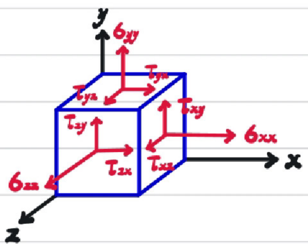
Thin-Walled Tube in Torsion

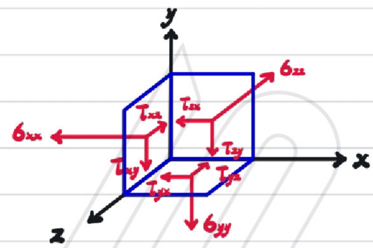
Basic Concepts

Stress

+ve face



$\longleftrightarrow$ -ve face

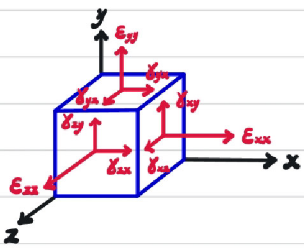


differential equation for equilibrium $\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} + \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = 0$ where complementary shear stress $\tau_{ij} = \tau_{ji}$

out-of plane load force per unit volume

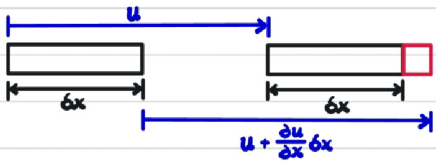
Strain

$$\begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_{zz} \end{bmatrix}$$

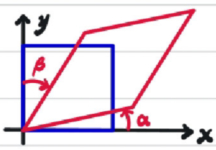


$$\xrightarrow{\text{Hooke's law under plane stress}} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

where Poisson's ratio $\nu \propto \frac{1}{\beta}$ ← compressibility of the soild



$$\varepsilon_{xx} = \frac{(u + \frac{\partial u}{\partial x}\delta x) - u}{\delta x} = \frac{\partial u}{\partial x} \longrightarrow \varepsilon_{ii} = \frac{\partial u_i}{\partial x_i}$$



$$\gamma_{xy} = \alpha + \beta = \tan^{-1}\left(\frac{\partial v}{\partial x}\right) + \tan^{-1}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \longrightarrow \gamma_{ij} = \varepsilon_{ij} = \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}$$

Geometric properties of an area

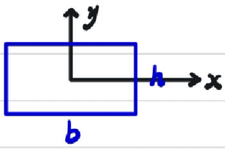
1st moment of area $Q_x = \int y \, dA$ and $Q_y = \int x \, dA$

centroid of area $\bar{x} = \frac{Q_y}{A} = \frac{1}{A}\int x \, dA$ and $\bar{y} = \frac{Q_x}{A} = \frac{1}{A}\int y \, dA$

2nd moment of area $\longleftrightarrow$ area moment of inertia

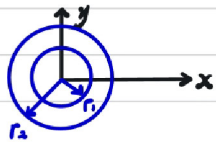
$I_x = \int y^2 \, dA$, $I_y = \int x^2 \, dA$ and $I_{xy} = \int xy \, dA$

e.g.



$I_x = \int y^2\,dA = \int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} y^2\,dx\,dy = \frac{bh^3}{12}$,

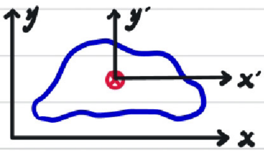
$I_y = \int x^2\,dA = \int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} x^2\,dx\,dy = \frac{b^3h}{12}$ and $I_{xy} = \int xy\,dA = \int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} xy\,dx\,dy = 0$



$I_x = \int y^2\,dA = \int_0^{2\pi}\int_{r_1}^{r_2} (r\sin\theta)^2\,r\,dr\,d\theta = \frac{\pi}{4}(r_2^4 - r_1^4)$,

$I_y = \int_0^{2\pi}\int_{r_1}^{r_2} (r\cos\theta)^2\,r\,dr\,d\theta = \frac{\pi}{4}(r_2^4 - r_1^4)$ and $I_{xy} = \int_0^{2\pi}\int_{r_1}^{r_2} (r\cos\theta \times r\sin\theta)\,r\,dr\,d\theta = 0$

parallel axis theorem



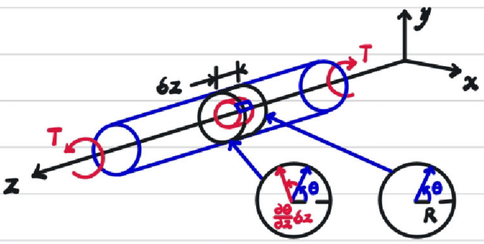
$I_x = I_x' + A d_y^2$, $I_y = I_y' + A d_x^2$ and $I_{xy} = I_{x'y'} + A d_x d_y$

where $d_x = x' - x$ and $d_y = y' - y$

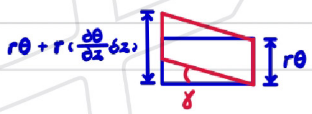
principle axes i.e. $I_{xy} = 0$

Torsion of circular section bar

assume the bar deform such way that a straight line on the cross-section remains straight after twisting



$\longrightarrow$



$\gamma = r\frac{\partial\theta}{\partial z} = r\theta' \longrightarrow \tau = G\gamma = Gr\theta'$

shear modulus $G = \frac{E}{2(1+\nu)}$

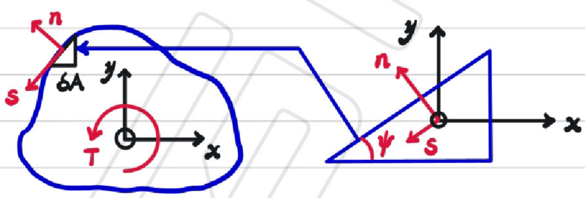
$F = \tau A \longrightarrow dF = \tau\,dA \xrightarrow{T = Fr} dT = r\cdot\tau\,dA \longrightarrow T = \frac{\tau}{r}\int r^2\,dA = \frac{\tau}{r}J = GJ\theta'$

where $J = \int r^2\,dA = \int_0^{2\pi}\int_0^{R} r^2\cdot r\,dr\,d\theta = \frac{\pi R^4}{2}$

Torsion of Non-Circular Open Section Bar

Free surface boundary condition

the complementary shear stress τ_{ij} acting normal to the free surface is zero $\longrightarrow$ wraping displacement



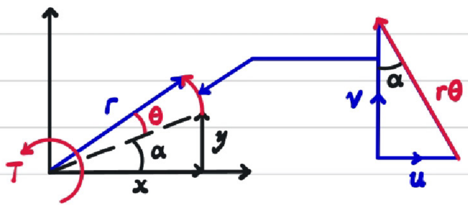
$\tau_{zn} = 0$ at the free surface

$\longrightarrow \tau_{zy}\cos\psi - \tau_{zx}\sin\psi = 0$

$\longrightarrow -\tau_{zy}\frac{\partial x}{\partial s} + \tau_{zx}\frac{\partial y}{\partial s} = 0$ ①

Saint-Venant's Assumptions

for the deformation of a bar in torsions $\frac{d\theta}{dz} = \theta' = \text{constant}$



$$\longrightarrow \begin{cases} u = -r\theta\sin\alpha = -y\theta = -y(\theta' z + C_1) \\ v = r\theta\cos\alpha = x\theta = x(\theta' z + C_2) \\ w = \phi(x,y)\theta' \end{cases}$$

$$\longrightarrow \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & (\frac{\partial\phi}{\partial x} - y)\theta' \\ 0 & 0 & (\frac{\partial\phi}{\partial y} + x)\theta' \\ (\frac{\partial\phi}{\partial x} - y)\theta' & (\frac{\partial\phi}{\partial y} + x)\theta' & 0 \end{bmatrix}$$

$$\xrightarrow{\tau = G\gamma} \begin{cases} \tau_{xz} = G\gamma_{xz} = G\theta'(\frac{\partial\phi}{\partial x} - y) \quad ② \\ \tau_{yz} = G\gamma_{yz} = G\theta'(\frac{\partial\phi}{\partial y} + x) \quad ③ \end{cases} \quad \text{and} \quad \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & 0 \end{bmatrix} \quad ④$$

Stress function

substitude ④ into differential equation for equilibrium

$$\begin{bmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & 0 \end{bmatrix}\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} + \cancel{\begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}}^{\text{ignore}} = 0 \longrightarrow \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} = 0 \quad ⑤$$

define stress function $\tau_{xz} = G\theta'\frac{\partial F}{\partial y}$ ⑥ and $\tau_{yz} = -G\theta'\frac{\partial F}{\partial x}$ ⑦

substitude ⑥, ⑦ into ②, ③ $\frac{\partial F}{\partial y} = \frac{\partial\phi}{\partial x} - y$ and $-\frac{\partial F}{\partial x} = \frac{\partial\phi}{\partial y} + x$

$$\longrightarrow \frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2 F}{\partial y^2} + 1 = -\frac{\partial^2 F}{\partial x^2} - 1 \longrightarrow \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = \nabla^2 F = -2 \quad ⑧$$

substitude ⑥, ⑦ into ① $-(-G\theta'\frac{\partial F}{\partial x})\frac{dx}{ds} + (G\theta'\frac{\partial F}{\partial y})\frac{dy}{ds} = 0 \longrightarrow \frac{\partial F}{\partial s} = 0$ ⑨ $\longrightarrow F = 0$ on free surface

$$F = \tau A \longrightarrow dF = \tau dA \xrightarrow{T = Fr} dT = r \cdot \tau dA$$

$$\longrightarrow T = \int r \cdot \tau\, dA = \iint (x, y) \times (\tau_{xz}, \tau_{yz})\, dx\,dy$$

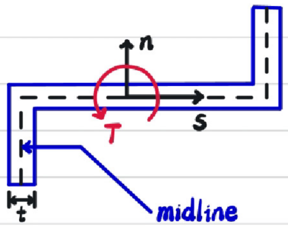
$$\xrightarrow{\text{substitude ⑥, ⑦}} \iint (-G\theta'\frac{\partial F}{\partial x})x - (G\theta'\frac{\partial F}{\partial y})y\, dx\,dy$$

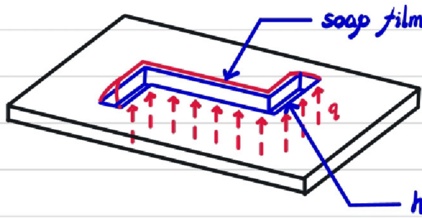
where $\iint \frac{\partial F}{\partial x} x\, dx\,dy = \int(\cancel{Fx}^{0} - \int F\,dx)\,dy$ and $\iint \frac{\partial F}{\partial y} y\, dx\,dy = \int(\cancel{Fy}^{0} - \int F\,dy)\,dx$

$$\longrightarrow T = 2G\theta' \iint F\,dx\,dy \xrightarrow{T = GJ\theta'} J = 2\iint F\,dx\,dy$$

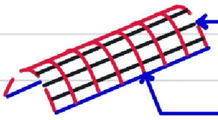
Membrane analogy

for a thin-walled open sections, stress function F and deflection of the soap film w have the same form





where $\nabla^2 w = -\frac{q}{T}$ and $w = 0$ at edge



contour line $\frac{\partial w}{\partial s} = 0 \longrightarrow \frac{\partial F}{\partial s} = 0 \longrightarrow \tau_{nz} = -G\theta' \frac{\partial F}{\partial s} = 0$

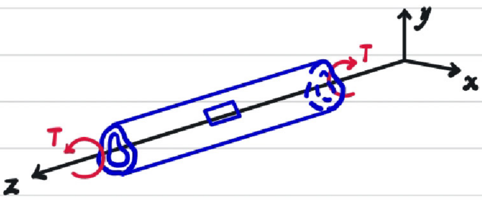
maximum slope $\left(\frac{\partial w}{\partial n}\right)_{max} \longrightarrow \left(\frac{\partial F}{\partial n}\right)_{max} \longrightarrow (\tau_{sz})_{max} = G\theta' \left(\frac{\partial F}{\partial n}\right)_{max}$

