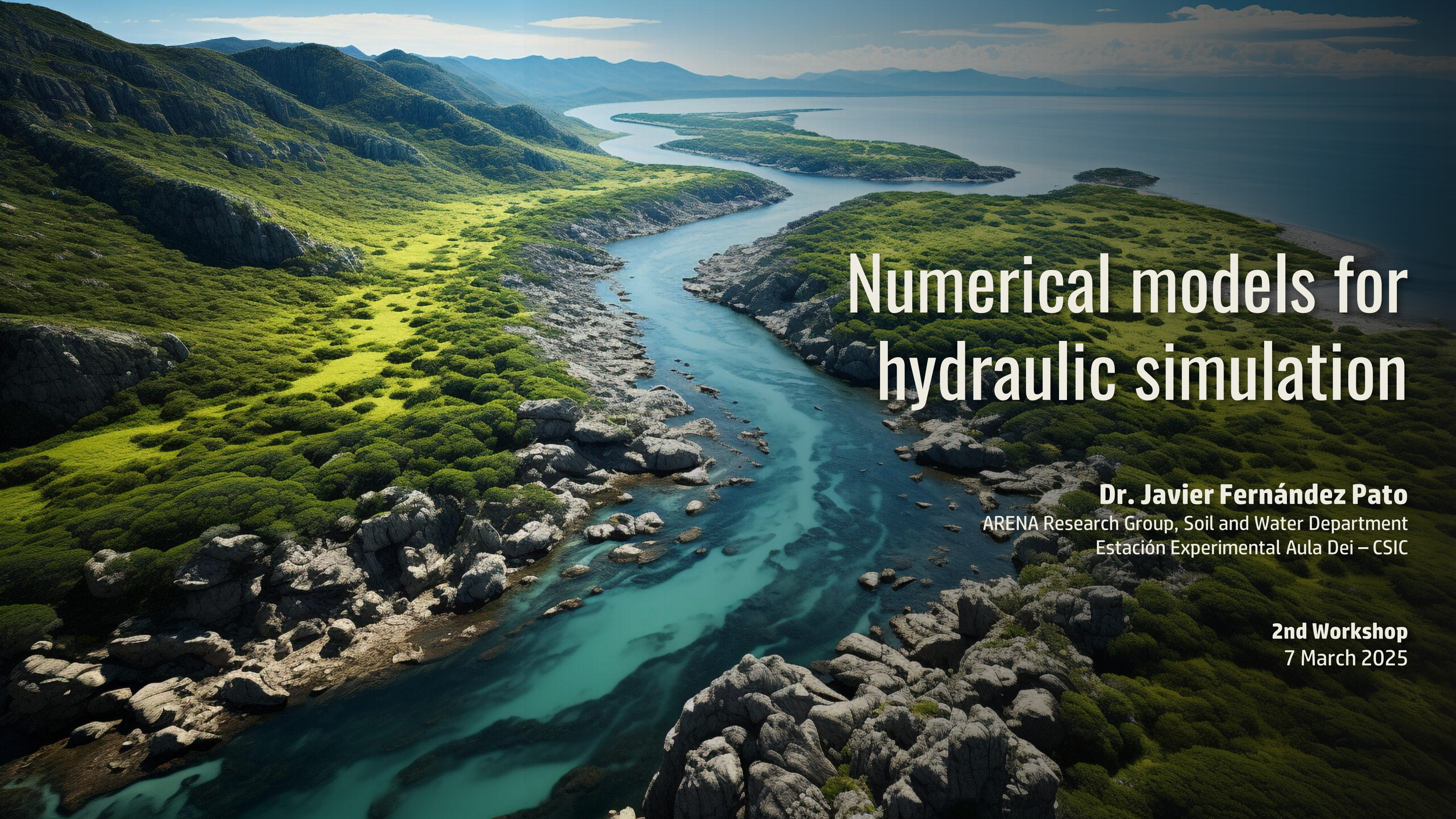

Numerical models for hydraulic simulation

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2nd Workshop
7 March 2025

OUTLINE

Motivation

Review of two-dimensional (2D) hydraulic models

- 2D Shallow Water Equations
- 2D Zero-Inertia Model

Numerical techniques

Computational meses

Examples of 2D test case simulation

- Urban hydrology
- Contaminant transport
- Groundwater Flow
- Volcanic lava flow

