The Laplace Transformation

Introdution

Time domain apporach

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n \\ y(t) = Cx(t) \end{cases}$$

$\longrightarrow$ let $x = x_h + x_p = e^{At}x_0 + x_p \longrightarrow \dot{x} = Ae^{At}x_0 + \dot{x}_p = A(e^{At}x_0 + x_p) + Bu$

$\longrightarrow \dot{x}_p = Ax_p + Bu \longrightarrow$ let $x_p = \int_0^t e^{A(t-s)} Bu(s)\, ds$

$$\xrightarrow{\frac{d}{dt}\int_a^b f(x,t)dx = \int_a^b \frac{\partial}{\partial t} f(x,t)dx + \frac{db}{dt}f(b,t) - \frac{da}{dt}f(a,t)} \dot{x}_p = \int_0^t Ae^{A(t-s)} Bu(s)\, ds + Bu(t) = Ax_p + Bu$$

$\longrightarrow y = Ce^{At}x_0 + \int_0^t Ce^{A(t-s)} Bu(s)\, ds$

Frequency domain apporach

$$u(t) \longrightarrow \boxed{\begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n \\ y(t) = Cx(t) \end{array}} \longrightarrow y(t)$$

Laplace transforms

$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \longrightarrow$ care only about $t \geq 0$ and $Re(s) > a$ (ROC)

linear $L\{af(t) + bg(t)\} = aF(s) + bG(s)$

derivatives $L\{f^{(n)}(t)\} = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0) = s^n F(s)$ for $f^k(0) = 0,\ k \in N$

derivatived $L\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$

s shift theorem $L\{e^{at} f(t)\} = F(s-a)$

t shift theorem $L\{f(t-a)H(t-a)\} = e^{-as}F(s)$

scale theorem $L\{f(at)\} = \frac{1}{a}F(\frac{s}{a})$

convolution theorem $L\{\int_0^t f(u)g(t-u)du\} = F(s)G(s)$

Transfer Function

The transfer function

second-order system $\ddot{y}(t) + 2\xi\omega_n\dot{y}(t) + \omega_n^2 y(t) = u(t),\ y(0) = \dot{y}(0) = 0$

$$\xrightarrow{\mathcal{L}} s^2 Y(s) + 2\xi\omega_n s Y(s) + \omega_n^2 Y(s) = U(s),\ s \in \mathbb{C}$$

$$\longrightarrow G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

general control system $\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t) + Du(t),\ x(0) = x_0$

$$\xrightarrow{\mathcal{L}} sX(s) = AX(s) + BU(s) \longrightarrow X(s) = (sI - A)^{-1} BU(s)$$

$$\longrightarrow G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1} B + D$$

$$\longrightarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The frequency response

$$u = re^{i\theta} \Rightarrow y = \mathcal{L}^{-1}\{G\} u = gre^{i\theta} = Ae^{i\phi} re^{i\theta} = Are^{i(\theta+\phi)}$$

$$\xrightarrow{Re} u = \cos(\omega t) \Rightarrow y = A\cos(\omega t + \phi)$$

frequency response $\omega \longmapsto G(i\omega)$

gain $A = |G(i\omega)|$

phase $\phi = \angle G(i\omega)$

Bode diagram

$L(s) = K(s)G(s)$ $\quad 20\log_{10}|L(i\omega)| = 20\log_{10}|K(i\omega)| + 20\log_{10}|G(i\omega)|$ and $\angle L(i\omega) = \angle K(i\omega) + \angle G(i\omega)$

Poles, Zeros and Performance

Poles and zeros

transfer function of a SISO system $G(s) = \dfrac{b(s)}{a(s)} = \dfrac{b_0 + b_1 s + \ldots + b_m s^m}{a_0 + a_1 s + \ldots + a_n s^n},\ m \leq n$

poles $a(s) = 0 \Leftrightarrow \mathrm{eig}(A)$ →
- $\mathrm{Re}(p) < 0$ → stable
- $\mathrm{Re}(p) = 0$ →
  - single → marginally stable
  - repeated → unstable
- $\mathrm{Re}(p) > 0$ → unstable