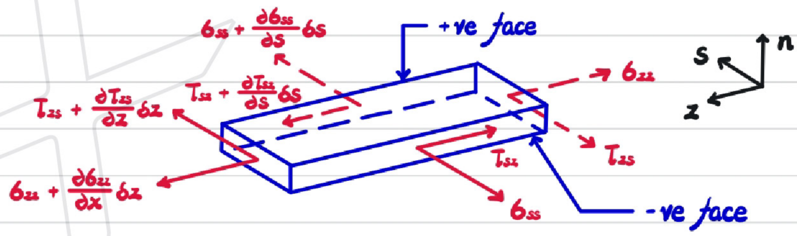
$\nabla^2 F = -2$ and $\frac{\partial F}{\partial s} = 0 \longrightarrow \frac{\partial^2 F}{\partial n^2} = -2 \longrightarrow F = -n^2 + Cn + D \xrightarrow{F = 0 \text{ at } n = \pm\frac{1}{2}t} F = -n^2 + \frac{1}{4}t^2$

$\longrightarrow \tau_{sz} = G\theta' \frac{\partial F}{\partial n} = -2nG\theta'$ and $J = 2\iint -n^2 + \frac{1}{4}t^2 \, ds\, dn = \frac{1}{3}\oint t^3 ds \longrightarrow (\tau_{sz})_{max} = -2\left(\pm\frac{t}{2}\right)G\theta' = \pm tG\theta'$

Close Single-Cell Tube in Torsion

Assumptions





where $\sigma_{nz} = \sigma_{zn} = 0$ normal to the free surface

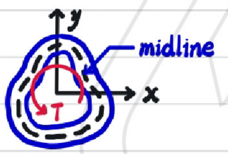
assume no axial constraint i.e. tube is free to deform in the z-direction $\sigma_{zz} = 0$, the cross sectional shape is unaltered by the torque loading $\sigma_{ss} = 0$ and wall thickness is small $\frac{\partial \tau_{sz}}{\partial n} = \frac{\partial \tau_{zs}}{\partial n} = 0$

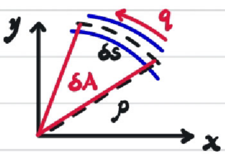
$\xrightarrow{\text{force equilibrium}} \tau_{sz} = \tau_{sz} + \frac{\partial \tau_{sz}}{\partial s}\delta s$ and $\tau_{zs} = \tau_{zs} + \tau_{zs}\frac{\partial \tau_{zs}}{\partial z}\delta z = 0 \longrightarrow \frac{\partial \tau_{sz}}{\partial s} = \frac{\partial \tau_{zs}}{\partial z} = 0$

Shear flow

define shear flow $q = \tau_{sz} t = \tau_{zs} t \longrightarrow \frac{\partial q}{\partial s} = \frac{\partial q}{\partial z} = 0$ i.e. q = constant

Brent - Batho formula





$T = Fp = p \cdot \tau \delta A = p \cdot \tau \cdot t\delta s = \oint q \cdot p\, ds$

$\xrightarrow{\oint p\, ds = 2\oint dA} T = 2q\oint dA = 2qA$

Virtual work

$$\bar{1}\cdot\theta = \int_V \bar{\tau}\cdot\gamma\, dV \longrightarrow \theta = \int_V \frac{q}{t}\cdot\frac{\bar{q}}{Gt}\, dV = \int\oint \frac{q}{t}\cdot\frac{1}{2AGt}(t\,ds)\,dz \longrightarrow \frac{d\theta}{dz} = \frac{1}{2AG}\oint\frac{q}{t}ds = \frac{T}{4A^2G}\oint\frac{1}{t}ds$$

virtual unit torque

Comparision of Torsional Behaviour

Shear stress

for open circular tube, $\tau = G\theta'\frac{dF}{dn} = -2nG\theta' = -2n\frac{T}{J} \xrightarrow{J=\frac{1}{3}\int t^3 ds = \frac{2}{3}\pi R t^3} \tau = -\frac{3nT}{\pi R t^3} \longrightarrow \tau_{max} = \frac{3T}{2\pi R t^2}$

for closed circular tube, $\tau = \frac{q}{t} = \frac{T}{2At} = \frac{T}{2\pi R^2 t}$

$$\frac{(\tau_{open})_{max}}{\tau_{closed}} = \frac{3R}{t} >> 1$$

Rate of twist

for open circular tube, $\theta' = \frac{T}{GJ} \xrightarrow{J=\frac{1}{3}\int t^3 ds = \frac{2}{3}\pi R t^3} \theta' = \frac{3T}{2G\pi R t^3}$

for closed circular tube, $\theta' = \frac{1}{2AG}\oint\frac{q}{t}ds = \frac{T}{4A^2G}\oint\frac{1}{t}ds = \frac{T}{2\pi R^3 G t}$

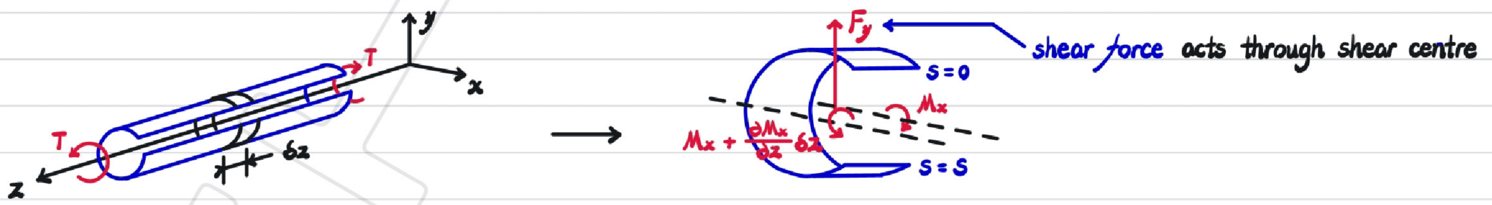
$$\frac{\theta'_{open}}{\theta'_{closed}} = \frac{3R^2}{t^2} >> 1$$

Single Cell Tube Loaded by Shear Forces

Shear of Thin-Walled Open Tubes

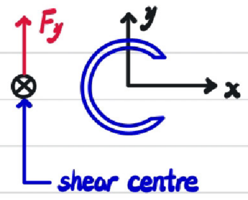
Assumptions

consider a thin-wall open section beam with constant cross-section loaded



assume the direct stress on cross section is constant across wall thickness $\frac{\partial\sigma_{zz}}{\partial t} = \frac{\partial\sigma_{ss}}{\partial t} = 0$ and shear stress on cross section is constant across wall thickness $\frac{\partial\tau_{sz}}{\partial t} = \frac{\partial\tau_{zs}}{\partial t} = 0 \longrightarrow$ concept of shear flow is applicable

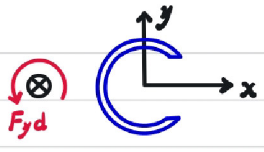
Shear centre



$\tau_1 = \frac{q}{t} \neq 0$

$\theta_1' = \frac{T}{GJ} = 0$

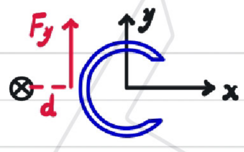
$+$



$\tau_2 = -2G\theta' n$

$\theta_2' = \frac{F_y d}{GJ}$

$=$

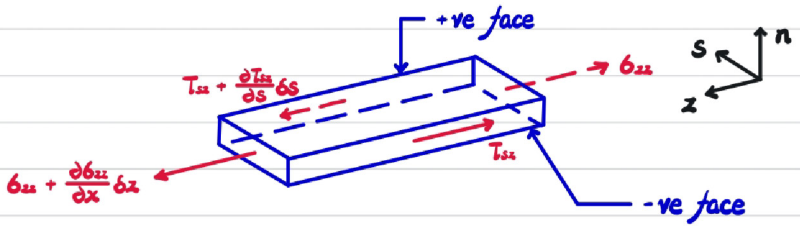


$\tau = \tau_1 + \tau_2$

$\theta' = \theta_1' + \theta_2'$

Shear flow

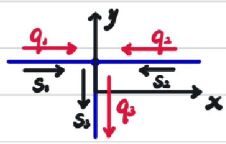
for loading in y-direction, force equilibrium in the z-direction, $\frac{\partial \sigma_{zz}}{\partial z}\delta z + \frac{\partial \tau_{sz}}{\partial s}\delta s = 0 \longrightarrow \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{t}\frac{\partial q}{\partial s} = 0$



$\xrightarrow{\sigma_{zz} = \frac{M_x}{I_x}y \text{ about principle axes}} \frac{\partial M_x}{\partial z}\frac{y}{I_x} + \frac{1}{t}\frac{\partial q}{\partial s} = 0 \xrightarrow{F_y = \frac{\partial M_x}{\partial z}} q = -\frac{F_y}{I_x}\int_0^s yt\, ds$

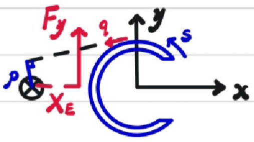
for loading in both direction on principle axes $q = \frac{F_y}{I_x}D_x + \frac{F_x}{I_y}D_y$ where $D_x = -\int_0^s yt\, ds$ and $D_y = -\int_0^s xt\, ds$

conservation of shear flow $\Sigma q_{out} = \Sigma q_{in}$ at junction, $F_y = \int q_y\, ds_y$ and $F_x = \int q_x\, ds_x$

e.g. 

$q_3 = q_1 + q_2$, $F_y = -\int q_3\, ds_3$ and $F_x = \int q_1\, ds_1 - \int q_2\, ds_2$

Location of the shear centre



take moment at shear centre, $\Sigma M = F_y X_E = \int_0^s p \cdot q\, ds$

$\longrightarrow F_x Y_E = \int_0^s p \cdot q\, ds$

if the section poesses an axis of symmetry $\longrightarrow$ shear centre lies along the axis

Shear of Thin-Walled Closed Tubes

Assumptions

consider a thin-wall closed section beam with constant cross-section loaded

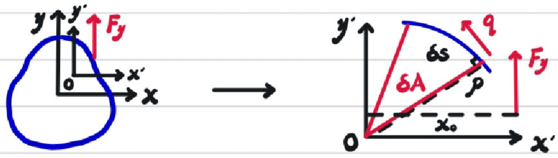
assume the direct stress on cross section is constant across wall thickness $\frac{\partial \sigma_{zz}}{\partial t} = \frac{\partial \sigma_{ss}}{\partial t} = 0$, shear stress on cross section is constant across wall thickness $\frac{\partial \tau_{sz}}{\partial t} = \frac{\partial \tau_{zs}}{\partial t} = 0 \longrightarrow$ concept of shear flow is applicable

and hoop direct stress is zero $\sigma_{ss} = 0$

Shear flow

for loading in y-direction, force equilibrium in the z-direction on principle axes, $q = -\frac{F_y}{I_x}\oint yt\,ds$

$\xrightarrow{\text{no free edge}} q = -\frac{F_y}{I_x}\int_0^s yt\,ds + q_0 \longrightarrow q = q_0 + q_b = q_0 + \frac{F_y}{I_x}D_x$ where $D_x = -\int_0^s yt\,ds$



take moment about O, $\Sigma M = F_y x_0 + \oint p\cdot(q_0 + q_b)\,ds$

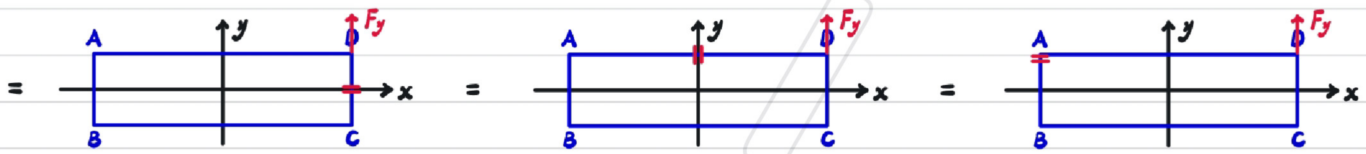
$\xrightarrow{\oint p\,ds = 2A} F_y x_0 = 2q_0 A + \int p\cdot q_b\,ds$

e.g.

