AERO40005 Materials 1

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SOLVED TUTORIAL QUESTIONS and DATASHEET

Q1. *(a) Define Poisson's ratio* ν *and dilatation* Δ *in the straining of an elastic solid; (b) calculate the dilatation in a uniaxial elastic extension of a bar in terms of* ν *and of the axial strain* ε *; find the value of* ν *for which the dilation is equal to zero; (c) Poisson's ratio for most metal is around 0.3, whereas for cork it is close to zero and for rubbers it is close to 0.5. What are the approximate volume changes for these materials during a uniaxial tensile strain as a function of the axial strain* ε *.*

(a) $\varepsilon_t = -\nu\varepsilon \quad \therefore -\nu = \dfrac{\varepsilon_t}{\varepsilon}$

$$\Delta = \varepsilon_v = \frac{\Delta V}{V_0}$$

(b) $\varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

$$= \varepsilon - \nu\varepsilon - \nu\varepsilon = (1-2\nu)\varepsilon$$

(c) $\Delta V = \varepsilon_v V_0 = (1-2\nu)\varepsilon V_0$

$$\therefore \frac{\Delta V}{V_0} = (1-2\nu)\varepsilon$$

For $\nu \approx 0.3$, $\dfrac{\Delta V}{V_0} \approx 0.4\varepsilon$

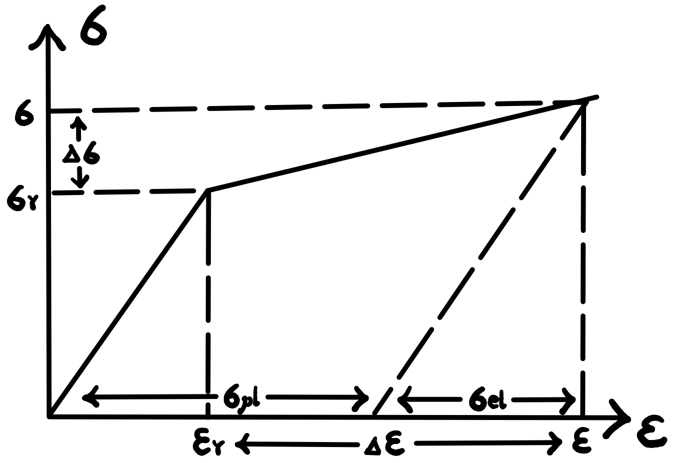
For $\nu \to 0$, $\dfrac{\Delta V}{V_0} \to \varepsilon$

For $\nu \to 0.5$, $\dfrac{\Delta V}{V_0} \to 0$

Q2. An aluminium rod of length 1m and rectangular cross-section of 5 x 20 mm is subjected to a tensile force. The aluminium has modulus 70 GPa, yield stress 250 MPa and linear strain hardening with hardening modulus $H = 200\,\text{MPa}$ *(nominal hardening modulus). If a tensile force of 30 kN is applied to one end of the rod, find the elongation of the bar and calculate the elastic and plastic strains.*

$L = 1m$, $A = 5 \times 20\,mm^2$, $E = 70GPa$, $\sigma_Y = 250MPa$, $H = 200MPa$

$F = 30kN$



$$\sigma = \frac{F}{A} = \frac{3 \times 10^4 N}{1 \times 10^{-4} m^2} = 3 \times 10^8\,Pa = 300MPa$$

In elastic state, $\sigma_Y = 250MPa$

$$\therefore \varepsilon_Y = \frac{\sigma_Y}{E} = \frac{250MPa}{70000MPa} = \frac{1}{280}\,m$$

In plastic state, $\Delta\sigma = \sigma - \sigma_Y = 50MPa$

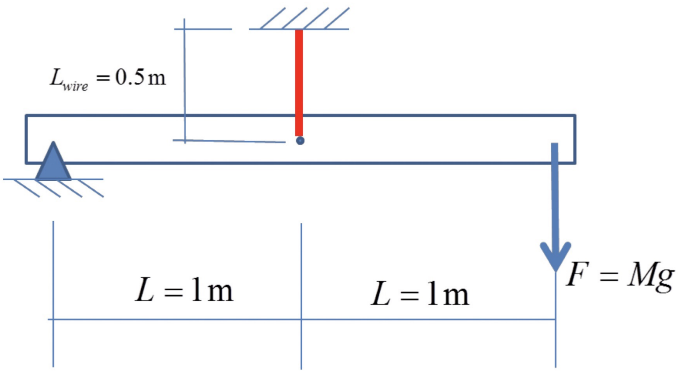
$$\therefore \Delta\varepsilon = \frac{\Delta\sigma}{H} = \frac{50MPa}{200MPa} = 0.25m$$

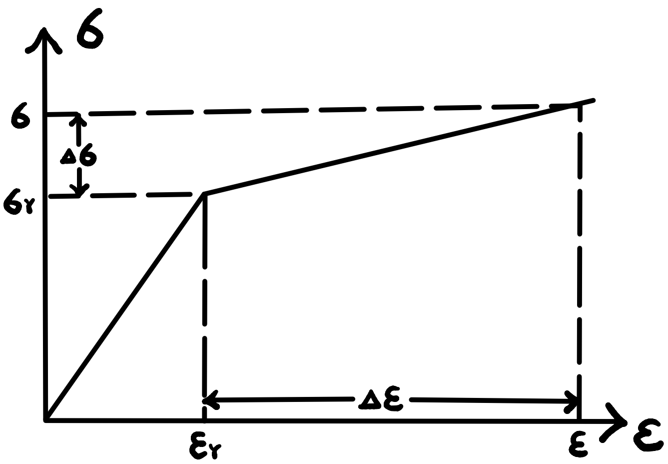
$$\therefore \varepsilon = \varepsilon_Y + \Delta\varepsilon = 0.2536\,m$$

$$\therefore \varepsilon_{el} = \frac{\sigma}{E} = \frac{300MPa}{70000MPa} = \frac{3}{700}\,m = 4.2857 \times 10^{-3}\,m$$

$$\therefore \varepsilon_{pl} = \varepsilon - \varepsilon_{el} = 0.2493m$$

Q3. *Consider a rigid bar hinged at a fixed point and suspended via a steel wire as indicated in the figure. The bar is loaded at the end by the weight of the mass M. The steel wire has an initial cross-sectional area of 4 mm² and is made from a material with a bilinear nominal stress/strain curve with properties* ($E = 210\,\mathrm{GPa};\ \sigma_Y = 400\,\mathrm{MPa};\ \ H = 100\,\mathrm{MPa}$)*. Calculate the value of M to cause an extension of the wire from 500 to 600 mm.*





$$\sigma = \frac{F}{A} = \frac{2Mg}{4\,\mathrm{mm}^2} = 5\times10^5 Mg$$

$$\varepsilon = \frac{\Delta L}{L_0} = \frac{600\,\mathrm{mm} - 500\,\mathrm{mm}}{500\,\mathrm{mm}} = 0.2$$

$$\therefore\ \varepsilon_Y = \frac{\sigma_Y}{E} = \frac{1}{525} \qquad \therefore\ \Delta\varepsilon = \varepsilon - \varepsilon_Y = \frac{104}{525}$$

$$\therefore\ \Delta\sigma = H\Delta\varepsilon = 1.98\times10^7\,Pa$$

$$\therefore\ \sigma = \sigma_Y + \Delta\sigma = 4.198\times10^8\,Pa = 5\times10^5 Mg$$

$$\therefore\ M = 85.6\,kg$$

Q4. *A material specimen of gauge length 40 mm and square cross-section of side 5 mm elongates by 10 μm when subjected to a tension of 1.25 kN. Assuming that the material responds elastically determine: (a) Young's modulus. (b) The change in volume (take Poisson's ratio ν = 0.3).*

