2D models and numerical techniques for erodible bed and sediment transport simulation

Workshop 2 – RESCUER MSCA Doctoral Network

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Module content

1. Sediment transport in Earth surface flows
2. Shallow-flow models for sediment transport
3. Complex flows with sediment transport
4. Sediment transport in rivers and estuaries
    4.1. Suspended load transport models
    4.2. Bedload transport models
5. Numerical discretization for erosive models

Sediment transport in the Earth water system

- Erosion in headwaters
- Hillslope erosion
- Gullies in bare soils

- Ravines and gorges
- Maintain rivers
- Meandering rivers

- Flood plains
- Lakes
- Wetlands and marshes
- Water reservoirs

- Estuaries
- River deltas

- Sediment supply to shore areas
- Coastal morphology

- Beach evolution
- Dune formation
- Cliffs

Sediment mobilization in hazardous surface flows

- Landslides
- Rock avalanches
- Submarine landslides.

- Debris flows
- Mudflows
- Pyroclastic flows

- Fast floods
- Hyper-concentrated flows
- Extreme river floods

- Dike overtopping erosion
- Dam piping and dambreach

- Mining tailings dam failure
- Extreme storm after massive wildfires

Sediment mobilization in hazardous surface flows

Ilgraben channel (Switzerland), June 29, 2020

- One of the most active debris-flow systems in the Alps, triggered by heavy rainfall or snowmelt.
- Transports sediment, rocks, and water, posing risks to downstream communities and infrastructure.
- Serves as a natural laboratory for studying alpine sediment transport and improving hazard management.
- Protective measures like debris barriers and sediment retention basins are in place.
- Highlights the need for effective hazard mitigation in mountainous regions.

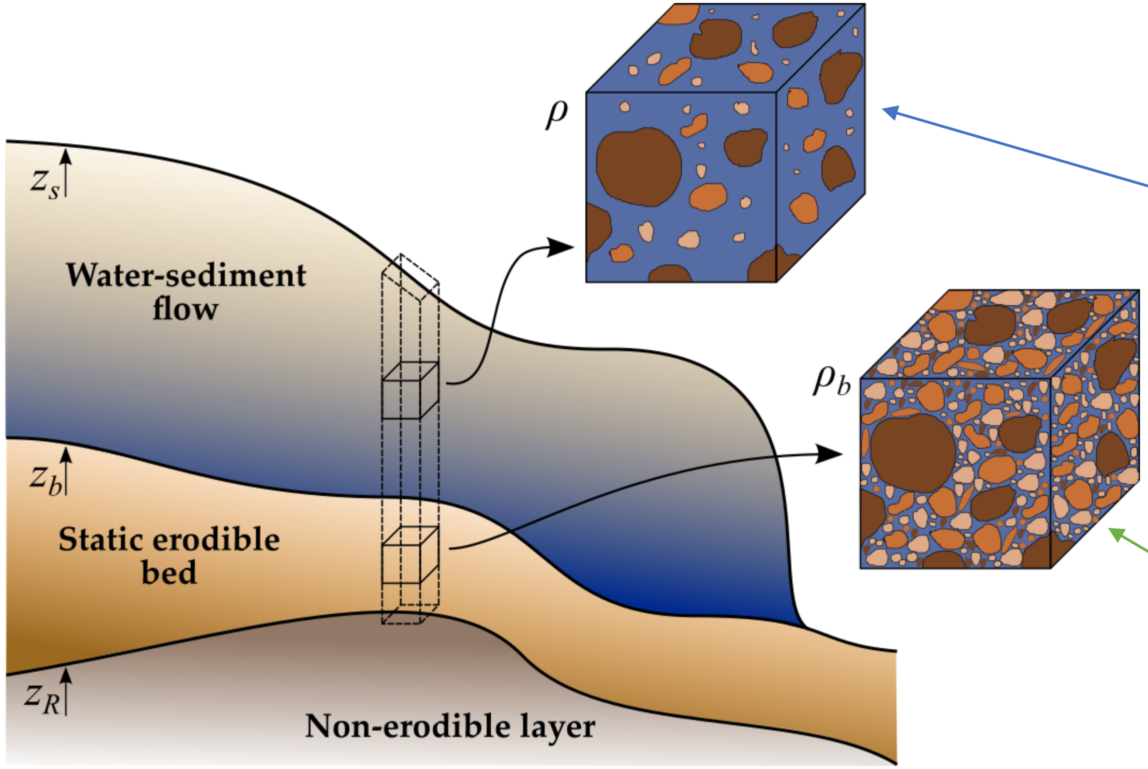
Sediment mobilization in hazardous surface flows

Swar Chaung dam collapse (Myanmar), August 29, 2018

- Heavy monsoon rains overwhelmed the dam.
- Potential design and construction flaws.
- Impacts:
  - ➢ Massive flooding destroyed homes, farmland, and infrastructure.
  - ➢ Over 70 deaths and 50000 displaced.
  - ➢ Environmental damage from sediment and debris.
  - ➢ Severe economic losses for local communities.
- Highlighted the need for stricter safety studies and robust engineering.

Module content

Water-sediment mixture properties



ρ_s $[ML^{-3}]$ Solid phase density

ρ_w $[ML^{-3}]$ Liquid phase density

$\phi = \dfrac{V_s}{V_s + V_w}$ $[-]$ Vol. solids concentration in the flow

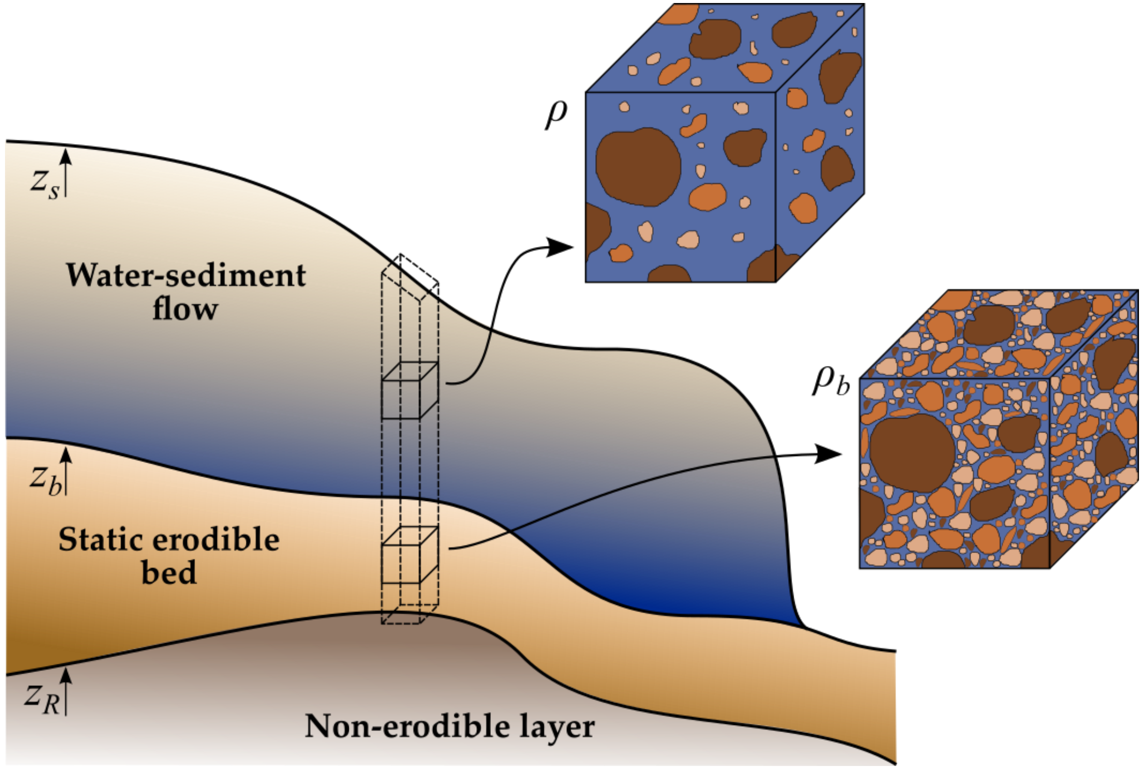
$n_w = 1 - \phi$ $[-]$ Vol. liquid concentration in the flow

$p = \dfrac{V_e}{V_T}$ $[-]$ Bed porosity

$\phi_b = 1 - p$ $[-]$ Bed solid vol. conc.

$c_w \leq p$ $[-]$ Bed water content (saturation)

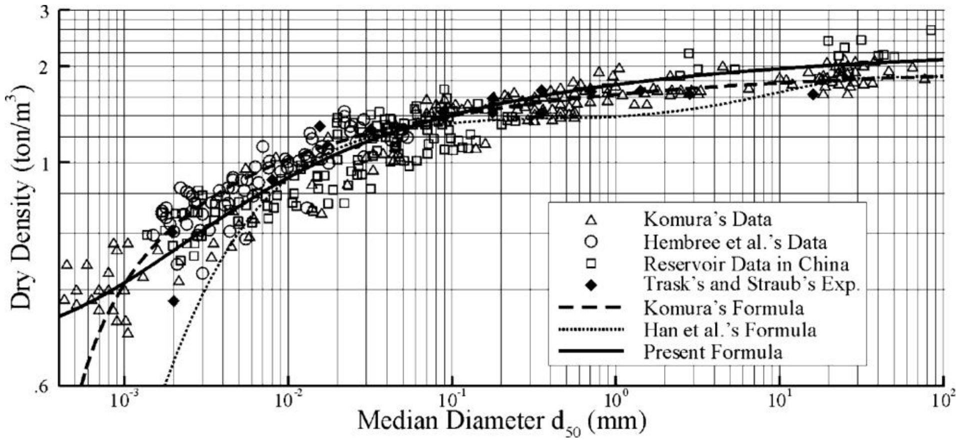
Water-sediment mixture properties



$d_s \ [L]$ Characteristic solid diameter

$\rho_{b,dry} = \rho_s(1-p) \ [ML^{-3}]$ Bed dry density

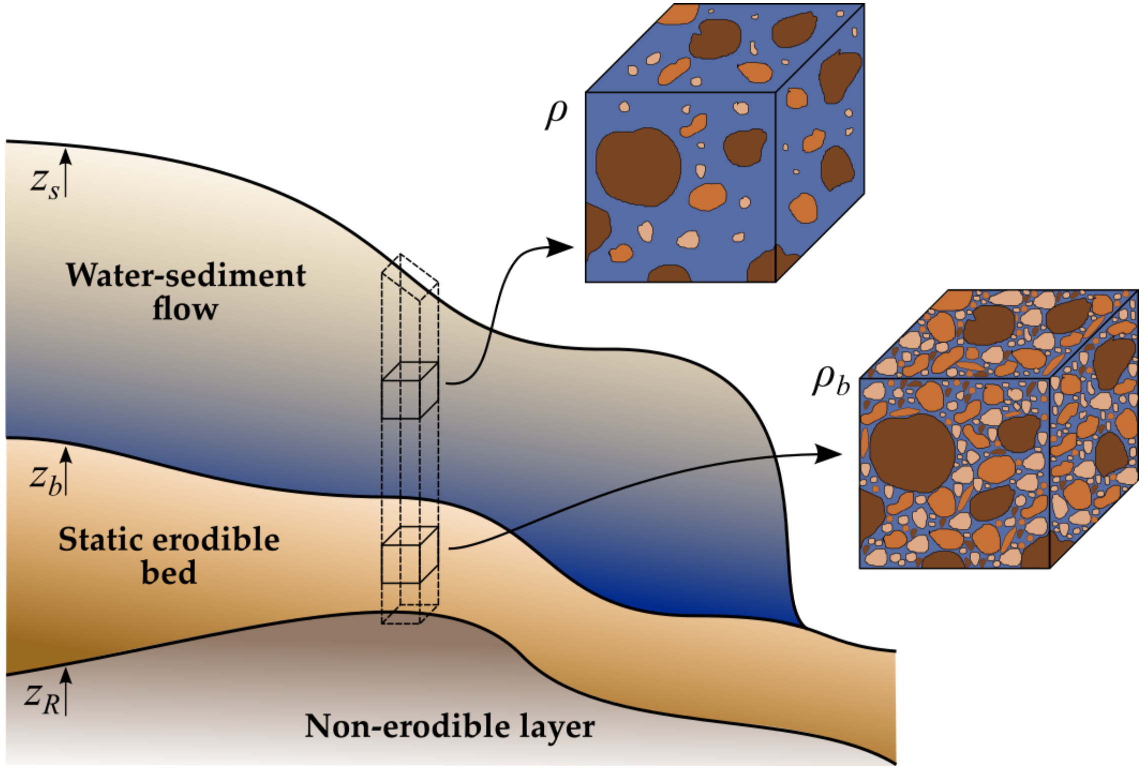
(Reproduced from Wu, 2009)



$$p = 0.245 + \frac{0.0864}{(d_s)^{0.21}}$$ *(Komura, 1963)*

$$p = 0.13 + \frac{0.21}{(d_s + 0.002)^{0.21}}$$ *(Wu & Wang, 2006)*

Water-sediment mixture properties



Mixture bulk density

$$\rho = \rho_w n + \rho_s \phi = \rho_w + (\rho_s - \rho_w)\,\phi \quad [ML^{-3}]$$

Mixture bulk velocity

$$\mathbf{u} = \frac{\rho_w n \mathbf{u_w} + \rho_s \phi \mathbf{u_s}}{\rho} \; [LT^{-1}]$$

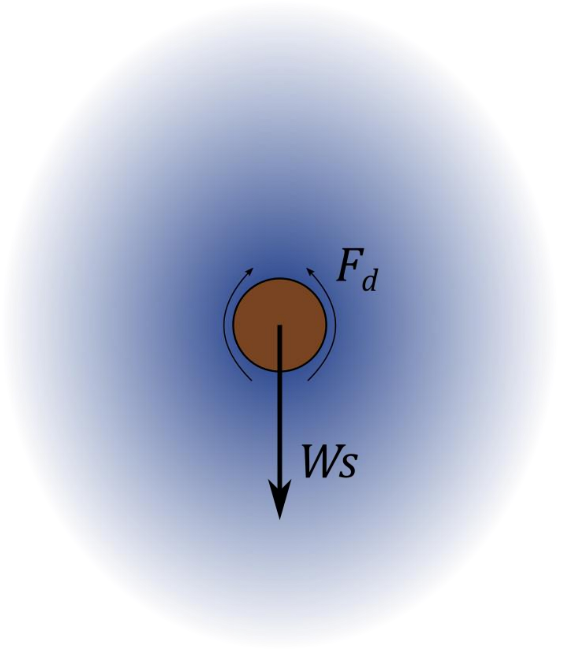
$\mathbf{u_w}$ $[LT^{-1}]$ Liquid phase velocity

$\mathbf{u_s}$ $[LT^{-1}]$ Solid phase velocity

Partially saturated bed density

$$\rho = \rho_w c_w + \rho_s (1-p) \; [ML^{-3}]$$

Settling of isolated sphere in clear water



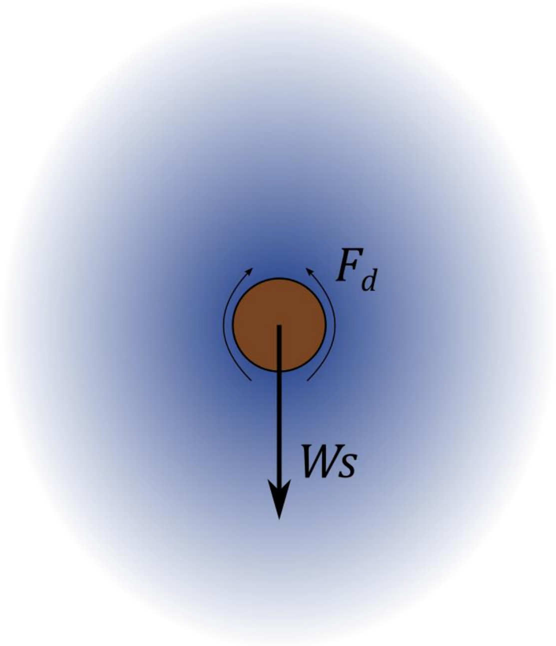
$Ws = (\rho_s - \rho_w) g \frac{\pi}{6} d_s^3$ Submerged weight of the particle

$F_d = C_d \frac{1}{2} \rho_w \frac{\pi}{4} d_s^2 \omega_s^2$ Drag force on the particle

Settling velocity Denotes the falling velocity of the sediment particles which balances the weight and the drag force of the particle.

$$F_d = Ws \quad \Rightarrow \quad \omega_s = \left(\frac{4}{3} \frac{1}{C_d} \frac{\rho_s - \rho_w}{\rho_w} g d_s \right)^{1/2}$$

Settling of isolated sphere in clear water



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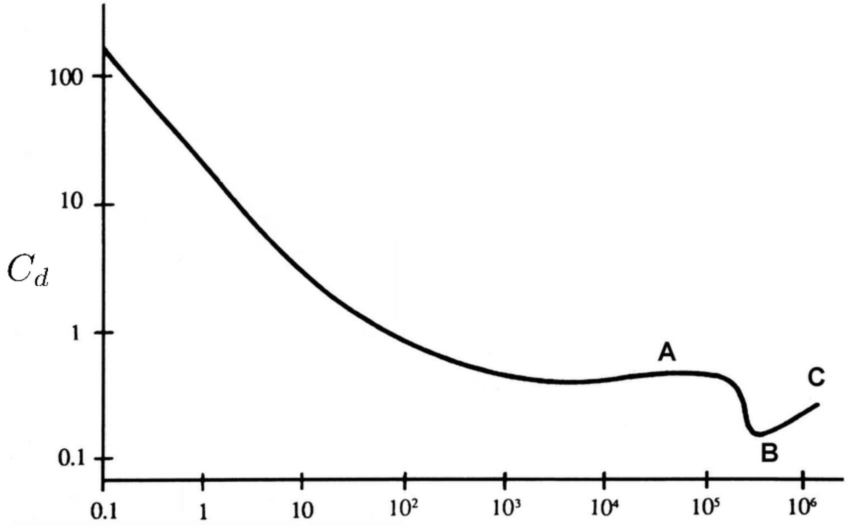
Settling velocity Denotes the falling velocity of the sediment particles which balances the weight and the drag force of the particle.