'cut' $q_b = 0$

find $q_{1b}, q_{2b}, q_{3b}, q_{4b}, q_{5b}$ where $F_y = -\int q_1\,ds_1 + \int q_3\,ds_3 - \int q_5\,ds_5$

$\longrightarrow$ take moment about F, $F_y\cdot 0 = 2q_0A + \int p\cdot q\,ds \longrightarrow$ find q_0



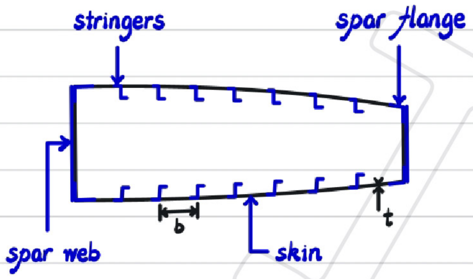
Location of the shear centre

the point shear flow produce no rate of twist $\theta' = 0 \longrightarrow \frac{1}{2AG}\oint\frac{q}{t}\,ds = 0 \longrightarrow \oint\frac{q}{t}\,ds = 0$

$\longrightarrow q_0\oint\frac{1}{t}\,ds + \oint\frac{q_b}{t}\,ds = 0 \longrightarrow q_0 = -\frac{\oint\frac{q_b}{t}\,ds}{\oint\frac{1}{t}\,ds} \longrightarrow F_y X_E = 2Aq_0 + \int_0^s p\cdot q_b\,ds$ and $F_x Y_E = 2Aq_0 + \int_0^s p\cdot q_b\,ds$

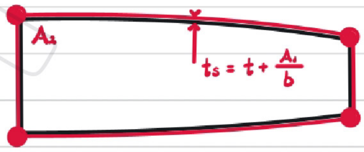
Structure Idealization

Typical wing cross-section



can be simplified with direct stress carrying material

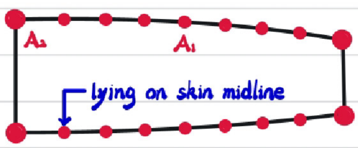
$\xrightarrow{\text{smeared stringers}}$



direct stress carrying capacity $I_x, D_x, \ldots$ by t_s

shear θ' by $t \longrightarrow \theta' = \frac{1}{2AG}\oint\frac{q}{t}\,ds$

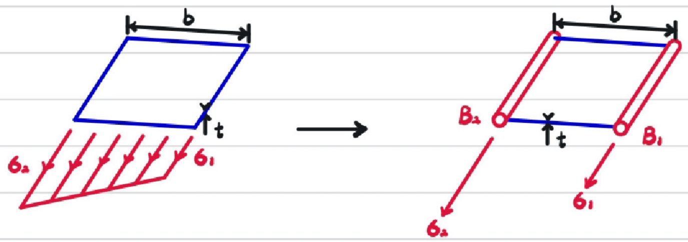
$\xrightarrow{\text{boom stiffened panel}}$



$\xrightarrow{\text{boom - shear panel}}$

Boom-shear panel idealization

to distribute direct stress carrying capacity of a skin panel to adjacent booms



direct stress carrying capacity I_x, D_x, ... by B

shear θ' by t $\longrightarrow$ $\theta' = \frac{1}{2AG}\oint \frac{q}{t}\,ds$

take moment about right edge, $\Sigma M = \int_0^b [\sigma_1 + (\sigma_2 - \sigma_1)\frac{l}{b}]\cdot l\,(t\cdot dl) = \sigma_2 B_2 b \longrightarrow \frac{1}{3}\sigma_2 b^2 t + \frac{1}{6}\sigma_1 b^2 t = \sigma_2 B_2 b$

take moment about left edge, $\Sigma M = \int_0^b [\sigma_2 - (\sigma_2 - \sigma_1)\frac{l}{b}]\cdot l\,(t\cdot dl) = \sigma_1 B_1 b \longrightarrow \frac{1}{3}\sigma_1 b^2 t + \frac{1}{6}\sigma_2 b^2 t = \sigma_1 B_1 b$

$\longrightarrow B_1 = \frac{tb}{6}(2 + \frac{\sigma_2}{\sigma_1})$ and $B_2 = \frac{tb}{6}(2 + \frac{\sigma_1}{\sigma_2})$ $\xrightarrow{\text{only subject to bending}}$

- about principle x-axis, $\sigma_z = \frac{M_x}{I_x}y$ $\longrightarrow$ $B_1 = \frac{tb}{6}(2 + \frac{y_2}{y_1})$
- about principle y-axis, $\sigma_z = \frac{M_y}{I_y}x$ $\longrightarrow$ $B_2 = \frac{tb}{6}(2 + \frac{x_2}{x_1})$

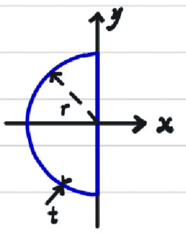
e.g. subjected to bending about principle x-axis M_x

$B_1 = \Delta B_1|_{1\to 2} + \Delta B_1|_{1\to 4} = \frac{tb}{6}(2+\frac{y_2}{y_1}) + \frac{tb}{6}(2+\frac{y_4}{y_1})$

$\longrightarrow$ find B_2, B_3, B_4 $\longrightarrow$ $I_x = A dy^2 = \sum_{i=1}^{4} B_i (y_i - \bar{y})^2$

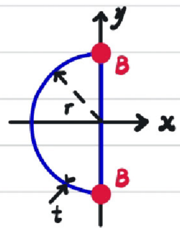
$\longrightarrow D_1 = 0,\ D_2 = D_1 - y_2 B_2,\ D_3 = D_2 - y_3 B_3,\ D_4 = D_3 - y_4 B_4$

for simple two boom idealizations which symmetric about the axis of bending



$$I_x = \frac{bh^3}{12} + \int (r\sin\theta)^2\,dA$$
$$= \frac{t(2r)^3}{12} + \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} (r\sin\theta)^2 (rt\,d\theta)$$
$$= \frac{2tr^3}{3} + \frac{\pi t r^3}{2}$$

$\longrightarrow$



$I_x = B(r-0)^2 + B(-r-0)^2 = 2Br^2$

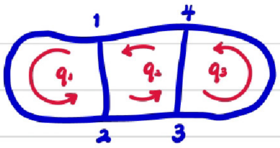
$\longrightarrow \frac{2tr^3}{3} + \frac{\pi t r^3}{2} = 2Br^2 \longrightarrow B = \frac{tr}{3} + \frac{\pi t r}{4} = (\frac{1}{3} + \frac{\pi}{4})tr$

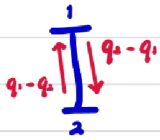
Multi-cell Thin-Walled Tube

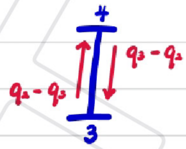
Torsion of Multi-Cell Tubes

Shear flow

from consideration of an element in a given wall, q will be constant along that wall but different in different walls



$\Sigma q_{out} = \Sigma q_{in}$ at junction $\longrightarrow$ 

and 

N cells $\longrightarrow$ N elements $\longrightarrow$ N unknowns

Determine the shear flow distribution

$T = \Sigma 2qA = 2q_1A_1 + 2q_2A_2 + 2q_3A_3$

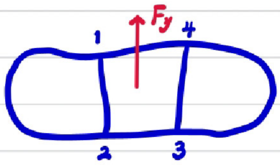
and $\theta'_{total} = \theta'_{cell} = \frac{1}{2A_1G}\oint_{cell\,1} \frac{q}{t}ds = \frac{1}{2A_1G}\left[q_1 \int_{(1\to2)ext} \frac{1}{t_{(1\to2)ext}} ds_{(1\to2)ext} + (q_1 - q_2) \int_{(2\to1)in} \frac{1}{t_{(2\to1)in}} ds_{(2\to1)in} \right]$

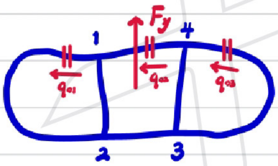
$= \frac{1}{2A_2G}\oint_{cell\,2} \frac{q}{t}ds = \frac{1}{2A_3G}\oint_{cell\,3} \frac{q}{t}ds$

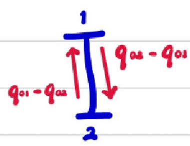
$\longrightarrow$ find $q_1, q_2, q_3 = f(T)$

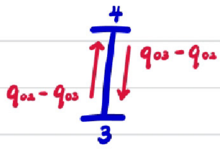
Multi-Cell Tube Subjected to Shear

Shear flow



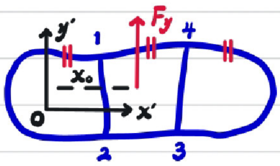
$\longrightarrow$ 

where 

and 

find $q_b = f(F_y)$, N cells $\longrightarrow$ N elements $\longrightarrow$ N unknowns

Determine the shear flow distribution



take moment about O, $\Sigma M = F_y x_0 = \Sigma 2q_{0i}A_i + \int p \cdot q_b \, ds$

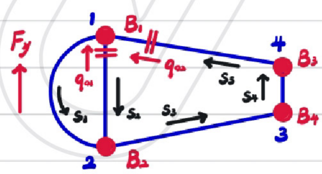
and $\theta'_{total} = \theta'_{cell} = \frac{1}{2A_iG}\oint_{cell\,i} \frac{q}{t}ds \longrightarrow$ find $q_{01}, q_{02}, q_{03} = f(F_y)$

$\longrightarrow$ find $q = f(F_y)$

Shear centre

find $q_b = F(y)$ and at shear centre, $\theta'_{total} = \theta'_{cell} = \frac{1}{2A_iG}\oint_{cell\,i} \frac{q}{t}ds = 0 \longrightarrow$ find $q_{01}, q_{02}, q_{03} = f(F_y)$

$\longrightarrow F_y X_E = \Sigma 2q_{0i}A_i + \int p \cdot q_b \, ds$

e.g. 

find B_i e.g. $B_1 = \Delta B_{1-2} + \Delta B_{1-4} = (\frac{1}{3}t_{in} + \frac{\pi}{4}t_{ext})r + \frac{tb}{6}(2 + \frac{y_4}{y_1}) \longrightarrow$ find q_{bi}

$\xrightarrow{\theta_i' = 0}$ find q_{0i} $\xrightarrow{\text{take moment about 2}}$ find X_E

Deflection of a Thin-Walled Tube

Translation of Cross Section

Virtual work

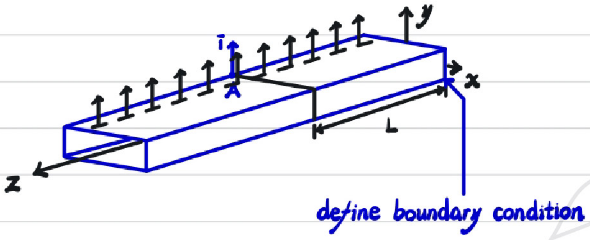
$$\bar{1}\cdot\delta = \int_V \bar{\sigma}^T \cdot \varepsilon \, dV = \int_V [\,\bar{\sigma}_{xx}\ \ \bar{\sigma}_{yy}\ \ \bar{\sigma}_{zz}\ \ \bar{\tau}_{xy}\ \ \bar{\tau}_{xz}\ \ \bar{\tau}_{yz}\,][\,\varepsilon_{xx}\ \ \varepsilon_{yy}\ \ \varepsilon_{zz}\ \ \gamma_{xy}\ \ \gamma_{xz}\ \ \gamma_{yz}\,]^T \, dV$$

$$\xrightarrow{\text{only membrane stress in thin-walled tube}} \delta = \int_V [\,\bar{\sigma}_{zz}\ \ \bar{\tau}_{sz}\,][\,\varepsilon_{zz}\ \ \gamma_{sz}\,]^T \, dV = \int_0^L (\bar{\sigma}_{zz}\cdot\varepsilon_{zz})\, dA\, dz + \int_0^L \oint (\bar{\tau}_{sz}\cdot\gamma_{sz})\, t\, ds\, dz$$