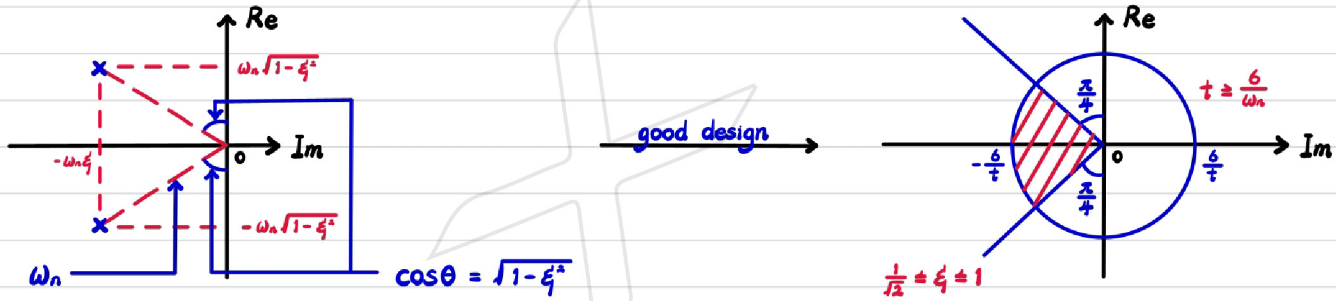
zeros $b(s) = 0$ → system unresposive to certain inputs

Poles of a second-order system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\longrightarrow s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$\xrightarrow{\text{lightly damped } (0<\xi<1)} s = \omega_n(-\xi \pm i\sqrt{1-\xi^2})$$

→ 

Routh-Hurwitz stability criterion

2nd order system $a(s) = s^2 + a_1 s + a_0$ is stable if $a_1, a_0 > 0$

3rd order system $a(s) = s^3 + a_2 s^2 + a_1 s + a_0$ is stable if $a_2 a_1 > a_0$ and $a_2, a_1, a_0 > 0$

4th order system

$a(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$ is stable if $a_3 a_2 a_1 > a_3^2 a_0 + a_1^2$, $a_3 a_2 > a_1$ and $a_3, a_2, a_1, a_0 > 0$

PID Control Using Transfer Function

The Final Value Theorem

$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$ only correct if system is stable

Step response

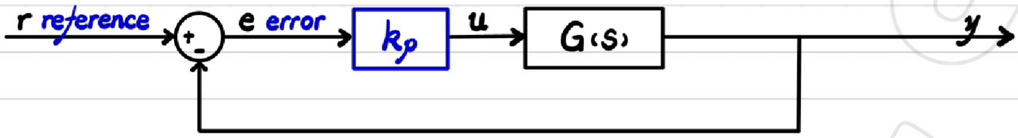
for a step input $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$\longrightarrow U(s) = \frac{1}{s}, \; Re(s) > 0$

$\longrightarrow \lim_{y \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sG(s)U(s) = \lim_{s \to 0} G(s) = \lim_{s \to 0} C(sI - A)^{-1}B + D$

$\longrightarrow \lim_{y \to \infty} y(t) = -CA^{-1}B + D$

Proportional control



$T_{ry}(s) = \frac{k_p G(s)}{1 + k_p G(s)}$

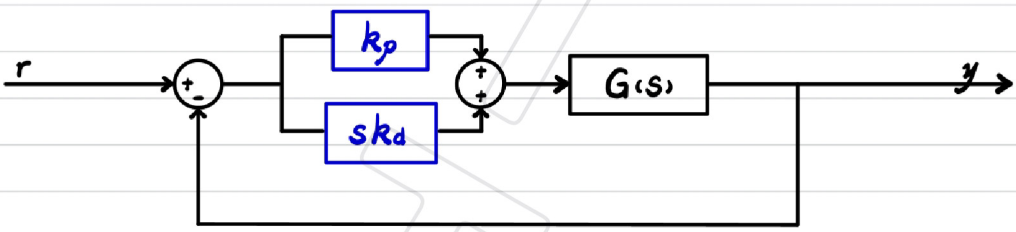
$G(s) = \frac{1}{(1+s)^2}$ $\quad T_{ry}(s) = \frac{k_p}{(1+s)^2 + k_p} = \frac{k_p}{s^2 + 2s + (1+k_p)} \longrightarrow \omega_n = \sqrt{k_p + 1}$ and $\xi = \frac{1}{\sqrt{k_p^2 + 1}}$

track step $\quad \lim_{t \to \infty} e(t) = \lim_{t \to \infty} [r(t) - y(t)] = \lim_{s \to 0} [1 - \frac{k_p}{(1+s)^2 + k_p}] = \frac{1}{1 + k_p} \neq 0$