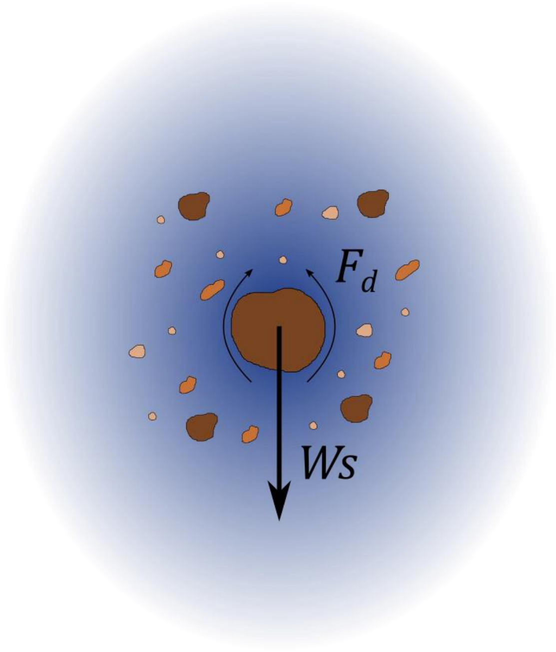
Laminar regime $C_d = 24/Re$ $\omega_s = \frac{1}{18} \frac{\rho_s - \rho_w}{\rho_w} g \frac{d_s^2}{\nu}$

Turbulent regime $C_d = 0.45$ $\omega_s = 1.72 \left(\frac{\rho_s - \rho_w}{\rho_w} d_s \right)^{1/2}$



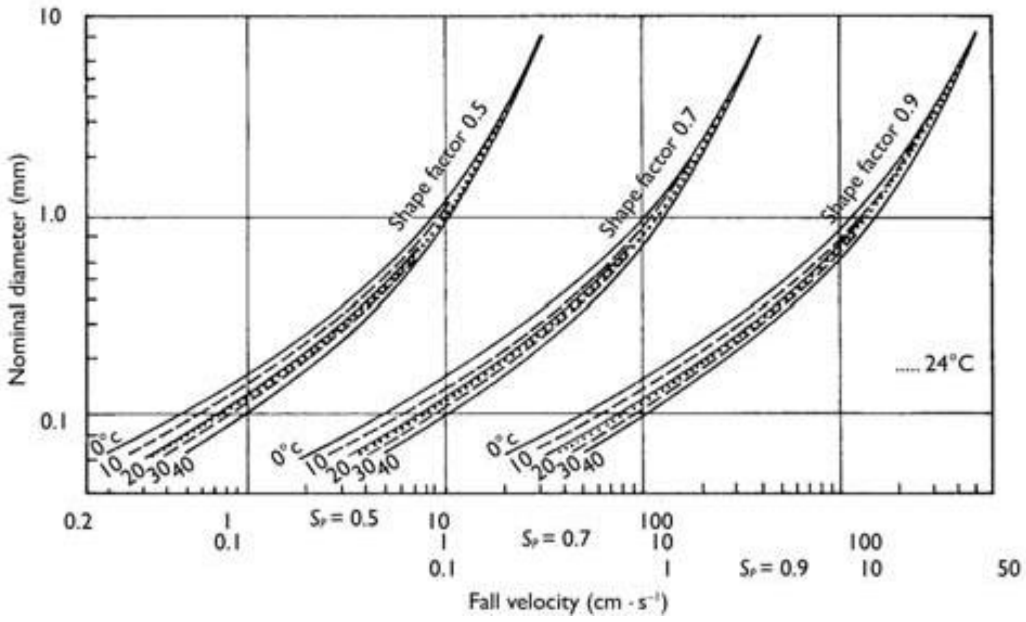
$$Re = \frac{\rho_w \, \omega_s \, d_s}{\mu} = \frac{\omega_s \, d_s}{\nu}$$

Settling velocity of sediment particles

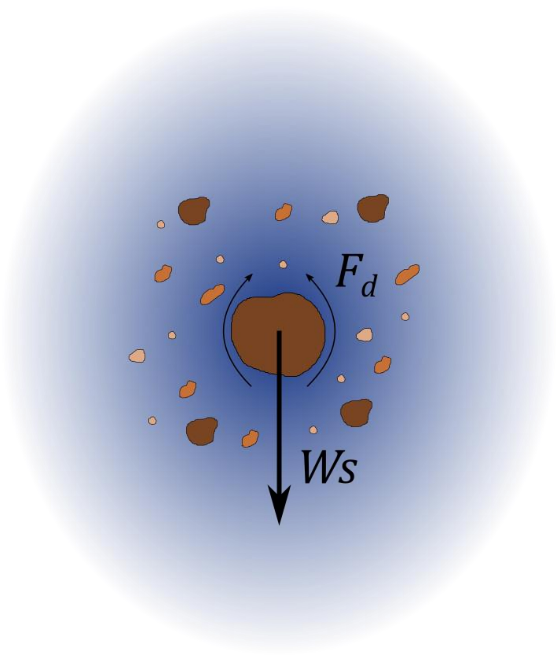


- Water temperature
- Particle surface roughness
- Particle shape

(Reproduced from Wu, 2009)



Settling velocity of sediment particles



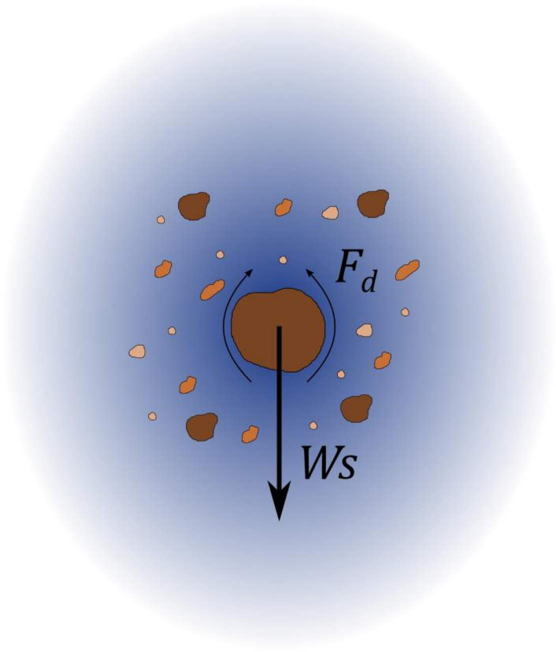
- Water temperature
- Particle surface roughness
- Particle shape

Van Rijn (1984) $$\omega_s = 1.1 \left(\frac{\rho_s - \rho_w}{\rho_w} d_s\right)^{1/2}$$

Zhanke (1977) $$\omega_s = 10 \frac{\nu}{d_s} \left[\left(1 + 0.01 \frac{\rho_s - \rho_w}{\rho_w} g \frac{d_s^3}{\nu^2}\right)^{1/2} - 1\right]$$

Zhang & Xie (1993) $$\omega_s = \left[\left(13.95 \frac{\nu}{d_s}\right)^2 + 1.09 \frac{\rho_s - \rho_w}{\rho_w} g\, d_s\right]^{1/2} - 13.95 \frac{\nu}{d_s}$$

Settling velocity of sediment particles



- Water temperature
- Particle surface roughness
- Particle shape
- Interaction between sediment particles

High concentrations of sediment hide/inhibit the particle deposition due to the presence of frictional forces between solid particles.

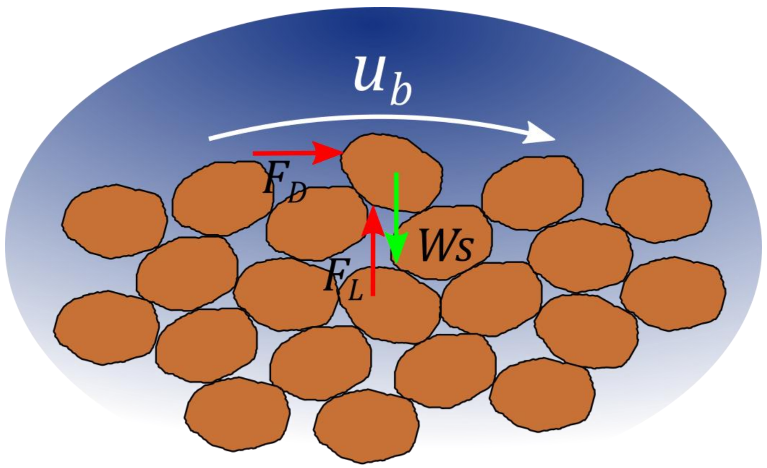
Richardson & Zaki (1954)

$$\omega_{sm} = (1 - \phi)^4 \, \omega_s$$

Sha (1965)

$$\omega_{sm} = \left(1 - \frac{\phi}{2\sqrt{d_s}}\right)^3 \omega_s$$

Incipient motion of the bed material



Movement initiation mechanisms

- Sliding
- Rolling
- Saltation

Stabilising forces

$$Ws = (\rho_s - \rho_w) g\, a_s d_s^3$$

Driven Forces

$$F_D = C_D \tfrac{1}{2} \rho_w\, a_D d_s^2\, u_b^2$$

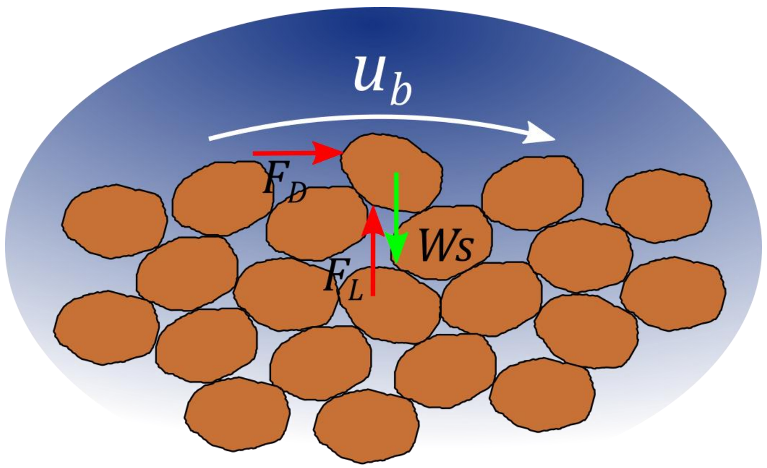
$$F_L = C_L \tfrac{1}{2} \rho_w\, a_L d_s^2\, u_b^2$$

Critical velocity

Flow velocity at the top layer of the bed for which the initiating momentum exceeds the stabilizing momentum and movement starts.

$$u_{b,c} = f(a_s, a_D, a_L, C_D, C_L) \sqrt{\frac{\rho_s - \rho_w}{\rho_w} g d_s} \;\Rightarrow\; u_{b,c} \propto \tau_c$$

Incipient motion of the bed material



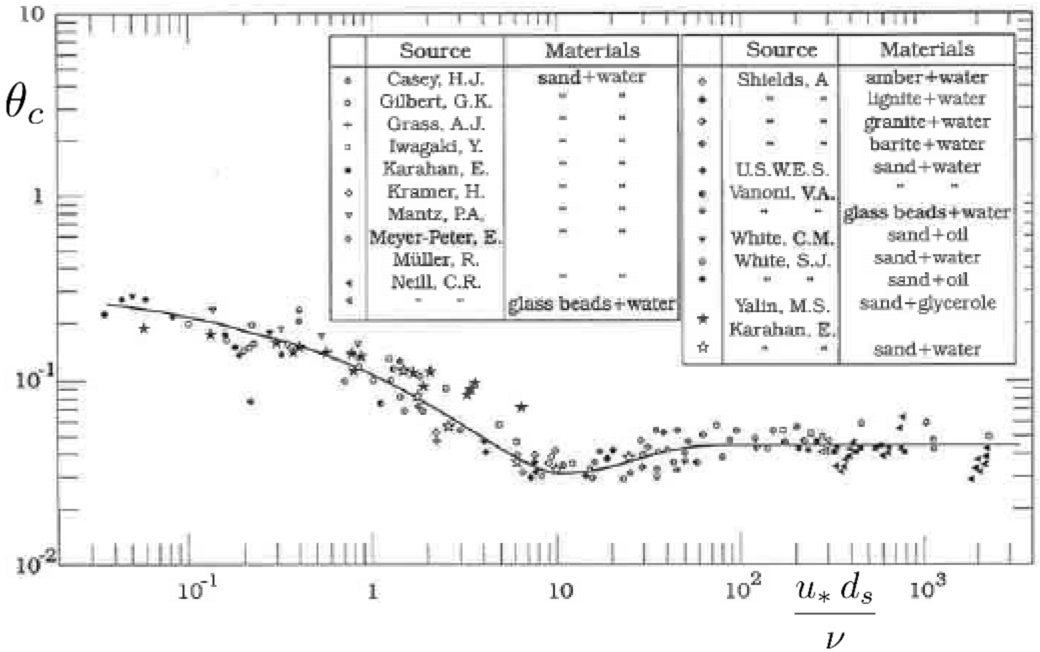
Critical shear stress criterion (Shields, 1936)

$$\frac{\tau_c}{(\rho_s - \rho_w) g d_s} = f(Re_*)$$

$$Re_* = \frac{u_* \, d_s}{\nu} \; [-]$$ Particle Reynolds number

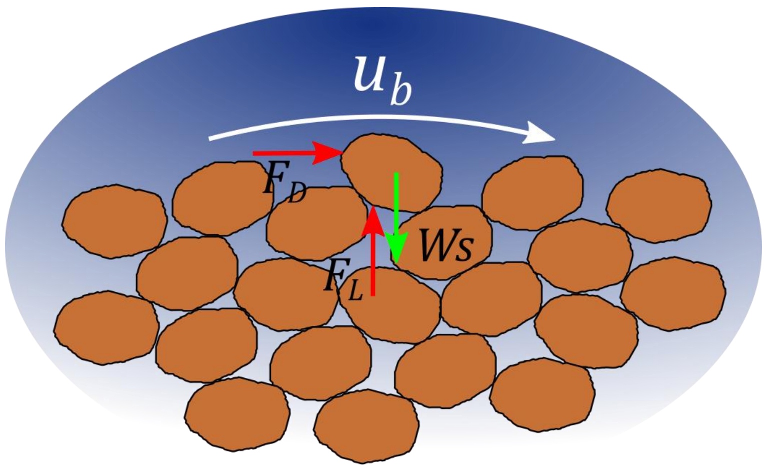
$$u_* = \sqrt{\tau_b / \rho_w} \; [LT^{-1}]$$ Shear velocity

(Reproduced from Fazeres-Ferradosa, 2012).



For particle Reynolds numbers larger than $Re \geq 10$ the critical shear stress remains almost constant. This high Reynolds number are common for a wide range of natural alluvial rivers and estuaries.