Motivation

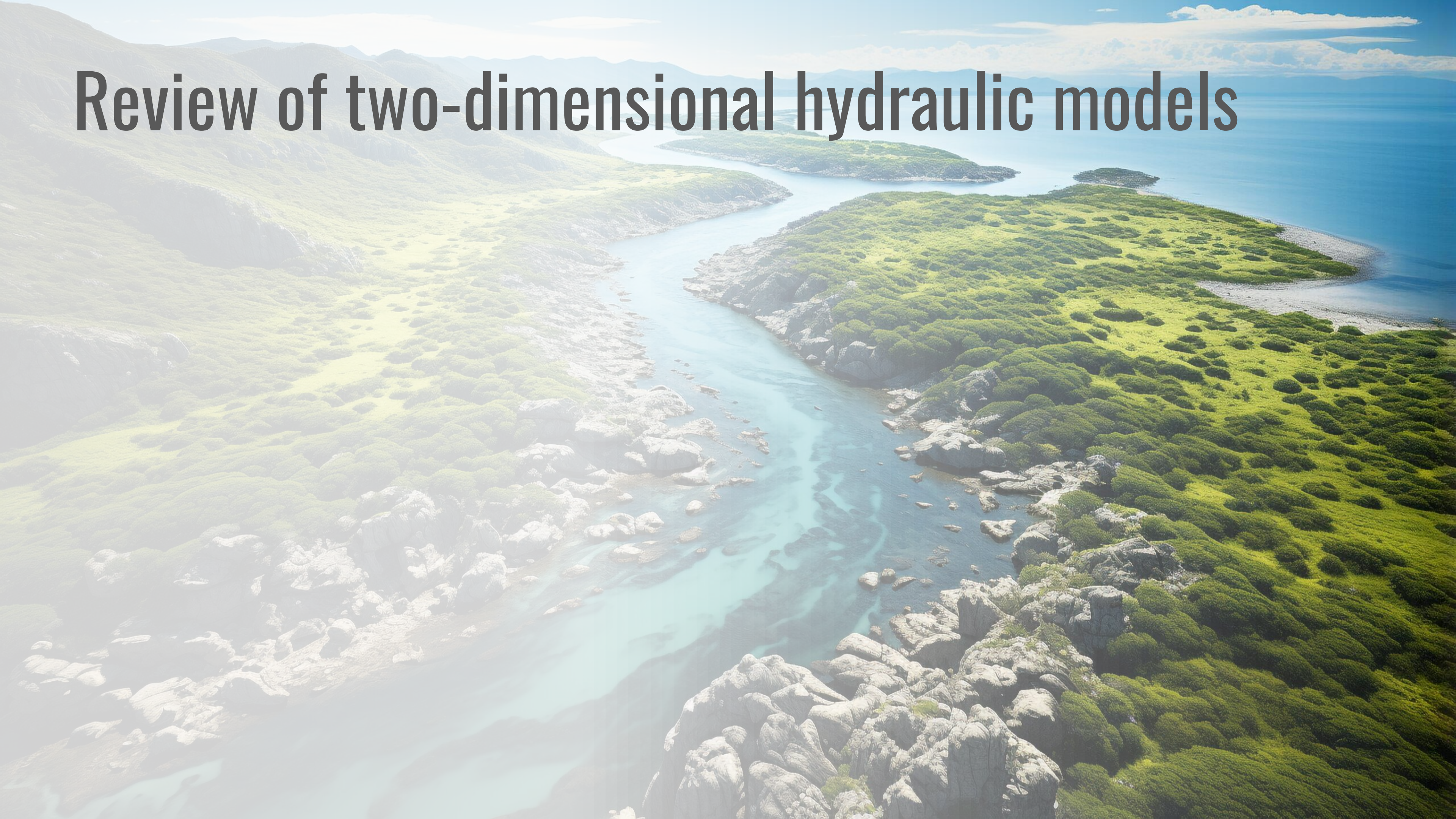
Motivation:

- Simulations allow us to predict the behavior of water (or other fluids) under adverse conditions, such as floods, flash floods, dam failures, etc.
- They help define water resource management regulations and design emergency plans.
- Proposals for corrective measures to mitigate flood damage, pollutant discharges, etc.






Review of two-dimensional hydraulic models



What can we consider in a comprehensive hydraulic simulator?

- Surface, groundwater, and urban drainage flows.
- Hydrological components: rainfall, evaporation and infiltration.
- Suspended and bed sediment transport.
- Solute/contaminant transport and water quality.
- Viscous flow (mud, oil, lava, etc.).
- Hydraulic structures: bridges, dikes, spillways, etc.

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Many simultaneous processes

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If the domain is large, many computational cells.

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High computational costs

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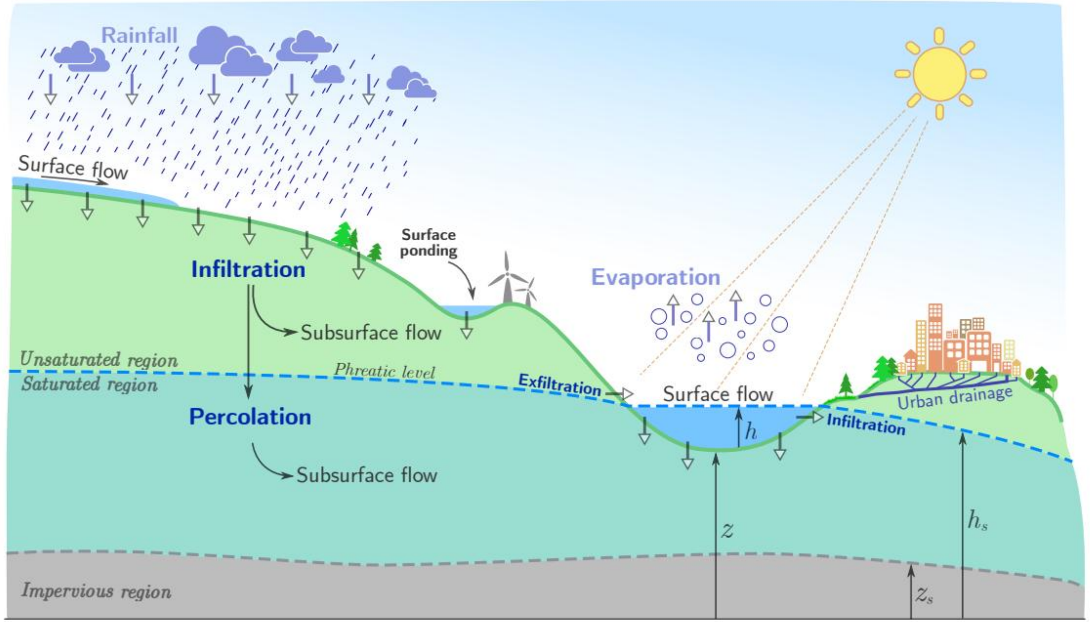
Optimization techniques