$\longrightarrow$ can not eliminate steady-state error

$\longrightarrow$ increase k_p reduce steady-state error $\Leftrightarrow$ increase oscillation

Proportional-Derivative (PD) control



$T_{ry}(s) = \frac{(k_p + sk_d)G(s)}{1 + (k_p + sk_d)G(s)}$

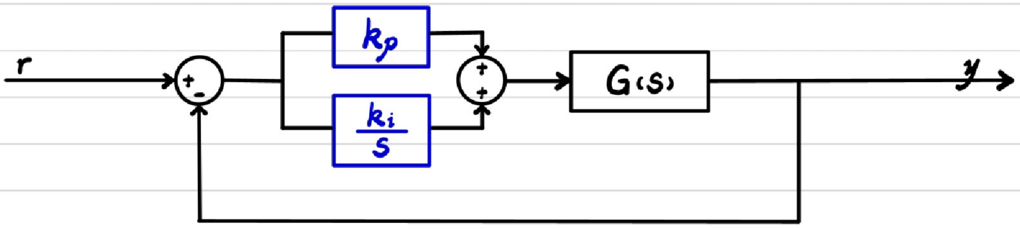
$G(s) = \frac{1}{(1+s)^2}$ $\quad T_{ry}(s) = \frac{k_p + sk_d}{(1+s)^2 + (k_p + sk_d)} = \frac{k_p + sk_d}{s^2 + (2+k_d)s + (1+k_p)} \longrightarrow \omega_n = \sqrt{k_p + 1}$ and $\xi = \frac{2 + k_d}{2\sqrt{k_p^2 + 1}}$

track step $\quad \lim_{t \to \infty} e(t) = \lim_{t \to \infty} [r(t) - y(t)] = \lim_{s \to 0} [1 - \frac{k_p + sk_d}{(1+s)^2 + (k_p + sk_d)}] = \frac{1}{1 + k_p} \neq 0$

$\longrightarrow$ can not eliminate steady-state error

$\longrightarrow$ increase k_d reduce oscillation $\Leftrightarrow$ more sensitive to noise

Proportional-Integral (PI) control



$$T_{ry}(s) = \frac{(k_p + k_i/s)G(s)}{1 + (k_p + k_i/s)G(s)} = \frac{(sk_p + k_i)G(s)}{s + (sk_p + k_i)G(s)}$$

track step $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} [r(t) - y(t)] = \lim_{s\to 0} [1 - \frac{(sk_p + k_i)G(s)}{s + (sk_p + k_i)G(s)}] = 1 - 1 = 0$

track ramp $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} [r(t) - y(t)] = \lim_{s\to 0} [\frac{1}{s} - \frac{(sk_p + k_i)G(s)}{s + (sk_p + k_i)G(s)}\frac{1}{s}] = \frac{1}{k_i G(0)} \neq 0$

⟶ might eliminate steady-state error depends on reference signal

Proportional-Integral-Derivative (PID) control

$K(s) = k_p + \frac{k_i}{s} + sk_d$

⟶ both damping and steady-state error can be controlled

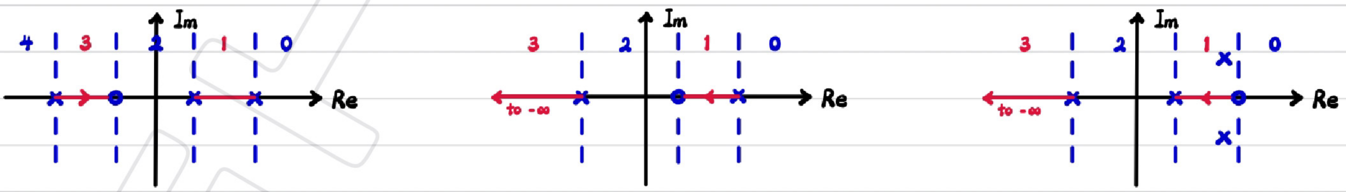
Root Locus Design

Root Locus Design

Plotting the root locus diagram

① for open-loop transfer function $G(s)$ ⟶
- P pole(s) : $p_1, p_2, \ldots, p_P$ ⟶ plot as x
- Z zero(s) : $z_1, z_2, \ldots, z_Z$ ⟶ plot as o