Incipient motion of the bed material



Critical shear stress criterion (Shields, 1936)

$$\frac{\tau_c}{(\rho_s - \rho_w) g d_s} = f(Re_*)$$

$$Re_* = \frac{u_* \, d_s}{\nu} \; [-]$$ Particle Reynolds number

$$u_* = \sqrt{\tau_b / \rho_w} \; [LT^{-1}]$$ Shear velocity

Dimensionless Shields stress

$$\theta = \frac{\tau_b}{(\rho_s - \rho_w) g d_s} \; [-]$$

Critical Shields stress for incipient motion

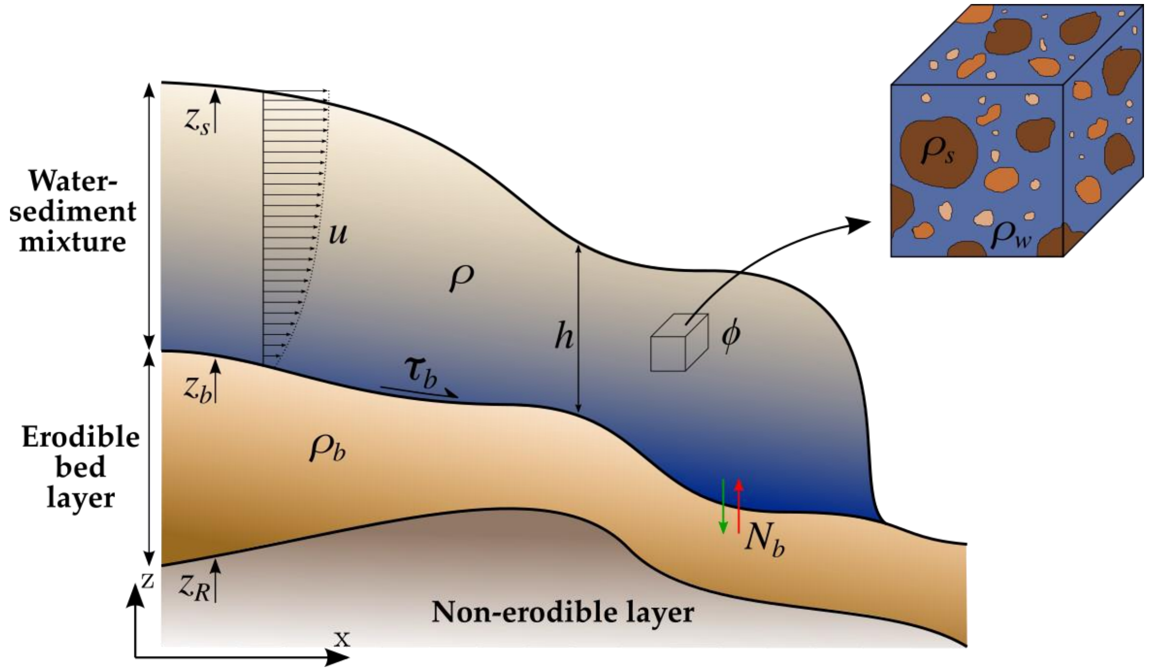
The sediment motion initialization was by a threshold value of the dimensionless Shields stress. This critical Shields stress is almost constant for a wide range of environmental flow conditions.

$$\theta_c = \frac{\tau_c}{(\rho_s - \rho_w) g d_s} [-] \approx \mathrm{Const} \, [0.3, 0.6]$$

$$\Delta\theta = \theta - \theta_c$$ Shields stress exceed

Module content

Effects of the sediment transport in the flow



Transport of two phases within the flow

- Solid phase: multiple particle types.
- Liquid phase: filling the pores.

Rheological changes in the fluid material

- Fine particles: changes in the viscosity of the liquid phase.
- Coarse particles: frictional effect due to particle-particle contact.
- Interaction between liquid and solid phases.
- Non-Newtonian (complex) behaviour.

Exchange of material between flow and bed

- Net erosion: increase in mass involved in the flow.
- Net deposition: reduction of the mass involved in the flow.
- Temporal evolution of bed elevation.

Depth-averaged models for sediment-laden flows

Two-phase models

Mixing or diffusive models (quasi-single-phase or mixing-phase)

Single-phase models

Depth-averaged models for sediment-laden flows

Two-phase models

- Resolve separately both liquid and solid phases within the flow layer.
- Conservation of the **mass of both phases**
- Conservation of the **linear momentum of both phases**.
- **Stress partitioning** between liquid phase, solid phase and liquid-solid interaction.
- **Non-Newtonian** (complex) rheological models for stress computation.
- **Does not include** exchange with the bed (normally).

Mixing or diffusive models (quasi-single-phase or mixing-phase)

Single-phase models

Depth-averaged models for sediment-laden flows

Two-phase models

Mixing or diffusive models (quasi-single-phase or mixing-phase)

- **Temporal-spatial evolution** of the properties of the flow layer**.**
- Conservation of the **mass** of the water-solid **mixture**.
- Conservation of the **linear momentum** of the water-solid **mixture**.
- Conservation of the **mass** of the **solid phase**, which control the flow layer properties.
- **Split-modelling of stresses** by means of non-Newtonian semi-empirical laws**.**
- **Closures formulas for the exchange** between the flow and the erodible bed.

Single-phase models

Depth-averaged models for sediment-laden flows

Two-phase models

Mixing or diffusive models (quasi-single-phase or mixing-phase)

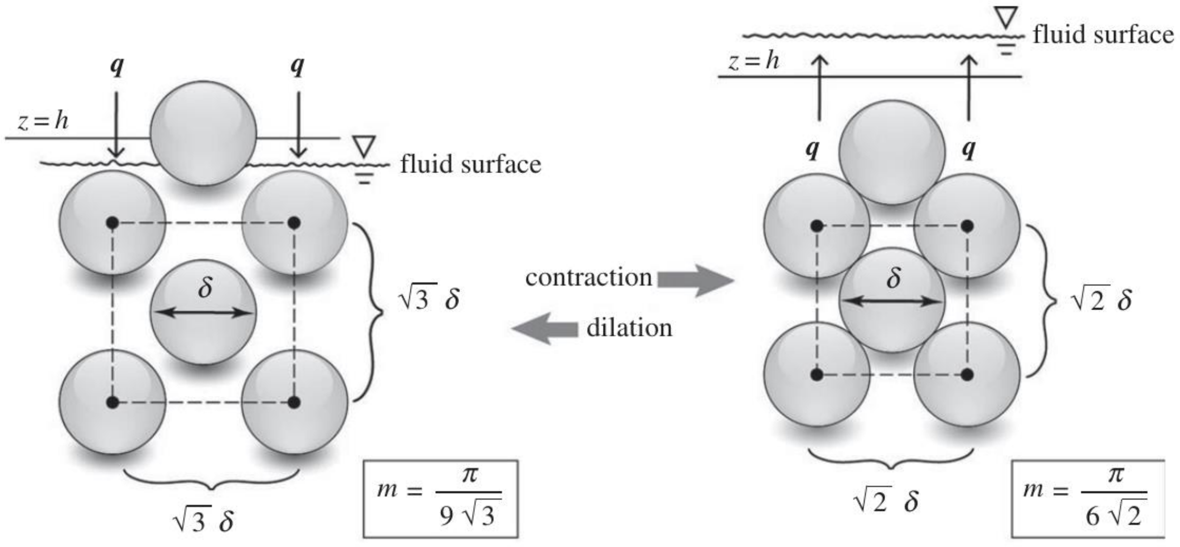
Single-phase models

- **Constant properties** of the flow in time and space.
- Conservation of the **mass** in the flow layer.
- Conservation of the **linear momentum** in the flow layer.
- Resolve the solid phase movement as a **passive transport.**
- Modelling the stresses in the flow by means of **pure turbulent semi-empirical laws**.
- **Closures formulas for the solid exchange** between flow and erodible bed.

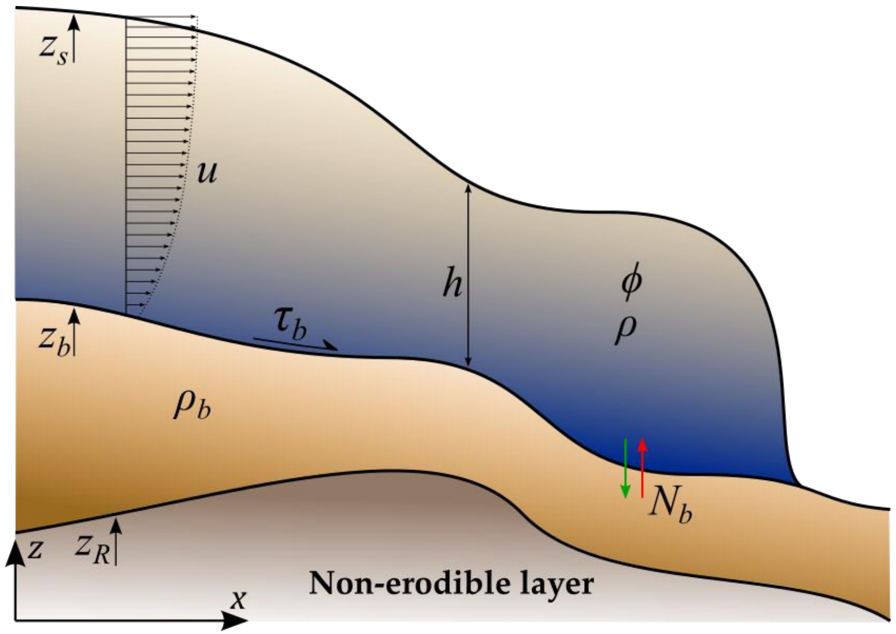
Depth-integration of the Navier-Stokes equations

- Mass conservation (1 eq.)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad\qquad \rho = \rho_w + (\rho_s - \rho_w)\,\phi$$



(Reproduced from Iverson & George, 2014)



Depth-integration of the Navier-Stokes equations

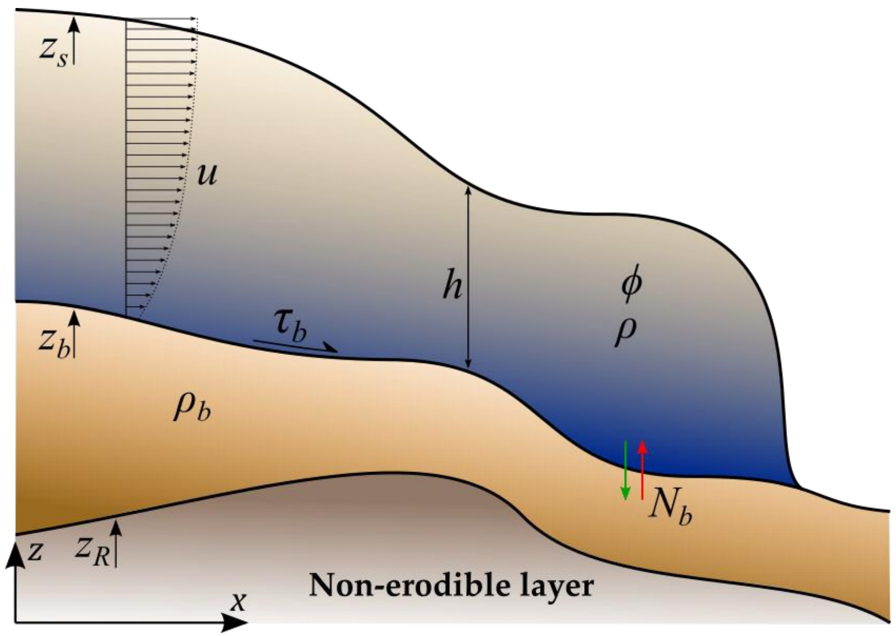
- Mass conservation (1 eq.)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad \rho = \rho_w + (\rho_s - \rho_w)\,\phi$$

- Linear momentum conservation (3 eq.)

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \mathbf{f} + \nabla \cdot \boldsymbol{\sigma}$$

$$\mathbf{f} = \rho \mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ -\rho g \end{pmatrix}$$ Vector of external forces



$$\boldsymbol{\sigma} = -p\,\mathbf{I} + \boldsymbol{\tau} = -\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} + \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

$p = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$ Pressure

$\boldsymbol{\tau}$ Viscous stress tensor

Depth-integration of the Navier-Stokes equations

- Mass conservation (1 eq.)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad \rho = \rho_w + (\rho_s - \rho_w)\,\phi$$

- Linear momentum conservation (3 eq.)

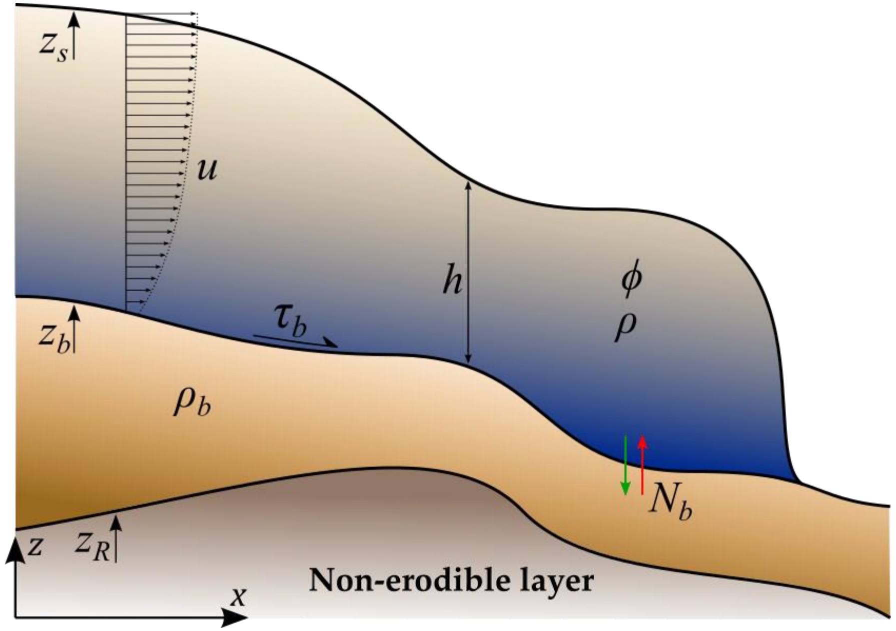
$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \rho \mathbf{g} - \nabla p \mathbf{I} + \nabla \cdot \boldsymbol{\tau}$$

- Solid phase transport in the flow (1 eq.)

$$\frac{\partial \rho_s \phi}{\partial t} + \nabla \cdot (\rho_s \phi\, \mathbf{u}) = \nabla \cdot [\rho_s \phi\, (\mathbf{u} - \mathbf{u_s})]$$

- Bed layer mass conservation (1 eq.)

$$\frac{\partial \rho_b}{\partial t} = 0$$



The conservation of the total solid mass in the domain must be ensured, although the exchange of bed material between layers may be allowed.

Depth-integration of the Navier-Stokes equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \rho \mathbf{g} - \nabla p \mathbf{I} + \nabla \cdot \boldsymbol{\tau}$$

$$\frac{\partial (\rho_s \phi)}{\partial t} + \nabla \cdot (\rho_s \phi \, \mathbf{u}) = \nabla \cdot [\rho_s \phi \, (\mathbf{u} - \mathbf{u_s})]$$

$$\frac{\partial \rho_b}{\partial t} = 0$$

Kinematic condition at the free surface

$$\frac{\partial z_s}{\partial t} + u_s \frac{\partial z_s}{\partial x} + v_s \frac{\partial z_s}{\partial y} - w_s = 0$$

Kinematic condition at the bed surface

$$\frac{\partial z_b}{\partial t} + u_b \frac{\partial z_b}{\partial x} + v_b \frac{\partial z_b}{\partial y} - w_b = \frac{N_b}{\phi_b}$$

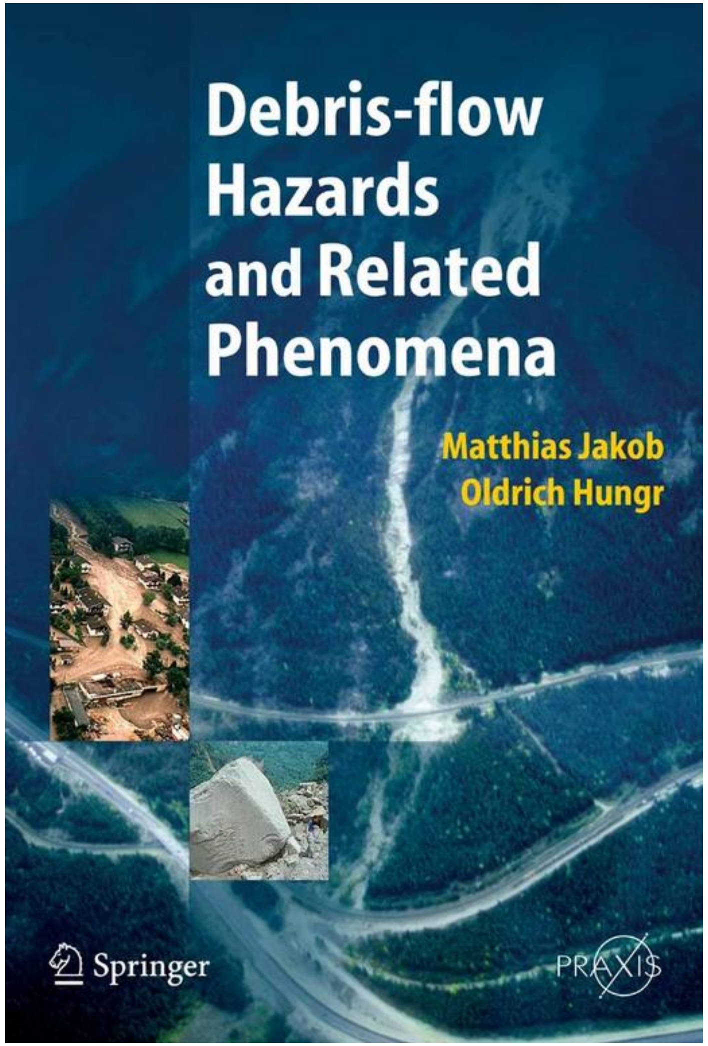
The term N_b accounts for the net volumetric flux of solid material throughout the bed surface.