Common Steps for Developing a Computational Hydraulic Model

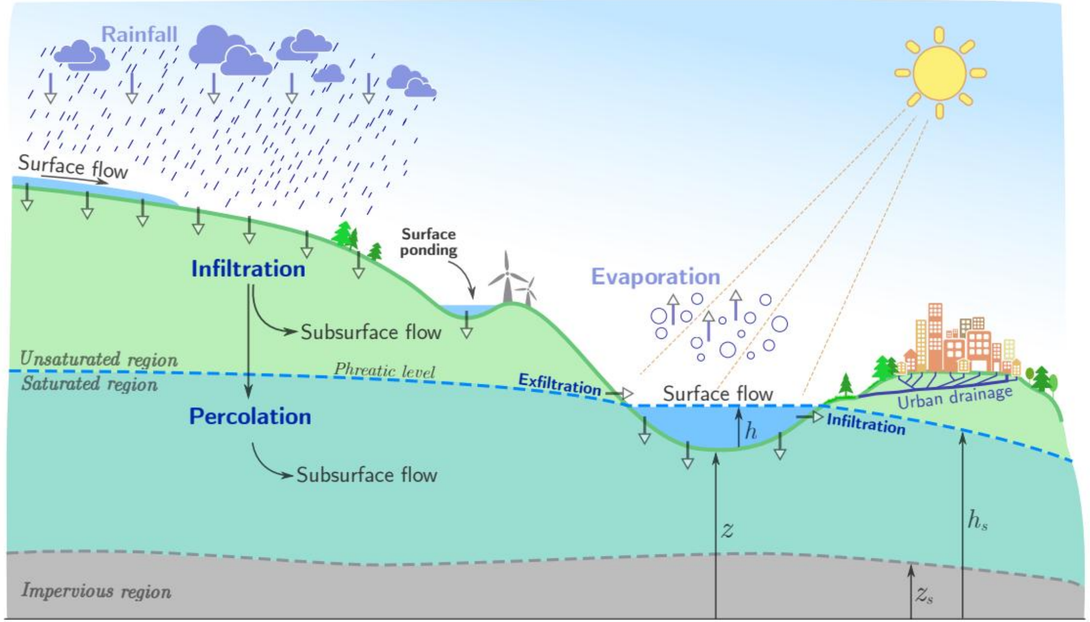
1. Identifying the physical processes that need to be considered.
2. Choosing the most suitable mathematical model:

   Higher or lower complexity?

   How many dimensions are required?
3. Solving the equations:

   Exact solution (rarely used).

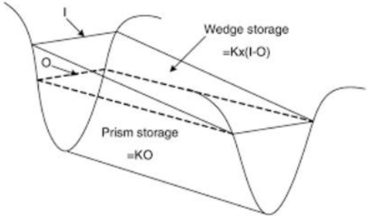
   Approximate solution (numerical methods).
4. Validation of numerical results.
5. Parameter calibration (if necessary).



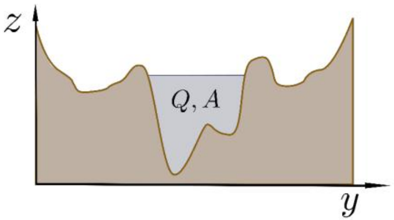
How many dimensions are required?

0D / Lumped models

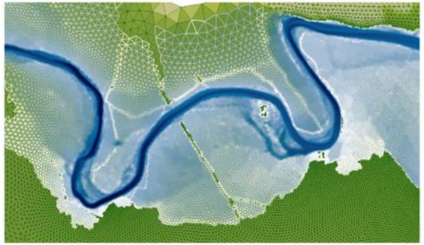
Simplified transport methods

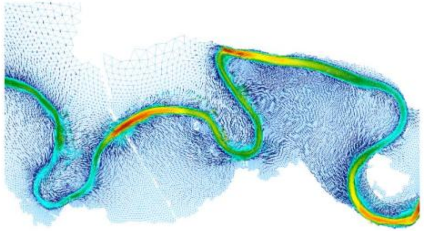


1D



2D

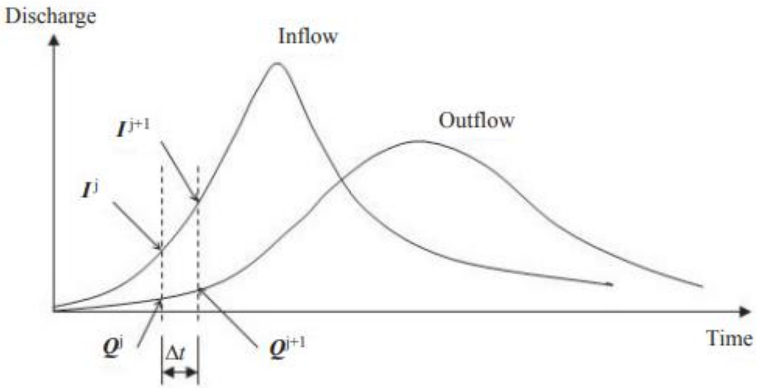




How many dimensions are required?