(a) $$\varepsilon = \frac{\Delta L}{L} = \frac{10\times10^{-6}\,m}{40\times10^{-3}\,m} = 2.5\times10^{-4}$$

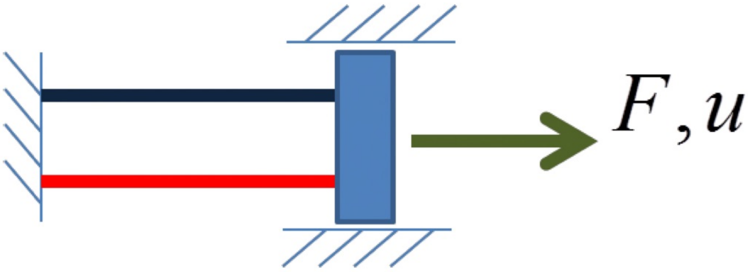
$$\sigma = \frac{F}{A} = \frac{1.25\times10^{3}\,N}{(5\times10^{-3}\,m)^2} = 5\times10^{7}\,Pa$$

$$\therefore E = \frac{\sigma}{\varepsilon} = 2\times10^{11}\,Pa = 200\,GPa$$

(b) $$\varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon - \nu\varepsilon - \nu\varepsilon = (1-2\nu)\varepsilon = \frac{\Delta V}{V_0}$$

$$\therefore \Delta V = V_0\varepsilon_v = (1-2\nu)\varepsilon V_0 = (1-2\nu)\varepsilon A_0 L_0$$

$$= 1\times10^{-10}\,m^3$$



Q5. *A rigid block is constrained to move horizontally and connected to a fixed wall by two metal wires as indicated in the figure. The wires have identical initial cross-section* $A_0 = 5\,\text{mm}^2$ *and length* $L_0 = 0.5\,\text{m}$ *; one of them is made of perfectly-plastic aluminium* $(E_1 = 70\,\text{GPa};\ \sigma_{Y1} = 300\,\text{MPa};\ H_1 = 0\,\text{MPa})$ *while the other is made of perfectly plastic steel* $(E_2 = 210\,\text{GPa};\ \sigma_{Y2} = 500\,\text{MPa};\ H_2 = 0\,\text{MPa})$*; stress quantities are intended to be nominal. A horizontal displacement of 10 mm is imposed such to place the wires in tension; construct a quantitative plot of the force versus displacement history recorded in this experiment.*

For Al wire, $F_1 = \sigma_{Y_1} A_0 = 1500\,N$, $\varepsilon_{Y1} = \dfrac{\sigma_{Y1}}{E_1} = \dfrac{3}{700}$

$$\therefore \Delta L_1 = \varepsilon_{Y_1} L_0 = \frac{3}{1400}\,m = \frac{15}{7}\,mm$$

For steel wire, $F_2 = \sigma_{Y_2} A_0 = 2500\,N$, $\varepsilon_{Y_2} = \dfrac{\sigma_{Y2}}{E_2} = \dfrac{1}{420}$

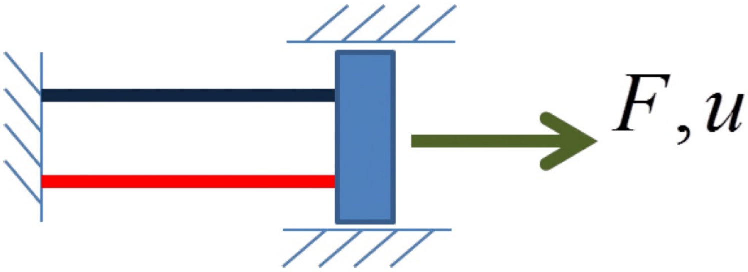
$$\therefore \Delta L_2 = \varepsilon_{Y_2} L_0 = \frac{1}{840}\,m = \frac{25}{21}\,mm$$

$\because \Delta L_1 > \Delta L_2$

$\therefore$ When $\Delta L = \Delta L_2$ for Al wire, $\varepsilon' = \varepsilon_{Y2} = \dfrac{1}{420}$

$\therefore \sigma' = \varepsilon' E_1 = 1.\dot{6} \times 10^8\,Pa$

$\therefore F' = \sigma' A_0 = \dfrac{2500}{3}\,N$

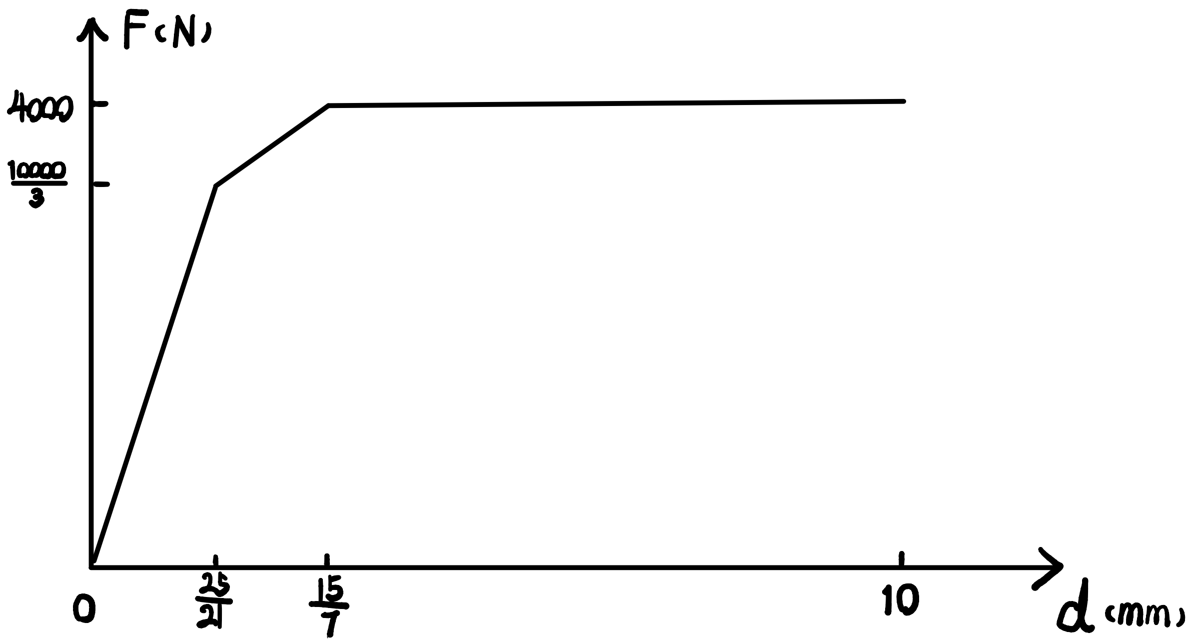


Q5. A rigid block is constrained to move horizontally and connected to a fixed wall by two metal wires as indicated in the figure. The wires have identical initial cross-section $A_0 = 5\,\text{mm}^2$ *and length* $L_0 = 0.5\,\text{m}$ *; one of them is made of perfectly-plastic aluminium* $(E_1 = 70\,\text{GPa};\ \ \sigma_{Y1} = 300\,\text{MPa};\ \ H_1 = 0\,\text{MPa})$ *while the other is made of perfectly plastic steel* $(E_2 = 210\,\text{GPa};\ \ \sigma_{Y2} = 500\,\text{MPa};\ \ H_2 = 0\,\text{MPa})$*; stress quantities are intended to be nominal. A horizontal displacement of 10 mm is imposed such to place the wires in tension; construct a quantitative plot of the force versus displacement history recorded in this experiment.*