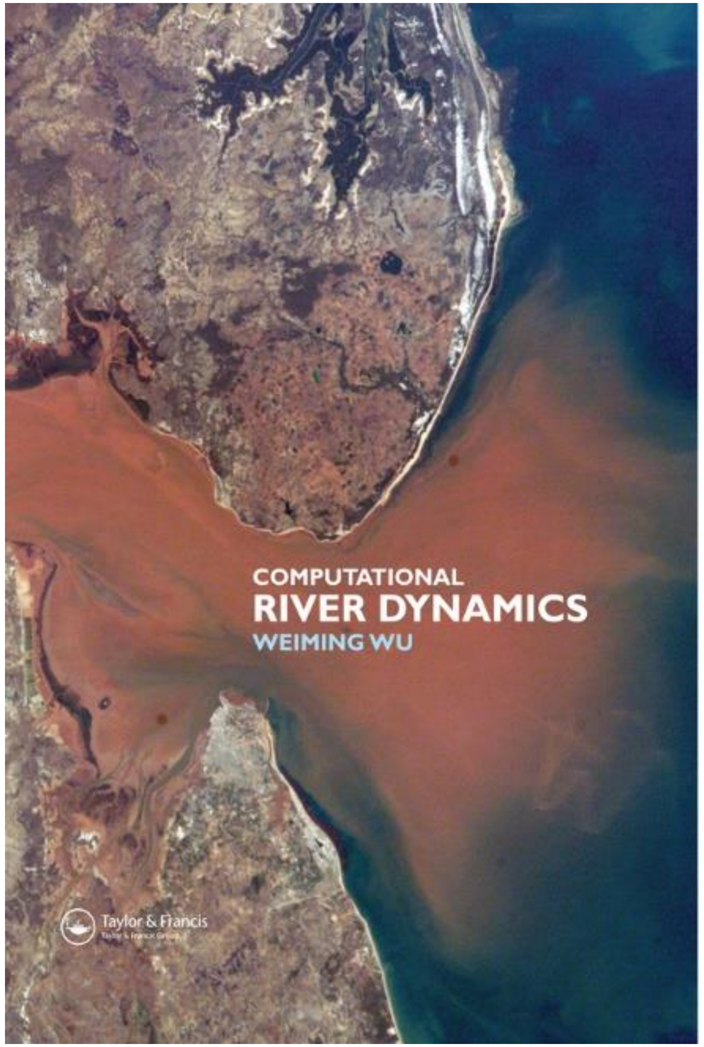
Hydrostatic pressure hypothesis

Assuming the shallow-flow scaling, the temporal, acceleration and stress terms can be neglected in the vertical momentum equation, allowing to assume a hydrostatic pressure distribution in the flow column.

$$0 = -\rho g - \frac{\partial p}{\partial z} \quad \longrightarrow \quad p(z) = p(z_s) + \int_z^{z_s} \rho g \, \mathrm{d}z$$

Depth-integration of the Navier-Stokes equations





Module content

Mixing-phase models for sediment-laden flows

- Mixture continuity

$$\frac{\partial(\rho h)}{\partial t}+\frac{\partial}{\partial x}(\rho hu)+\frac{\partial}{\partial y}(\rho hv)=-\rho_b\frac{N_b}{1-p}$$

- Mixture momentum equations

$$\frac{\partial(\rho hu)}{\partial t}+\frac{\partial}{\partial x}(\rho hu^2+\frac{1}{2}g\rho h^2)+\frac{\partial}{\partial y}(\rho huv)=-g\rho h\frac{\partial z_b}{\partial x}-\tau_{bx}+\frac{\partial}{\partial x}[\rho h(D_{xx}+T_{xx})]+\frac{\partial}{\partial y}[\rho h(D_{xy}+T_{xy})]$$

$$\frac{\partial(\rho hv)}{\partial t}+\frac{\partial}{\partial x}(\rho huv)+\frac{\partial}{\partial y}(\rho hv^2+\frac{1}{2}g\rho h^2)=-g\rho h\frac{\partial z_b}{\partial y}-\tau_{by}+\frac{\partial}{\partial x}[\rho h(D_{yx}+T_{yx})]+\frac{\partial}{\partial y}[\rho h(D_{yy}+T_{yy})]$$

Time derivative

Conservative momentum flux

Bed pressure

Depth-averaged normal and shear (Reynolds) stresses
Dispersion momentum transport

Bed shear force

Mixing-phase models for sediment-laden flows

- **Mixture continuity**

$$\frac{\partial(\rho h)}{\partial t} + \frac{\partial}{\partial x}(\rho h u) + \frac{\partial}{\partial y}(\rho h v) = -\rho_b \frac{N_b}{1-p}$$

- **Mixture momentum equations**

$$\frac{\partial(\rho h u)}{\partial t} + \frac{\partial}{\partial x}(\rho h u^2 + \frac{1}{2} g \rho h^2) + \frac{\partial}{\partial y}(\rho h u v) = -g \rho h \frac{\partial z_b}{\partial x} - \tau_{bx} + \frac{\partial}{\partial x}[\rho h (D_{xx} + T_{xx})] + \frac{\partial}{\partial y}[\rho h (D_{xy} + T_{xy})]$$

$$\frac{\partial(\rho h v)}{\partial t} + \frac{\partial}{\partial x}(\rho h u v) + \frac{\partial}{\partial y}(\rho h v^2 + \frac{1}{2} g \rho h^2) = -g \rho h \frac{\partial z_b}{\partial y} - \tau_{by} + \frac{\partial}{\partial x}[\rho h (D_{yx} + T_{yx})] + \frac{\partial}{\partial y}[\rho h (D_{yy} + T_{yy})]$$

- **Solid phase transport equation**

$$\rho_s \frac{\partial(h\phi)}{\partial t} + \rho_s \frac{\partial}{\partial x}(h u \phi) + \rho_s \frac{\partial}{\partial y}(h v \phi) = -\rho_s N_b + \rho_s \frac{\partial}{\partial x}\left[h\left(\epsilon_s \frac{\partial \phi}{\partial x} + D_{sx}\right)\right] + \rho_s \frac{\partial}{\partial y}\left[h\left(\epsilon_s \frac{\partial \phi}{\partial y} + D_{sy}\right)\right]$$

Time derivative | Advective sediment flux | Bed exchange | Turbulent diffusion sediment flux, Dispersion sediment flux

Mixing-phase models for sediment-laden flows

- Mixture continuity

$$\frac{\partial(\rho h)}{\partial t} + \frac{\partial}{\partial x}(\rho h u) + \frac{\partial}{\partial y}(\rho h v) = -\rho_b \frac{N_b}{1-p}$$

- Mixture momentum equations

$$\frac{\partial(\rho h u)}{\partial t} + \frac{\partial}{\partial x}(\rho h u^2 + \frac{1}{2} g \rho h^2) + \frac{\partial}{\partial y}(\rho h u v) = -g\rho h \frac{\partial z_b}{\partial x} - \tau_{bx} + \frac{\partial}{\partial x}[\rho h(D_{xx} + T_{xx})] + \frac{\partial}{\partial y}[\rho h(D_{xy} + T_{xy})]$$

$$\frac{\partial(\rho h v)}{\partial t} + \frac{\partial}{\partial x}(\rho h u v) + \frac{\partial}{\partial y}(\rho h v^2 + \frac{1}{2} g \rho h^2) = -g\rho h \frac{\partial z_b}{\partial y} - \tau_{by} + \frac{\partial}{\partial x}[\rho h(D_{yx} + T_{yx})] + \frac{\partial}{\partial y}[\rho h(D_{yy} + T_{yy})]$$

- Solid phase transport equation

$$\rho_s \frac{\partial(h\phi)}{\partial t} + \rho_s \frac{\partial}{\partial x}(hu\phi) + \rho_s \frac{\partial}{\partial y}(hv\,\phi) = -\rho_s N_b + \rho_s \frac{\partial}{\partial x}\left[h\left(\epsilon_s \frac{\partial \phi}{\partial x} + D_{sx}\right)\right] + \rho_s \frac{\partial}{\partial y}\left[h\left(\epsilon_s \frac{\partial \phi}{\partial y} + D_{sy}\right)\right]$$

- Bed layer mass conservation

$$\frac{\partial(\rho_b\, z_b)}{\partial t} = \rho_b \frac{N_b}{1-p}$$

Simplified mixing-phase models (I)

- Mixture continuity

$$\frac{\partial(\rho h)}{\partial t} + \frac{\partial}{\partial x}(\rho h u) + \frac{\partial}{\partial y}(\rho h v) = -\rho_b \frac{N_b}{1-p}$$

- Mixture momentum equations

$$\frac{\partial(\rho h u)}{\partial t} + \frac{\partial}{\partial x}(\rho h u^2 + \frac{1}{2} g \rho h^2) + \frac{\partial}{\partial y}(\rho h u v) = -g \rho h \frac{\partial z_b}{\partial x} - \tau_{bx}$$

$$\frac{\partial(\rho h v)}{\partial t} + \frac{\partial}{\partial x}(\rho h u v) + \frac{\partial}{\partial y}(\rho h v^2 + \frac{1}{2} g \rho h^2) = -g \rho h \frac{\partial z_b}{\partial x} - \tau_{by}$$

Depth-averaged Reynolds stresses and dispersion terms are neglected

- Solid phase transport equation

$$\frac{\partial(h\phi)}{\partial t} + \frac{\partial}{\partial x}(hu\phi) + \frac{\partial}{\partial y}(hv\,\phi) = -N_b$$

Turbulent diffusion and dispersion fluxes are neglected

- Bed layer mass conservation

$$\frac{\partial z_b}{\partial t} = \frac{N_b}{1-p}$$

Uniform and constant bed density

Simplified mixing-phase models (II)

- Mixture continuity

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = -\frac{N_b}{1-p}$$

Decoupling density and flow depth in the continuity and momentum equation

- Mixture momentum equations

$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}(hu^2 + \frac{1}{2}gh^2) + \frac{\partial}{\partial y}(huv) = -gh\frac{\partial z_b}{\partial x} - \frac{\tau_{bx}}{\rho} \quad - \frac{1}{2}g\frac{h^2}{\rho}\frac{\partial \rho}{\partial x} \quad + u\frac{\rho_b - \rho}{\rho}\frac{N_b}{1-p}$$

$$\frac{\partial (hv)}{\partial t} + \frac{\partial}{\partial x}(huv) + \frac{\partial}{\partial y}(hv^2 + \frac{1}{2}gh^2) = -gh\frac{\partial z_b}{\partial y} - \frac{\tau_{by}}{\rho} \quad - \frac{1}{2}g\frac{h^2}{\rho}\frac{\partial \rho}{\partial y} \quad + v\frac{\rho_b - \rho}{\rho}\frac{N_b}{1-p}$$

Density gradient term

Erosion/sedimentation momentum term

Self-adjustment principle in fluvial dynamics:

- Net entrainment $N_b < 0$ consumes momentum
- Net deposition $N_b > 0$ leads to momentum increments

- Solid phase transport equation

$$\frac{\partial (h\phi)}{\partial t} + \frac{\partial}{\partial x}(hu\phi) + \frac{\partial}{\partial y}(hv\,\phi) = -N_b$$

- Bed layer mass conservation

$$\frac{\partial z_b}{\partial t} = \frac{N_b}{1-p}$$

Mixing-phase models vs. classical SWE

- Mixture continuity

$$\frac{\partial(\rho h)}{\partial t} + \frac{\partial}{\partial x}(\rho h u) + \frac{\partial}{\partial y}(\rho h v) = -\rho_b \frac{N_b}{1-p}$$

- Mixture momentum equations

$$\frac{\partial(\rho h u)}{\partial t} + \frac{\partial}{\partial x}\left(\rho h u^2 + \frac{1}{2} g \rho h^2\right) + \frac{\partial}{\partial y}(\rho h u v) = -g\rho h \frac{\partial z_b}{\partial x} - \tau_{bx}$$

$$\frac{\partial(\rho h v)}{\partial t} + \frac{\partial}{\partial x}(\rho h u v) + \frac{\partial}{\partial y}\left(\rho h v^2 + \frac{1}{2} g \rho h^2\right) = -g\rho h \frac{\partial z_b}{\partial x} - \tau_{by}$$

- Solid phase transport equation

$$\frac{\partial(h\phi)}{\partial t} + \frac{\partial}{\partial x}(hu\phi) + \frac{\partial}{\partial y}(hv\,\phi) = -N_b$$

- Bed layer mass conservation

$$\frac{\partial z_b}{\partial t} = \frac{N_b}{1-p}$$

1) Density gradients have effects on the flow dynamics.

Mixing-phase models vs. classical SWE

- Mixture continuity

$$\frac{\partial(\rho h)}{\partial t} + \frac{\partial}{\partial x}(\rho h u) + \frac{\partial}{\partial y}(\rho h v) = -\boldsymbol{\rho_b}\,\frac{\boldsymbol{N_b}}{\mathbf{1}-\boldsymbol{p}}$$

- Mixture momentum equations

$$\frac{\partial(\rho h u)}{\partial t} + \frac{\partial}{\partial x}\left(\rho h u^2 + \frac{1}{2} g\rho h^2\right) + \frac{\partial}{\partial y}(\rho h u v) = -g\rho h \frac{\partial z_b}{\partial x} - \tau_{bx}$$

$$\frac{\partial(\rho h v)}{\partial t} + \frac{\partial}{\partial x}(\rho h u v) + \frac{\partial}{\partial y}\left(\rho h v^2 + \frac{1}{2} g\rho h^2\right) = -g\rho h \frac{\partial z_b}{\partial x} - \tau_{by}$$

- Solid phase transport equation

$$\frac{\partial(h\phi)}{\partial t} + \frac{\partial}{\partial x}(h u \phi) + \frac{\partial}{\partial y}(h v\,\phi) = -\boldsymbol{N_b}$$

- Bed layer mass conservation

$$\frac{\partial z_b}{\partial t} = \frac{\boldsymbol{N_b}}{\mathbf{1}-\boldsymbol{p}}$$

1) Density gradients have effects on the flow dynamics.

2) Net exchange of bed material modifies the flow mass and momentum

Mixing-phase models vs. classical SWE

- Mixture continuity

$$\frac{\partial(\rho h)}{\partial t} + \frac{\partial}{\partial x}(\rho h u) + \frac{\partial}{\partial y}(\rho h v) = -\rho_b \frac{N_b}{1-p}$$

- Mixture momentum equations

$$\frac{\partial(\rho h u)}{\partial t} + \frac{\partial}{\partial x}(\rho h u^2 + \frac{1}{2} g \rho h^2) + \frac{\partial}{\partial y}(\rho h u v) = -g\rho h \frac{\partial z_b}{\partial x} - \boldsymbol{\tau_{bx}}$$

$$\frac{\partial(\rho h v)}{\partial t} + \frac{\partial}{\partial x}(\rho h u v) + \frac{\partial}{\partial y}(\rho h v^2 + \frac{1}{2} g \rho h^2) = -g\rho h \frac{\partial z_b}{\partial x} - \boldsymbol{\tau_{by}}$$

- Solid phase transport equation

$$\frac{\partial(h\phi)}{\partial t} + \frac{\partial}{\partial x}(h u \phi) + \frac{\partial}{\partial y}(h v\, \phi) = -N_b$$

- Bed layer mass conservation

$$\frac{\partial z_b}{\partial t} = \frac{N_b}{1-p}$$

1) Density gradients have effects on the flow dynamics.

