closed contour of the annulus C, that winds exactly once round z_0

$\longrightarrow C_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$

Singularities of complex functions

removable singularity Laurent series contains no negative power

pole of order m Laurent series contains finite negative power with $(-m)$th numerically greatest

$\longrightarrow$ if and only if $\lim_{z\to z_0}(z-z_0)^m f(z) = D$, which is finite and non-zero

essential singularity Laurent series contains infinite many negative power

The Residual Theorem

Cauchy's Residue Theorem

if $f(z)$ is analytic on the simple closed contour C, in whose interior it has isolated singularities at $z_1, z_2, \dots, z_N$

$\longrightarrow \oint_C f(z)\,dz = 2\pi i \sum_{n=1}^{N} \operatorname*{Res}_{z=z_n} f(z)$

isolated simple pole $\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0}\{(z-z_0)f(z)\}$

isolated pole of order $m \geq 2$ $\quad \underset{z=z_0}{\operatorname{Res}} f(z) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-z_0)^m f(z)\} \Big|_{z=z_0}$

$f(z) = \frac{p(z)}{q(z)}$ with p, q analytic at z_0

- $\xrightarrow{q(z_0)=0,\ q'(z_0)\neq 0}$ $\underset{z=z_0}{\operatorname{Res}} f(z) = \frac{p(z_0)}{q'(z_0)}$
- $\xrightarrow{q(z_0)=q'(z_0)=0,\ q''(z_0)\neq 0}$ $\underset{z=z_0}{\operatorname{Res}} f(z) = \frac{2p}{q''}\left(\frac{p'}{p} - \frac{q'''}{3q''}\right)\Big|_{z=z_0}$

Complex potential

if the flow is irrotational $\longrightarrow$ velocity potential $\phi(x,y)$ $\longrightarrow$ $u = \frac{d\phi}{dx}$ and $v = \frac{d\phi}{dy}$

if the flow is incompressible $\longrightarrow$ streamline function $\psi(x,y)$ $\longrightarrow$ $u = \frac{d\psi}{dy}$ and $v = -\frac{d\psi}{dx}$

if the flow is both irrotational and incompressible

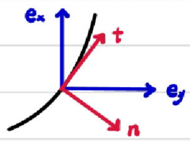
$\longrightarrow$ complex potential $\Omega(x,y) = \phi(x,y) + i\psi(x,y)$

$\longrightarrow$ complex velocity $\frac{d\Omega}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = u - iv$ $\longrightarrow$ true velocity $\overline{\frac{d\Omega}{dz}} = u + iv$

Flow past an obstacle

complex velocity $V(z) = \Omega'(z) = \overline{U_\infty} + \frac{C_1}{z} + \frac{C_2}{z^2} + \dots$

unit vectors



$\vec{t} = \frac{dx}{ds}\vec{e_x} + \frac{dy}{ds}\vec{e_y}$

$\vec{n} = \frac{dy}{ds}\vec{e_x} - \frac{dx}{ds}\vec{e_y}$

flux $F_\gamma = \oint_\gamma \vec{v}\cdot\vec{n}\, ds = \oint_\gamma (u\,dy - v\,dx)$

$\xrightarrow{\text{Green's Theorem}} \iint \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx\,dy = \iint \nabla\cdot\vec{v}\, dx\,dy = 0$ for incompressible flow

circulation $C_\gamma = \oint_\gamma \vec{v}\cdot\vec{t}\, ds = \oint_\gamma (u\,dx + v\,dy)$

$\longrightarrow \oint_\gamma V(z)\,dz = C_\gamma + iF_\gamma = C_\gamma$

The Blasius Laws

Bernoulli equation $P = p_0 + \frac{1}{2}\rho(u^2+v^2) = p_0 + \frac{1}{2}\rho\left|\frac{d\Omega}{dz}\right|^2$

complex force $F = F_x - iF_y = -\int P\, ds\,\vec{n} = -\oint_\gamma P\,dy + i\oint_\gamma P\,dx = \oint_\gamma P(-dy + i\,dx) = -i\oint_\gamma P\,d\bar{z}$

$\longrightarrow -i\oint_\gamma \left(p_0 + \frac{1}{2}\rho\left|\frac{d\Omega}{dz}\right|^2\right)d\bar{z} = -\frac{1}{2}\rho i\oint_\gamma \frac{d\Omega}{dz}\overline{\frac{d\Omega}{dz}}\,d\bar{z} = -\frac{1}{2}\rho i\oint_\gamma \frac{d\Omega}{dz}\,\overline{d\Omega}$