0D / Lumped models

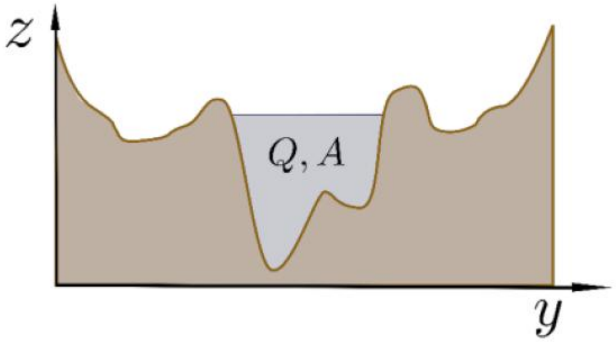
- The transport of a flow across a domain can be classified as lumped or distributed.
- In lumped models, the flow is computed as a function of time at a specific location, whereas in distributed models, it is calculated simultaneously at multiple cross-sections or cells within the domain.
- **Physical processes are assumed to be constant** throughout the entire domain (watershed) or within subdomains (sub-watersheds).



How many dimensions are required?

1D

- Predominantly one-dimensional flow.
- Depth and velocity are only functions of (x,t).
- Velocity is considered uniform across the entire cross-section.



1D Shallow Water Equations (reminder from yesterday!)

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A} + gI_1\right) = gA\left(S_0 - S_f\right) + gI_2$$

$$S_0 = -\frac{\partial z_b}{\partial x} = tg\theta$$

$$S_f = \frac{n^2 Q|Q|}{R_h^{4/3} A^2}$$

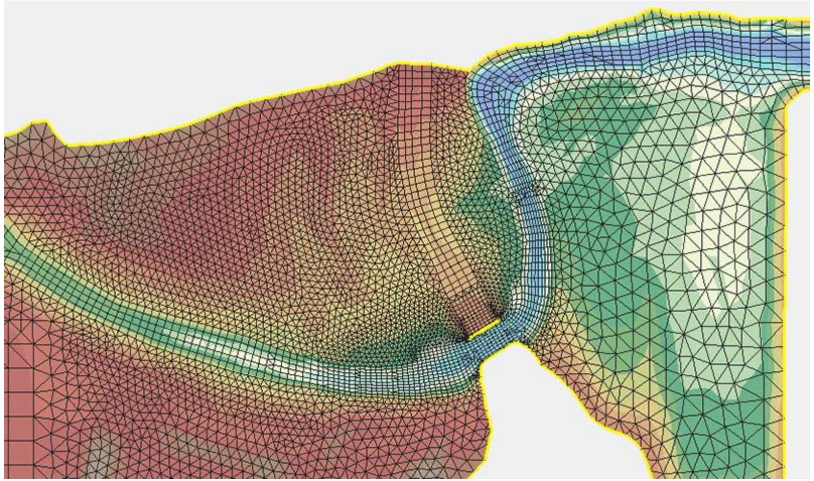
$$I_1 = \int_0^h (h - \eta) \cdot \sigma(x, \eta) d\eta$$

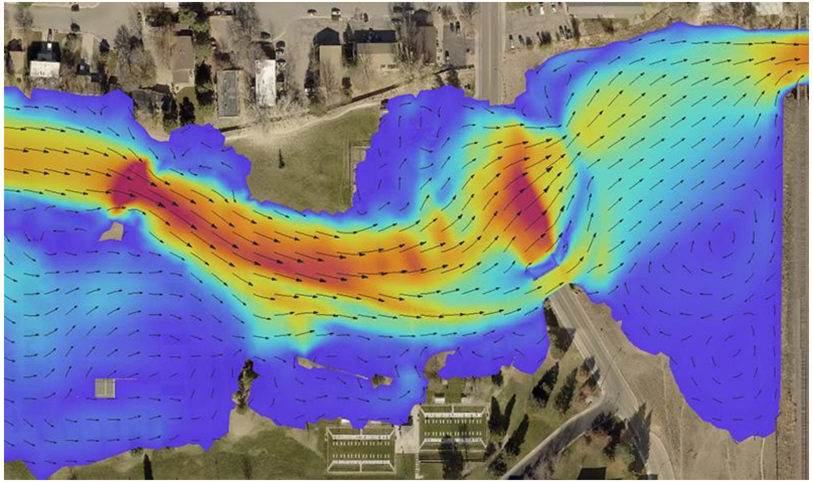
$$I_2 = \int_0^h (h(x) - \eta) \frac{\partial \sigma(x, \eta)}{\partial x} d\eta$$

How many dimensions are required?

2D

- The simulation covers the entire domain, not just the main channel → Computational mesh.
- All hydraulic variables are averaged in the vertical.
- It allows for a more realistic representation of natural topographies, such as a watershed.
- In each cell, localized input (rainfall, infiltration, roughness, etc.) and output (water depths, velocities, infiltrated water, etc.) can be defined.
- Much higher computational cost:
  - Many more cells.
  - Many more calculations per cell.