At $d = \frac{25}{21}\,\text{mm}$, $F = F' + F_2$

At $d = \frac{15}{7}\,\text{mm}$, $F = F_1 + F_2$

$\therefore$



Q6. *An elastic bar of length 1 m is fixed at one end and loaded in uniaxial tension at the opposite end. It has to withstand a force F of 100 N with a deflection d < 10 mm. Neglecting the possibility of material failure, determine a material merit index to maximise in order to achieve a minimum mass design.*

$$\varepsilon = \frac{\Delta L}{L_0} < 0.01$$

$$\sigma = \frac{F}{A_0} = \frac{FL_0}{V_0} = \frac{100\,\mathrm{Nm}}{V_0} = \frac{100\rho}{m}$$

$$E = \frac{\sigma}{\varepsilon}$$

$$\therefore\ \sigma = E\varepsilon = \frac{100\rho}{m} < 0.01E$$

$$\therefore\ m > 10000\,\frac{\rho}{E}$$

$$\therefore\ m \propto \frac{\rho}{E}$$

To minimize m, minimize $\frac{\rho}{E}$

$$\therefore \text{ maximize } \left(\frac{\rho}{E}\right)^{-1} = \frac{E}{\rho}$$

$$\therefore \text{ material merit index} = \frac{E}{\rho}$$

Q7. *An elastic bar of length 1 m has to be loaded in uniaxial tension by a force F of 1000 N without failure. Determine a material merit index to maximise in order to achieve a minimum mass design.*

$$\sigma = \frac{F}{A_0} = \frac{1000N}{A_0} = \frac{1000L_0}{V_0} = \frac{1000Nm}{V_0} = \frac{1000\rho}{m}$$

$$\sigma = \frac{1000\rho}{m} \leq \sigma_Y$$

$$\therefore\ m \geq \frac{1000\rho}{\sigma_Y}$$

$$\therefore\ m \propto \frac{\rho}{\sigma_Y}$$

To minimize m, minimize $\frac{\rho}{\sigma_Y}$

$\therefore$ maximize $\frac{\sigma_Y}{\rho}$

$\therefore$ merit index $= \frac{\sigma_Y}{\rho}$

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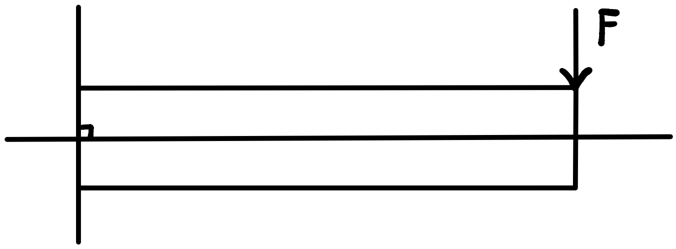
Q8. *A metallic cantilever of square cross-section $t \times t$ and known length L has to withstand a transverse load F at the free end without failure. Determine a material merit index to maximise in order to achieve a design of minimum mass or minimum cost.*



$$\sigma = \frac{F}{A_o} = \frac{6FL}{t^3} \text{ (beam theory)}$$

$$\therefore \sigma \leq \sigma_Y$$

$$\therefore \frac{6FL}{t^3} \leq \sigma_Y$$

$$\because m = \rho V = \rho t^2 L$$

$$\therefore t = \sqrt{\frac{m}{\rho L}}$$

$$\therefore \frac{6F\rho^{\frac{3}{2}} L^{\frac{5}{2}}}{m^{\frac{3}{2}}} \leq \sigma_Y$$

$$\therefore m^{\frac{3}{2}} \geq \frac{6F\rho^{\frac{3}{2}} L^{\frac{5}{2}}}{\sigma_Y}$$

$$\therefore m \geq \frac{6^{\frac{2}{3}} F^{\frac{2}{3}} \rho L^{\frac{5}{3}}}{\sigma_Y^{\frac{2}{3}}} \qquad \therefore m \propto \frac{\rho}{\sigma_Y^{\frac{2}{3}}} \qquad \therefore M_I = \frac{\sigma_Y^{\frac{2}{3}}}{\rho}$$

$$\because \text{cost} = mc \qquad \therefore mc \propto \frac{\rho c}{\sigma_Y^{\frac{2}{3}}} \qquad \therefore M_I = \frac{\sigma_Y^{\frac{2}{3}}}{\rho c}$$

Q14. *A composite construction brick is made from a mixture of sand and natural rubber. Each brick is obtained by thoroughly mixing 4 kg of sand (* $\rho_S = 2500\,\text{kg}\,\text{m}^{-3}$ *) with 2.5 kg of natural rubber (* $\rho_R = 950\,\text{kgm}^{-3}$ *); in the process, a porosity (volume fraction of pores) of 10% is introduced in the aggregate. Calculate the density of the composite brick and the volume fractions of sand and rubber. If a better manufacturing method is used, allowing eliminating porosity completely, calculate the percent increase in the density of the brick.*

$$90\% V = \frac{m_s}{\rho_s} + \frac{m_R}{\rho_R} = \frac{201}{47500}\,m^3 = 4.2316 \times 10^{-3}\,m^3$$

$$\therefore V = \frac{67}{14250}\,m^3 = 4.7018 \times 10^{-3}\,m^3$$

$$\therefore \rho = \frac{m}{V} = \frac{m_s + m_R}{V} = 1382\,kg/m^3$$

$$\therefore \varphi_s = \frac{V_s}{V} = \frac{114}{335} = 0.3403$$

$$\varphi_R = \frac{V_R}{V} = \frac{75}{134} = 0.5597$$

$$If \quad V' = 90\% V = \frac{201}{47500}\,m^3$$

$$\therefore \rho' = \frac{m}{V'} = 1536\,kg/m^3$$

$$\therefore \Delta\rho = \frac{\rho' - \rho}{\rho} \times 100\% = 11.1\%$$

Q15. *A void-free, high performance unidirectional CFRP composite lamina is obtained by mixing high strength carbon fibres ($E = 280\,\text{GPa}$; $\rho = 1800\,\text{kgm}^{-3}$) with an epoxy matrix ($E = 2.5\,\text{GPa}$; $\rho = 1300\,\text{kgm}^{-3}$). Volume fractions are 70% for the fibres and 30% for the matrix; calculate the density of the composite. A square specimen of this lamina of dimensions $1\times1\,\text{m}$ and thickness 1 mm is loaded by a tensile force of 30 kN directed along the fibres. Calculate the extension of the square plate in the direction of loading, the maximum tensile stress in the fibres and the weight of the panel. Will the panel fail if the tensile strength of carbon fibres is 3GPa?*