$\xrightarrow{\text{along } \gamma,\ d\psi = 0} -\frac{1}{2}\rho i\oint_\gamma \frac{d\Omega}{dz}\,d\Omega = -\frac{1}{2}\rho i\oint_\gamma \frac{d\Omega}{dz}\frac{d\Omega}{dz}\,dz$

$\longrightarrow$ Blasius law for hydrodynamic force $-\frac{1}{2}\rho i\oint_\gamma \left(\frac{d\Omega}{dz}\right)^2 dz$

turning moment $M = \oint_\gamma P(x\,dx + y\,dy) = \mathrm{Re}\{-\frac{1}{2}\rho\oint_\gamma |\frac{d\Omega}{dz}|^2 z\,d\bar{z}\} = \mathrm{Re}\{-\frac{1}{2}\rho\oint_\gamma (\frac{d\Omega}{dz})^2 z\,dz\}$

Laplace Transforms

Laplace transforms

$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \longrightarrow$ care only about $t \geq 0$ and $\mathrm{Re}(s) > a$ (ROC)

linear $L\{af(t) + bg(t)\} = aF(s) + bG(s)$

derivatives $L\{f^{(n)}(t)\} = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$

derivatived $L\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$

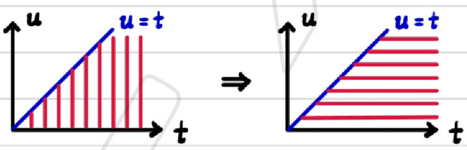
s shift theorem $L\{e^{at} f(t)\} = F(s-a)$

t shift theorem $L\{f(t-a)H(t-a)\} = e^{-as}F(s)$

scale theorem $L\{f(at)\} = \frac{1}{a}F(\frac{s}{a})$

convolution theorem $L\{\int_0^t f(u)g(t-u)\,du\}$

$\longrightarrow \int_0^\infty e^{-st}\int_0^t f(u)g(t-u)\,du\,dt$

$\longrightarrow$ 

$\longrightarrow \int_0^\infty e^{-st}\int_u^\infty f(u)g(t-u)\,dt\,du$

$\longrightarrow \int_0^\infty e^{-su}f(u)\int_u^\infty e^{-s(t-u)}g(t-u)\,dt\,du$

$\longrightarrow \int_0^\infty e^{-su}f(u)\int_0^\infty e^{-s(t-u)}g(t-u)\,d(t-u)\,du$

$\longrightarrow F(s)G(s)$

Inverting the Laplace transforms

F is defined for complex s and have the property of tending uniformly to 0 as $s \to \infty$ provided $\mathrm{Re}(s) > 0$

$\longrightarrow$ exists some vertical line $\mathrm{Re}(s) = \gamma$ lies to the right of all sigularities of F

$\longrightarrow f(t) = \frac{1}{2\pi i}\oint_C \frac{F(z)}{z-s}\,dz = \lim_{R\to\infty}\left(\frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR}\frac{F(z)}{z-s}\,dz + \frac{1}{2\pi i}\int_{C_L}\frac{F(z)}{z-s}\,dz\right)$

where $\lim_{R\to\infty}\left|\int_{C_L}\frac{F(z)}{z-s}dz\right| \le \lim_{R\to\infty}\int_{C_L}\frac{|F(z)|}{|(z-\gamma)-(s-\gamma)|}|dz| \le \lim_{R\to\infty}\left(\frac{M}{R-|s|-|\gamma|}\pi R\right) \xrightarrow{M\to 0} 0$

$\longrightarrow f(t) = \frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR} F(z)\,L^{-1}\left(\frac{1}{z-s}\right)dz \xrightarrow{z=s} f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}\,ds$

Bromwich contour along C_L, $|F(s)| \le \frac{M}{|s|^k}$ for some constant $k>0$

$$\longrightarrow \lim_{R\to\infty}\left(\frac{1}{2\pi i}\int_{C_L}\frac{F(z)}{z-s}dz\right) \longrightarrow 0$$

$$\longrightarrow f(t) = \sum_{n=1}^{N}\operatorname*{Res}_{z=z_n} f(z)$$

Conformal Mapping

Laplace's equation

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

polar form $x = r\cos\theta$ and $y = r\sin\theta$

$$\frac{\partial\phi}{\partial r} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial\phi}{\partial x}\cos\theta + \frac{\partial\phi}{\partial y}\sin\theta$$

$$\frac{\partial\phi}{\partial\theta} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial\theta} = -\frac{\partial\phi}{\partial x}r\sin\theta + \frac{\partial\phi}{\partial y}r\cos\theta$$