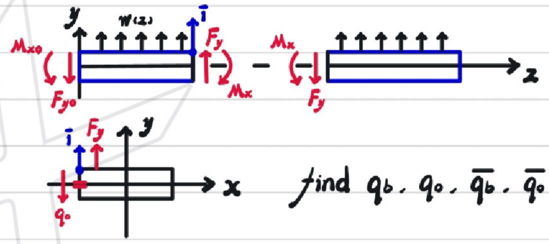
$$\xrightarrow{\text{about principle axis}} \delta = \int_0^L \left(\frac{\bar{m}_x}{EI_x}y + \frac{\bar{m}_y}{EI_y}x\right)\left(\frac{M_x}{I_x}y + \frac{M_y}{I_y}x\right) dA\, dz + \int_0^L \oint \frac{\bar{q}}{t}\cdot\frac{q}{Gt}\, t\, ds\, dz$$

Vertical displacement

$$v = \int_0^L \frac{M_x \bar{m}_x}{EI_x^2}\left(\int y^2 dA\right) dz + \int_0^L \oint \frac{q\bar{q}}{Gt^2}\, t\, ds\, dz = \int_0^L \frac{M_x \bar{m}_x}{EI_x}\, dz + \int_0^L \oint \frac{q\bar{q}}{Gt}\, ds\, dz \longleftarrow \text{moving upward}$$



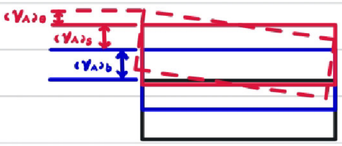
$\longrightarrow$



find F_y, M_x, $\bar{m}_x$

find q_b, q_0, $\bar{q}_b$, $\bar{q}_0$ $\longrightarrow$ find q, $\bar{q}$

$\longrightarrow$



where $(v_A)_b = \int_0^L \frac{M_x \bar{m}_x}{EI_x}\, dz$, $(v_A)_s = \int_0^L \oint \frac{q_b \bar{q}_b}{Gt}\, ds\, dz$ and $(v_A)_\theta = \int_0^L q_0 \bar{q}_0 \oint \frac{1}{Gt}\, ds\, dz$

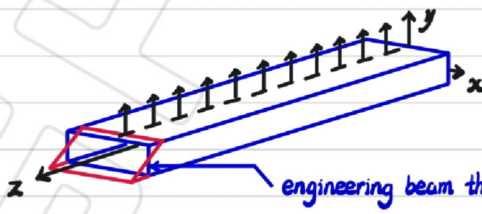
when $\oint q_b\, ds = \oint \bar{q}_b\, ds = 0$ cut at the symmetric line

Warping of a Closed Tube

Warping

displacement in z-direction results in an originally planer section becoming non-planer

due to bending

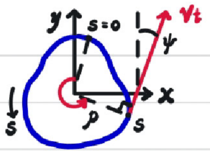


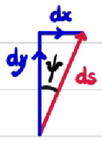
engineering beam theory no warping

and due to torque

Shear strain

$$q \longrightarrow \tau_{sz} = \frac{q}{t} \longrightarrow \gamma_{sz} = \frac{\tau_{sz}}{G} = \frac{q}{Gt} \quad \text{and} \quad \gamma_{sz} = \varepsilon_{sz} = \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z} \longleftarrow \text{tangential displacement}$$

where 

$\longrightarrow$ 

$$v_t = p\theta + v\cos\psi + u\sin\psi = p\theta + v\frac{\partial y}{\partial s} + u\frac{\partial x}{\partial s}$$

θ: angle of rotation of the cross section

$$\longrightarrow \gamma_{sz} = \frac{q}{Gt} = \frac{\partial w}{\partial s} + p\theta' + \frac{\partial v}{\partial z}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial x}{\partial s}$$

$$\longrightarrow \int_0^s \gamma_{sz}\,ds = \int_0^s \frac{q}{Gt}\,ds = [w]_0^s + \int_0^s p\theta'\,ds + \frac{\partial v}{\partial z}[y]_0^s + \frac{\partial u}{\partial z}[x]_0^s$$

$$\longrightarrow \int_0^s \gamma_{sz}\,ds = \frac{1}{G}\int_0^s \frac{q}{t}\,ds = (w_s - w_0) + 2A_{os}\theta' + \frac{dv}{dz}(y_s - y_0) + \frac{du}{dz}(x_s - x_0)$$

A_{os}: area swept out by the line joining the origin of the x-y axes to a point on the wall midline as it move from s = 0 to some point s

$$\longrightarrow w_s - w_0 = \underbrace{\left(\frac{1}{G}\int_0^s \frac{q}{t}\,ds - \frac{A_{os}}{AG}\oint \frac{q}{t}\,ds\right)}_{\text{potentially warping}} + \underbrace{\left[-\frac{du}{dz}(x_s - x_0) - \frac{dv}{dz}(y_s - y_0)\right]}_{\text{planar rotation}}$$

$$\longrightarrow w^* = \frac{1}{G}\int_0^s \frac{q}{t}\,ds - \frac{A_{os}}{AG}\oint \frac{q}{t}\,ds$$

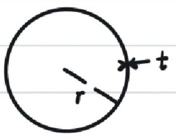
Warping of closed tube in torsion

for constant shear flow in torsion, $q = \frac{T}{2A}$ $\longrightarrow$ $w^* = \frac{T}{2AG}\int_0^s \frac{1}{t}\,ds - \frac{TA_{os}}{2A^2G}\oint \frac{1}{t}\,ds$

Section do not warp in torsion

$$w^* = 0 \longrightarrow \int_0^s \frac{1}{t}\,ds - \frac{A_{os}}{A}\oint \frac{1}{t}\,ds = 0 \longrightarrow \frac{\int_0^s \frac{1}{t}\,ds}{\oint \frac{1}{t}\,ds} - \frac{A_{os}}{A} = 0 \longrightarrow \frac{\int_0^s \frac{1}{t}\,ds}{\oint \frac{1}{t}\,ds} - \frac{\frac{1}{2}\int_0^s p\,ds}{\frac{1}{2}\oint p\,ds} = 0$$

for Neuber tubes, pt = constant = k $\longrightarrow$ $\frac{\int_0^s \frac{1}{t}\,ds}{\oint \frac{1}{t}\,ds} - \frac{k\int_0^s \frac{1}{t}\,ds}{k\oint \frac{1}{t}\,ds} = 0$

e.g. 

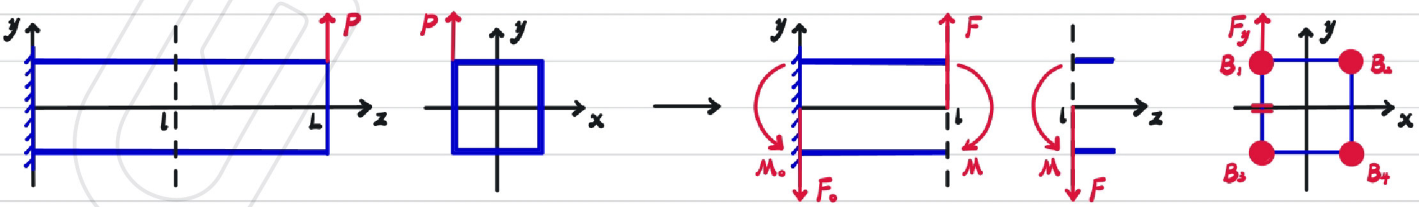
$k = rt$ and 

$k = rt$

Tapered Beams

Boom-Shear Panel Idealisations of Tapered Beams

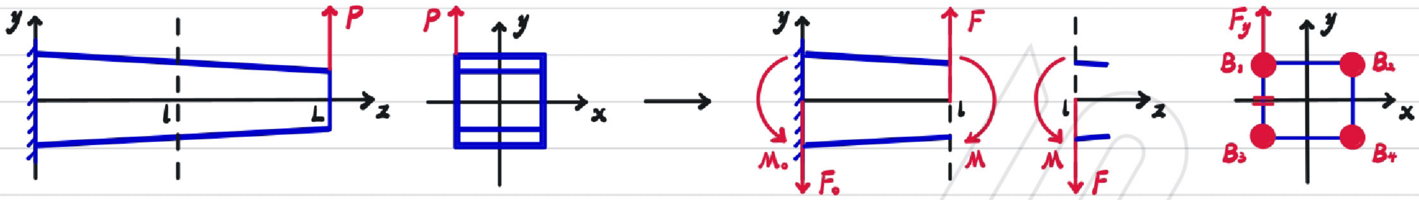
Single web beam



$\Sigma F = 0 \xrightarrow{\text{at } z=0} F_0 = P \xrightarrow{\text{at } z=l} F = F_0 = P$

$\longrightarrow F_y = F \longrightarrow$ find $q_b \longrightarrow$ find $q_0 \longrightarrow$ find $q = q_0 + q_b$

Tapered single cell tube



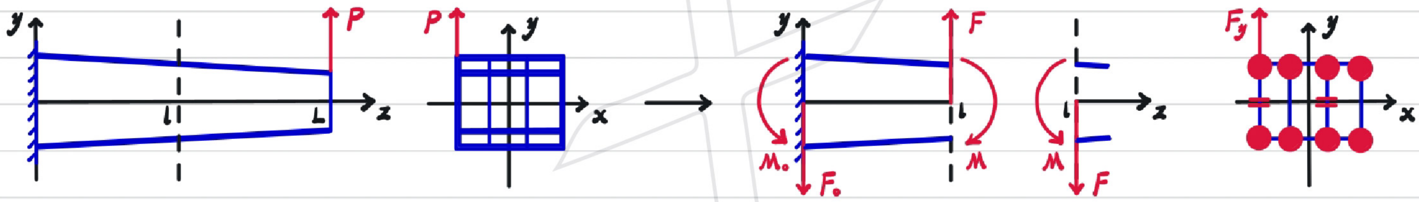
$\Sigma F = 0 \longrightarrow F = P$ and $\Sigma M = 0 \xrightarrow{\text{at } z=0} M_0 = -PL \xrightarrow{\text{at } z=l} M = F_0 l + M_0 = P(l - L)$

$F = F_y - P_{yi} = F_y - \Sigma P_{zi} \tan\alpha$

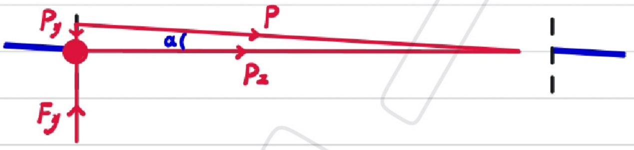
where $P_{zi} = \sigma_{zi} A = (\frac{M_x}{I_x} y) B_i$

$\longrightarrow$ find $q_b \longrightarrow$ find q_0 where $F x_0 = 2Aq_0 + \oint p \cdot q \, ds - \Sigma p \cdot P_{yi} \longrightarrow$ find $q = q_0 + q_b$

Tapered multi-cell tube



$\Sigma F = 0 \longrightarrow F = P$ and $\Sigma M = 0 \xrightarrow{\text{at } z=0} M_0 = -PL \xrightarrow{\text{at } z=l} M = F_0 l + M_0 = P(l - L)$



$F = F_y - P_y = F_y - \Sigma P_{zi} \tan\alpha$

where $P_{zi} = \sigma_{zi} A = (\frac{M_x}{I_x} y) B_i$

$\longrightarrow$ find $q_b \longrightarrow$ find $q_{01}, q_{02}, \ldots$ where $F x_0 = 2\Sigma A q_0 + \oint p \cdot q \, ds - \Sigma p \cdot P_{yi}$ and $\theta'_{01} = \theta'_{02} = \ldots$

