Universidad de Zaragoza

1D dam-break simulation using a finite volume scheme

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1 CONTENTS

2 OBJECTIVES

1. To become familiar with the properties of an explicit, upwind, conservative finite volume scheme to solve the 1D shallow water equations.
2. To complete, compile and run a simulation code for cases of 1D dam-break flow.
3. To evaluate the sensitivity of the results to the grid size and the CFL.

3 INTRODUCTION

The 1D shallow water or Saint-Venant equations for unsteady flow in open channels are the 1D approximation of free-surface flow and are a simple model of complex phenomena, incorporating only the most important influences. Strictly one-dimensional flow and hydrostatic pressure distribution are considered. The differential equations for conservation of mass and momentum in the x direction are:

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0 \tag{1}$$

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right)+gA\frac{\partial h}{\partial x}=gA(S_0-S_f) \tag{2}$$

Where *(A,Q)* are the wetted cross-sectional area and the flow rate through it respectively, *h* is the maximum water depth at each section and g is the acceleration due to gravity. The terms on the right in the momentum equation are the bed slope S_0 defined as the derivative (with opposite sign) of the bed elevation z_b in the *x* direction and the friction loss slope which we will formulate using Manning's law as a function of the roughness coefficient n and the hydraulic radius $R=A/P_m$ where P_m is the wetted perimeter.

$$S_0=-\frac{\partial z_b}{\partial x} \qquad\qquad S_f=\frac{n^2 Q|Q|}{A^2 R^{\frac{4}{3}}} \tag{3}$$

If we **simplify the equations** by assuming that we have a rectangular section channel ($A = Bh$), prismatic (B=cte), with a flat bottom ($S_0 = 0$) and without friction ($n = 0$, $S_f = 0$), the system of equations written per unit of width is:

$$\frac{\partial h}{\partial t}+\frac{\partial hu}{\partial x}=0 \tag{4}$$

$$\frac{\partial (hu)}{\partial t}+\frac{\partial}{\partial x}\left(hu^2+g\frac{h^2}{2}\right)=0 \tag{5}$$

3.1 Exercise:

1) Check that system (4-5) is hyperbolic by identifying the vector of conserved variables **U** and the flux vector **F** as well as their Jacobian matrix and computing their eigenvalues and eigenvectors.

4 IDEAL DAM-BREAK

The ideal dam-break is one of the most classic examples of transient problems with an exact solution and one of the most used to compare the performance of numerical schemes. This problem, solved for the first time by Stoker (1957), is a particular case, applied to the SW equations, of what is called the Riemann problem. Considering the theory of characteristics and the theory of discontinuity propagation, an exact solution to the evolution of the problem can be calculated.

Initial conditions with a discontinuity that imaginarily separates two regions with zero velocity allow us to obtain a solution to the movement of the water once the imaginary separating wall disappears. The evolution of the water surface is schematized in Figure 1. It can be seen that the side where there was a higher water level empties while the side where there was a lower level fills. For a given time t_0 >0, four regions of space can be distinguished, delimited by the characteristic curves and by the path of the discontinuity propagation. This second part is filled by means of a mobile hydraulic jump type wave that advances at a constant speed in the absence of obstacles and friction.

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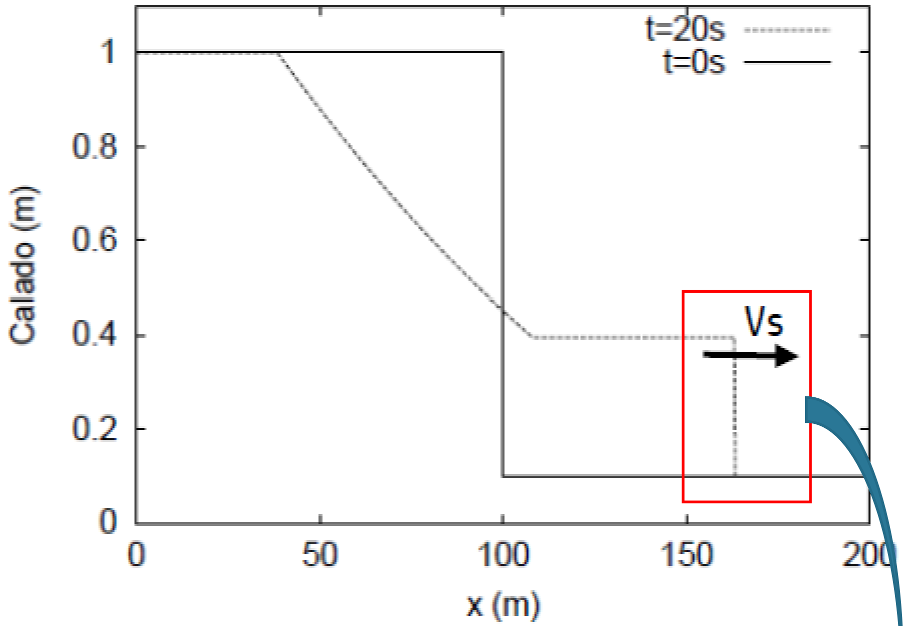