2) Net exchange of bed material modifies the flow mass and momentum

3) Non-Newtonian rheological stresses

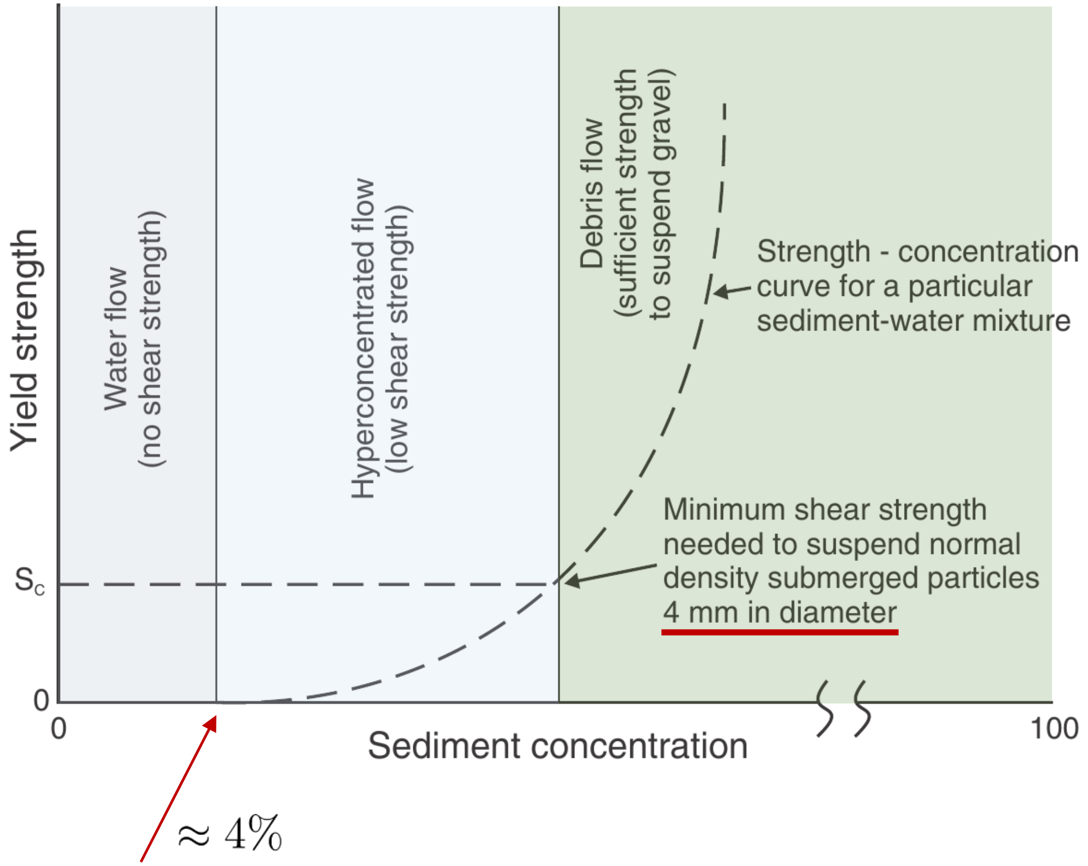
Rheological models for bed shear stress

High concentrations of solid particles, especially fine plastics, create a permanent **yield strength** in the mixture.

For concentrations above 4% of the volume, the shear stress starts to affect the behavior of the fluid, becoming a **hyper-concentrated fluidized** material.

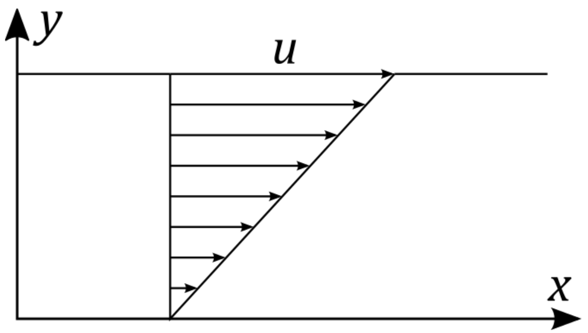
When the shear stress is able to suspend the coarser solid particles even though there is no flow, the material is called **mud or debris**.

(Reproduced from Calhoun & Clague, 2017)

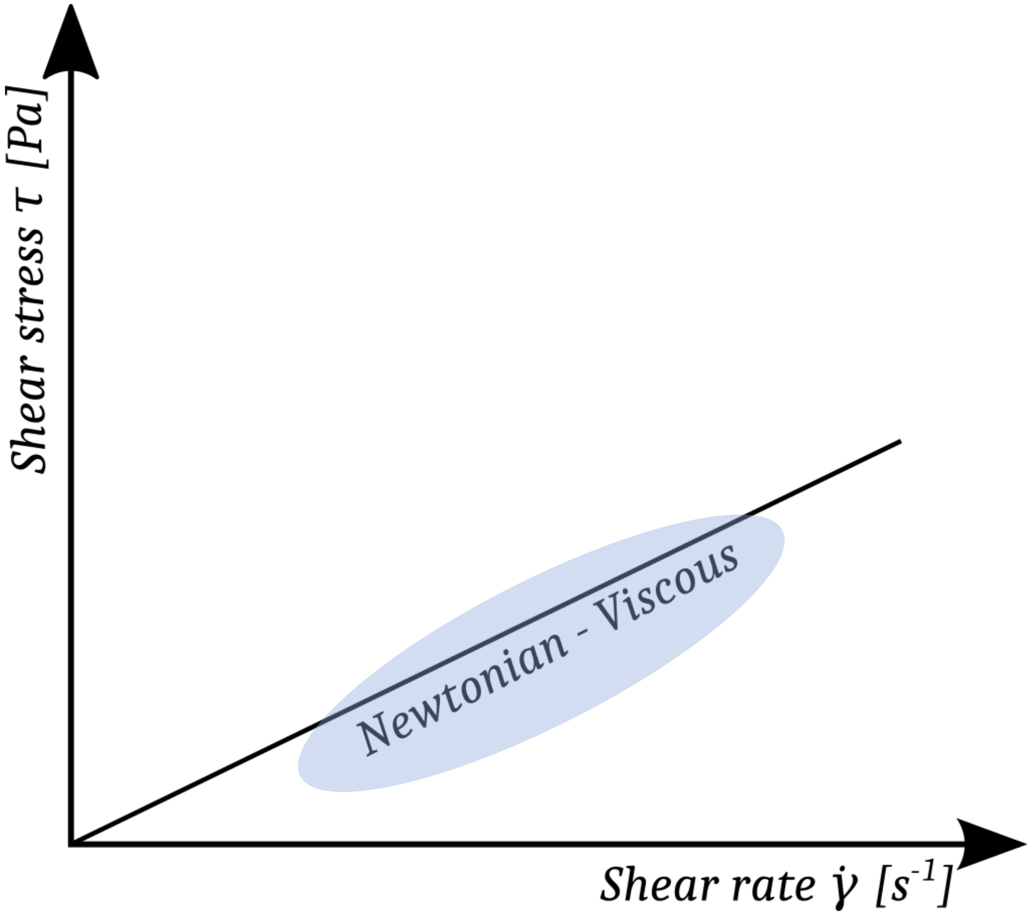


Rheological models for bed shear stress

Newtonian behaviour

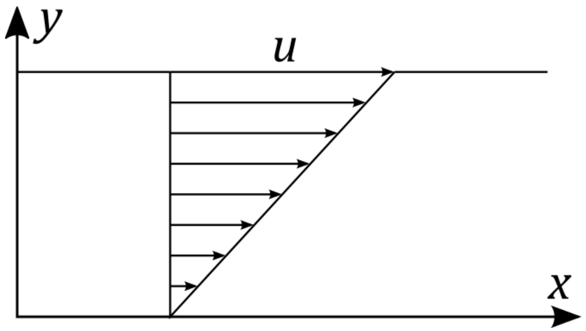


$$\tau_{xy} = \mu\dot{\gamma} = \mu\frac{\partial u}{\partial y} \qquad \mu \equiv const$$



Rheological models for bed shear stress

Newtonian behaviour



$$\tau_{xy} = \mu\dot{\gamma} = \mu\frac{\partial u}{\partial y} \qquad \mu \equiv const$$

Non-Newtonian behaviour

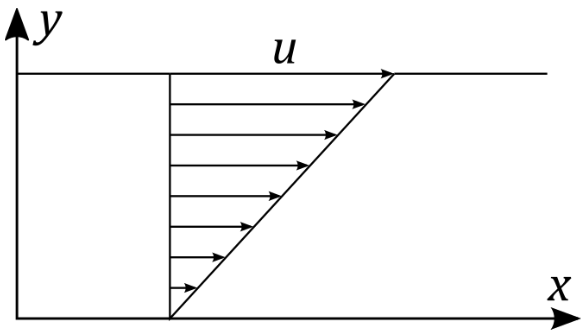
$$\tau_{xy} = \tau_y + \mu\dot{\gamma} = \tau_y + \mu(\dot{\gamma})\frac{\partial u}{\partial y}$$

Yield strength

$$\mu(\dot{\gamma}) \equiv f(\dot{\gamma})$$

Rheological models for bed shear stress

Newtonian behaviour



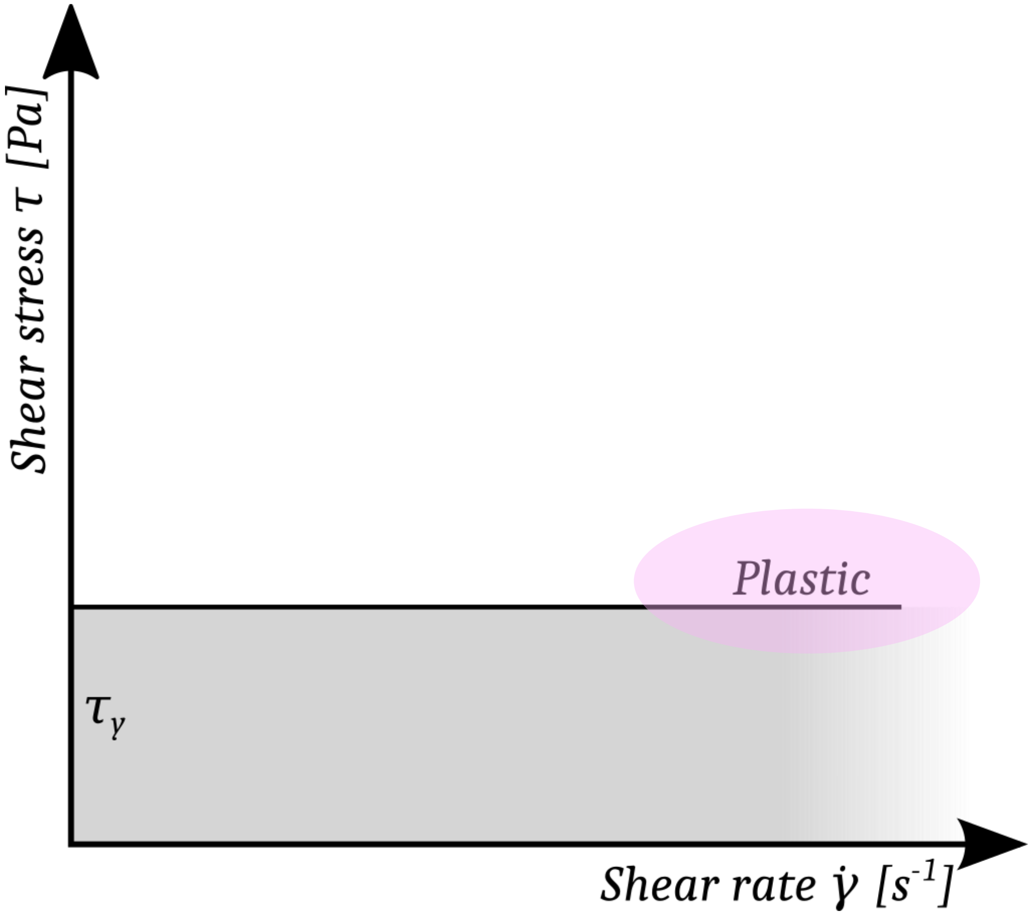
$$\tau_{xy} = \mu\dot{\gamma} = \mu\frac{\partial u}{\partial y} \qquad \mu \equiv const$$

Non-Newtonian behaviour

$$\tau_{xy} = \tau_y + \mu\dot{\gamma} = \tau_y + \mu(\dot{\gamma})\frac{\partial u}{\partial y}$$

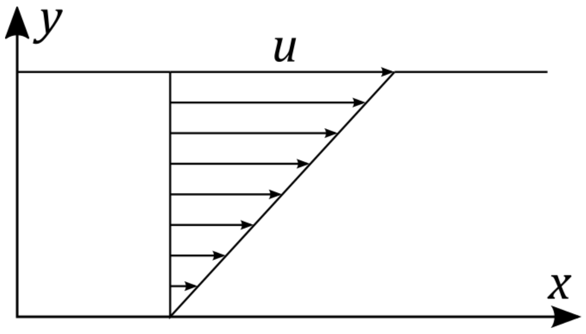
Yield strength

$$\mu(\dot{\gamma}) \equiv f(\dot{\gamma})$$



Rheological models for bed shear stress

Newtonian behaviour



$$\tau_{xy} = \mu\dot{\gamma} = \mu\frac{\partial u}{\partial y} \qquad \mu \equiv const$$

Non-Newtonian behaviour

$$\tau_{xy} = \tau_y + \mu\dot{\gamma} = \tau_y + \mu(\dot{\gamma})\frac{\partial u}{\partial y}$$

Yield strength

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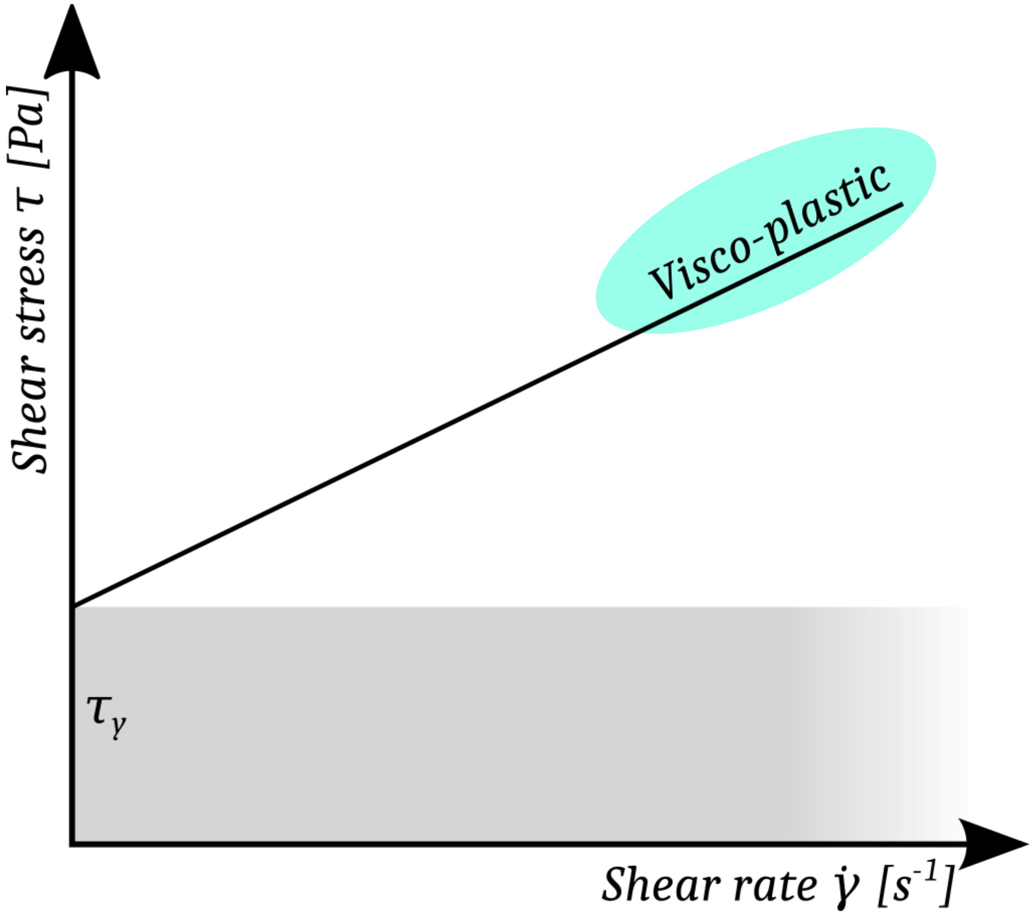
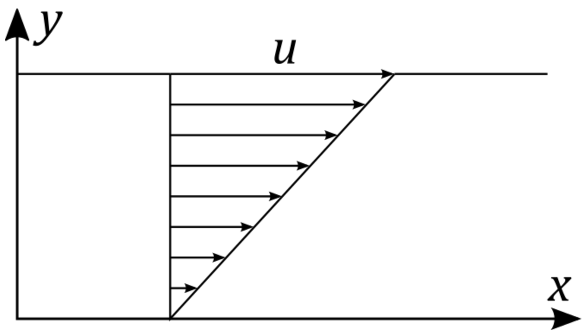