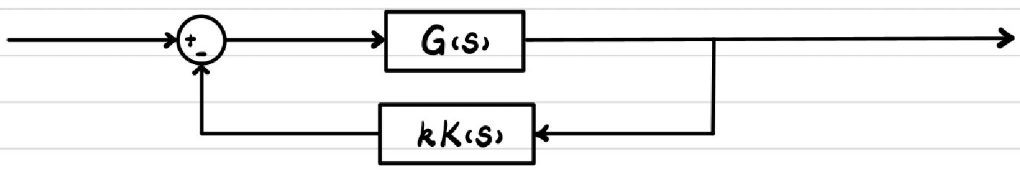
② real axis pole(s) converge to real axis zero(s) or conjugate poles or $-\infty$ at 'odd region(s)'



③ find asymptotes with asymptote centre $\frac{1}{P-Z}(\sum_{i=1}^{P} p_i - \sum_{j=1}^{Z} z_j)$ at evenly-spaced angles $\frac{(2n+1)\pi}{P-Z}$

④ find break point(s) at $G'(s) = 0$

Feedback controller



$$G_{cl}(s) = \frac{G(s)}{1-[-G(s)kK(s)]} = \frac{G(s)}{1+kG(s)K(s)} = \frac{b_G/a_G}{1+k(b_G/a_G)(b_K/a_K)} = \frac{a_G b_K}{a_G a_K + k b_G b_K}$$

$\longrightarrow$ closed-loop pole(s) at $a_{GK}(s) + kb_{GK}(s) = 0$ $\xrightarrow{k=0}$ open-loop pole(s) at $a_{GK}(s) = 0$

$\longrightarrow$ plotting the root locus for $G(s)K(s)$ to show the movement of closed-loop poles as $k>0$ increases

The Nyquist Criterion

The Nyquist Diagram

Plotting the Nyquist diagram

$\omega \longmapsto G(i\omega)$, $-\infty < \omega < \infty$ run from $-\infty$ up to ∞ on the complex plane

① $\overline{G(i\omega)} = \overline{\dfrac{b(i\omega)}{a(i\omega)}} = \dfrac{\overline{b(i\omega)}}{\overline{a(i\omega)}} = \dfrac{b(-i\omega)}{a(-i\omega)} = G(-i\omega) \longrightarrow$ symmetric about x-axis

② calculate asympotic behaviour as $\omega \to 0$ and $\omega \to \infty$

$\longrightarrow$ find $\angle G(\pm i\infty) = \mp \frac{2n}{\pi}$ at asymptotic values for nth order system

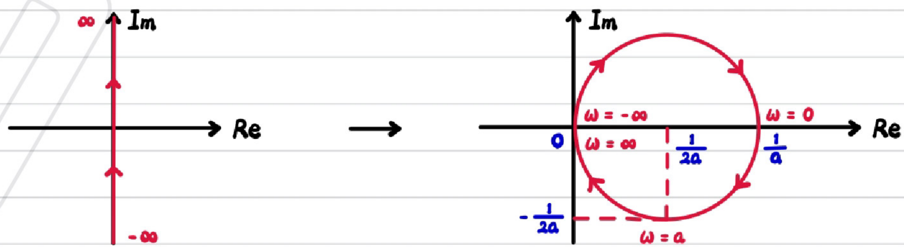
③ multiply transfer function by a factor k $\longrightarrow$ scaling the entire Nyquist plot by a factor k

$G(s) = -(2+s)$

$G(s) = \dfrac{1}{s+a}$ $\quad \lim_{\omega\to\infty} G(i\omega) = 0 \longrightarrow \lim_{\omega\to\infty} \angle G(i\omega) = -\frac{\pi}{2}$

$G(0) = \frac{1}{a} \longrightarrow \angle G(0) = 0$

$G(ia) = \frac{1}{2a} - i\frac{1}{2a} \longrightarrow \angle G(ia) = -\frac{\pi}{4}$