$$\frac{\partial^2\phi}{\partial r^2} = \frac{\partial}{\partial r}\frac{\partial\phi}{\partial r} = \left(\frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r}\right)\times\left(\frac{\partial\phi}{\partial x}\cos\theta + \frac{\partial\phi}{\partial y}\sin\theta\right)$$

$$\frac{\partial^2\phi}{\partial\theta^2} = \frac{\partial}{\partial\theta}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\phi}{\partial x}r\cos\theta - \frac{\partial\phi}{\partial y}r\sin\theta + \left(\frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial\theta}\right)\times\left(-\frac{\partial\phi}{\partial x}r\sin\theta + \frac{\partial\phi}{\partial y}r\cos\theta\right)$$

$$\frac{\partial^2\phi}{\partial r^2} = \frac{\partial^2\phi}{\partial x^2}\cos^2\theta + 2\frac{\partial^2\phi}{\partial x\partial y}\cos\theta\sin\theta + \frac{\partial^2\phi}{\partial y^2}\sin^2\theta$$

$$\frac{\partial^2\phi}{\partial\theta^2} = -\frac{\partial\phi}{\partial x}r\cos\theta - \frac{\partial\phi}{\partial y}r\sin\theta + \frac{\partial^2\phi}{\partial x^2}r^2\sin^2\theta - 2\frac{\partial^2\phi}{\partial x\partial y}r^2\cos\theta\sin\theta + \frac{\partial^2\phi}{\partial y^2}r^2\cos^2\theta$$

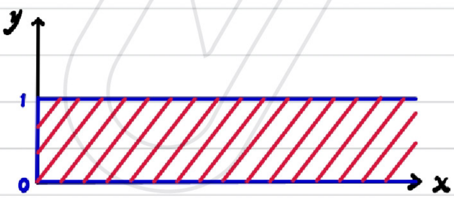
$$\longrightarrow \nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

The Dirichlet problem

unique function ϕ satisfy the Laplace's equation on the interior of region, given boundary conditions

the Method of seperation of variables

ϕ satisfies boundary conditions $\phi(x,0) = \phi(x,1) = 0$, $x>0$ and $\phi(0,y) = \sin(\pi y)$, $0<y<1$



$\longrightarrow$ assume $\phi(x,y) = F(x)G(y)$

$\longrightarrow$ $\nabla^2\phi = F''G + FG'' = 0 \longrightarrow$ let $\frac{F''}{F} = -\frac{G''}{G} = k^2$

$\longrightarrow$ let $F(x) = Ae^{kx} + Be^{-kx}$ and $G(y) = C\cos(ky) + D\sin(ky)$

$\longrightarrow$ for $\phi(x,0) = 0$, $C = 0$

$\longrightarrow$ for $\phi(x,1) = 0$, $D\sin(k) = 0 \longrightarrow k = n\pi$, $n = 1, 2, 3, \dots$

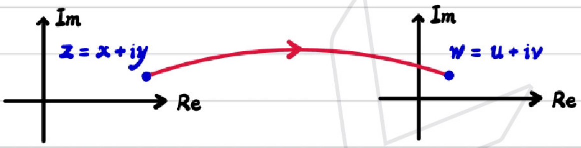
$\longrightarrow$ for boundedness of ϕ when $x \to \infty$, $A = 0$

$\xrightarrow{D = 1 \text{ without loss of generality}}$ $\phi = \sum_{n=1}^{\infty} B_n e^{-n\pi x}\sin(n\pi y)$

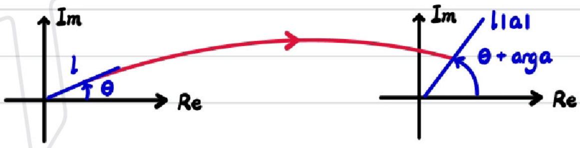
$\longrightarrow$ for $\phi(0,y) = \sin(\pi y)$, $B_1 = 1$ and $B_n|_{n\neq 1} = 0 \longrightarrow \phi = e^{-\pi x}\sin(\pi y)$

Complex mappings

locally one-to-one



$w = az$ $\quad Re^{i\phi} = |a|e^{i\arg a}\, re^{i\theta} = |a|re^{i(\theta + \arg a)}$