$\longrightarrow$ find $q_1, q_2, \ldots$

Plate Bending

Physical Model

Simplifying assumptions

real world problem $\xrightarrow{\text{simplify assumptions}}$ physical model e.g. beam model $\xrightarrow{\text{governing equations}}$ mathematical model

1D Elements

strings  $\xrightarrow{\text{curved}}$ slacked strings

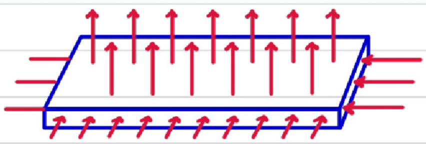
columns  $\xrightarrow{\text{curved}}$ brick arches

bars and rods and structs 

beams  $\xrightarrow{\text{curved}}$ curved beams

some of the transverse load are carried by in-plane (membrane) loads increase load carrying capacity

2D Elements

plates  $\xrightarrow{\text{curved}}$ shells

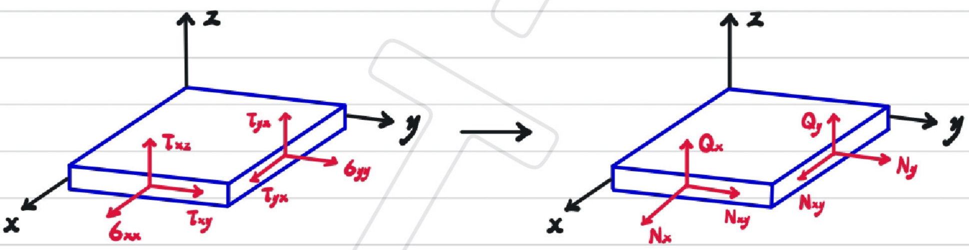
Fundemental Plate Equation

Stress resultants

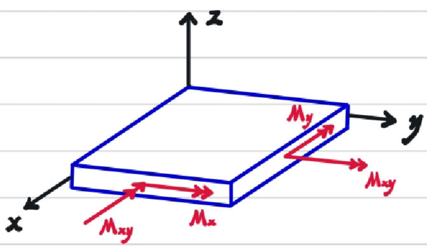
for elastic, isentropic, homogenous, thin $\frac{h}{L} \ll 1$, flat plates

where the deflection at z-direction w of the midplane at $\frac{1}{2}h$ is small compared with the thickness h

assume middle surface remains unstrained after bending



and

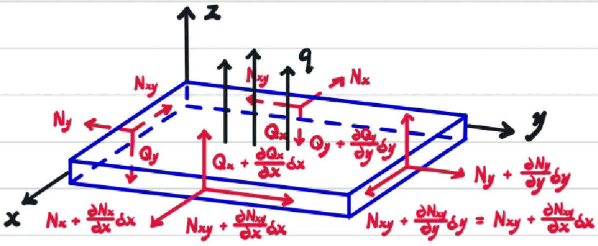


per unit width

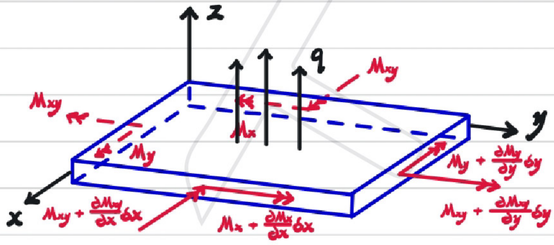
- shear stress resultant $Q = [\, Q_x \; Q_y \,]^T = \int [\, \tau_{xz} \; \tau_{yz} \,]^T \, dz$
- membrane stress resultant $N = [\, N_x \; N_y \; N_{xy} \,]^T = \int \sigma \, dz$ where $\sigma = [\, \sigma_x \; \sigma_y \; \tau_{xy} \,]^T$
- bending stress resultant $M = [\, M_x \; M_y \; M_{xy} \,]^T = \int \sigma z \, dz$ where $\sigma = [\, \sigma_x \; \sigma_y \; \tau_{xy} \,]^T$

Plate equilibrium equations

for out-of-plane load (transverse force per unit area) q (acting out of the plate)



and



force equilibrium $\longrightarrow (Q_x + \frac{\partial Q_x}{\partial x}\delta x)\delta y - Q_x\delta y + (Q_y + \frac{\partial Q_y}{\partial y}\delta y)\delta x - Q_y\delta x + q\delta x\delta y = 0 \longrightarrow \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0$ ①

momentum equilibrium

- *about y-axis* $\longrightarrow (\frac{\partial M_x}{\partial x}\delta x)\delta y + (\frac{\partial M_{xy}}{\partial y}\delta y)\delta x - [(Q_x + \frac{\partial Q_x}{\partial x}\delta x)\delta y]\delta x - [(\frac{\partial Q_y}{\partial y}\delta y)\delta x]\frac{1}{2}\delta y - (q\,\delta x\delta y)\frac{1}{2}\delta x = 0$ ②
- *about x-axis* $\longrightarrow -(\frac{\partial M_y}{\partial y}\delta y)\delta x - (\frac{\partial M_{xy}}{\partial x}\delta x)\delta y + [(Q_y + \frac{\partial Q_y}{\partial y}\delta y)\delta x]\delta y + [(\frac{\partial Q_x}{\partial x}\delta x)\delta y]\frac{1}{2}\delta x + (q\,\delta x\delta y)\frac{1}{2}\delta y = 0$ ③

substitude ①

- *into ②* $\longrightarrow \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x - \frac{1}{2}\frac{\partial Q_x}{\partial x}\delta x = 0 \xrightarrow{\text{dropping higher order terms}} \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0$ ④
- *into ③* $\longrightarrow -\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} + Q_y + \frac{1}{2}\frac{\partial Q_y}{\partial y}\delta y = 0 \xrightarrow{\text{dropping higher order terms}} \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0$ ⑤

$\frac{d}{dx}④ + \frac{d}{dy}⑤ \longrightarrow (\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial y\partial x} - \frac{\partial Q_x}{\partial x}) + (\frac{\partial^2 M_y}{\partial y^2} + \frac{\partial^2 M_{xy}}{\partial x\partial y} - \frac{\partial Q_y}{\partial y}) = 0 \xrightarrow{\text{substitude ①}} \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$ ⑥

Kirchhoff's Plate Theory

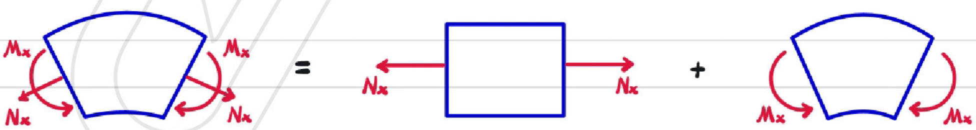
Strains

Kirchhoff hypothesis assume no plane stress $\sigma_z = \tau_{xz} = \tau_{yz} = 0 \longrightarrow \gamma_{xz} = \gamma_{yz} = 0$ and $w(x,y) = w(x,y,z)$

where $\theta_y \approx -\frac{\partial w}{\partial x}$

$\longrightarrow u = z\theta_y \approx -z\frac{\partial w}{\partial x}$ and $v \approx -z\frac{\partial w}{\partial y}$

$$\xrightarrow{\text{Kirchhoff hypothesis}} \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} -z\frac{\partial^2 w}{\partial x^2} & -2z\frac{\partial^2 w}{\partial x\partial y} & 0 \\ -2z\frac{\partial^2 w}{\partial x\partial y} & -z\frac{\partial^2 w}{\partial y^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} -z\frac{\partial^2 w}{\partial x^2} \\ -z\frac{\partial^2 w}{\partial y^2} \\ -2z\frac{\partial^2 w}{\partial x\partial y} \end{bmatrix} = zK$$



$\longrightarrow \varepsilon = \varepsilon_0 + zK$

Stresses

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} -z\frac{\partial^2 w}{\partial x^2} \\ -z\frac{\partial^2 w}{\partial y^2} \\ -2z\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} = -\frac{E}{1-\nu^2} z \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} \\ \nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \\ (1-\nu)\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix}$$

Moments

$$M = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} z\,dz = -D\begin{bmatrix} \frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} \\ \nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \\ (1-\nu)\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix}$$

where $D = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{E}{1-\nu^2} z\right) z\,dz = \frac{Eh^3}{12(1-\nu^2)}$

D: flexural stiffness (EI for beam)

$$\xrightarrow{\text{substitude into ⑥}} -D\left[\frac{\partial^4 w}{\partial x^4} + \nu\frac{\partial^4 w}{\partial x^2 \partial y^2} + 2(1-\nu)\frac{\partial^4 w}{\partial x^2 \partial y^2} + \nu\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] + p = 0$$

$$\longrightarrow \nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} = q\left[\frac{Eh^3}{12(1-\nu^2)}\right]^{-1}$$

Exact Solution of the Plate Equation

Boundary conditions

8 boundary conditions

at each side

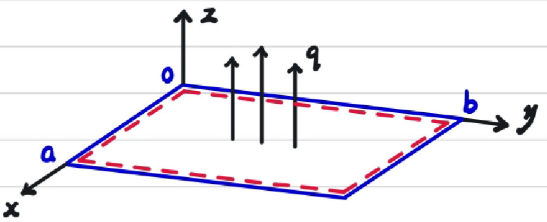
- essential (geometrical) boundary condition: w (deflection; Q_n) and w' (rotation) $\left(\frac{\partial w}{\partial n}, \frac{\partial w}{\partial t}\right)$ (M_n, M_{nt})
- natural (force) boundary condition: $w''\left(\frac{\partial^2 w}{\partial n^2}, \frac{\partial^2 w}{\partial t^2}\right) \propto M_n$ (culvature) and $w'''\left(\frac{\partial^3 w}{\partial n^3}, \frac{\partial^3 w}{\partial t^3}\right) \propto Q_n$ (shear (vertical) force)

where w, w''' or w', w'' cannot be defined at the same time

- clamped (fixed) edge: $w = 0$ and $\frac{\partial w}{\partial n} = 0$
- simply supported edge: $w = 0$ and $M_n = 0 \longrightarrow \frac{\partial^2 M}{\partial n^2} + \nu\frac{\partial^2 M}{\partial t^2} = 0 \xrightarrow{\left.\frac{\partial w}{\partial t}\right|_{edge} = 0} \frac{\partial^2 M}{\partial n^2} = 0$
- free edge: $\underbrace{Q_n = M_n = M_{nt} = 0}_{\text{12 boundary conditions}} \longrightarrow \underbrace{V_n}_{\text{effective shear force}} = 0$ and $M_n = \frac{\partial^2 w}{\partial n^2} = 0$

  where $V_n = Q_n + \frac{\partial M_{nt}}{\partial t} \xrightarrow{\text{substitude ④}} \frac{\partial M_n}{\partial n} + 2\frac{\partial M_{nt}}{\partial t} = \left(\frac{\partial^3 w}{\partial n^3} + \frac{\partial^3 w}{\partial n \partial t^2}\right) + 2(1-\nu)\frac{\partial^3 w}{\partial n \partial t^2} = \frac{\partial^3 w}{\partial n^3} + (2-\nu)\frac{\partial^3 w}{\partial n \partial t^2}$
- sliding edge: $\frac{\partial w}{\partial n} = 0$ and $V_n = \frac{\partial M_n}{\partial n} + 2\frac{\partial M_{nt}}{\partial t} = \frac{\partial^3 w}{\partial n^3} + (2-\nu)\frac{\partial^3 w}{\partial n \partial t^2} = 0$

Navier's solution for a simply supported plate



for $w|_{edge} = 0$ and $M_n|_{edge} = \frac{\partial^2 w}{\partial n^2}|_{edge} = 0$

$\xrightarrow{x|_{edge} = 0,\ a}$ $w(0,y) = \frac{\partial^2 w}{\partial x^2}(0,y) = 0$ and $w(a,y) = \frac{\partial^2 w}{\partial x^2}(a,y) = 0$

