2013 Question 2:

2. (a) State in words the conservation of mass for a two-dimensional control volume.

(b) In many engineering applications one is interested in the transport of a contaminant, such as a chemical pollutant, by the fluid flow. Let C be the concentration of the contaminant (i.e. mass per unit volume of fluid). Considering a two-dimensional rectangular control volume and assuming no contaminant is produced within the control volume and the contaminant is sufficiently dilute to leave the fluid flow unchanged, show that the transport equation for the contaminant is given by

$$\frac{\partial C}{\partial t} + \frac{\partial Cu}{\partial x} + \frac{\partial Cv}{\partial y} = 0.$$

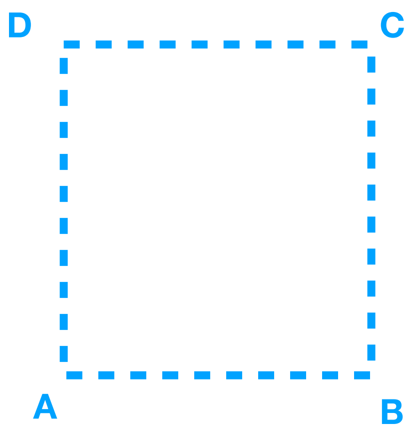
2(a) From lecture notes in Section 2-1

(a) the rate of change of mass in control volume plus the net flux mass out of the control volume equal to zero.

2(b)

Can apply exactly the same derivation we have used for mass conservation to the conservation of the contaminant. Start by completing a sketch of the control volume:

$C = \frac{m}{V} = \rho$



Rate of Change of Contaminant C:

$$\frac{\partial C}{\partial t} \Delta x \Delta y \Delta z$$

Flux of C though:

AD: $u_1 C \Delta y \Delta z$

BC: $u_2 C \Delta y \Delta z$

AB: $v_1 C \Delta x \Delta z$

DC: $v_2 C \Delta x \Delta z$

Conservation of C is therefore given by:

$$\frac{\partial C}{\partial t} \Delta x \Delta y \Delta z + (u_2 - u_1) C \Delta x \Delta y \Delta z + (v_2 - v_1) C \Delta x \Delta y \Delta z = 0$$

Dividing by Δx Δy Δx:

$$\frac{\partial C}{\partial t} + \frac{C \Delta u}{\Delta x} + \frac{C \Delta v}{\Delta y} = 0$$

Finally taking limit as CV size goes to zero we get:

$$\frac{\partial C}{\partial t} + \frac{\partial C u}{\partial x} + \frac{\partial C v}{\partial y} = 0$$

(c) Consider the control volume shown in figure 1 where b is the wing span. The trailing vortices from the wing induce the velocity behind the wing to exit the control volume at an angle ϵ producing a downwash velocity w. Use the vertical and horizontal momentum balance in the control volume to show that

$$C_{Di} = \frac{C_L^2 S}{\pi b^2}$$

$\frac{\partial}{\partial t}(\ ) = 0$

where S is the wing area, C_{Di} is the induced drag coefficient and C_L is the lift coefficient. Assume steady conditions, ignore pressure forces on the control volume and consider the control volume to be a tube of diameter b where flow only crosses the circular ends of the control volume. As part of your derivation you should apply the small angle approximations $\sin(\epsilon) = \epsilon$ and $\cos(\epsilon) = 1 - \epsilon^2/2$.