$G(s) = \frac{1}{s}$

avoid singularity at $\omega = 0 \longrightarrow s = \begin{cases} i\omega, & -\infty < \omega < -\varepsilon \\ \varepsilon e^{i\theta}, & -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ i\omega, & \varepsilon < \omega < \infty \end{cases}$

$$\lim_{\omega\to\infty} G(i\omega) = 0 \longrightarrow \lim_{\omega\to\infty} \angle G(i\omega) = \lim_{\omega\to\infty} \angle(1) - \angle(s) = 0 - \frac{\pi}{2} = -\frac{\pi}{2}$$

$$G(0) = \frac{1}{\varepsilon} \longrightarrow \angle G(0) = 0$$

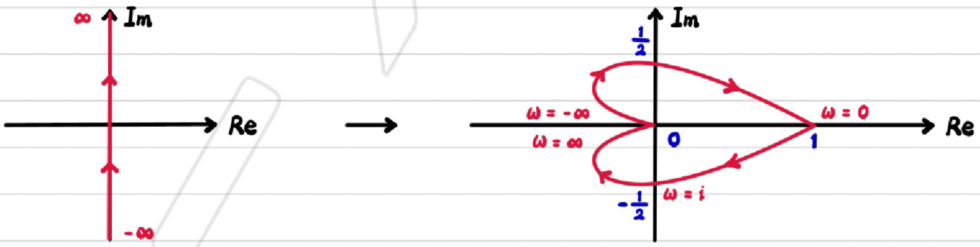
$$G(i\varepsilon) = -i\frac{1}{\varepsilon} \longrightarrow \angle G(i\varepsilon) = -\frac{\pi}{4}$$

$G(s) = \frac{1}{s^2+2s+1}$

$$\lim_{\omega\to\infty} G(i\omega) = 0 \longrightarrow \lim_{\omega\to\infty} \angle G(i\omega) = \lim_{\omega\to\infty} \angle(1) - 2\angle(s+1) = -\pi$$

$$G(0) = 1 \longrightarrow \angle G(0) = 0$$

$$G(i) = -\frac{1}{2}i \longrightarrow \angle G(i) = -\frac{\pi}{2}$$



$G(s) = \frac{1}{s(s+a)^2}$

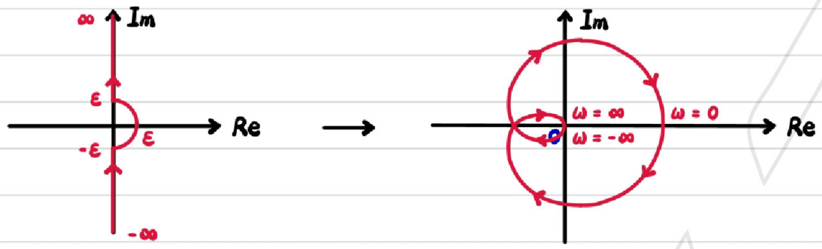
$$G(i\omega) = \frac{1}{i\omega(a^2-\omega^2+2ia\omega)} = \frac{1}{-2a\omega^2 + i(a^2\omega-\omega^3)} = -\frac{2a}{(a^2+\omega^2)^2} - i\frac{a^2-\omega^2}{\omega(a^2+\omega^2)^2}$$

$$\lim_{\omega\to\infty} G(i\omega) = 0 \longrightarrow \lim_{\omega\to\infty} \angle G(i\omega) = \lim_{\omega\to\infty} \angle(1) - [\angle(s) + 2\angle(s+a)] = -\frac{3}{2}\pi$$

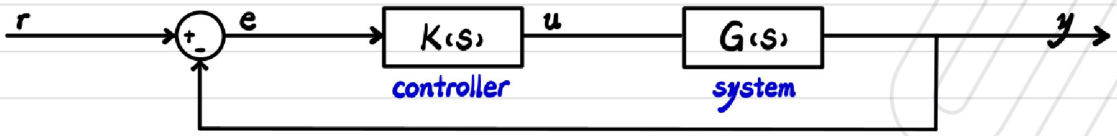
$$G(0) = -\frac{2}{a^3} - i\infty \longrightarrow \angle G(0) = -\frac{3}{2}\pi$$