$\xrightarrow{y|_{edge} = 0,\ b}$ $w(x,0) = \frac{\partial^2 w}{\partial y^2}(x,0) = 0$ and $w(x,b) = \frac{\partial^2 w}{\partial x^2}(x,b) = 0$

$\longrightarrow$ let $w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$ and $q = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$

$$\longrightarrow \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \left[\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right] \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$= \pi^4\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$\xrightarrow{\text{Kirchhoff equilibrium equation}} \pi^4\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = \frac{1}{D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$\longrightarrow \int_0^a\int_0^b \pi^4\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\left(\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}\right) dxdy = \frac{1}{D}\int_0^a\int_0^b \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\left(\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}\right) dxdy$$

$$\xrightarrow{\text{orthogonality}} \pi^4\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2\left(A_{mn}\frac{a}{2}\frac{b}{2}\right) = \frac{1}{D}\left(B_{mn}\frac{a}{2}\frac{b}{2}\right)$$

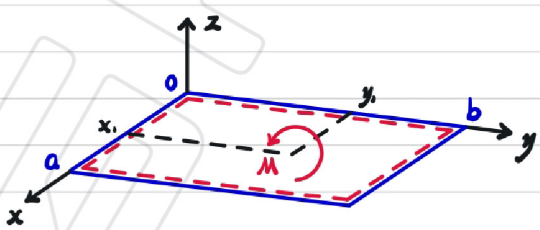
$$\longrightarrow A_{mn} = \frac{B_{mn}}{D\pi^4\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2} \quad \text{where} \quad D = \frac{Eh^3}{12(1-\nu^2)}$$

where $\int_0^a\int_0^b q\left(\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}\right) dxdy = \int_0^a\int_0^b \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\left(\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}\right) dxdy$

$$\longrightarrow B_{mn} = \frac{4}{ab}\int_0^a\int_0^b q \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\, dxdy$$

e.g. for $q = F(x_0, y_0)$, $B_{mn} = \frac{4}{ab}\int_0^a\int_0^b F\delta(x-x_0)\delta(y-y_0)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\, dxdy = \frac{4F}{ab}\sin\frac{m\pi x_0}{a}\sin\frac{n\pi y_0}{b}$

for M shown below, $q = \lim_{e\to 0}[P(x_0, y_0) - P(x_0, y_0 - e)]$ where $P = \lim_{e\to 0}\left(\frac{M}{e}\right)$



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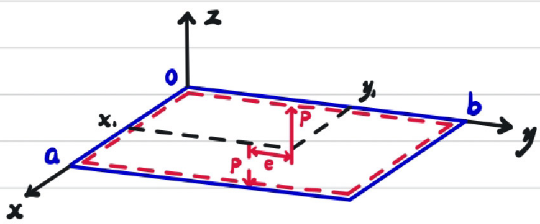


$$\longrightarrow B_{mn} = \frac{ab}{4}\lim_{e\to 0}\int_0^a\int_0^b [P\delta(x-x_0)\delta(y-y_0) - P\delta(x-x_0)\delta(y-y_0+e)]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\, dxdy$$

Energy Method for Plate Problems

Strain energy in bending

$$U = \frac{1}{2}\int \sigma \cdot \varepsilon \, dV = \frac{1}{2}\int_0^a\int_0^b\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz} \, dx\,dy\,dz$$

$$\xrightarrow{\text{Kirchhoff hypothesis}} \frac{1}{2}\int_0^a\int_0^b \left(\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_x z\,dz\right)\frac{\varepsilon_x}{z} + \left(\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_y z\,dz\right)\frac{\varepsilon_y}{z} + \left(\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{xy} z\,dz\right)\frac{\gamma_{xy}}{z} \, dx\,dy$$

$$\longrightarrow \frac{1}{2}\int_0^a\int_0^b -M_x\frac{\varepsilon_x}{z} - M_y\frac{\varepsilon_y}{z} - M_{xy}\frac{\gamma_{xy}}{z} \, dx\,dy$$

$$\longrightarrow \frac{D}{2}\int_0^a\int_0^b \left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\frac{\partial^2 w}{\partial x^2} + \left(\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\frac{\partial^2 w}{\partial y^2} + \left[(1-\nu)\frac{\partial^2 w}{\partial x\partial y}\right]2\frac{\partial^2 w}{\partial x\partial y} \, dx\,dy$$

$$\longrightarrow U = \frac{1}{2}D\int_0^a\int_0^b \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right] dx\,dy$$

External work in bending

work done in z-direction $W_{op} = \int_0^a\int_0^b q(x,y)\,w(x,y)\,dx\,dy$

work done by edge forces and moments $W_s = \oint_s \left(M_n\frac{\partial w}{\partial n} + M_{nt}\frac{\partial w}{\partial t} + Q_n w\right) ds$

typically zero except cases where loaded edges are allowed to move in loading direction

$$\longrightarrow W = W_{op} + W_s \xrightarrow{\text{typically}} W = W_{op}$$

Potential energy in bending

$$\Pi = U - W = U - (W_{op} + W_s) \xrightarrow{\text{typically}} U - W_{op}$$

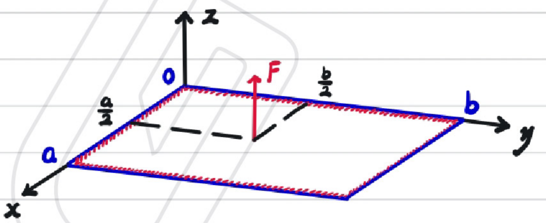
Rayleigh-Ritz method

equilibrium solutions w correspond to minimum in the potential energy $\longrightarrow \frac{\partial \Pi}{\partial w} = 0$

$\xrightarrow{\text{Rayleigh-Ritz method}}$ assumes an approximate solution $w(x,y) = \sum_m\sum_n W_{mn}(x,y)$

where displacement function W_{xy} must satisfy essential (geometrical) boundary condition

e.g. clamped plate, assume $w = \frac{1}{4}W_0\left(1-\cos\frac{2\pi x}{a}\right)\left(1-\cos\frac{2\pi y}{b}\right) = W_0\sin^2\frac{2\pi x}{a}\sin^2\frac{2\pi y}{b}$



$$\longrightarrow \begin{cases} \frac{\partial w}{\partial x} = \frac{\pi}{2a}W_0\sin\frac{2\pi x}{a}\left(1-\cos\frac{2\pi y}{b}\right) \\ \frac{\partial w}{\partial y} = \frac{\pi}{2b}W_0\left(1-\cos\frac{2\pi x}{a}\right)\sin\frac{2\pi y}{b} \end{cases}$$

check essential boundary condition

$\xrightarrow{\text{along } x \text{ edge}} w(0,y) = \frac{\partial w}{\partial x}(0,y) = 0$ and $w(a,y) = \frac{\partial w}{\partial x}(a,y) = 0$

$\xrightarrow{\text{along } y \text{ edge}} w(x,0) = \frac{\partial w}{\partial y}(x,0) = 0$ and $w(x,b) = \frac{\partial w}{\partial x}(x,b) = 0$

$$\rightarrow \frac{\partial^2 w}{\partial x^2} = \left(\frac{\pi}{a}\right)^2 w_0 \cos\frac{2\pi x}{a}\left(1-\cos\frac{2\pi y}{b}\right)$$

$$\rightarrow \frac{\partial^2 w}{\partial y^2} = \left(\frac{\pi}{b}\right)^2 w_0 \left(1-\cos\frac{2\pi x}{a}\right)\cos\frac{2\pi y}{b}$$

$$\rightarrow \frac{\partial^2 w}{\partial x \partial y} = \frac{\pi^2}{ab} w_0 \sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b}$$

$$\longrightarrow U = \frac{1}{2}D\int_0^a\int_0^b \left(\frac{\pi}{a}\right)^4 w_0^2\cos^2\frac{2\pi x}{a}\left(1-2\cos\frac{2\pi y}{b}+\cos^2\frac{2\pi y}{b}\right) + \left(\frac{\pi}{b}\right)^4 w_0^2\left(1-2\cos\frac{2\pi x}{a}+\cos^2\frac{2\pi x}{a}\right)\cos^2\frac{2\pi y}{b}$$

$$+ 2\nu\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2 w_0^2\left(\cos\frac{2\pi x}{a}-\cos^2\frac{2\pi x}{a}\right)\left(\cos\frac{2\pi y}{b}-\cos^2\frac{2\pi y}{b}\right) + 2(1-\nu)\left(\frac{\pi^2}{ab}\right)^2 w_0^2\sin^2\frac{2\pi x}{a}\sin^2\frac{2\pi y}{b}\,dx\,dy$$

$$= \frac{1}{2}D\int_0^b \left(\frac{\pi}{a}\right)^4 w_0^2\left(\frac{a}{2}\right)\left(1-2\cos\frac{2\pi y}{b}+\cos^2\frac{2\pi y}{b}\right) + \left(\frac{\pi}{b}\right)^4 w_0^2\left(a-0+\frac{a}{2}\right)\cos^2\frac{2\pi y}{b}$$

$$+ 2\nu\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2 w_0^2\left(0-\frac{a}{2}\right)\left(\cos\frac{2\pi y}{b}-\cos^2\frac{2\pi y}{b}\right) + 2(1-\nu)\left(\frac{\pi^2}{ab}\right)^2 w_0^2\left(\frac{a}{2}\right)\sin^2\frac{2\pi y}{b}\,dy$$

$$= \frac{1}{2}D\left[\left(\frac{\pi}{a}\right)^4 w_0^2\left(\frac{a}{2}\right)\left(b-0+\frac{b}{2}\right) + \left(\frac{\pi}{b}\right)^4 w_0^2\left(\frac{3}{2}a\right)\left(\frac{b}{2}\right) + 2\nu\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2 w_0^2\left(-\frac{a}{2}\right)\left(0-\frac{b}{2}\right) + 2(1-\nu)\left(\frac{\pi^2}{ab}\right)^2 w_0^2\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\right.$$

$$= \frac{1}{2}D\pi^4 w_0^2\left(\frac{3b}{4a^3}+\frac{3a}{4b^3}+\frac{\nu}{2ab}+\frac{1-\nu}{2ab}\right) = \frac{1}{2}D\pi^4 w_0^2\left(\frac{3b}{4a^3}+\frac{3a}{4b^3}+\frac{1}{2ab}\right)$$

and $W_{op} = \int_0^a\int_0^b F\delta\left(x-\frac{a}{2}\right)\delta\left(y-\frac{b}{2}\right)\frac{1}{4}w_0\left(1-\cos\frac{2\pi x}{a}\right)\left(1-\cos\frac{2\pi y}{b}\right)dx\,dy$

$$= \frac{1}{4}Fw_0(1+1)(1+1) = Fw_0$$

$$\longrightarrow \Pi = U - W_{op} = \frac{1}{2}D\pi^4 w_0^2\left(\frac{3b}{4a^3}+\frac{3a}{4b^3}+\frac{1}{2ab}\right) - Fw_0$$

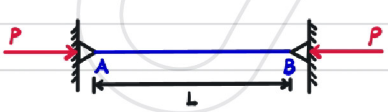
$$\longrightarrow \frac{d\Pi}{dw_0} = D\pi^4 w_0\left(\frac{3b}{4a^3}+\frac{3a}{4b^3}+\frac{1}{2ab}\right) - F = 0$$

$\longrightarrow w_0 = \frac{F}{D\pi^4}\left(\frac{3b}{4a^3}+\frac{3a}{4b^3}+\frac{1}{2ab}\right)^{-1}$ where $D = \frac{Eh^3}{12(1-\nu^2)}$

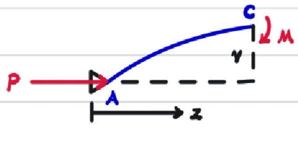
Plate Buckling

Beam Buckling

Elementary beam



$\longrightarrow$



$\xrightarrow{\text{take moment at } C} M = Pv \xrightarrow{M = -EIv''} -EIv'' = Pv$