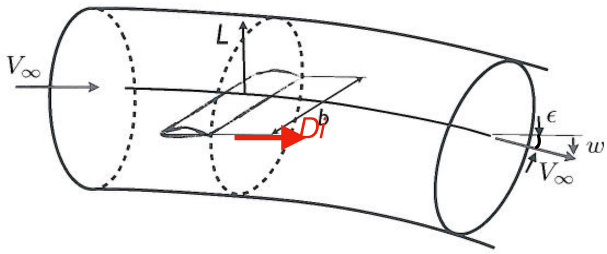


Figure 1:

Conservation of momentum for the case of a steady flow implies:

the rate of change of momentum in control volume plus the net flux momentum out of c.v equal to the applied force acting on the c.v

First consider vertical momentum balance where applied force is Lift L , No vertical component of flux into CV but there is negative vertical component ($-w$) of flux out of CV due to normal velocity V:

$$\rho V_\infty (-w) A = L \qquad (1)$$

(Note that w is pointing downwards and so we have negative flux out of CV from this term)

Next consider the horizontal momentum balance.

$$\rho V_\infty (V_\infty \cos\epsilon) A - \rho V_\infty V_\infty A = D_i \tag{2}$$

Since A = $\frac{\pi}{4} b^2$

$w = \quad V_\infty \sin(\epsilon) = \epsilon V_\infty$

we are also given $\cos(\epsilon) = 1 - \epsilon^2/2.$

So equation (1) becomes: L = $-\rho \epsilon V_\infty^2 A = \frac{-\pi \rho \varepsilon V_\infty^2 b^2}{4}$

Which can be re-arranged to give

$$\varepsilon = \frac{-4L}{\pi \rho V_\infty^2 b^2} \tag{3}$$

Next consider equation (2)

$D_{induced} = \rho V_\infty^2 (\frac{1-\varepsilon^2}{2}) A - \rho V_\infty^2 A$

$= -\frac{1}{8} \pi \rho V_\infty^2 \varepsilon^2 b^2$ (using A = π^2 $b^2/4$)

Substituting ε from equation (3):

$D_{induced} = -\frac{1}{8} \pi \rho V_\infty^2 b^2 (\frac{-4L}{\pi \rho V_\infty^2 b^2})^2 = \frac{-2L^2}{\pi \rho V_\infty^2 b^2}$

and so

$$C_{D_{induced}} = \frac{D_{induced}}{\frac{1}{2}\rho V_\infty^2 S} = \frac{-2L^2}{\pi \rho V_\infty^2 b^2} \cdot \frac{2}{\rho V_\infty^2 S} = \frac{L^2}{\frac{1}{4}\rho^2 V_\infty^4 S^2} \cdot \frac{S}{\pi b^2} = \frac{C_L^2 S}{\pi b^2}$$

Note that the negative sign is due to this being the force on the control volume which is equal and opposite of the force of the flow on the wing.

2014 Question 1:

(b) The velocity components in a two-dimensional flow are given by the following equations

$$u = 3x^2 - 4xy$$
$$v = Ay^2 - 3x^2 - 6xy.$$

If this flow is to satisfy continuity, find the constant A.

(c) Define total, dynamic, static and hydrostatic pressure. Explain, with the aid of a labelled diagram, the use of a Pitot-static tube and manometer for the measurement of air velocity in incompressible flow.

(b) Recall from section 2-2 that for an incompressible flow we have the continuity condition

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = 6x - 4y \qquad \frac{\partial v}{\partial y} = 2Ay - 6x$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = (2A - 4)y = 0 \qquad \therefore A = 2$$

(c)
$$P_0 = p + \frac{1}{2}\rho U^2$$

Total pressure = P_0

Static pressure = p

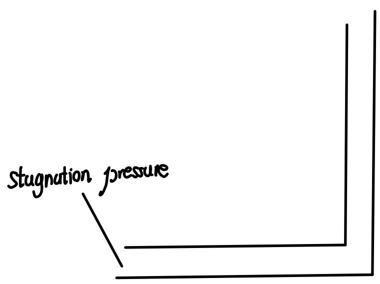
Dynamics Pressure = $\frac{1}{2}\rho U^2$

Hydrostatic pressure: $\Delta p = \rho g \Delta h$

(Δh = height of water column)

(Δp = pressure above the local state pressure)

From lecture 2-6:



$$\Delta p = P_0 - p$$
$$= \rho g \Delta h$$
$$= \frac{1}{2}\rho U^2$$
$$\therefore U = \sqrt{2g\Delta h}$$