Rheological models for bed shear stress

Newtonian behaviour



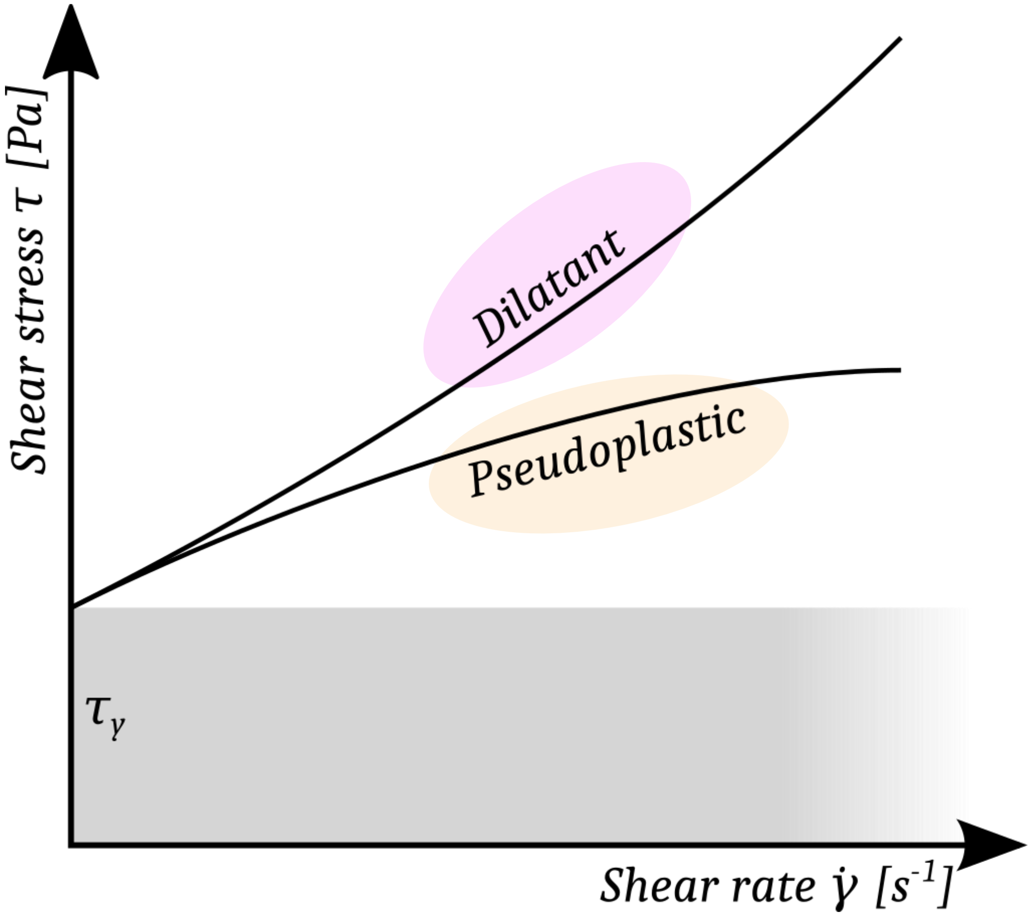
$$\tau_{xy} = \mu\dot{\gamma} = \mu\frac{\partial u}{\partial y} \qquad \mu \equiv const$$

Non-Newtonian behaviour

$$\tau_{xy} = \tau_y + \mu\dot{\gamma} = \tau_y + \mu(\dot{\gamma})\frac{\partial u}{\partial y}$$

Yield strength

$$\mu(\dot{\gamma}) \equiv f(\dot{\gamma})$$



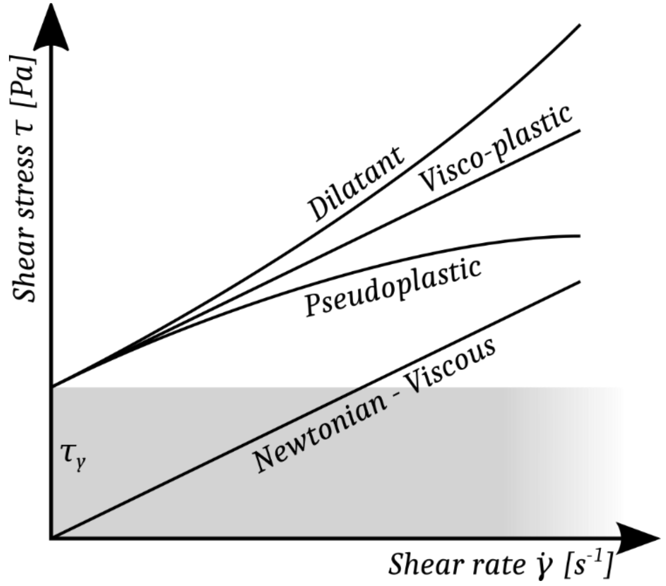
Rheological models for bed shear stress

Turbulent-dispersive stresses $\tau_t = \rho g h C_f |\mathbf{u}|^2$

Viscous stresses $\tau_\mu = 3\mu \dfrac{|\mathbf{u}|}{h}$

Yield strength $\tau_y \approx 10 - 400\, Pa$

Intergranular frictional stresses $\tau_f = (\rho g h - \mathcal{P}_b) \tan\varphi + c$



$$|\boldsymbol{\tau_b}| = f(\tau_t, \tau_\mu, \tau_y, \tau_f, \ldots)$$

$$\boldsymbol{\tau_b} = \mathrm{sgn}(u, v)\, |\boldsymbol{\tau_b}|$$

	Formulation	Flow resistance relation
PT	Pure turbulent	$\lvert\boldsymbol{\tau_b}\rvert = \tau_t$
TC	Turbulent & Coulomb	$\lvert\boldsymbol{\tau_b}\rvert = \tau_t + \tau_f$
FB	Bingham viscoplastic	$2\lvert\boldsymbol{\tau_b}\rvert^3 - 3(\tau_y + 2\tau_\mu)\lvert\boldsymbol{\tau_b}\rvert^2 + \tau_y^3 = 0$
SB	Simplified Bingham	$\lvert\boldsymbol{\tau_b}\rvert = 1.5\tau_y + 3\tau_\mu$
QD	Quadratic	$\lvert\boldsymbol{\tau_b}\rvert = \tau_y + \tau_t + \kappa/8\, \tau_\mu$
CV	Coulomb–Viscous	$2\lvert\boldsymbol{\tau_b}\rvert^3 - 3(\tau_f + 2\tau_\mu)\lvert\boldsymbol{\tau_b}\rvert^2 + \tau_f^3 = 0$

Synthetic dambreak experiment

$$\frac{\partial(\rho h u)}{\partial t} + \frac{\partial}{\partial x}(\rho h u^2 + \frac{1}{2} g \rho h^2) = -g\rho h \frac{\partial z_b}{\partial x} - \tau_{bx}$$

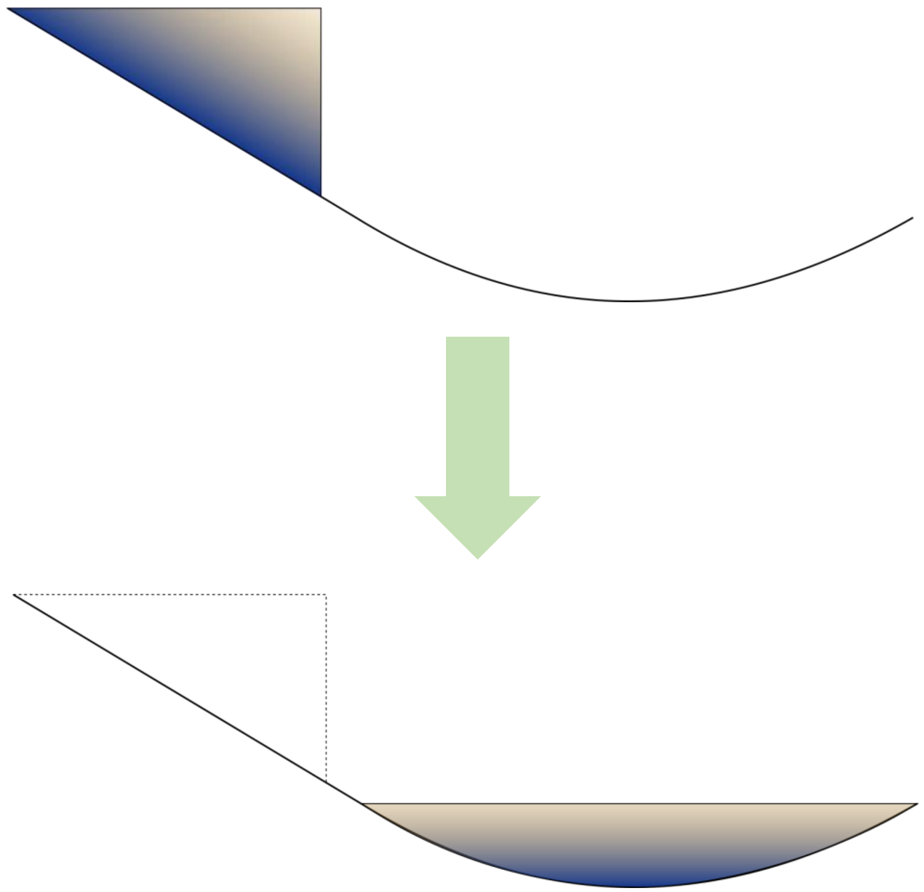
Quiescent equilibrium state:

$$\begin{cases} \partial/\partial t = 0 \\ u = 0 \end{cases} \quad \rightarrow \quad g\rho h \frac{\partial h}{\partial x} = -g\rho h \frac{\partial z_b}{\partial x} - \tau_{bx}$$

Newtonian shear stress

$$\tau_{bx} = \mu \left(\frac{\partial u}{\partial z} \right)_b = 0$$

$$g\rho h \frac{\partial (h + z_b)}{\partial x} = 0 \quad \rightarrow \quad z_s \equiv \text{const}$$



Synthetic dambreak experiment

$$\frac{\partial(\rho h u)}{\partial t} + \frac{\partial}{\partial x}(\rho h u^2 + \frac{1}{2} g \rho h^2) = -g\rho h \frac{\partial z_b}{\partial x} - \tau_{bx}$$

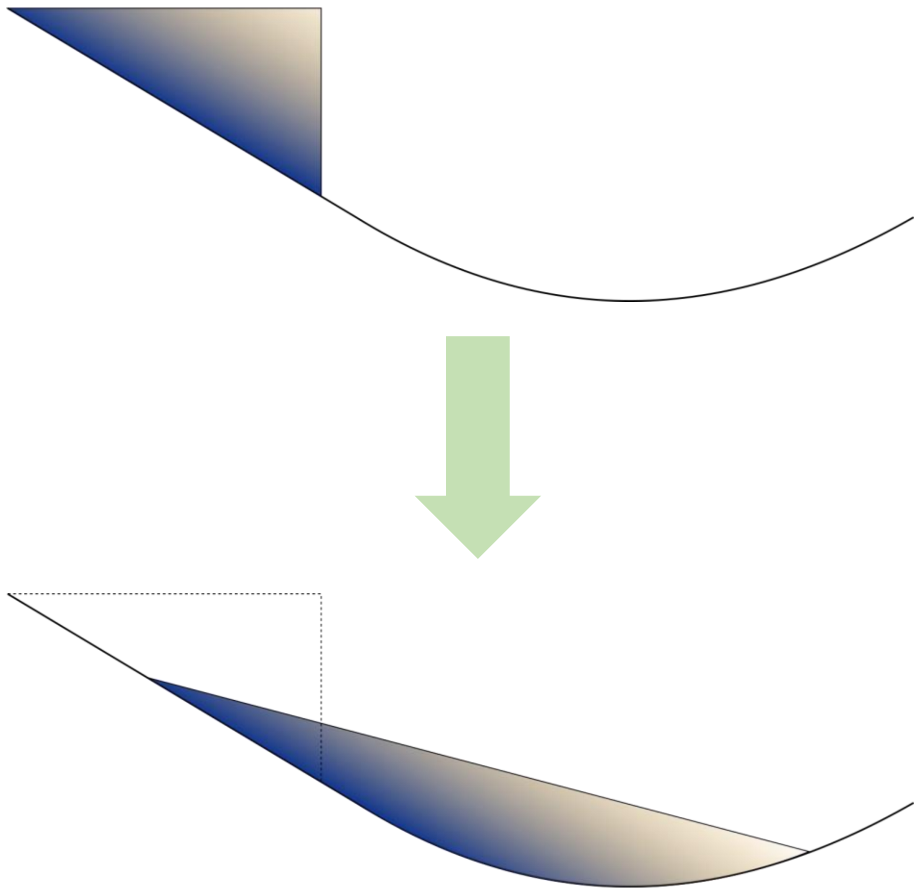
Quiescent equilibrium state:

$$\begin{cases} \partial/\partial t = 0 \\ u = 0 \end{cases} \quad \rightarrow \quad g\rho h \frac{\partial h}{\partial x} = -g\rho h \frac{\partial z_b}{\partial x} - \tau_{bx}$$

Non-Newtonian shear stress

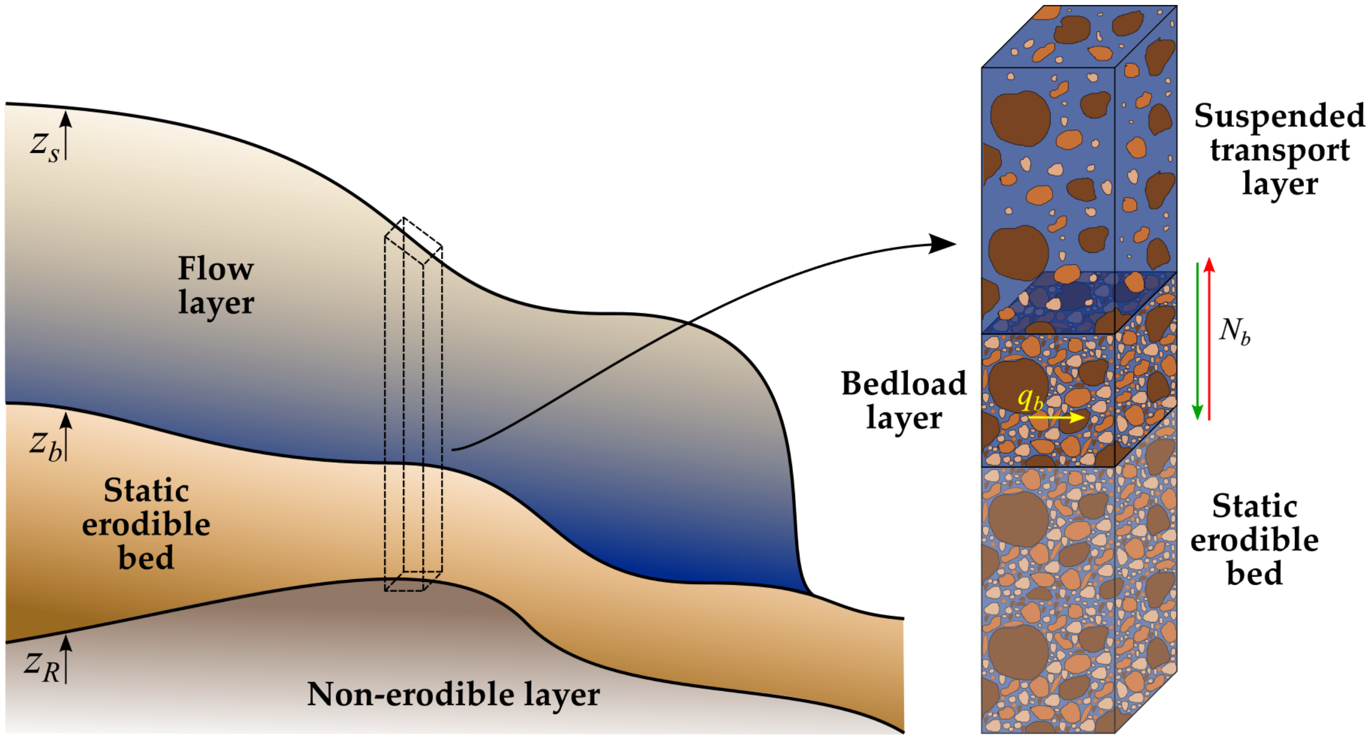
$$\tau_{bx} = \tau_y + \left(\mu(\dot{\gamma})\frac{\partial u}{\partial z}\right)_b = \tau_y \qquad \tau_y = \rho g h \tan\varphi$$

$$g\rho h \frac{\partial (h + z_b)}{\partial x} = -\rho g h \tan\varphi \quad \rightarrow \quad \tan z_s = \tan\varphi$$



Module content

Sediment load in rivers and estuaries



1) The total sediment load is composed of bed-material and wash load.

2) Wash load is composed of very fine particles which move mainly in suspension and do not deposit. Lower than 10% of the sediment load in natural rivers (Einstein, 1950) .

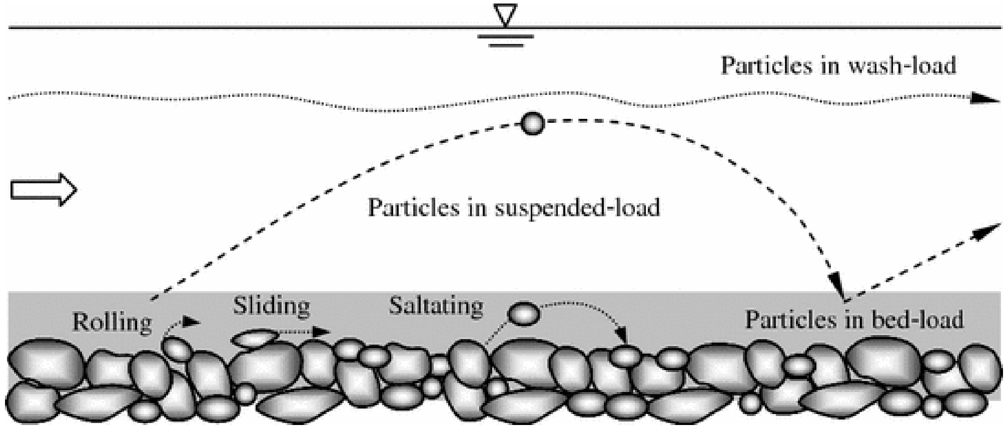
3) The bed-material load is composed of particles found in the erodible bed layer. It is exchanged constantly between the flow and the bed layer. It is divided into bedload and suspended load.