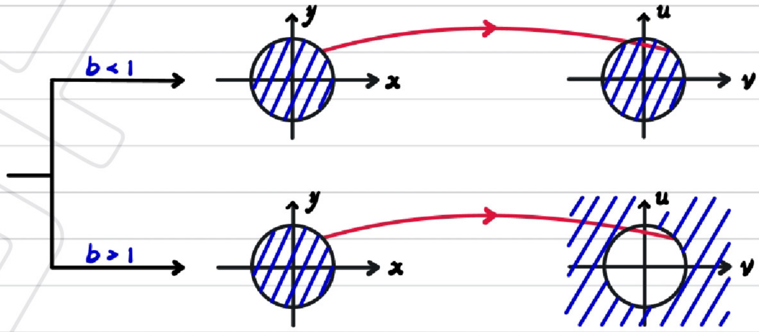


$w = \frac{az - b}{cz + d}$ $\quad u + iv = \frac{(ax-b) + iay}{(cx+d) + icy} = \frac{(ax-b)+iay}{(cx+d)+icy}\,\frac{(cx+d)-icy}{(cx+d)-icy} = \frac{acx^2 + (ad-bc)x - bd + acy^2}{(cx+d)^2 + c^2y^2} + i\,\frac{(ad+bc)y}{(cx+d)^2+c^2y^2}$

$w = \frac{z - b}{bz - 1}$, $|w| < 1$

$u + iv = \frac{(x-b) + iy}{(bx-1) + iby} \longrightarrow |w|^2 = w\bar{w} = \frac{(x-b)^2 + y^2}{(bx-1)^2 + by^2} < 1$

$\longrightarrow (x-b)^2 + y^2 < (bx-1)^2 + b^2y^2 \longrightarrow (1-b^2)x^2 + (1-b^2)y^2 \leq 1 - b^2$



Conformal mappings

a mapping $w = f(z)$ is said to be conformal if it preserves magnitude and sense of the angle

$\longrightarrow$ if $f(z)$ is analytic with $f'(z) \neq 0$ in a region D, $w = f(z)$ is locally one-to-one and conformal there

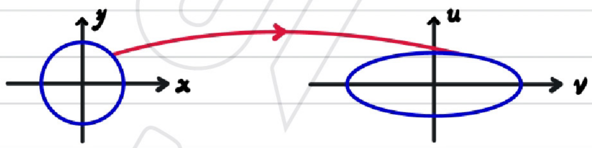
harmonic $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \Rightarrow \frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2} = 0$

The Joukowski transformation

$w = z + \frac{c^2}{z} \longrightarrow u + iv = re^{i\theta} + \frac{c^2}{r}e^{-i\theta} = (r + \frac{c^2}{r})\cos\theta + i(r - \frac{c^2}{r})\sin\theta$

$\xrightarrow{r=a} (a + \frac{c^2}{a})\cos\theta + i(a - \frac{c^2}{a})\sin\theta$

$\longrightarrow \frac{u^2}{(a + c^2/a)} + \frac{v^2}{(a + c^2/a)} = 1$

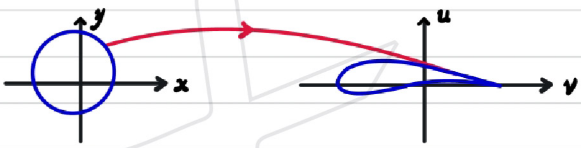


$z = -d + (a+d)e^{i\theta}$

$\longrightarrow u \approx 2a\cos\theta - d(1 - \cos2\theta)$

$\longrightarrow v \approx 2a\sin\theta - d\sin2\theta$

$z = -d + ic + (a + \sqrt{d^2 + c^2})e^{i\theta}$



complex potential assume far field velocity is 1, $\Omega = ze^{-i\alpha} + \frac{a^2}{z}e^{i\alpha}$

complex velocity $V = \frac{d\Omega}{dz} = e^{-i\alpha} - \frac{a^2}{z^2}e^{i\alpha}$

The Kutta-Joukowski lifting force

$F = -\frac{1}{2}\rho i \oint_\gamma (\frac{d\Omega}{dw})^2 dw$

$\longrightarrow \oint_\gamma (\frac{d\Omega}{dw})^2 dw = \oint_\gamma (\frac{d\Omega}{dz}\frac{dz}{dw})^2 \frac{dw}{dz} dz = \oint_\gamma (\frac{d\Omega}{dz})^2 (\frac{dw}{dz})^{-1} dz$

$\longrightarrow \oint_\gamma (e^{-i\alpha} - \frac{a^2}{z^2}e^{i\alpha})^2 (1 - \frac{c^2}{z^2})^{-1} dz$

$\longrightarrow \oint_\gamma \frac{z^2}{z^2 - c^2}(e^{-2i\alpha} - 2\frac{a^2}{z^2} + \frac{a^4}{z^4}e^{2i\alpha}) dz$