$$G(ia) = -\frac{1}{2a^3} \longrightarrow \angle G(ia) = \pi$$

⇔ consider singularity at $\omega = 0$



Feedback stability



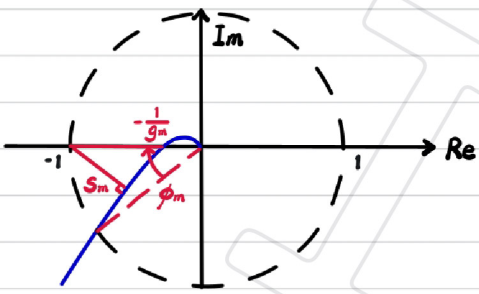
loop transfer function $L(s) = G(s)K(s) \longrightarrow T_{ry}(s) = \frac{L(s)}{1+L(s)} = \frac{a_L(s)}{a_L(s)+b_L(s)}$

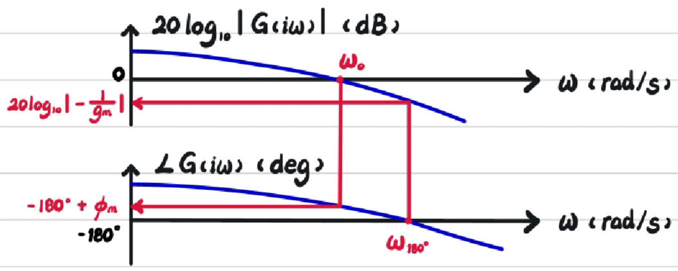
The Nyquist stability criterion

N_w number of times -1 is encircled in the anticlockwise direction by the Nyquist contour

- → for stable open-loop system, closed-loop system is stable if $N_w = 0$
- → for unstable open-loop system, closed-loop system is stable if N_w = number of unstable poles

Stability margins





gain margin g_m

Bode stable region if $\angle L(i\omega)$ cross 180° from above only once at $\omega_{180°}$, system is stable if $|L(i\omega_{180°})| < 1$

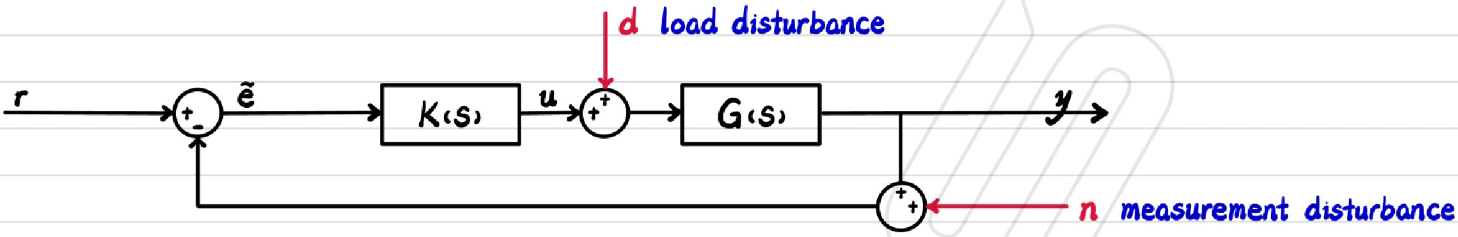
phase margin ϕ_m

stability margin s_m

$G(s)$ is an approximation of true system and $K(s)$ cannot be implemented directly $\longrightarrow$ $L(i\omega)$ in reality might be $gL(i\omega)e^{i\phi}$

Feedback System Performance and Design

Feedback system



$$\bar{e}(s) = \bar{r}(s) - Y(s)$$

$$= \bar{r}(s) - G(s)[U(s) + \bar{d}(s)]$$

$$= \bar{r}(s) - G(s)K(s)\tilde{e}(s) + G(s)\bar{d}(s)$$

$$= \bar{r}(s) - G(s)K(s)[\bar{e}(s) - \bar{n}(s)] + G(s)\bar{d}(s)$$

$$\longrightarrow \bar{e}(s) = \frac{1}{1+L(s)}\bar{r}(s) + \frac{G(s)}{1+L(s)}\bar{d}(s) - \frac{L(s)}{1+L(s)}\bar{n}(s)$$

sensitivity function $S(s) = \frac{1}{1+L(s)}$

load sensitivity $S(s)G(s) = \frac{G(s)}{1+L(s)}$

complementary sensitivity $T(s) = \frac{L(s)}{1+L(s)}$

$\longrightarrow$ $S(s) + T(s) = 1$ $\longrightarrow$ trade-off
- $\longrightarrow$ make $S(i\omega)$ small at low ω $\longrightarrow$ $|L(i\omega)| \gg 1$
- $\longrightarrow$ make $T(i\omega)$ small at high ω $\longrightarrow$ $|L(i\omega)| \ll 1$