$$\varphi_f = 0.7 = \frac{V_f}{V} \qquad \therefore V_f = 0.7V$$

$$\varphi_m = 0.3 = \frac{V_m}{V} \qquad \therefore V_m = 0.3V$$

$$\therefore V = V_f + V_m$$

$$m = \rho_f V_f + \rho_m V_m = (0.7\rho_f + 0.3\rho_m)V$$

$$\therefore \rho = \frac{m}{V} = (0.7\rho_f + 0.3\rho_m) = 1650\,kg/m^3$$

$$\therefore m = \rho V = 1.65kg$$

$$\therefore 6 = \frac{F}{A_0} = 3\times10^7\,Pa$$

$$\therefore E_1 = \varphi_f E_f + \varphi_m E_m = 196.75\,GPa$$

$$\therefore \varepsilon_1 = \frac{6}{E_1} = 1.525\times10^{-4}$$

$$\therefore \Delta L_1 = \varepsilon_1 L = 1.525\times10^{-4}\,m$$

$$\therefore 6_f = E_f\varepsilon_1 = 4.269\times10^7\,Pa < 3GPa$$

Q17. *Pure, solid aluminium (*$E_s = 70\,\text{GPa};\ \sigma_{Ys} = 200\,\text{MPa};\ \rho_s = 2700\,\text{GPa};$*) can be foamed by various processes to any relative density in the range* $0.05 < \bar{\rho} < 1$. *For a certain foaming process, scaling laws are* $E_{foam} = E_s\bar{\rho};\ \sigma_{Yfoam} = \sigma_{Ys}\bar{\rho}^2$. *Calculate the ranges of moduli and compressive yield stress of the foams that can be manufactured from this solid material.*

$E_{foam} = E_s \bar{\rho}$

$\therefore\ 3.5\,\text{GPa} \leq E_{foam} \leq 70\,\text{GPa}$

$\sigma_{Y foam} = \sigma_{Ys} \bar{\rho}^2$

$\therefore\ 0.5\,\text{MPa} \leq \sigma_{Y foam} \leq 200\,\text{MPa}$

Q18. *A crash absorber for a train has to be designed, to be fitted at the date end of a railway line. The component should be able to slow down and arrest gradually a train, of mass 1000 tonnes, impacting at a velocity of 10 km/h. The maximum allowable deceleration for safety of people and goods is 10 m/s^2. If the absorber is a foam cylinder of cross-section* $A = 0.1\,\mathrm{m}^2$ *and length L, calculate the yield stress of the foam needed and L, assuming a densification strain of 0.8.*

$$F = ma = 1 \times 10^7 N$$

$$\varepsilon = 0.8$$

$$\Delta L = \frac{v^2 - u^2}{2a} = 0.3858m$$

$$\therefore \sigma_Y = \frac{F}{A_0} = 1 \times 10^8 Pa$$

$$L = \frac{\Delta L}{\varepsilon} = 0.4823m$$

The following datasheet will be available at the exam

DATASHEET – AERO40005

$$\sigma_n = \frac{F}{A_0};$$

$$\sigma_t = \frac{F}{A} = \sigma_n \ln\left(1+\varepsilon_n\right) \qquad \left(\text{for plastically incompressible solids}\right)$$

$$\varepsilon_n = \frac{\Delta L}{L_0};\ \varepsilon_t = \ln\left(1+\varepsilon_n\right);\quad \varepsilon_{V_n} = \frac{\Delta V}{V_0}$$

$$\tau = G\gamma;\quad G = \frac{E}{2\left(1+\nu\right)};$$

$$\sigma_H = -p = \frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3};\qquad \sigma_H = K\varepsilon_V;\qquad K = \frac{E}{3\left(1-2\nu\right)}$$

$$U\left(r\right) = -\frac{A}{r^m} + \frac{B}{r^n},\quad n>m;\ F\left(r\right) = \frac{dU}{dr};\qquad S\left(r\right) = \frac{dF}{dr} = \frac{d^2U}{dr^2};\quad S_0 = S\left(r_0\right)$$

$$\varepsilon^{th} = \alpha\,\Delta T$$

$$\dot{\varepsilon} = \frac{d\varepsilon}{dt} = \frac{dl}{ldt} = V/l$$

$$\rho = \varphi_{\mathrm{f}}\rho_{\mathrm{f}} + \left(1-\varphi_{\mathrm{f}}\right)\rho_{\mathrm{m}};$$

$$E_1 = \varphi_f E_f + \left(1-\varphi_f\right)E_m;$$

$$\frac{1}{E_2} = \frac{\varphi_f}{E_f} + \frac{\left(1-\varphi_f\right)}{E_m};\ \frac{1}{G_2} = \frac{\varphi_f}{G_f} + \frac{\left(1-\varphi_f\right)}{G_m}$$

$$\bar{\rho} = \frac{\rho_{foam}}{\rho_S};\quad f = \frac{V_{pores}}{V_{total}} = 1-\bar{\rho}$$

SOLUTIONS

Q1. *(a) Define Poisson's ratio* ν *and dilatation* Δ *in the straining of an elastic solid; (b) calculate the dilatation in a uniaxial elastic extension of a bar in terms of* ν *and of the axial strain* ε *; find the value of* ν *for which the dilation is equal to zero; (c) Poisson's ratio for most metal is around 0.3, whereas for cork it is close to zero and for rubbers it is close to 0.5. What are the approximate volume changes for these materials during a uniaxial tensile strain as a function of the axial strain* ε *.*

Q1 - solution

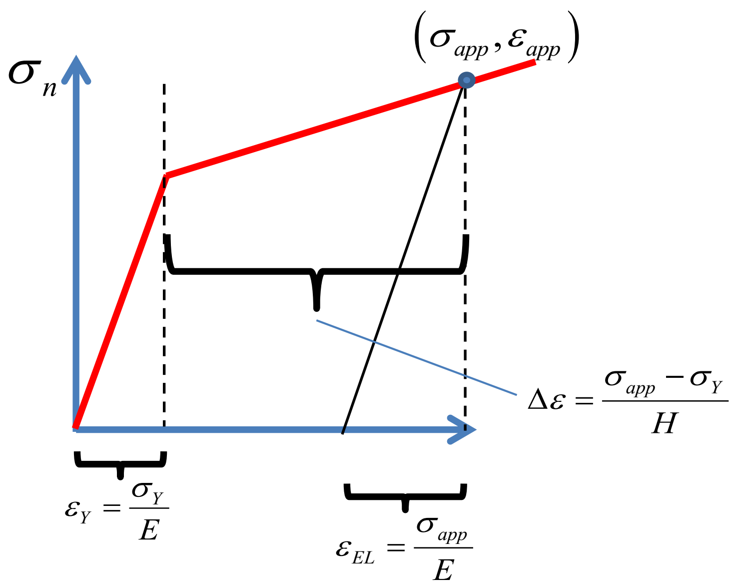
$a)$ $\quad \nu = -\dfrac{\varepsilon_{transverse}}{\varepsilon_{axial}}; \quad \Delta = \varepsilon_V = \dfrac{\Delta V}{V_0}$

$b)$ $\quad \varepsilon_V = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_{axial} - \nu\varepsilon_{axial} - \nu\varepsilon_{axial} = (1-2\nu)\varepsilon_{axial}$

$c)$ $\quad$ substitute given Poisson's ratios into previous equation

Q2. *An aluminium rod of length 1m and rectangular cross-section of 5 x 20 mm is subjected to a tensile force. The aluminium has modulus 70 GPa, yield stress 250 MPa and linear strain hardening with hardening modulus* $H = 200\,\text{MPa}$ *(nominal hardening modulus). If a tensile force of 30 kN is applied to one end of the rod, find the elongation of the bar and calculate the elastic and plastic strains.*

Q2 – solution



With reference to the figure above:

$$\sigma_{app} = \frac{F}{A_0} = \frac{30\ 10^3 N}{10^2 \text{mm}^2} = 300 \text{ MPa}; \ \varepsilon_Y = \frac{\sigma_Y}{E} = \frac{250}{70\ 10^3} = 3.6\ 10^{-3}; \ \varepsilon = \varepsilon_{app} = \frac{\Delta L}{L_0}$$

$$\Delta\varepsilon = \frac{\Delta\sigma}{H} = 0.25; \ \ \varepsilon_{app} = \varepsilon_Y + \Delta\varepsilon = 0.2536; \ \ \Delta L = 0.2536 \text{ m}$$

$\varepsilon_{EL} = \frac{300}{70000} = 4.28\ 10^{-3}$; $\varepsilon_{PL} = \varepsilon_{app} - \varepsilon_{EL} = 0.24932$; note that elastic contributions are always minor at large strains.