$\longrightarrow \operatorname{Res}_{z=c} f(z) = \lim_{z\to c} \frac{z^2}{z+c}(e^{-2i\alpha} - 2\frac{a^2}{z^2} + \frac{a^4}{z^4}e^{2i\alpha}) = \frac{1}{2}(e^{-2i\alpha} - 2\frac{a^2}{c^2} + \frac{a^4}{c^4}e^{2i\alpha})$

$\longrightarrow \operatorname{Res}_{z=-c} f(z) = \lim_{z\to -c} \frac{z^2}{z-c}(e^{-2i\alpha} - 2\frac{a^2}{z^2} + \frac{a^4}{z^4}e^{2i\alpha}) = -\frac{1}{2}(e^{-2i\alpha} - 2\frac{a^2}{c^2} + \frac{a^4}{c^4}e^{2i\alpha})$

$\longrightarrow$ at $z = 0$, $-\frac{z^2}{c^2}(1 + \frac{z^2}{c^2} + \frac{z^4}{c^4} + \ldots)(e^{-2i\alpha} - 2\frac{a^2}{z^2} + \frac{a^4}{z^4}e^{2i\alpha}) \longrightarrow \operatorname{Res}_{z=0} f(z) = c_{-1} = 0$

$\longrightarrow \oint_\gamma (\frac{d\Omega}{dw})^2 dw = 2\pi i[\operatorname{Res}_{z=1} f(z) + \operatorname{Res}_{z=-1} f(z) + \operatorname{Res}_{z=0} f(z)] = 0 \longrightarrow F = 0$

The Kutta-Joukowski turning moment

$$M = -\frac{1}{2}\rho \, \mathrm{Re}\left\{ \oint_\gamma w \left(\frac{d\Omega}{dw}\right)^2 dw \right\}$$

$$\longrightarrow \oint_\gamma w\left(\frac{d\Omega}{dw}\right)^2 dw = \oint_\gamma w\left(\frac{d\Omega}{dz}\frac{dz}{dw}\right)^2 \frac{dw}{dz} dz = \oint_\gamma w\left(\frac{d\Omega}{dz}\right)^2 \left(\frac{dw}{dz}\right)^{-1} dz$$

$$\longrightarrow \oint_\gamma \left(z + \frac{c^2}{z}\right)\left(e^{-i\alpha} - \frac{a^2}{z^2} e^{i\alpha}\right)^2 \left(1 - \frac{c^2}{z^2}\right)^{-1} dz$$

$$\longrightarrow \oint_\gamma \frac{z^3 + zc^2}{z^2 - c^2}\left(e^{-2i\alpha} - 2\frac{a^2}{z^2} + \frac{a^4}{z^4} e^{2i\alpha}\right) dz$$

$$\longrightarrow \operatorname*{Res}_{z=c} f(z) = \lim_{z \to c} \frac{z^3 + zc^2}{z + c}\left(e^{-2i\alpha} - 2\frac{a^2}{z^2} + \frac{a^4}{z^4}e^{2i\alpha}\right) = c^2\left(e^{-2i\alpha} - 2\frac{a^2}{c^2} + \frac{a^4}{c^4}e^{2i\alpha}\right)$$

$$\longrightarrow \operatorname*{Res}_{z=-c} f(z) = \lim_{z \to -c} \frac{z^3 + zc^2}{z - c}\left(e^{-2i\alpha} - 2\frac{a^2}{z^2} + \frac{a^4}{z^4}e^{2i\alpha}\right) = c^2\left(e^{-2i\alpha} - 2\frac{a^2}{c^2} + \frac{a^4}{c^4}e^{2i\alpha}\right)$$

$\longrightarrow$ at $z = 0$. $-\frac{z^3 + zc^2}{c^2}\left(1 + \frac{z^2}{c^2} + \frac{z^4}{c^4} + \dots\right)\left(e^{-2i\alpha} - 2\frac{a^2}{z^2} + \frac{a^4}{z^4}e^{2i\alpha}\right) \longrightarrow \operatorname*{Res}_{z=0} f(z) = c_{-1} = 2a^2 - 2\frac{a^4}{c^2}e^{2i\alpha}$

$$\longrightarrow \oint_\gamma \left(\frac{d\Omega}{dw}\right)^2 dw = 2\pi i \left[\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=-1} f(z) + \operatorname*{Res}_{z=0} f(z)\right] = 2\pi i\left(-2a^2 + 2c^2 e^{-2i\alpha}\right)$$

$$\longrightarrow \mathrm{Re}\left\{\oint_\gamma w\left(\frac{d\Omega}{dw}\right)^2 dw\right\} = 4\pi c^2 \sin(2\alpha)$$

$$\longrightarrow M = -2\rho\pi c^2 \sin(2\alpha)$$