4) The non-erodible layer (bedrock) does not allow bed-material entrainment, but deposition is possible.

Sediment load in rivers and estuaries

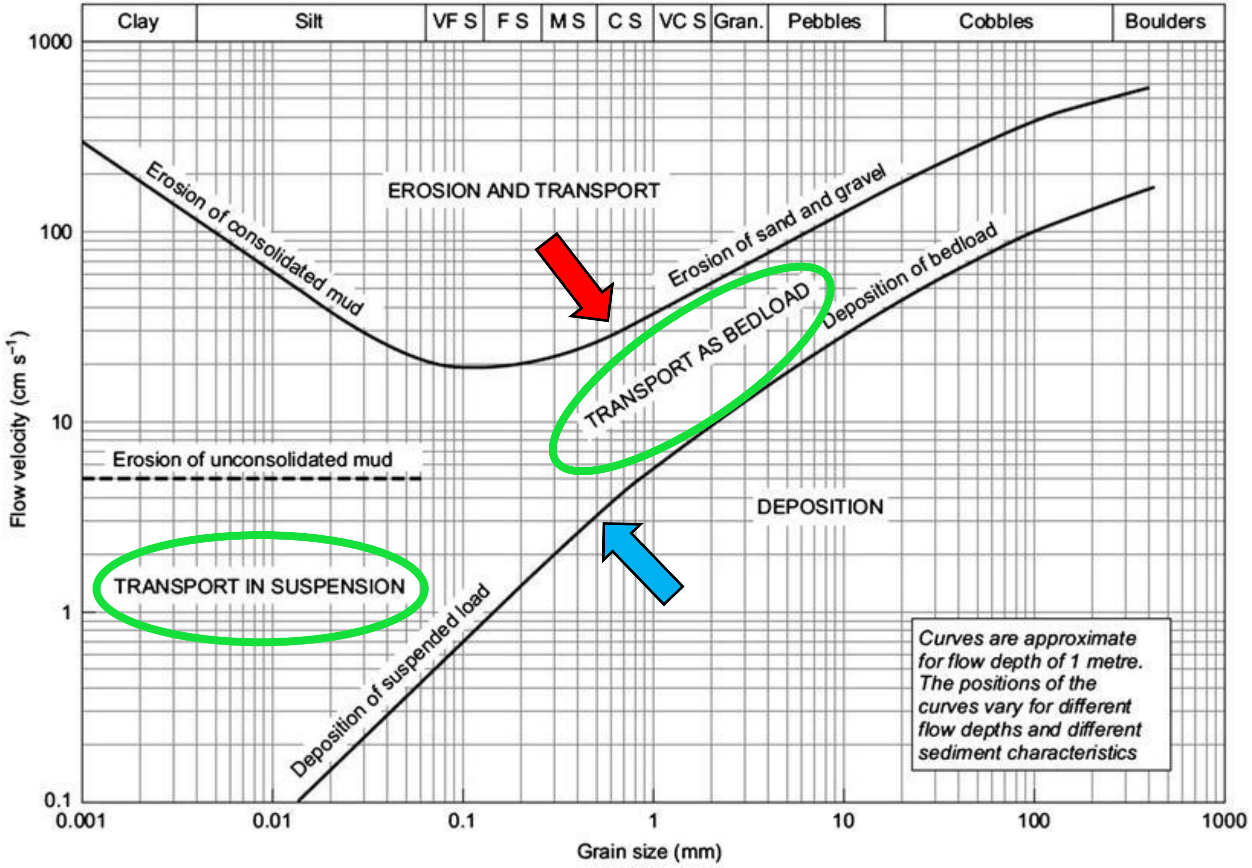
Bedload load is transported by rolling, sliding and saltation mechanisms in a thin layer over the bed surface. It accounts for 25-100% of total load for medium and coarse particles (Wu, 2007).

Suspended load is composed mainly of fine materials transported by suspension mechanism (supported by the flow turbulence) in the flow column.



(Reproduced from Dey, 2014)

Hjulström-Sundborg diagram (1935)



(Reproduced from Press & Siever, 1986)

Suspended transport equations

- Flow continuity

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = -\frac{N_b}{1-p}$$

- Flow momentum equations

$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}(hu^2 + \frac{1}{2}gh^2) + \frac{\partial}{\partial y}(huv) = -gh\frac{\partial z_b}{\partial x} - ghC_f\mathrm{U}u$$

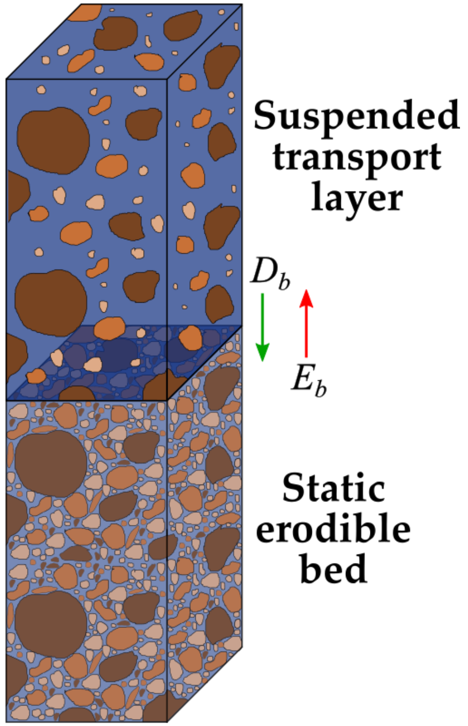
$$\frac{\partial (hv)}{\partial t} + \frac{\partial}{\partial x}(huv) + \frac{\partial}{\partial y}(hv^2 + \frac{1}{2}gh^2) = -gh\frac{\partial z_b}{\partial y} - ghC_f\mathrm{U}v$$

- Suspended load transport

$$\frac{\partial (h\phi)}{\partial t} + \frac{\partial}{\partial x}(hu\phi) + \frac{\partial}{\partial y}(hv\,\phi) = -N_b$$

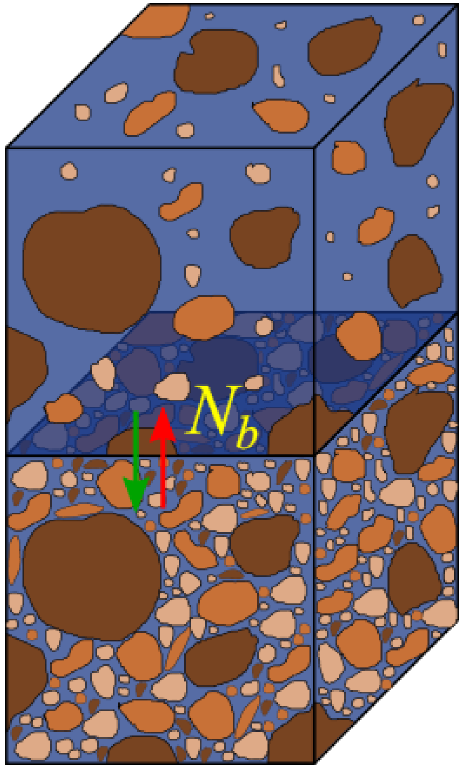
- Bed surface evolution

$$\frac{\partial z_b}{\partial t} = \frac{N_b}{1-p}$$



The lag between the flow and suspended sediment velocity is neglected.
Modelling the net exchange flux between the flow and the erodible bed requires a closure formulation.

Net solid exchange flux



The net exchange flux per unit area is usually expressed as the unbalance between the sediment deposition rate D_b and the sediment entrainment rate E_b at the bed surface.

$$N_b = D_b - E_b \; [LT^{-1}]$$

Deposition rate

$$D_b = \omega_s \, \phi\Big|_{z=z_b} = \alpha_d \, \omega_s \, \phi$$

The adaptation coefficient α_e relates the near-bed concentration to the depth-averaged suspended concentration.

Entrainment rate

$$E_b = \omega_s \, \phi^*\Big|_{z=z_b} = \alpha_e \, \omega_s \, \phi^*$$

$$\phi^* = \frac{q_s^*}{h\mathrm{U}}$$

The entrainment rate is related to the depth-averaged equilibrium concentration $\phi^*[-]$. This equilibrium concentration depends on the suspended load carrying capacity of the flow per unit length q_s^* $[L^2T^{-1}]$.

Solid transport capacity for suspended load

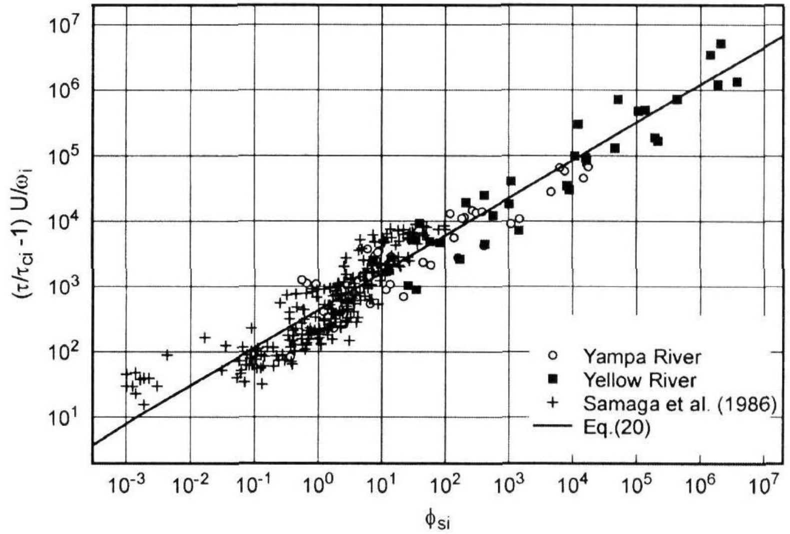
Closure formulations for the estimation of the suspended transport capacity q_s^* has been derived from empirical data. The suspended capacity depends on the local flow conditions and the sediment characteristics

Wu et at. (2000)

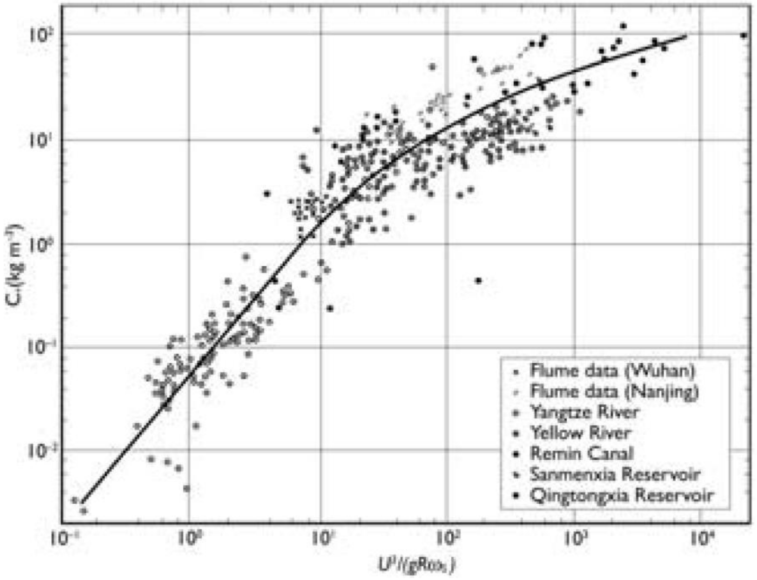
$$q_s^* = 2.62{\cdot}10^{-5}\left[\left(\frac{\theta}{\theta_c}-1\right)\frac{\mathrm{U}}{\omega_s}\right]^{1.74}\sqrt{\left(\frac{\rho_s}{\rho_w}-1\right)gd_s^3}$$

Zhang & Xie (1993)

$$q_s^* = h\mathrm{U}\frac{\frac{1}{20}\left(\frac{\mathrm{U}^3}{gh\omega_s}\right)^{1.5}}{\rho_s\left[1+\left(\frac{1}{45}\frac{\mathrm{U}^3}{gh\omega_s}\right)^{1.15}\right]}$$



(Reproduced from Wu, 2007)



(Reproduced from Wu, 2007)

Bedload transport equations

- **Flow continuity**

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = -\frac{N_b}{1-p}$$

- **Flow momentum equations**

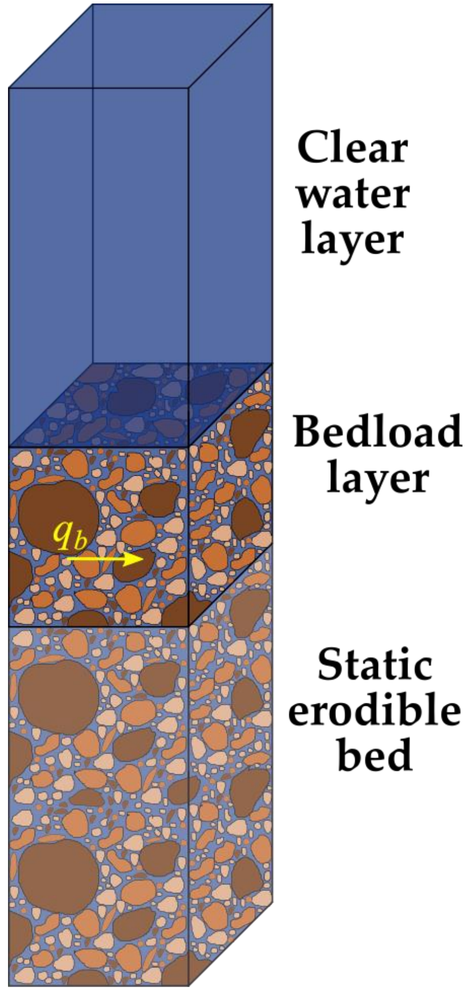
$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}(hu^2 + \frac{1}{2}gh^2) + \frac{\partial}{\partial y}(huv) = -gh\frac{\partial z_b}{\partial x} - ghC_f\mathrm{U}u$$

$$\frac{\partial (hv)}{\partial t} + \frac{\partial}{\partial x}(huv) + \frac{\partial}{\partial y}(hv^2 + \frac{1}{2}gh^2) = -gh\frac{\partial z_b}{\partial y} - ghC_f\mathrm{U}v$$

- **Capacity bedload transport (Exner equation)**

$$\frac{\partial z_b}{\partial t} + \frac{1}{1-p}\frac{\partial q_{bx}}{\partial x} + \frac{1}{1-p}\frac{\partial q_{by}}{\partial y} = 0$$

The classical approach assumes that the bedload transport is always in equilibrium conditions, i.e. the actual bedload rate adapts instantaneously to the flow capacity transport for bedload.



Bedload transport rate

Bedload moves mainly by saltation, while rolling and sliding occurs near the threshold of entrainment. The sediment velocity in the transport layer is lower than the local flow velocity.

$$\mathrm{q_b} = \sqrt{q_{bx}^2 + q_{by}^2} = (1-p)\eta\,\mathrm{U_b}$$

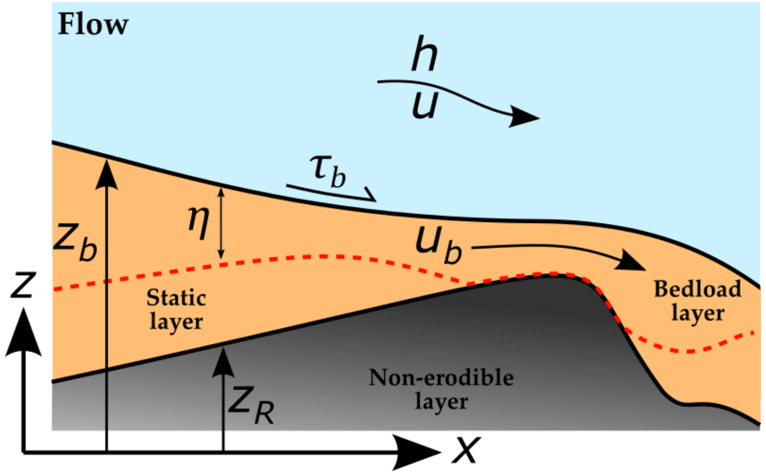
Van Rijn (1984)

$$\mathrm{U_b} = 1.5(\theta/\theta_c - 1)^{0.6}\sqrt{(\rho_s/\rho - 1)gd_s}$$

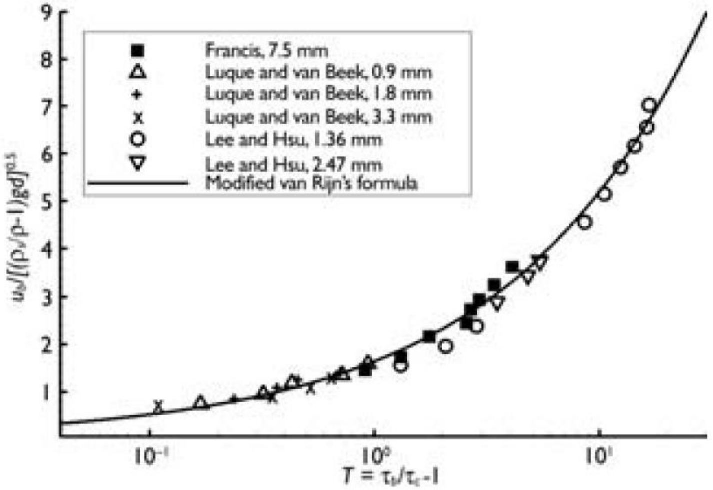
Wu et al. (2006)

$$\mathrm{U_b} = 1.64(\theta/\theta_c - 1)^{0.5}\sqrt{(\rho_s/\rho - 1)gd_s}$$

The lag between the flow and the bedload velocity increases as the sediment size increases.
Estimating the transport layer thickness is also a challenge.