NOTE: **<u>The solution method above is only valid if the stress/strain curve is provided in terms of engineering stress and strains</u>**, as only in this case the use of engineering stress and strain is justified.

If the stress/strain curve (i.e. the set of given yield stress, elastic and hardening moduli) is interpreted in terms of 'true' stress/strain quantities, we need to work in terms of true stress and strain! For the data given it turns out that <u>this question has no solution if the data is interpreted in true stress/strain terms</u>. Let's see how this is possible.

<u>Assume now that the data given is interpreted in terms of true stress/strain quantities</u> and let's solve the question again; the subscripts n and t refer to nominal and true quantities, respectively. For the yield stress and the yield strain (as well as modulus), we can assume that these do not change as they are defined at low strains, where there is only a little difference between nominal and true stress/strain:

$\varepsilon_Y = \varepsilon_{Y_t} \approx \varepsilon_{Y_n} = 0.0036$ (note: ln(1.0036)=~0.0036)

Yield stress: $\sigma_Y = \sigma_{Y_t} \approx \sigma_{Y_n} = 250$ MPa as yield occurs at low strains in metals. On the other hand if the hardening modulus is provided in terms of true stress/strain, this means that

$H = \frac{\sigma_{app_t} - \sigma_Y}{\varepsilon_{app_t} - \varepsilon_Y}$ (rather than $H = \frac{\sigma_{app_n} - \sigma_Y}{\varepsilon_{app_n} - \varepsilon_Y}$ if the hardening modulus is specified in terms of nominal stress/strain). We can still calculate the applied nominal stress as

$\sigma_{app_n} = \dfrac{F}{A_0} = 300\,\text{MPa}$; however we do not know the value of true applied stress, as this depends on the applied strain. We know that

$$\sigma_{app_t} = \sigma_{app_n}\left(1+\varepsilon_{app_n}\right);\ \ \varepsilon_{app_t} = \ln\left(1+\varepsilon_{app_n}\right) \Rightarrow \sigma_{app_t} = \sigma_{app_n}\exp\left(\varepsilon_{app_t}\right)$$

The true applied stress and strain are obtained solving the system of equations

$$\begin{cases} \sigma_{app_t} = \sigma_{app_n}\exp\left(\varepsilon_{app_t}\right) \\ H = \dfrac{\sigma_{app_t} - \sigma_Y}{\varepsilon_{app_t} - \varepsilon_Y} \end{cases} \quad \text{(to be solved numerically).}$$

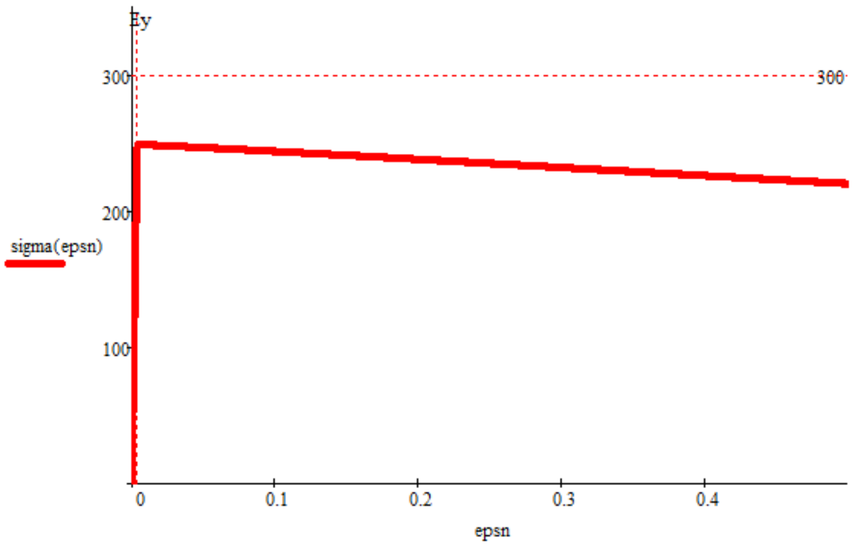
For the particular set of data given, this system has no solution; it is therefore impossible to apply a force of 30 kN to this bar reaching equilibrium. How is this physically possible? Imagine converting this bi-linear true stress/strain curve to a 'nominal' or 'engineering' stress strain curve, using the equations $\sigma_t = \sigma_n\left(1+\varepsilon_n\right);\ \varepsilon_t = \ln\left(1+\varepsilon_n\right)$. The bilinear true stress/strain curve is defined by

$$\sigma^t = \begin{cases} E\varepsilon^t & \text{for } \varepsilon^t < \varepsilon_Y \\ \sigma_Y + H\left(\varepsilon^t - \varepsilon_Y\right) & \text{for } \varepsilon^t \geq \varepsilon_Y \end{cases}$$

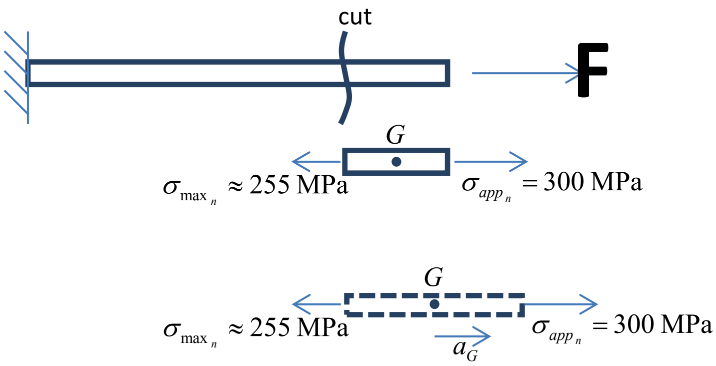
The nominal stress/strain curve is therefore defined by

$$\sigma_n = \frac{\sigma_t}{1+\varepsilon_n} = \begin{cases} \dfrac{E\ln\left(1+\varepsilon_n\right)}{1+\varepsilon_n} & \text{for } \varepsilon_n < \varepsilon_Y \\ \left[\sigma_Y + H\left(\ln(1+\varepsilon_n) - \varepsilon_Y\right)\right]/\left(1+\varepsilon_n\right) & \text{for } \varepsilon_n \geq \varepsilon_Y \end{cases}$$

Plotting the above equations the nominal stress/strain curve is obtained, graphed below to scale