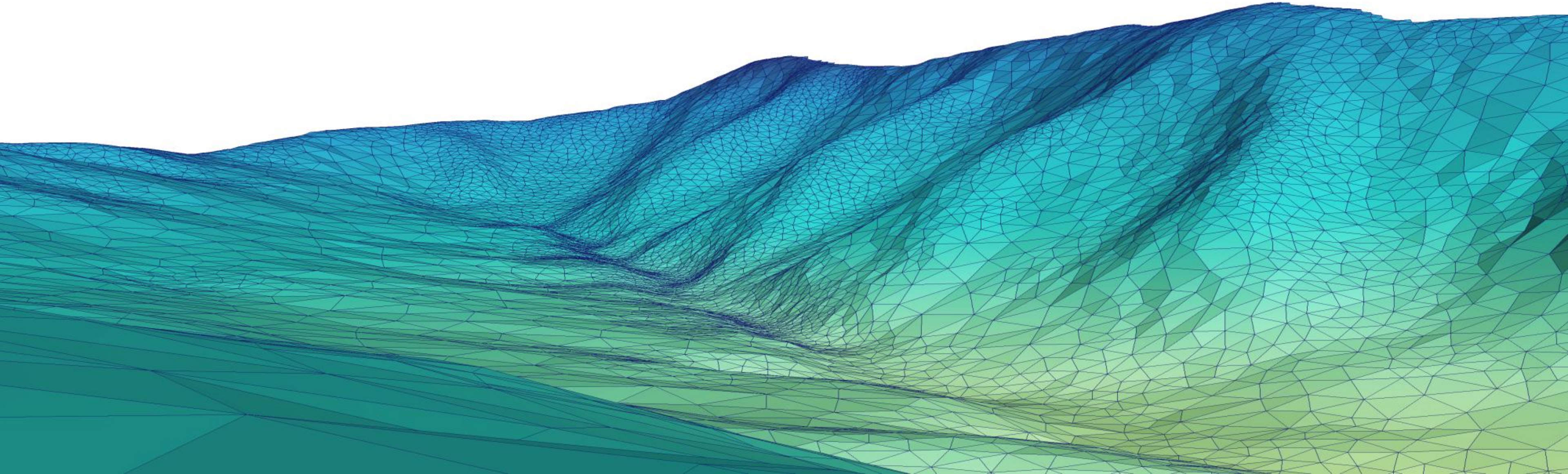
2D: Differences from 1D

- Different variables to solve. For example, in the shallow water model:
$$(A, Q) \rightarrow (h, q_x, q_y)$$
- Steady flow is much more difficult to characterize.
- The concepts of normal depth, critical depth, and Froude number have a local meaning.
- The computational domain is dynamic depending on how many cells are wetted.

Pros and cons:

1D models	2D models
Suitable for long river reaches	Requires river bathymetry reconstruction
Fast computations	Detailed topographic information (raster)
Detailed topographic information based on cross-sections	Slow computations
Not valid for unconfined flow	Valid for any type of geometry and scenarios: Floodplains, estuaries, deltas, etc.
	Channels with bidimensional flow.
	Flood simulation.

Mathematical formulation of surface flow:
2D Shallow Water Equations



Mathematical formulation of surface flow:

2D Shallow Water Equations

Generalization of the 1D Saint-Venant equations to 2D. They can be derived from the Navier-Stokes equations (physical conservation laws of mass and momentum) for incompressible flow by applying a series of assumptions:

- *Vertical dynamics* ***can be neglected compared to horizontal dynamics***.
- *The momentum equations are* ***vertically integrated*** *along the water column, eliminating*
    - *The vertical coordinate as an independent variable.*
    - *The free surface is part of the solution boundary.*
- *Surface waves vary smoothly, which is equivalent to assuming a* ***hydrostatic pressure distribution in the vertical***.
- *The* ***average bed slope*** *is so* ***small*** *that the tangent can be approximated by the angle, and measurements at the bed are equivalent to measurements in the horizontal plane.*

Shallow Water: It does not necessarily have to be shallow (e.g., tsunamis) nor does it have to be water (e.g., atmospheric flow, oil flow).

Mathematical formulation of surface flow:
2D Shallow Water Equations

Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0$$

Conservation of linear momentum:

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = \rho g_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

Mathematical formulation of surface flow:
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Highly complex equations → Approximations:

- Compressibility is neglected → Density is no longer an unknown variable
- Viscosity is neglected
- Gravity acts only in the z direction

Mathematical formulation of surface flow:
2D Shallow Water Equations

Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \quad \rightarrow \quad \frac{\partial u_i}{\partial x_i} = 0 \qquad (\nabla \cdot \vec{u} = 0)$$

Conservation of linear momentum:

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = \rho g_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \quad \rightarrow \quad \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = g_i - \frac{1}{\rho} \frac{\partial p}{\partial x_i}$$

Highly complex equations → Approximations:

- Compressibility is neglected → Density is no longer an unknown variable
- Viscosity is neglected
- Gravity acts only in the z direction

We can write them in expanded form:

→

Mathematical formulation of surface flow:
2D Shallow Water Equations

Conservation of mass:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Conservation of linear momentum:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Mathematical formulation of surface flow:
2D Shallow Water Equations

Conservation of mass:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

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We have 4 equations with 4 unknowns

↓

We could solve the system if we had the appropriate initial and boundary conditions, but...

↓

... we do not know the position of the free surface.