$\longrightarrow v'' + \frac{P}{EI}v = v'' + \mu^2 v = 0 \longrightarrow$ let $v = v_1 \cos(\mu z) + v_2 \sin(\mu z) \xrightarrow{\text{at A, } v = v'' = 0} v_1 = 0$

$\xrightarrow{\text{at B, } v = v'' = 0} v_2 \sin(\mu L) = 0 \xrightarrow{v_2 \neq 0} \mu L = n\pi \longrightarrow \sqrt{\frac{P}{EI}}L = n\pi \longrightarrow P = \frac{n^2\pi^2 EI}{L^2} \xrightarrow{\text{lowest } P} P_{crit} = \frac{\pi^2 EI}{L^2}$

$\longrightarrow P_{crit} = \frac{\pi^2 EI}{L_{eff}^2}$ where

- → fixed-free end $L_{eff} = 2L$
- → pin-pin end $L_{eff} = L$
- → fixed-pin end $L_{eff} = \frac{L}{\sqrt{2.05}}$
- → fixed-fixed end $L_{eff} = \frac{1}{2}L$

Shortening of a beam due to buckling

$u = \Delta L = L\varepsilon = \frac{L\sigma}{E} = \frac{PL}{AE} \xrightarrow{\text{buckling}} u_1 = \frac{P_1 L}{AE}$

for small deflection $\frac{u}{L} \ll 1$ and $(\frac{dw}{dz})^2 \ll 1$, $ds = \sqrt{dz^2 + dw^2} = \sqrt{1 + (\frac{dw}{dz})^2}\,dz \xrightarrow{\text{Taylor expansion}} [1 + \frac{1}{2}(\frac{dw}{dz})^2]\,dz$

$\longrightarrow ds = \{1 + \frac{1}{2}[\frac{d}{dz}(v_2 \sin \mu L)]^2\}\,dz \xrightarrow{\mu L = n\pi \longrightarrow \pi} [1 + \frac{1}{2}v_2^2 \frac{\pi^2}{L_{eff}^2}\cos^2(\frac{\pi}{L_{eff}}z)]\,dz$

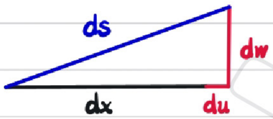
$\longrightarrow L - u_1 = \int ds = \int_0^{L-u} [1 + \frac{1}{2}v_2^2 \frac{\pi^2}{L_{eff}^2}\cos^2(\frac{\pi}{L_{eff}}z)]\,dz = [z + \frac{\pi v_2^2}{4L_{eff}}(\frac{\pi z}{L_{eff}} + \frac{1}{2}\sin\frac{2\pi z}{L_{eff}})]_0^{L-u} = (L-u) + \frac{\pi^2 v_2^2}{4L_{eff}}$

$\xrightarrow{\frac{u}{L} \ll 1} u - u_1 = \frac{\pi^2 v_2^2}{4L_{eff}} \longrightarrow v_2 \approx \frac{2}{\pi}\sqrt{(u - u_1)L_{eff}}$

Energy Method for finding Buckling Loads

External work done by in-plane normal forces

by 1st order deflection theory $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} < 1$ and assume small strains $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \ll 1$



$ds = \sqrt{(dx + du)^2 + dw^2} = \sqrt{dx^2 + 2dxdu + du^2 + dw^2} = \sqrt{1 + 2\frac{du}{dx} + (\frac{du}{dx})^2 + (\frac{dw}{dx})^2}\,dx$

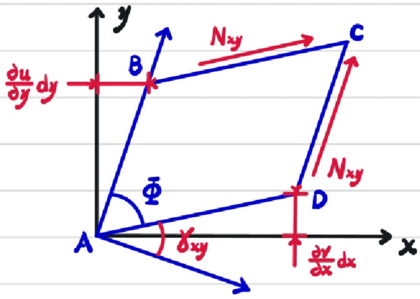
$\xrightarrow{\frac{du}{dx} \ll 1} \sqrt{1 + 2\frac{\partial u}{\partial x} + (\frac{\partial w}{\partial x})^2}\,dx \xrightarrow{\text{Taylor expansion}} [1 + \frac{\partial u}{\partial x} + \frac{1}{2}(\frac{\partial w}{\partial x})^2]\,dx$

$\longrightarrow ds - dx = [\frac{\partial u}{\partial x} + \frac{1}{2}(\frac{\partial w}{\partial x})^2]\,dx$

$\longrightarrow dW_x = (N_x\,dy)(ds - dx) = N_x[\frac{\partial u}{\partial x} + \frac{1}{2}(\frac{\partial w}{\partial x})^2]\,dx\,dy$

$\xrightarrow{\text{similarily}} dW_y = N_y[\frac{\partial u}{\partial y} + \frac{1}{2}(\frac{\partial w}{\partial y})^2]\,dx\,dy$

External work done by in-plane shear forces



$$dW_{xy} = (N_{xy}\,dx)(\frac{\partial u}{\partial y}dy) + (N_{xy}\,dy)(\frac{\partial v}{\partial x}dx) = N_{xy}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})\,dxdy$$

$$= N_{xy}\gamma_{xy}\,dxdy \approx N_{xy}\sin(\gamma_{xy})\,dxdy = N_{xy}\cos(\Phi)\,dxdy$$

$$= N_{xy}\frac{AD\cdot AB}{|AD||AB|}\,dxdy \quad \text{where } AD = (1, \frac{\partial v}{\partial x}, \frac{\partial w}{\partial x})dx \text{ and } AB = (\frac{\partial u}{\partial y}, 1, \frac{\partial w}{\partial y})dy$$

$$= N_{xy}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y})\,dxdy$$

Potential energy in buckling

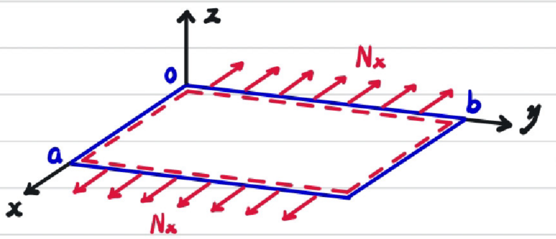
$$\Pi = U - W = U - (W_{op} + W_{ip})$$

neglected membrane strains $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} \approx 0 \longleftarrow$ membrane stress and strain are stationary at buckling

$$\Pi = \frac{1}{2}D\int_0^a\int_0^b [(\frac{\partial^2 w}{\partial x^2})^2 + (\frac{\partial^2 w}{\partial y^2})^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)(\frac{\partial^2 w}{\partial x\partial y})^2]\,dxdy - \int_0^a\int_0^b q(x,y)\,w(x,y)\,dxdy$$

$$- \oint_s (M_n\frac{\partial w}{\partial n} + M_{nt}\frac{\partial w}{\partial t} + Q_n w)\,ds - \frac{1}{2}\int_0^a\int_0^b [N_x(\frac{\partial w}{\partial x})^2 + N_y(\frac{\partial w}{\partial y})^2 + 2N_{xy}(\frac{\partial w}{\partial x})(\frac{\partial w}{\partial y})]\,dxdy$$

Buckling of a simply supported plate



$\xrightarrow{\text{Navier's solution}}$ let $w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

$$\frac{\partial w}{\partial x} = \frac{m}{a}\pi\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\frac{\partial w}{\partial y} = \frac{n}{b}\pi\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$\frac{\partial^2 w}{\partial x^2} = -(\frac{m}{a})^2\pi^2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\frac{\partial^2 w}{\partial y^2} = -(\frac{n}{b})^2\pi^2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\frac{\partial^2 w}{\partial x\partial y} = \frac{m}{a}\frac{n}{b}\pi^2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$\longrightarrow U = \frac{1}{2}D\pi^4 A_{mn}^2[(\frac{m}{a})^4(\frac{a}{2})(\frac{b}{2}) + (\frac{n}{b})^4(\frac{a}{2})(\frac{b}{2}) + 2\nu(\frac{m}{a})^2(\frac{n}{b})^2(\frac{a}{2})(\frac{b}{2}) + 2(1-\nu)(\frac{m}{a})^2(\frac{n}{b})^2(\frac{a}{2})(\frac{b}{2})]$$

$$= \frac{1}{2}D\pi^4 A_{mn}^2(\frac{ab}{4})[(\frac{m}{a})^2 + (\frac{n}{b})^2]^2$$

and $W = \frac{1}{2}\int_0^a\int_0^b N_x(\frac{\partial w}{\partial x})^2\,dxdy = \frac{1}{2}\int_0^a\int_0^b N_x(\frac{m}{a})^2\pi^2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2\cos^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}\,dxdy = \frac{1}{2}N_x\pi^2 A_{mn}^2(\frac{ab}{4})(\frac{m}{a})^2$

$$\longrightarrow \Pi = U - W = \frac{ab}{8}\pi^2 A_{mn}^2\{D\pi^2[(\frac{m}{a})^2 + (\frac{n}{b})^2]^2 - N_x(\frac{m}{a})^2\}$$

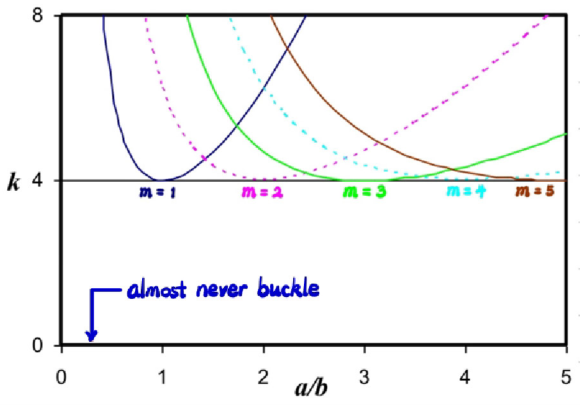
$$\longrightarrow \frac{d\Pi}{dA_{mn}} = \frac{ab}{4}\pi^2 A_{mn}\{D\pi^2[(\frac{m}{a})^2 + (\frac{n}{b})^2]^2 - N_x(\frac{m}{a})^2\} = 0$$

$$\xrightarrow{A_{mn}\neq 0} D\pi^2(\frac{m^2}{a^2} + \frac{n^2}{b^2})^2 - N_x\frac{m^2}{a^2} = 0 \longrightarrow N_x = D\pi^2\frac{a^2}{m^2}(\frac{m^2}{a^2} + \frac{n^2}{b^2})^2 \quad \text{where } D = \frac{Eh^3}{12(1-\nu^2)}$$

one buckle in y-direction, n=1 → $N_{xB} = D\pi^2 \frac{a^2}{m^2}\left(\frac{m^2}{a^2}+\frac{1}{b^2}\right)^2 = \frac{D\pi^2}{b^2}\left(\frac{mb}{a}+\frac{a}{mb}\right)^2 = \frac{D\pi}{b^2}k_{ss}$

half-wave (buckles) in x-direction

the length of the half-wave $\lambda = \frac{a}{m}$ → $k_{ss} = \left(\frac{b}{\lambda}+\lambda b\right)^2 = b^2\left(\frac{1}{\lambda}+\lambda\right)^2 = \left(\frac{1}{\bar{\lambda}}+\bar{\lambda}\right)^2$



for minimum buckling loads $\frac{\partial k_{ss}}{\partial \bar{\lambda}} = 2\left(1-\frac{1}{\bar{\lambda}^2}\right)\left(\frac{1}{\bar{\lambda}}+\bar{\lambda}\right)$