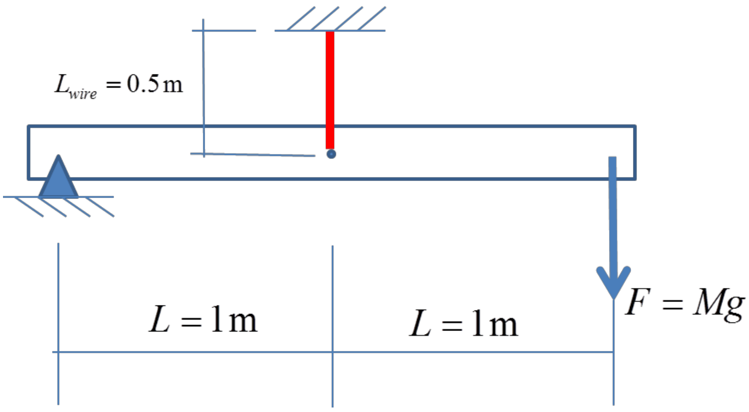
(Note how small elastic strains actually are, compared with inelastic deformation). The curve displays softening with a peak nominal stress below the applied value of 300 MPa; this means that if we apply a tensile force of 30 kN to this bar (resulting in a nominal stress of 300 MPa), the material will never be able to achieve equilibrium as it cannot react with a nominal stress greater than ~255 MPa. Consider equilibrium of the loaded end of the bar:



$$a_G = \frac{A_0\left(\sigma_{app_n} - \sigma_{\max_n}\right)}{m} > 0$$

From the free body diagram above we can conclude that the bar will have to stretch indefinitely to achieve dynamic equilibrium; this is referred to as ‘plastic collapse’, and will ultimately trigger tensile necking and subsequent fracture of the bar. (Don’t be afraid of the above… in the exams you will deal with a nominal stress VS nominal strain curve – easy life for you).

Q3. *Consider a rigid bar hinged at a fixed point and suspended via a steel wire as indicated in the figure. The bar is loaded at the end by the weight of the mass M. The steel wire has an initial cross-sectional area of 4 mm*^{2} *and is made from a material with a bilinear nominal stress/strain curve with properties* ($E = 210\,\text{GPa};\ \sigma_Y = 400\,\text{MPa};\ \ H = 100\,\text{MPa}$ *). Calculate the value of M to cause an extension of the wire from 500 to 600 mm.*



Q3 – solution

Rotational equilibrium of the rigid bar (moments about hinge = 0) provides $2F = T = 2Mg$, where T is the tension in the wire. An extension of 100m of the steel wire correspond to a nominal strain of 100/500=0.2 (20%). This will correspond to a nominal stress

$$\sigma_{app\ n} = T / A_0 = 2Mg / A_0$$

By making use of the constitutive equation (nominal stress/strain curve) we can write

$$\sigma_{app_n} = \sigma_Y + H\left(\varepsilon_{app_n} - \varepsilon_Y\right) = \sigma_Y + H\left(\varepsilon_{app_n} - \sigma_Y / E\right) = 2Mg / A_0 \Rightarrow$$

$$M = \frac{A_0}{2g}\left(\sigma_Y + H\left(\varepsilon_{app_n} - \sigma_Y / E\right)\right) = \frac{410^{-6}}{2\;9.81}\left(40010^{6} + 10010^{6}\left(0.2 - \frac{40010^{6}}{21010^{9}}\right)\right) \Rightarrow$$

$$M = 85.6\ \text{kg}$$

Note that the question above cannot be solved if the material is perfectly plastic ($H = 0$): for these idealised materials, there is no unique relation between applied stress and applied strain – if the stress reaches the yield stress and the load is not removed the material will strain indefinitely, plastic collapse will be reached.

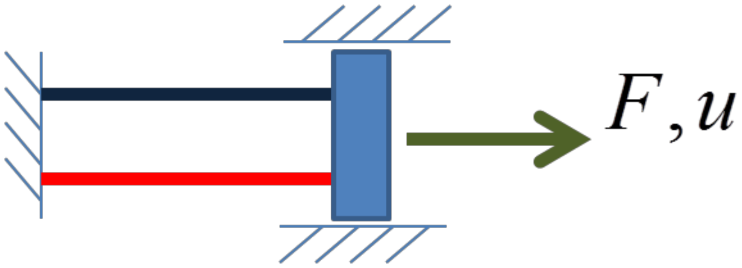
Q4. *A material specimen of gauge length 40 mm and square cross-section of side 5 mm elongates by 10 μm when subjected to a tension of 1.25 kN. Assuming that the material responds elastically determine: (a) Young's modulus. (b) The change in volume (take Poisson's ratio ν = 0.3).*

<u>Q4 – solution</u>

$$A_0 = 5 \cdot 5 = 25\,\text{mm}^2; \quad \sigma = F / A_0 = \frac{1.25\ 10^3}{25} = 50\,\text{MPa};$$

$$\varepsilon = \frac{\Delta L}{L_0} = \frac{10\ 10^{-6}}{40\ 10^{-3}} = 0.025; \qquad E = \sigma / \varepsilon = \frac{50}{0.025} = 20\,\text{GPa}$$

$$\Delta V = V_0 \varepsilon_V = A_0 L_0 (1 - 2\nu)\varepsilon = 25 \cdot 40 \cdot (1 - 2 \cdot 0.3) 0.025 = 1\,\text{mm}^3$$



Q5. *A rigid block is constrained to move horizontally and connected to a fixed wall by two metal wires as indicated in the figure. The wires have identical initial cross-section* $A_0 = 5\,\text{mm}^2$ *and length* $L_0 = 0.5\,\text{m}$ *; one of them is made of perfectly-plastic aluminium* $(E_1 = 70\,\text{GPa}; \ \ \sigma_{Y1} = 300\,\text{MPa}; \ \ H_1 = 0\,\text{MPa})$ *while the other is made of perfectly plastic steel* $(E_2 = 210\,\text{GPa}; \ \ \sigma_{Y2} = 500\,\text{MPa}; \ \ H_2 = 0\,\text{MPa})$*; stress quantities are intended to be nominal. A horizontal displacement of 10 mm is imposed such to place the wires in tension; construct a quantitative plot of the force versus displacement history recorded in this experiment.*

Q5 – solution

The wires will experience different tensile forces but the same strain due to the boundary conditions of the problem; the common nominal strain will be $\varepsilon = u / L_0$. Initially both wires will respond elastically with

$$F = F_1 + F_2 = A_0\left(E_1\varepsilon + E_2\varepsilon\right) = A_0\left(E_1 + E_2\right)\frac{u}{L_0} = \frac{5\;10^{-6}\,(210+70)10^{9}}{0.5}u = 2.8\;10^{6}u\;\text{[N]}$$

with force in Newtons and displacement in metres. The yield strains of the two materials are

$$\varepsilon_{Y1} = \frac{\sigma_{Y1}}{E_1} = \frac{300}{70000} = 4.28\;10^{-3};\;\; \varepsilon_{Y2} = \frac{\sigma_{Y2}}{E_2} = \frac{500}{210000} = 2.38\;10^{-3}$$

Clearly steel will yield first, at a displacement such to cause the yield strain, $u_{Y2} = \varepsilon_{Y2}L_0 = 2.38\;10^{-3}\;0.5 = 0.00119\,\text{m} = 1.19\,\text{mm}$