Mathematical formulation of surface flow:
2D Shallow Water Equations

Let's continue applying assumptions → We are considering that the vertical scales are << the horizontal ones → We can neglect the local acceleration and the convective terms in the z-axis:

$$\cancel{\frac{\partial w}{\partial t}} + u\cancel{\frac{\partial w}{\partial x}} + v\cancel{\frac{\partial w}{\partial y}} + w\cancel{\frac{\partial w}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \quad \rightarrow \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \Rightarrow p = p_0 + \rho g h$$

(Hydrostatic pressure distribution in the vertical)

This is also known as the "boundary layer hypothesis." → $\frac{Dw}{Dt} = 0$

Mathematical formulation of surface flow:
2D Shallow Water Equations

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(Hydrostatic pressure distribution in the vertical)

This is also known as the "boundary layer hypothesis." → $\frac{Dw}{Dt} = 0$

Now we can calculate the pressure gradient in x and y:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\left.\begin{aligned} \frac{\partial p}{\partial x} &= \rho g \frac{\partial h}{\partial x} \\ \frac{\partial p}{\partial y} &= \rho g \frac{\partial h}{\partial y} \end{aligned}\right\} \rightarrow \left\{\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -g\frac{\partial h}{\partial x} \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -g\frac{\partial h}{\partial y} \end{aligned}\right.$$

Mathematical formulation of surface flow:

2D Shallow Water Equations

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← The horizontal velocities *u* and *v* are independent of *z* and can be interpreted as average velocities.

Mathematical formulation of surface flow:

2D Shallow Water Equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g\frac{\partial h}{\partial y}$$

We average in the vertical by integrating:

$$\int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dz = 0$$

$\rightarrow$

Leibniz's rule

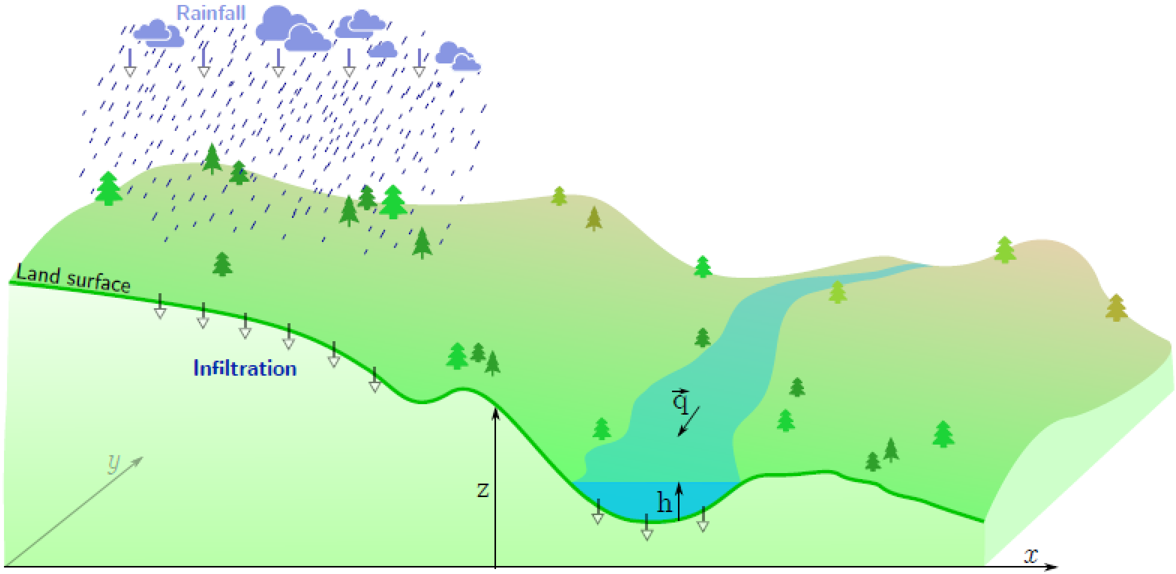
$$\left.\begin{aligned}
&\frac{\partial h}{\partial t} + u\frac{\partial hu}{\partial x} + v\frac{\partial hv}{\partial y} = 0\\
&\frac{\partial hu}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2\right) + \frac{\partial}{\partial y}(huv) = -gh\frac{\partial z_b}{\partial x}\\
&\frac{\partial hv}{\partial t} + \frac{\partial}{\partial x}(huv) + \frac{\partial}{\partial y}(hv^2 + \frac{1}{2}gh^2) = -gh\frac{\partial z_b}{\partial y}
\end{aligned}\right\}$$

2D Shallow Water Equations (without fricction)

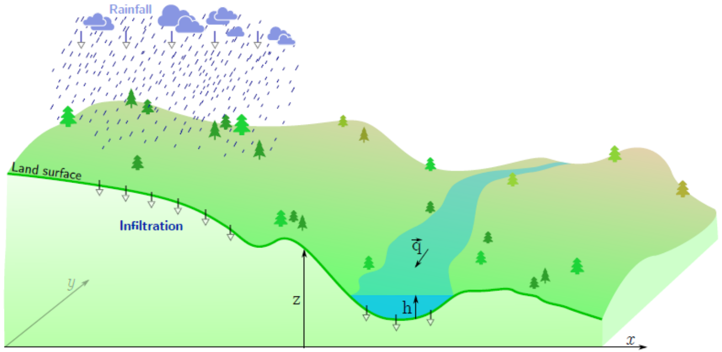
Mathematical formulation of surface flow:
2D Shallow Water Equations

Variables to solve:
- *Depth h*
- *Unit discharges qx and qy*

Source terms:
- *Rainfall*
- *Evaporation*
- *Infiltration*
- *Bed slope*
- *Friction*



Mathematical formulation of surface flow:
2D Shallow Water Equations



- Conservation of mass:

$$\frac{\partial h}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = R - e - f$$