Sensitivity function

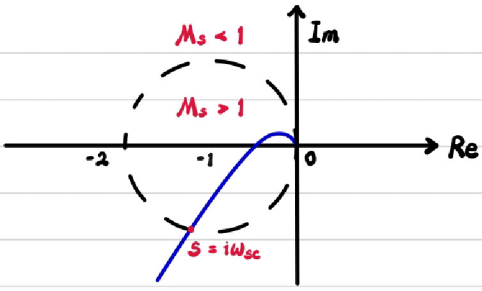
$$Y_{ol}(s) = L(s)\bar{r}(s) + G(s)\bar{d}(s)$$

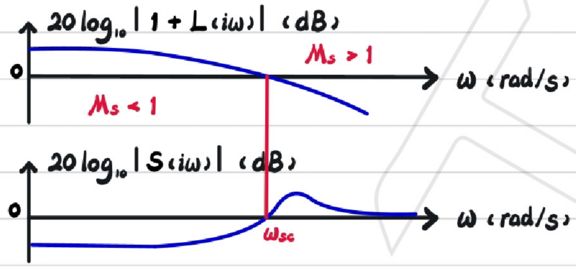
$$Y_{cl}(s) = \frac{L(s)\bar{r}(s) + G(s)\bar{d}(s)}{1+L(s)}$$

$$\longrightarrow \frac{Y_{cl}(s)}{Y_{ol}(s)} = S(s) = \frac{1}{1+L(s)}$$

maximum sensitivity $M_s = \max_{\omega} |S(i\omega)| = \frac{1}{|1+L(i\omega)|} = \frac{1}{S_m}$

sensitivity crossover frequency ω_{sc} $\longrightarrow$ $|S(i\omega_{sc})| = 1$





Bode's integral formula

supposed $L(s)$ has no poles on the RHP and tends to 0 faster than $\frac{1}{s}$ as $s \to \infty$

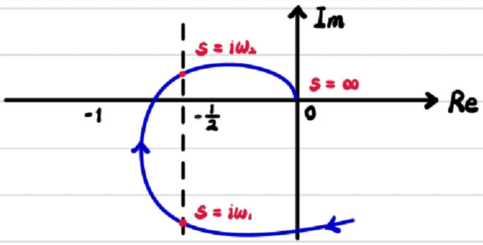
$\longrightarrow \int_0^\infty \ln|S(i\omega)|\, d\omega = 0$

$\longrightarrow$ Waterbed effect if sensitivity is pushed down for some ω, it must increase at some other ω

Complementary sensitivity

$$T(s) = \frac{L(s)}{1+L(s)}$$

at $|T(i\omega)| = \left|\frac{L(i\omega)}{1+L(i\omega)}\right| = 1 \longrightarrow |L(i\omega)| = |1+L(i\omega)| \longrightarrow Re\{L(i\omega)\} = -\frac{1}{2}$

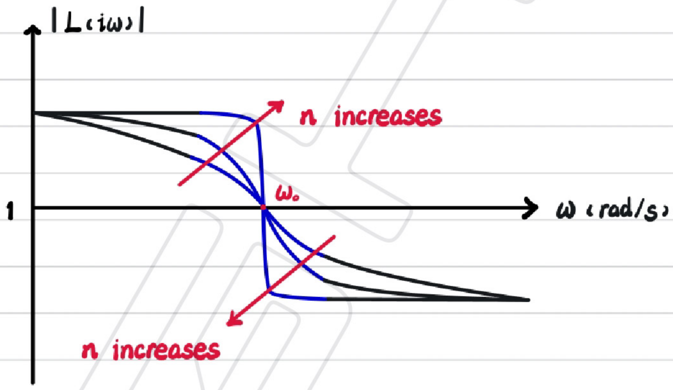


$$\begin{cases} |T(i\omega)| < 1, & -i\omega_1 < s < i\omega_1 \\ |T(i\omega)| > 1, & i\omega_1 < s < i\omega_2 \\ |T(i\omega)| < 1, & i\omega_2 < s < \infty \end{cases}$$