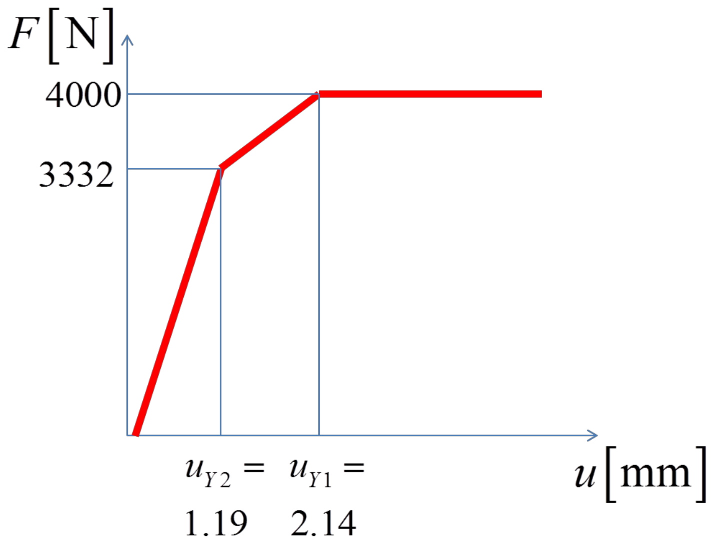
This will correspond to a force $F_{Y2} = 2.8\;10^{6}\;0.00119 = 3332\,\text{N}$. As the displacement increases, the yielded, perfectly-plastic steel wire will provide a force equal to the nominal yield stress times the area, $A_0\sigma_{Y2} = 5\;10^{-6}\;500\;10^{6} = 2500\text{ N}$, and independent of displacement, while the aluminium wire will respond elastically until its yield strain is reached. This will occur at a displacement

$$u_{Y1} = \varepsilon_{Y1}L_0 = 4.28\;10^{-3}\;0.5 = 2.14\;10^{-3}\,\text{m} = 2.14\,\text{mm}$$

When the aluminium wire yields, the force it is carrying is $A_0\sigma_{Y1} = 5\;10^{-6}\;300\;10^{6} = 1500\text{ N}$ and the total force will be, by horizontal equilibrium

$$F_{Y1} = A_0\left(\sigma_{Y1} + \sigma_{Y2}\right) = 2500 + 1500 = 4000\,\text{N}$$

With increased displacement the total force will remain constant. The requested force/displacement curve is sketched below



Q6. *An elastic bar of length 1 m is fixed at one end and loaded in uniaxial tension at the opposite end. It has to withstand a force F of 100 N with a deflection d < 10 mm. Neglecting the possibility of material failure, determine a material merit index to maximise in order to achieve a minimum mass design.*

<u>Q6 – solution</u>

The condition on stiffness provides

$$\varepsilon = \frac{\delta}{L_0} \le \frac{10}{1000} = 0.01 \qquad \text{or equivalently} \qquad \varepsilon = \frac{\sigma}{E} = \frac{F}{A_0 E} \le 0.01 \Rightarrow A_0 \ge \frac{F}{0.01E}$$

The mass of the bar is given by $m = A_0 L_0 \rho$, increasing linearly with A_0. Consequently, we take the minimum value of A_0 allowed by the condition on stiffness, i.e.

$$A_0 = \frac{F}{0.01E} \Rightarrow m = \frac{F L_0 \rho}{0.01E} = \frac{100 \cdot 1}{0.01} \frac{\rho}{E} = 10^4 \frac{\rho}{E}$$

In order to minimise the mass, we therefore have to maximise the ratio (merit index) $M_I = E / \rho$.

If the cost of the bar is proportional to its mass, it can be expressed as $C = mc = \frac{FL_0}{0.01}\frac{\rho c}{E}$. For a cost-driven design, it will be necessary to maximise the merit index $M_I = E/(\rho c)$, with c the specific material cost [£/kg].

Q7. *An elastic bar of length 1 m has to be loaded in uniaxial tension by a force F of 1000 N without failure. Determine a material merit index to maximise in order to achieve a minimum mass design.*

Q7 – solution

For the bar not to fail, we should satisfy the condition

$\sigma_{appl} = \frac{F}{A_0} \le \sigma_Y \Rightarrow A_0 \ge F/\sigma_Y$ where σ_Y is the material's strength. Similar to the previous question, the mass of the bar may be written as

$$A_0 = \frac{F}{\sigma_Y} \Rightarrow m = \frac{FL_0\rho}{\sigma_Y} = 1000 \cdot 1\frac{\rho}{\sigma_Y} = 10^3\frac{\rho}{\sigma_Y}.$$

Therefore, in a strength-driven design (of a bar in tension!) we have to maximise the material merit index $M_I = \sigma_Y/\rho$ (specific strength). Cost can be expressed as $C = 10^3\frac{\rho c}{\sigma_Y}$; in a cost-driven design, we will have to maximise the merit index $M_I = \sigma_Y/(\rho c)$.

Q8. *A metallic cantilever of square cross-section* $t \times t$ *and known length L has to withstand a transverse load F at the free end without failure. Determine a material merit index to maximise in order to achieve a design of minimum mass or minimum cost.*

Q8 – solution

Beam theory provides expressions for the maximum stress in a cantilever beam loaded transversely at the end as $\sigma = 6FL/t^3$. For the beam not to fail it is necessary to satisfy the condition

$\sigma \le \sigma_Y \Rightarrow t \ge \sqrt[3]{\frac{6FL}{\sigma_Y}}$. The mass of the beam is given by $m = t^2 L\rho$ and scales with *t*. For both minimum weight and minimum cost designs it is necessary to minimise *t.* Assume the limiting condition

$$t = \sqrt[3]{\frac{6FL}{\sigma_Y}} \Rightarrow m = \left(\frac{6FL}{\sigma_Y}\right)^{2/3} L\rho = \left[(6FL)^{2/3} L\right]\frac{\rho}{\sigma_Y{}^{2/3}}.$$

If *F* and *L* are known, in order to minimise the mass it will be necessary to maximise the material merit index $M_I = \frac{\sigma_Y{}^{2/3}}{\rho}$. For a design driven by minimum cost, the corresponding merit index to maximise will be $M_I = \frac{\sigma_Y{}^{2/3}}{\rho c}$.

Note that contours of constant merit index can be represented as trajectories in the material design space, i.e. the space whose dimensions are the material properties that are relevant to our design. Each available material can be represented by a point in such space. For example, consider the merit index $M_I = \frac{\sigma_Y{}^{2/3}}{\rho c}$ in the cost-driven design of a square cantilever loaded transversely. Note that

$$M_I = \frac{\sigma_Y{}^{2/3}}{\rho c} = const = \bar{M}_I \Rightarrow \log\left(\frac{\sigma_Y{}^{2/3}}{\rho c}\right) = \log \bar{M}_I = \frac{2}{3}\log\sigma_Y - \log(\rho c) \Rightarrow$$
$$\log\sigma_Y = \frac{3}{2}\log\bar{M}_I + \frac{3}{2}\log(\rho c)$$

In a double logarithmic material selection chart with axes '$\log\sigma_Y$' and '$\log(\rho c)$', a trajectory of constant merit index is a straight line of slope 3/2(=1.5). This consideration, as discussed in the lectures, can help identifying graphically the materials with maximum M_I.

Q9. *What types of atomic bonds are found in polymeric materials? Give brief definitions of the following terms, relating to polymeric materials: glass transition temperature, degree of polymerisation, cross-linking, branching.*

Q9 – solution

Note: when answering this type of questions in the exam, please be as brief as possible, and use the minimum amount of words to answer the questions directly. You do not need to write an essay! Answers longer than necessary will be penalised. Example of a concise answer:

Polymers are held together by covalent bonds along the polymeric chain; different chains interact via secondary bonds.

Tg: temperature at which the secondary bonds melt

DP: number of repetitions of the monomer unit in a polymer chain

Cross-linking: connection of 2 different chains via transverse covalent bonds

Branching: bifurcation of a polymer chain.