- Conservation of linear momentum in *x* and *y* directions:

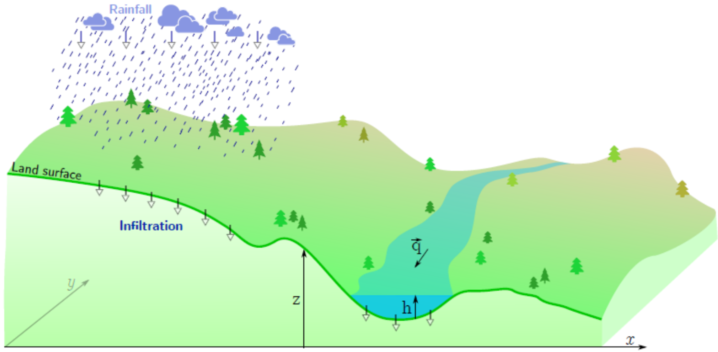
$$\frac{\partial q_x}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q_x^2}{h} + \frac{1}{2}gh^2\right) + \frac{\partial}{\partial y}\left(\frac{q_x q_y}{h}\right) = gh\left(S_{0x} - S_{fx}\right)$$

$$\frac{\partial q_y}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q_x q_y}{h}\right) + \frac{\partial}{\partial y}\left(\frac{q_y^2}{h} + \frac{1}{2}gh^2\right) = gh\left(S_{0y} - S_{fy}\right)$$

where

$$q_x = hu,\ q_y = hv.$$

Mathematical formulation of surface flow:
2D Shallow Water Equations



- Mass source terms:
  Rainfall-evaporation: R-e (*defined as input*)
  Infiltration: f (*formulated by means of semi-empirical models*)

- Linear momentum source terms:
  Bed slope:

$$S_{0y} = -\frac{\partial z_b}{\partial y} \qquad S_{0x} = -\frac{\partial z_b}{\partial x}$$

Friction:

$$S_{fx} = \frac{\tau_x}{\rho g h}, \qquad S_{fy} = \frac{\tau_y}{\rho g h}$$

$$\tau_x = \rho g h \frac{n^2 u \sqrt{u^2+v^2}}{h^{4/3}}, \qquad \tau_y = \rho g h \frac{n^2 v \sqrt{u^2+v^2}}{h^{4/3}}$$

← Shear stress on the bed
(usually formulated in terms of Manning's coefficient *n*)

Mathematical formulation of surface flow:
2D Shallow Water Equations

The shallow water equations in 2D form a nonlinear hyperbolic system of equations that can also be written in conservative form:

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{E} = \mathbf{S} \qquad \mathbf{E} = (\mathbf{F}, \mathbf{G})$$

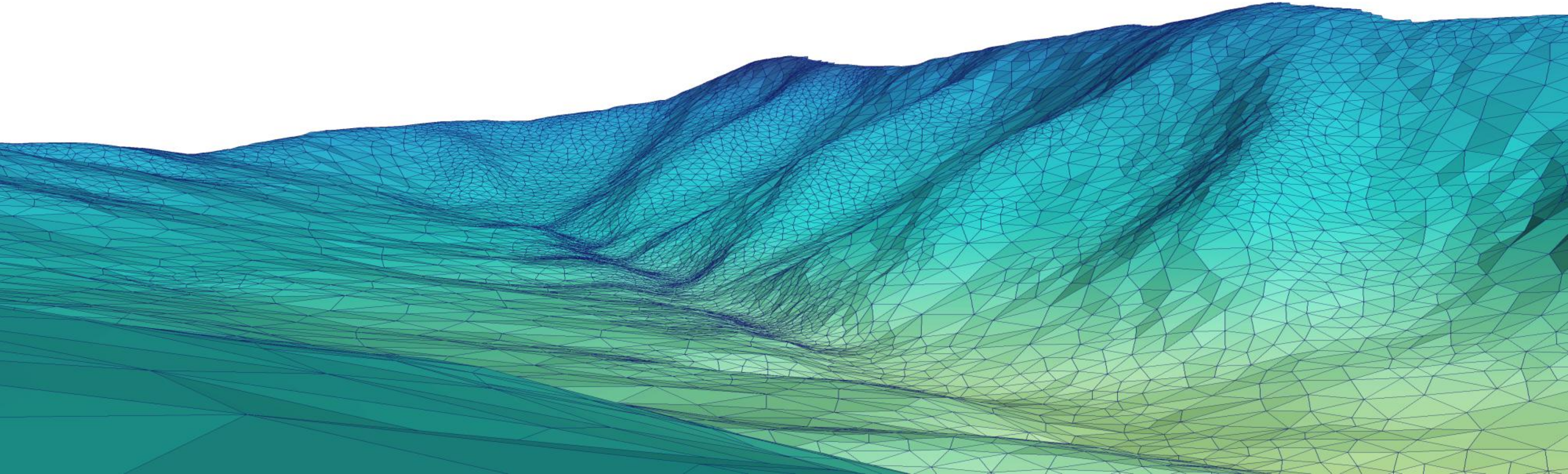
where

$$\mathbf{U} = (h, q_x, q_y)^T, \qquad q_x = hu, \qquad q_y = hv$$

$$\mathbf{F} = \left(q_x, \frac{q_x^2}{h} + \frac{1}{2}gh^2, \frac{q_x q_y}{h}\right)^T, \qquad \mathbf{G} = \left(q_y, \frac{q_x q_y}{h}, \frac{q_y^2}{h} + \frac{1}{2}gh^2\right)^T$$

$$\mathbf{S} = \left(R - f, gh\left(S_{0x} - S_{fx}\right), gh\left(S_{0y} - S_{fy}\right)\right)^T$$