Figure 1: Dam-break

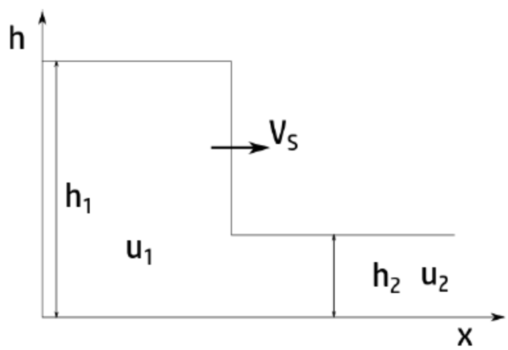

Figure 2: Detail of the shock wave propagation

The propagation of a discontinuity in water (traveling shock wave) is physically characterized by the depths and velocities on either side of the discontinuity and by the discontinuity propagation velocity V_S, as shown in Figure 2.

The relationship between these quantities is linked by the Rankine-Hugoniot equations of mass and momentum balance. Using integral equations of mass and momentum in a control volume from an observer moving with velocity V_S:

$$\delta\left[h(u-V_S)\right]=0 \tag{6}$$

$$\delta\left[h(u-V_S)^2+g\frac{h^2}{2}\right]=0 \tag{7}$$

where $\delta(x)=x_2-x_1$ applied to a given variable x and the velocity u- V_S would be relative to an observer moving with the velocity V_S. Therefore, having two equations and five unknowns (h_1, h_2, u_1, u_2 and V_S), the system will have three degrees of freedom. Knowing the depths h_1 and h_2, and the downstream velocity u_1, it is possible to theoretically obtain from (6) and (7) the upstream velocity u_2 and the propagation velocity of the discontinuity V_S:

$$V_S=u_2+\sqrt{\frac{g(h_1-h_2)}{2}\frac{h_1}{h_2}}$$
$$u_1=V_S\frac{h_1-h_2}{h_1}+\frac{h_2u_2}{h_1} \tag{8}$$

5 NUMERICAL SIMULATION USING ROE'S SCHEME

We are going to develop a simulation program that solves the system using Roe numerical scheme in finite volumes. We proceed to formulate the quantities necessary to construct the approximate Roe solver. The decomposition of the spatial variation vector of the conserved variables in the basis of eigenvectors:

$$\delta\mathbf{U}_{i+1/2}=\mathbf{U}_{i+1}-\mathbf{U}_i=\sum_{m=1}^{2}(\tilde{\alpha}^m\ \tilde{\mathbf{e}}^m)_{i+1/2}, \tag{9}$$

where the coefficients are

$$\tilde{\alpha}_1=\frac{\tilde{\lambda}_2\,\delta A-\delta Q}{2\tilde{c}},\quad \tilde{\alpha}_2=\frac{-\tilde{\lambda}_1\,\delta A+\delta Q}{2\tilde{c}}, \tag{10}$$

the approximate eigenvectors: $\tilde{\mathbf{e}}_1^T=(1,\tilde{\lambda}_1),\ \tilde{\mathbf{e}}_2^T=(1,\tilde{\lambda}_2)$

the approximation of the eigenvalues is: $\tilde{\lambda}_1=\tilde{u}-\tilde{c},\quad \tilde{\lambda}_2=\tilde{u}+\tilde{c},$

and the Roe's averages at the cell edge i+1/2 between cells i , i+1 are defined:

$$\tilde{u}_{i+1/2} = \frac{\sqrt{A_i}u_i + \sqrt{A_{i+1}}u_{i+1}}{\sqrt{A_i} + \sqrt{A_{i+1}}}, \quad \tilde{c}_{i+1/2} = \sqrt{g\frac{A_i + A_{i+1}}{B_i + B_{i+1}}}$$