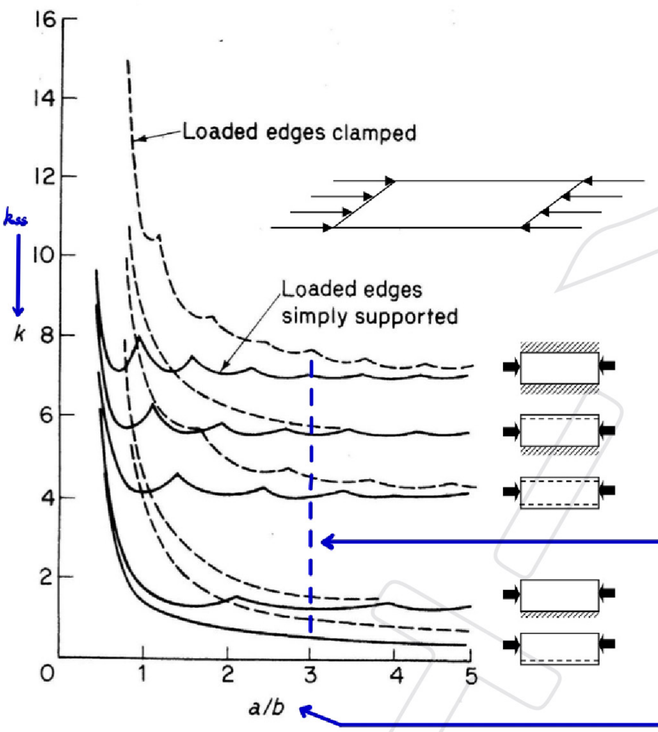
→ when $\bar{\lambda} = \frac{mb}{a} = 1$, $(k_{ss})_{min} = 4$ → $m = \frac{a}{b}$ ← square buckles

for maximum buckling loads $\sqrt{(k_{ss})_{max}} = \frac{mb}{a}+\frac{a}{mb} = \frac{(m+1)b}{a}+\frac{a}{(m+1)b}$

→ $\frac{a}{b} = \sqrt{m(m+1)}$ → $(k_{ss})_{max} = \left(\sqrt{\frac{m}{m+1}}+\sqrt{\frac{m+1}{m}}\right)^2$

→ when $\frac{a}{b} \geq 3$, $(k_{ss})_{max} \approx (k_{ss})_{min} = 4$

Buckling load for other boundary conditions



$N_{xB} = \frac{D\pi^2}{b^2}\left(\frac{mb}{a}+\frac{a}{mb}\right) = \frac{D\pi^2}{b^2}k_{ss}$ where $D = \frac{Eh^3}{12(1-\nu^2)}$

→ $\sigma_{xB} = \frac{N_{xB}}{h} = \frac{D\pi^2}{b^2 h}k_{ss} = KE\left(\frac{h}{b}\right)^2$

where $K = \frac{1}{Eh^3}D\pi^2 k_{ss} = \frac{k_{ss}\pi^2}{12(1-\nu^2)} \xrightarrow{\nu = 0.3} 0.9k$

→ $\sigma_{xB} = (0.9k)E\left(\frac{h}{b}\right)^2$

$= 3.62E\left(\frac{h}{b}\right)^2$ for simply supported plate

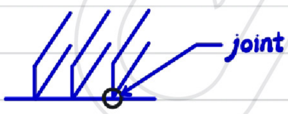
boundary condition at the loaded edge converged to the simply supported case as $\frac{a}{b}$ increase approximately for $\frac{a}{b} \geq 3$

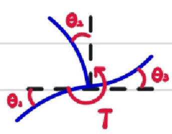
aspect ratio

Interaction Between Buckling Modes

Equilibrium at joint

$\sigma_{xB} \propto \frac{1}{b^2}$ → introduce longitudinal stiffeners (stringers) to seperate panel into regions with smaller width





→ $T = \Sigma k\theta = k_{\theta1}\theta_1 + k_{\theta2}\theta_2 + k_{\theta3}\theta_3$

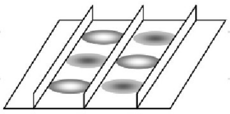
$\xrightarrow{\text{continuity, } \theta = \text{constant}} (k_{\theta1}+k_{\theta2}+k_{\theta3})\theta = 0$

- $\theta = 0$ clamped (on unloaded edges) plates ← highest buckling load
- $k_{\theta 1} = k_{\theta 2} = k_{\theta 3} = 0$ simply supported plates ← lowest buckling load
- $\Sigma k = 0$ elastically restrained plates →
  - $k_{\theta i} > 0$ resist rotation $\xrightarrow{\text{stable}}$ post buckling
  - $k_{\theta i} = 0$ simply supported $\xrightarrow{\text{neutrally (un)stable}}$ at buckling
  - $k_{\theta i} < 0$ creat moment that need to twist back $\xrightarrow{\text{unstable}}$ after buckling

  ↑ intermediate buckling load

Buckling modes of stiffened panel

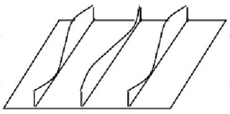
skin buckling



→ might allowed

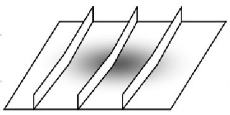
→ reduce aerodynamic performance

stiffener buckling



→ avoid

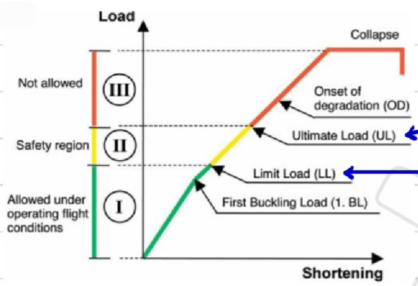
global panel buckling



→ optimal use of the material ← very sensitive to imperfections

→ no addition load carrying capacity once buckling started

Conservative design approach



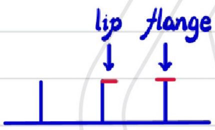
beef up stiffeners → acceptable safety margin to global buckling

1.5 × limit load stiffener and global buckling allowed above it

limit load skin buckling allowed above it

consider element seperately and assume they behave simply supported at the joint

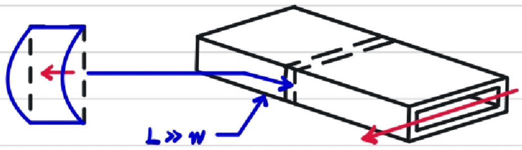
- stiffener $\xrightarrow{\text{3 simply supported edge + 1 free edge}}$ $\sigma_B \approx 0.385E(\frac{h_s}{b_s})^2$
- skin between two stiffeners $\xrightarrow{\text{4 simply supported edge}}$ $\sigma_B \approx 3.62E(\frac{h}{b})^2$



$\xrightarrow{\text{4 simply supported edge}}$ $\sigma_B = 3.62E(\frac{h_s}{b_s})^2$

Buckling Interaction in Rectangular Tubes

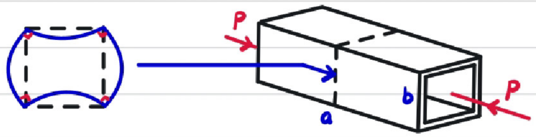
Global buckling of rectangular tube



in the direction of lowest buckling load $P_{crit} = \frac{\pi^2 EI}{L_{eff}^2} \propto I = \frac{tw^3}{12} \propto w$

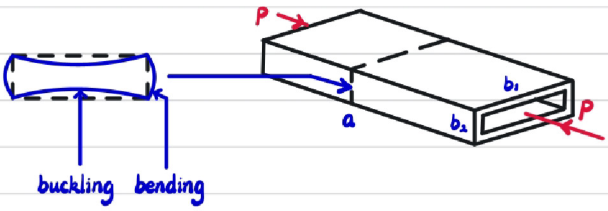
⟶ in the direction of tube have smallest width

Local buckling in square tube



for $\frac{a}{b} \geq 3$, $\sigma_o = \sigma_B = 3.62E(\frac{h}{b})^2$

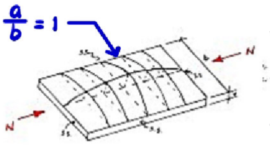
Local buckling in rectangular tube



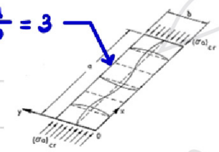
for $\frac{a}{b} \geq 3$, $3.62E(\frac{h}{b_1})^2 < \sigma_o < 3.62E(\frac{h}{b_2})^2$

ESDU data sheets for buckling interaction

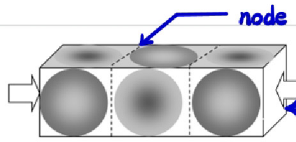
$\frac{a}{b} = 1$



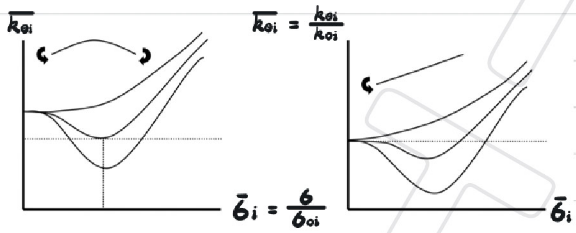
$\frac{a}{b} = 3$



and



⟶ periodic buckling pattern ⟶ edge moment varies sinusoidally along the edge



guess m ⟶ find $\bar{\lambda} = \frac{\lambda}{b_i} = \frac{a_i}{m b_i}$ ⟶ find $k_{ss} = (\frac{1}{\bar{\lambda}} + \bar{\lambda})^2$

⟶ find $\sigma_i = (0.9k)E(\frac{h}{b})^2$ ⟶ $\bar{\sigma}_i = \frac{\sigma_i}{\sigma_{oi}} = \frac{\sigma_i}{3.62E(\frac{h}{b})^2}$

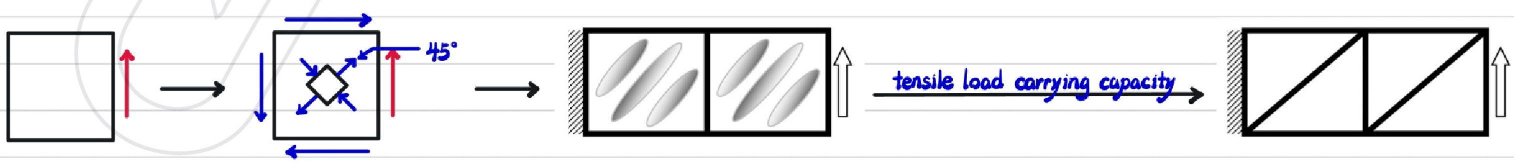
⟶ find $k_{o1}, k_{o2}, k_{o3}\ldots$ $\xrightarrow{\Sigma k = 0}$ find σ_B

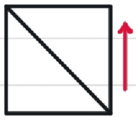
$\Sigma k \neq 0$ (back to guess m)

⟶ verify with $m \pm 1$

Other Forms of Buckling

Shear buckling in phase



tensile stress balanced by compressive stresses $\longrightarrow$  increase buckling load

$$\tau_{xyB} = \frac{N_{xyB}}{h} = \frac{D\pi^2}{hb^2}k_\tau = KE\left(\frac{h}{b}\right)^2$$

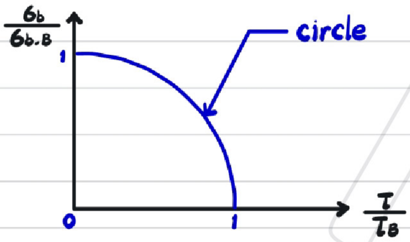
$\rightarrow$ $k_\tau = 5.35$, $K = 4.83$ for simply supported edges

$\rightarrow$ $k_\tau = 8.98$, $K = 8.11$ for clamped edge

Interactive buckling in phase

bending - shear interaction

$$\longrightarrow \left(\frac{\sigma_b}{\sigma_{b.B}}\right)^2 + \left(\frac{\tau}{\tau_B}\right)^2 = 1$$



compression - bending - shear interaction

$$\longrightarrow \left[\frac{\sigma}{\sigma_B} + \left(\frac{\sigma_b}{\sigma_{b.B}}\right)^2\right] + \left(\frac{\tau}{\tau_B}\right)^2 = 1$$

compression - shear interaction

$$\longrightarrow \frac{\sigma}{\sigma_B} + \left(\frac{\tau}{\tau_B}\right)^2 = 1$$