Robustness

$2 \le g_m \le 5$, $30^\circ \le \phi_m \le 60^\circ$ and $0.5 \le S_m \le 0.8$

Bode's Gain-Phase relationship $\angle L(i\omega) \approx 90^\circ \frac{d\ln|L(i\omega)|}{d\ln\omega}$



at cross-over region, $|L(i\omega)| = \frac{1}{\omega^n}$

$\angle L(i\omega) \approx 90^\circ \frac{d\ln\omega^{-n}}{d\ln\omega} = 90^\circ \frac{(-n)d\ln\omega}{d\ln\omega} = -90^\circ n$

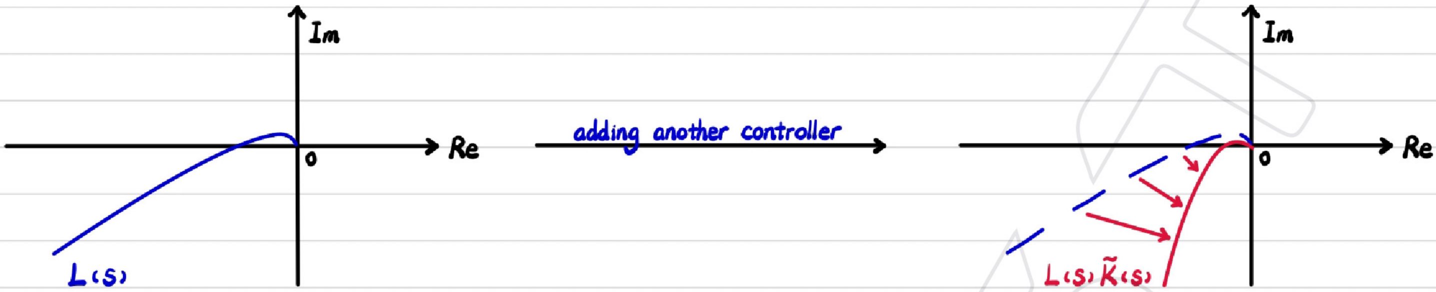
$\longrightarrow \angle L(i\omega_0) \approx -90^\circ n$

$\longrightarrow -180^\circ + \phi_m \approx -90^\circ n \longrightarrow \phi_m \approx 180^\circ - 90^\circ n$

$\longrightarrow n \le 2$

$\longrightarrow$ transition must happen gradually to ensure robustness

Loop shaping



increasing phase margin if $\omega_{new,o} = \omega_o$, $\phi_{new,m} = \phi_m + \angle\tilde{K}(i\omega_o)$

attenuating load disturbances at low frequencies

choose PI controller $\longrightarrow$ $\tilde{K}(s) = 1 + \frac{b}{s}$

$$\lim_{s \to 0} |\tilde{L}(s)| = \lim_{s \to 0} |L(s)| \lim_{s \to 0} |K(s)| = \infty \longrightarrow \lim_{s \to 0} |\tilde{S}(s)| = 0$$

$$\lim_{s \to \pm\infty} |\tilde{L}(s)| = \lim_{s \to \pm\infty} |L(s)| \lim_{s \to \pm\infty} |K(s)| = 0 \longrightarrow \lim_{s \to \pm\infty} |\tilde{S}(s)| = \infty$$

attenuating measurement disturbances at high frequencies

choose PD controller $\longrightarrow$ $\tilde{K}(s) = 1 + as$

$$\lim_{s \to \pm\infty} |\tilde{L}(s)| = \lim_{s \to \pm\infty} |L(s)| \lim_{s \to \pm\infty} |K(s)| = \infty \longrightarrow \lim_{s \to \pm\infty} |\tilde{S}(s)| = 0$$

$$\lim_{s \to 0} |\tilde{L}(s)| = \lim_{s \to 0} |L(s)| \lim_{s \to 0} |K(s)| = 0 \longrightarrow \lim_{s \to 0} |\tilde{S}(s)| = \infty$$

$\longrightarrow$ avoid infinite gain by use phase compensators $\frac{s+a}{s+b}$

- $\xrightarrow{a > b}$ PI controller
- $\xrightarrow{a < b}$ PD controller