(Reproduced from Wu, 2007)



Closure formula for bedload transport rate

The bedload capacity rate is estimated from the local flow conditions and the sediment characteristics using closure relationships derived using empirical and field data.

$$\mathrm{q_b} = \sqrt{q_{bx}^2 + q_{by}^2} = \Phi_b \sqrt{(\rho_s/\rho_w - 1) g d_s^3}$$

$$\Phi_b = C\,\theta^{m_1}\,(\theta - \theta_c)^{m_2} \;\; [-]$$

Dimensionless bedload transport rate (Einstein, 1950)

Meyer, Peter & Müller (1948)

$$\Phi_b = 8\,(\theta - \theta_c)^{1.5}$$

Meyer, Peter & Müller (1948)

$$\Phi_b = 12\,(\theta - \theta_c)^{1.5}$$

Nielsen (1992)

$$\Phi_b = 12\,\sqrt{\theta}\,(\theta - \theta_c)$$

Ashida & Michiue (1972)

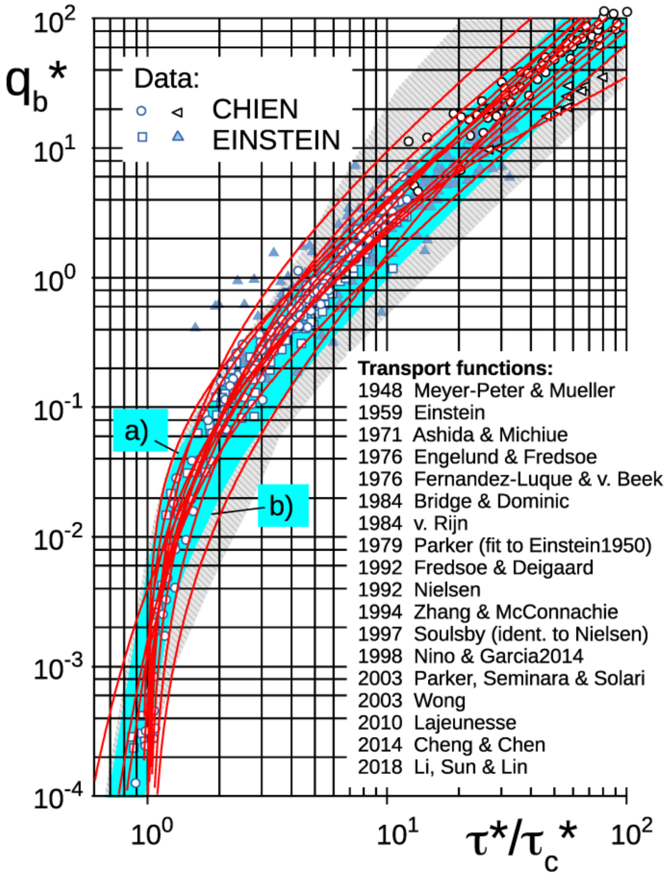
$$\Phi_b = 17\,(\theta - \theta_c)(\sqrt{\theta} - \sqrt{\theta_c})$$

Wong (2004)

$$\Phi_b = 4.93\,(\theta - \theta_c)^{1.6}$$

Fernández-Luque & van Beek (1976)

$$\Phi_b = 5.7\,(\theta - \theta_c)^{1.5}$$



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Module content

Suspended load transport models

The system of 5 equations is written in vector form as:

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{E}(\mathbf{U}) = \mathbf{S_b}(\mathbf{U}) + \mathbf{S_{\boldsymbol{\tau}}}(\mathbf{U}) + \mathbf{E_b}(\mathbf{U})$$

Conservative variables

$$\mathbf{U} = \begin{pmatrix} h \\ hu \\ hv \\ h\phi \\ z_b \end{pmatrix}$$

Conservative fluxes

$$\mathbf{E}(\mathbf{U}) = \big(\mathbf{F}(\mathbf{U}) \,|\, \mathbf{G}(\mathbf{U})\big) = \left(\begin{array}{c|c} hu & hv \\ hu^2 + \frac{1}{2}gh^2 & huv \\ huv & hv^2 + \frac{1}{2}gh^2 \\ hu\phi & hv\phi \\ 0 & 0 \end{array}\right)$$

Momentum source terms

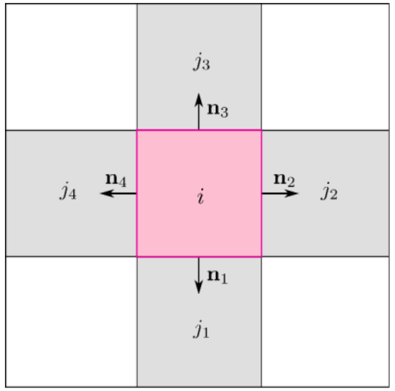
$$\mathbf{S_b}(\mathbf{U}) = \begin{pmatrix} 0 \\ -gh\frac{\partial z_b}{\partial x} \\ -gh\frac{\partial z_b}{\partial y} \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{S_{\boldsymbol{\tau}}}(\mathbf{U}) = \begin{pmatrix} 0 \\ -ghC_f \mathrm{U}u \\ -ghC_f \mathrm{U}v \\ 0 \\ 0 \end{pmatrix}$$

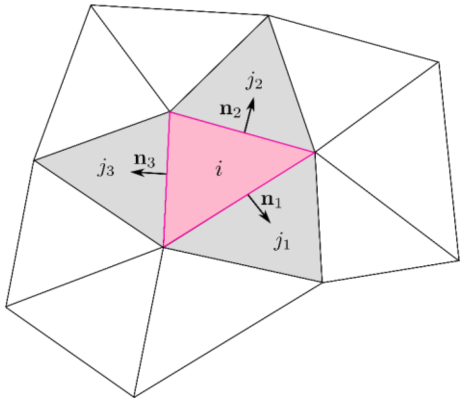
Exchange source terms

$$\mathbf{E_b}(\mathbf{U}) = \begin{pmatrix} -\frac{D_b - E_b}{1-p} \\ 0 \\ 0 \\ -(D_b - E_b) \\ -\frac{D_b - E_b}{1-p} \end{pmatrix}$$

Split method for suspended transport model

The split method solves separately the flow and the suspended transport components of the system.





1) The hydrodynamic conservative variables are updated to an intermediate state $(n+1/2)$ using a classical SWE solver.

$$\frac{d}{dt}\int_{\Omega_i} \mathbf{U}\, d\Omega = -\sum_{k=1}^{\mathrm{NE}} (\mathbf{E}\,\mathbf{n} - \widetilde{\mathbf{S}}_{\mathbf{b}} - \widetilde{\mathbf{S}}_{\boldsymbol{\tau}})_k\, l_k$$

The cell updating formula is written using the explicit numerical fluxes at the intercell walls.

$$\mathbf{U}_i^{n+1/2} = \mathbf{U}_i^n - \frac{\Delta t}{A_i}\sum_{k=1}^{\mathrm{NE}} \boldsymbol{\mathcal{F}}_k^{\downarrow -} l_k$$

$$\boldsymbol{\mathcal{F}}_k^{\downarrow -} = \begin{pmatrix} q_n^{\downarrow -} \\ m_x^{\downarrow -} \\ m_y^{\downarrow -} \end{pmatrix}_k^n$$

The numerical water flux $q_n^{\downarrow -}$ used for the updating of the flow mass equation must be stored for solving the sediment transport.

The time step is limited using a CFL condition

$$\Delta t = \mathrm{CFL}\, \min_k \left[\frac{\min(A_i, A_j)}{l_k(|\widetilde{u}_{nk}| + g\widetilde{h}_k)} \right]$$

Split method for suspended transport models

2) The suspended sediment mass at the cells is updated to the intermediate state $(n+1/2)$ using an upwind approximation for the intercell advective solid flux.

$$(h\phi)_i^{n+1/2} = (h\phi)_i^n - \frac{\Delta t}{A_i}\sum_{k=1}^{\mathrm{NE}}(q_n^{\downarrow -}\,\phi^{\downarrow -})_k\, l_k \qquad \phi^{\downarrow -} = \frac{\phi_i^n + \phi_j^n}{2} - \operatorname{sgn}(q_n^{\downarrow -})\frac{\phi_j^n - \phi_i^n}{2}$$

3) The exchange source terms are included using a centered integration to obtain the updated conservative variables at the new time $(n+1)$.

$$h_i^{n+1} = h_i^{n+1/2} - \Delta t\frac{(D_b - E_b)_i^{n+1/2}}{1-p} \qquad (D_b - E_b)_i^{n+1/2} = \alpha_d\,\omega_s\,\phi_i^{n+1/2} - \alpha_e\,\omega_s\left(\frac{q_s^*}{h\mathrm{U}}\right)_i^{n+1/2}$$

$$(h\phi)_i^{n+1} = (h\phi)_i^{n+1/2} - \Delta t(D_b - E_b)_i^{n+1/2}$$

$$(z_b)_i^{n+1} = (z_b)_i^{n+1/2} + \Delta t\frac{(D_b - E_b)_i^{n+1/2}}{1-p}$$

Bedload transport models

The system of 4 equations is written in vector form as:

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{E}(\mathbf{U}) = \mathbf{S_b}(\mathbf{U}) + \mathbf{S_{\tau}}(\mathbf{U})$$

Conservative variables

$$\mathbf{U} = \begin{pmatrix} h \\ hu \\ hv \\ z_b \end{pmatrix}$$

Conservative fluxes

$$\mathbf{E}(\mathbf{U}) = \big(\mathbf{F}(\mathbf{U}) \,|\, \mathbf{G}(\mathbf{U})\big) = \left(\begin{array}{c|c} hu & hv \\ hu^2 + \frac{1}{2}gh^2 & huv \\ huv & hv^2 + \frac{1}{2}gh^2 \\ \frac{1}{1-p}q_{bx} & \frac{1}{1-p}q_{by} \end{array}\right)$$

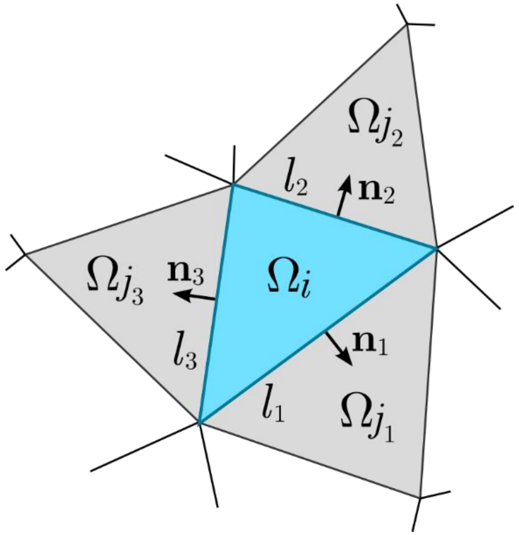
Momentum source terms

$$\mathbf{S_b}(\mathbf{U}) = \begin{pmatrix} 0 \\ -gh\frac{\partial z_b}{\partial x} \\ -gh\frac{\partial z_b}{\partial y} \\ 0 \end{pmatrix} \quad \mathbf{S_{\tau}}(\mathbf{U}) = \begin{pmatrix} 0 \\ -ghC_f \mathrm{U}u \\ -ghC_f \mathrm{U}v \\ 0 \end{pmatrix}$$

Split method for bedload models

1) The hydrodynamic conservative variables are updated to the next time $(n+1)$ using a classical SWE solver.

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{A_i} \sum_{k=1}^{\mathrm{NE}} \boldsymbol{\mathcal{F}}_k^{\downarrow -} l_k \qquad \boldsymbol{\mathcal{F}}_k^{\downarrow -} = \begin{pmatrix} q_n^{\downarrow -} \\ m_x^{\downarrow -} \\ m_y^{\downarrow -} \end{pmatrix}_k^n$$



2) The bed surface elevation is updated to the next time $(n+1)$ using an upwind approximation for the intercell bedload flux. The upwind behavior is given the sign of the virtual bedload celerity $\tilde{\lambda}_b$ at the intercell edges.

$$\frac{d}{dt}\int_{\Omega_i} z_b \, d\Omega = -\sum_{k=1}^{\mathrm{NE}} \left(\frac{1}{1-p} q_{bn}\right)_k l_k$$

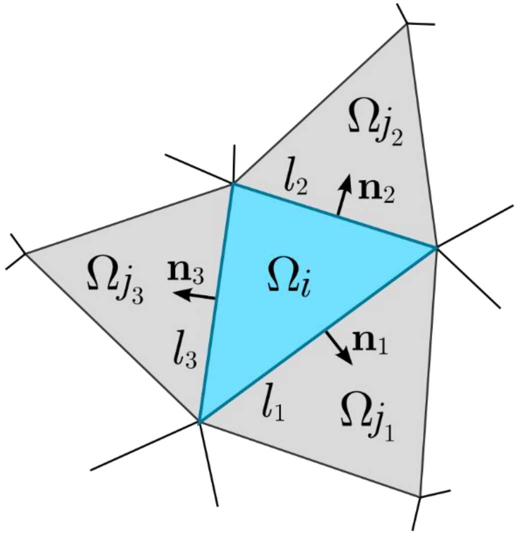
$$(z_b)_i^{n+1} = (z_b)_i^n - \frac{\Delta t}{A_i} \sum_{k=1}^{\mathrm{NE}} \left(\frac{1}{1-p} q_{bn}^{\downarrow -}\right)_k l_k \qquad q_{bn}^{\downarrow -} = \frac{(q_{bn})_i^n + (q_{bn})_j^n}{2} - \mathrm{sgn}(\lambda_{bk}) \frac{(q_{bn})_j^n - (q_{bn})_i^n}{2}$$

$$(q_{bn})_i^n = (q_{bx})_i^n \, n_{xk} + (q_{by})_i^n \, n_{yk}$$

Split method for bedload models

3) The virtual bedload celerity $\tilde{\lambda}_b$ at the cell edges is calculated from the discrete bedload transport rates at the neighbouring cells:

$$\widetilde{\lambda}_{bk} = \frac{1}{1-p}\frac{(q_{bn})_j^n - (q_{bn})_i^n}{(z_b)_j^n - (z_b)_i^n}$$



4) An additional limitation of the time step Δt must be included using the virtual bedload celerity $\tilde{\lambda}_b$ at the intercell edges. This gives the weakly-coupled character to the split scheme.

Flow time step

$$\Delta t_f = \min_k \left[\frac{\min(A_i, A_j)}{l_k(|\widetilde{u}_{nk}| + g\widetilde{h}_k)} \right]$$

Bedload time step

$$\Delta t_b = \min_k \left[\frac{\min(A_i, A_j)}{l_k\,|\widetilde{\lambda}_{bk}|} \right]$$

Then, the global time step is limited using a CFL condition $\Delta t = \mathrm{CFL}\,\min\left[\Delta t_f, \Delta t_b\right]$

References

Wu, W. (2007). Computational River Dynamics. NetLibrary, Inc, CRC Press .

Calhoun, N.C. & Clague, J.J. (2018). Distinguishing between debris flows and hyperconcentrated flows: an example from the eastern Swiss Alps, Earth Surface Processes and Landforms 43 (6), pp. 1280-1294.

Iverson, R.M. (1997). The physics of debris flows, Reviews of Geophysics 35 (3), pp. 245-296.

Jakob, M. & Hungr, O. (2005). Debris-flow Hazards and Related Phenomena, Springer Praxis Books, Springer Berlin Heidelberg.

George, D.L. & Iverson, R.M. (2014). A depth-averaged debris-flow model that includes the efects of evolving dilatancy. II. Numerical predictions and experimental tests, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 470 (2170), pp. 20130820.

Dey S. (2014) Bed-Load Transport. In: Fluvial Hydrodynamics. GeoPlanet: Earth and Planetary Sciences, Springer Berlin Heidelberg.

Zanke U. & Roland A. (2020) Sediment Bed-Load Transport: A Standardized Notation, Geosciences 10 (9), pp. 368.

Martínez-Aranda, S., Murillo, J. & García-Navarro, P. (2019). A comparative analysis of capacity and non-capacity formulations for the simulation of unsteady flows over finite-depth erodible beds. Advances in Water Resources 130, pp. 91–112

Martínez-Aranda, S., Murillo, J. & García-Navarro, P. (2020). A robust two-dimensional model for highly sediment-laden unsteady flows of variable density over movable beds. Journal of Hydroinformatics 22 (5), pp. 1138–1160.

2D models and numerical techniques for erodible bed and sediment transport simulation

Workshop 2 – RESCUER MSCA Doctoral Network

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