Q14. *A composite construction brick is made from a mixture of sand and natural rubber. Each brick is obtained by thoroughly mixing 4 kg of sand (* $\rho_S = 2500\,\mathrm{kg\,m^{-3}}$ *) with 2.5 kg of natural rubber (* $\rho_R = 950\,\mathrm{kgm^{-3}}$ *); in the process, a porosity (volume fraction of pores) of 10% is introduced in the aggregate. Calculate the density of the composite brick and the volume fractions of sand and rubber. If a better manufacturing method is used, allowing eliminating porosity completely, calculate the percent increase in the density of the brick.*

Q14 – solution

Volume occupied by the sand: $4/2500 = 0.0016\,\mathrm{m^3}$

Volume occupied by the rubber: $2.5/950 = 0.0027\,\mathrm{m^3}$; sum of the two: $0.0043\mathrm{m^3}$ must equal 90% of the total volume, if 10% is the volume fraction of pores:

$V_{tot} = 0.0043/0.9 = 0.0048\,\mathrm{m^3}$. The brick density is equal to the total mass divided by the total volume

$\rho_B = \dfrac{4+2.5}{0.0048} = 1354\,\text{kgm}^{-3}$ and volume fractions are

$\varphi_S = \dfrac{0.0016}{0.0048} = 33.\bar{3}\%$; $\varphi_R = \dfrac{0.0027}{0.0048} = 56.3\%$; $\varphi_P = 10\%$. Note that $\varphi_S + \varphi_R + \varphi_P \approx 100\%$ (not exactly equal because of rounding-off of the numbers), and density could be still calculated via a rule of mixture expression:

$$\rho = \varphi_S\rho_S + \varphi_R\rho_R + \varphi_P\rho_P = 0.333\cdot 2500 + 0.563\cdot 950 + 0.1\cdot 0 \approx 1360\,\text{kgm}^{-3}$$

If porosity is eliminated, the new volume fractions are

$\varphi_S = \dfrac{0.0016}{0.0043} = 37.2\%$; $\varphi_R = 1-\varphi_S = 62.8\%$ and the new density is

$$\rho = \varphi_S\rho_S + \left(1-\varphi_S\right)\rho_R = 0.372\cdot 2500 + 0.628\cdot 950 = 1527\,\text{kgm}^{-3}.$$

Percent increase of density: $\Delta\rho_{\%} = \dfrac{1527-1360}{1360} = 0.123 = +12.3\%$

Q15. *A void-free, high performance unidirectional CFRP composite lamina is obtained by mixing high strength carbon fibres (* $E = 280\,\text{GPa}$; $\rho = 1800\,\text{kgm}^{-3}$ *) with an epoxy matrix (* $E = 2.5\,\text{GPa}$; $\rho = 1300\,\text{kgm}^{-3}$ *). Volume fractions are 70% for the fibres and 30% for the matrix; calculate the density of the composite. A square specimen of this lamina of dimensions* $1\times 1\,\text{m}$ *and thickness 1 mm is loaded by a tensile force of 30 kN directed along the fibres. Calculate the extension of the square plate in the direction of loading, the maximum tensile stress in the fibres and the weight of the panel. Will the panel fail if the tensile strength of carbon fibres is 3GPa?*

Q15 – solution

Rule of mixtures: $\rho = 0.7\cdot 1800 + 0.3\cdot 1300 = 1650\,\text{kgm}^{-3}$ and the weight of the panel is

$1650\cdot 1\cdot 1\cdot 0.001 = 1.65\,\text{kg}$.

The modulus in the fibre directions is

$E_1 = \varphi_F E_F + (1-\varphi_F) E_M = 0.7 \cdot 280 + 0.3 \cdot 2.5 = 196.75\,\text{GPa}$ and the average applied stress is $\sigma = 30000/(1000 \cdot 1) = 30\,\text{MPa}$. Strain and elongation in the fibre direction are

$\Delta L_1 = \varepsilon_1 L_0 = \dfrac{\sigma}{E_1} L_0 = \dfrac{30 \cdot 1000}{196750} = 0.152\,\text{mm}$. The stress in the fibres is

$\sigma_F = E_F \varepsilon_1 = E_F \dfrac{\sigma}{E_1} = \dfrac{280}{196.75} 30 = 42.7\,\text{MPa} \ll 3000\ \text{MPa}$. The composite is very far from failing. Assuming the matrix will fail after the fibres (matrix is subject to a much lower stress, 42.7*2.5/280=0.42 MPa), we will have to increase the applied force by a factor 3000/42.7 =~ 70 in order to fail the composite. This means that this panel, weighing less than 2 kg, can support a force of 30000 x 70 = 2.1 MN = 210 tonnes before it fails. That is 200 cars hanging from it. Is it now clearer why we want to use composites in aeronautics?

Q17. *Pure, solid aluminium (*$E_s = 70\,\text{GPa};\ \sigma_{Ys} = 200\,\text{MPa};\ \rho_s = 2700\,\text{GPa};$*) can be foamed by various processes to any relative density in the range* $0.05 < \bar{\rho} < 1$. *For a certain foaming process, scaling laws are* $E_{foam} = E_s \bar{\rho};\ \sigma_{Yfoam} = \sigma_{Ys} \bar{\rho}^2$. *Calculate the ranges of moduli and compressive yield stress of the foams that can be manufactured from this solid material.*

Q17 – solution

Clearly the maximum achievable modulus and strength are those of the solid aluminium. The minima are obtained for the lowest achievable relative density 0.05, since the scaling laws are monotonic.

$$E_{\min} = E_s 0.05 = 3.5\,\text{GPa};\ \text{range } 3.5 - 70\,\text{GPa}$$
$$\sigma_{Y\min} = \sigma_{Ys} 0.05^2 = 0.5\,\text{MPa};\ \text{range } 0.5 - 200\,\text{MPa}$$

Q18. *A crash absorber for a train has to be designed, to be fitted at the dead end of a railway line. The component should be able to slow down and arrest gradually a train, of mass 1000 tonnes, impacting at a velocity of 10 km/h. The maximum allowable deceleration* *for safety of people and goods is 10 m/s². If the absorber is a foam cylinder of cross-section* $A = 0.1\,\mathrm{m}^2$ *and length L, calculate the yield stress of the foam needed and L, assuming a densification strain of 0.8.*

Q18 – solution

Note 1000 tonnes = 1000000 kg; 10km/h=2.777 m/s.

The kinetic energy to absorb is $KE = \frac{1}{2}mV^2 = 0.5\ 10^6\ 2.\overline{7}^2 = 3.858\,\mathrm{MJ}$.

The train deceleration will be $a = \frac{F}{m} = \frac{\sigma_Y A}{m} \le 10 \Rightarrow \sigma_{Y\max} = \frac{10 \cdot 10^6}{0.5} = 20\,\mathrm{MPa}$ due to the limitations in cross-section.

The volume of the crash absorber will need to be large enough to absorb all the kinetic energy before densification strain is reached. This translates into

$$\sigma_Y \varepsilon_D AL > KE \Rightarrow L > \frac{KE}{\sigma_Y A \varepsilon_D} = \frac{3.858\ 10^6}{20\ 10^6\ 0.5\ 0.8} = 0.48\,\mathrm{m}$$

The desired yield stress of 20 MPa can be achieved (from CES) with a steel foam of density 3500 kg/m^3, or equivalently, of relative density 3500/7850=44.5%. The crash absorber will weigh only 0.5x0.48x3500 = 840 kg, yet it will be adequate to absorb the huge kinetic energy of the train.