In case there are source terms **H** in the system, it can be written:

$$\frac{\partial \mathbf{U}(x,t)}{\partial t} + \mathbf{J}(x,\mathbf{U})\frac{\partial \mathbf{U}(x,t)}{\partial x} = \mathbf{H}'(x,\mathbf{U}) \qquad \mathbf{H}'(x,\mathbf{U}) = \mathbf{H}(x,\mathbf{U}) - \frac{\partial \mathbf{F}(x,\mathbf{U})}{\partial x}$$

They can also be expressed in the eigenvectors basis

$$(\widetilde{\mathbf{H}}'\ \delta x)_{i+1/2} = \sum_{m=1}^{2} (\tilde{\beta}_m\ \tilde{\mathbf{e}}_m)_{i+1/2}$$

With the coefficients, for the general case:

$$\tilde{\beta}_1 = -\frac{1}{2\tilde{c}}\{g\widetilde{A}[(\widetilde{S}_0 - \widetilde{S}_f)\delta x - \delta h + \frac{1}{\tilde{B}}\delta A]\}, \quad \tilde{\beta}_2 = -\tilde{\beta}_1,$$

In case of a prismatic frictionless channel:

$$\tilde{\beta}_1 = -\frac{1}{2\tilde{c}}\left\{g\tilde{A}\left[\left(\tilde{S}_0 - \tilde{S}_f\right)\delta x \quad \right]\right\}$$

Finally, the scheme for the variables update can be compactly written:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\left[\left(\sum_m \tilde{\lambda}^+\tilde{\gamma}\tilde{\mathbf{e}}\right)_{i-1/2}^m + \left(\sum_m \tilde{\lambda}^-\tilde{\gamma}\tilde{\mathbf{e}}\right)_{i+1/2}^m\right]^n \tag{14}$$

$$\tilde{\lambda}_{i+1/2}^{\pm m} = \frac{1}{2}(\tilde{\lambda} \pm |\tilde{\lambda}|)_{i+1/2}^m \qquad \tilde{\gamma}_{i+1/2}^m = \left(\tilde{\alpha} - \frac{\tilde{\beta}}{\tilde{\lambda}}\right)_{i+1/2}^m$$

Note the upwind character and the correction in case of transcritical flow (entropy correction)

$\lambda_i^m < 0 < \lambda_j^m$.

$$\bar{\lambda}_k^m = \lambda_i^m \frac{(\lambda_j^m - \widetilde{\lambda}_k^m)}{(\lambda_j^m - \lambda_i^m)} \qquad \widehat{\lambda}_k^m = \lambda_j^m \frac{(\widetilde{\lambda}_k^m - \lambda_i^m)}{(\lambda_j^m - \lambda_i^m)}$$

$\bar{\lambda}_k^m + \widehat{\lambda}_k^m = \widetilde{\lambda}_k^m$, and $\bar{\lambda}_k^m < 0$ and $\widehat{\lambda}_k^m > 0$ by definition.

EXERCISE

1. Verify that using the coefficients (10) it is possible to write the linear decomposition in the eigenvector basis (9).
2. Complete, compile and run the program in the folder *swe1D_fellow.* For that purpose, the formulation of the numerical model must be completed in *h_compute_wall_fluxes* within *shallow_water.c*
3. Use the code to obtain the numerical solution for the wave propagation in a closed domain both upstream and downstream under different configurations. The setting can be modified in *geometry.input* and *simulation.input* which are located inside the folder *input*
    a. Flat frictionless bed using different initial water surface discontinuities.
    b. Flat bed with friction.
    c. Sloping bed with/without friction.
    d. Horizontal bed with a step in the bottom.
    e. Horizontal bed with an adverse slope.