Mathematical formulation of surface flow:
2D Zero-Inertia model or Diffusion Wave model



Mathematical formulation of surface flow:
2D Zero-Inertia model or Diffusion Wave model

- Widely used in hydrological modeling
- Neglects the acceleration terms in the shallow water equations:

$$\frac{\partial h}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = R - e - f$$

$$\cancel{\frac{\partial q_x}{\partial t}} + \frac{\partial}{\partial x}\left(\cancel{\frac{q_x^2}{h}} + \frac{1}{2} g h^2\right) + \frac{\partial}{\partial y}\cancel{\left(\frac{q_x q_y}{h}\right)} = gh\left(S_{0x} - S_{fx}\right)$$

$$\cancel{\frac{\partial q_y}{\partial t}} + \frac{\partial}{\partial x}\cancel{\left(\frac{q_x q_y}{h}\right)} + \frac{\partial}{\partial y}\left(\cancel{\frac{q_y^2}{h}} + \frac{1}{2} g h^2\right) = gh\left(S_{0y} - S_{fy}\right)$$

Mathematical formulation of surface flow:
2D Zero-Inertia model or Diffusion Wave model

- The resulting system can be expressed as follows:

$$\frac{\partial h}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = R - f$$

$$\frac{\partial h}{\partial x} = S_{0x} - S_{fx}$$

$$\frac{\partial h}{\partial y} = S_{0y} - S_{fy}$$

- Rewriting the unit discharges as a function of the free surface slope: $\mathbf{S} = -\nabla(h+z)$:

we obtain the final equation of the zero-inertia model:

$$q_x = \left(\frac{h^{5/3}}{n\sqrt{|\mathbf{S}|}} S_x\right), \qquad q_y = \left(\frac{h^{5/3}}{n\sqrt{|\mathbf{S}|}} S_y\right)$$

Highly complex term

$$\frac{\partial h}{\partial t} + \nabla\left(\boxed{\frac{h^{5/3}}{n\sqrt{|\mathbf{S}|}}}\mathbf{S}\right) = R - f$$

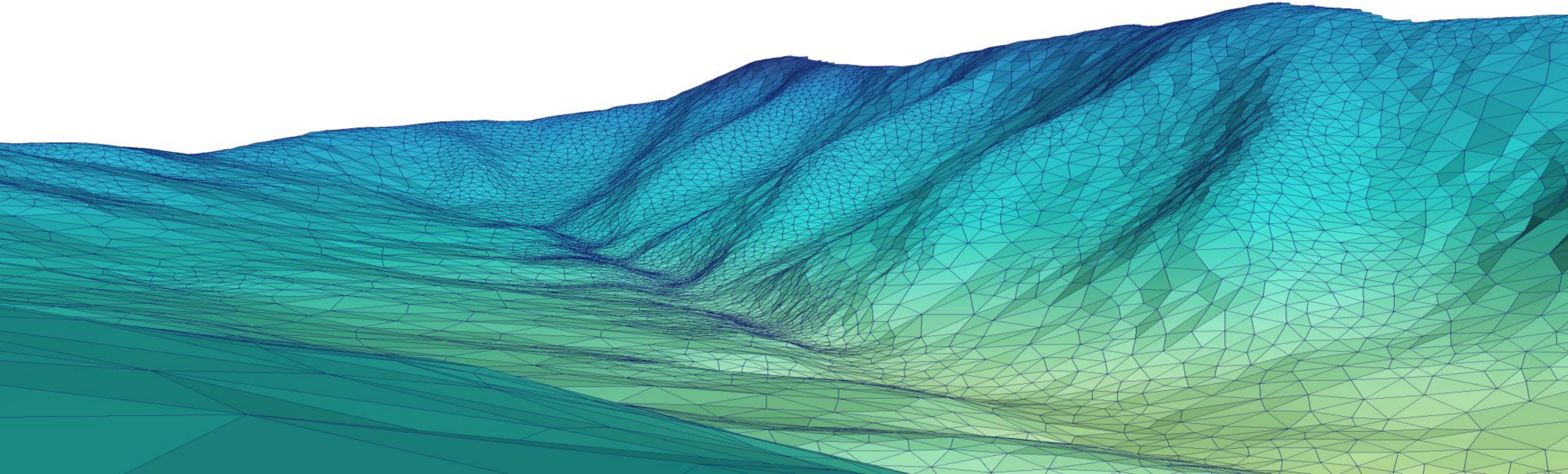
Mathematical formulation of surface flow:
2D Zero-Inertia model or Diffusion Wave model

- The 2D Zero-Inertia model can also be written in vector form:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \mathbf{S}$$

	2D Shallow Water Equations	2D Zero-Inertia
Variables (**U**)	$(h,\ q_x,\ q_y)^T$	$(h,\ 0,\ 0)^T$
Fluxes in x (**F**)	$\left(q_x,\ \frac{q_x^2}{h} + \frac{1}{2}gh^2,\ \frac{q_x q_y}{h}\right)^T$	$(q_x, h, 0)^T$
Fluxes in y (**F**)	$\left(q_y,\ \frac{q_x q_y}{h},\ \frac{q_y^2}{h} + \frac{1}{2}gh^2\right)^T$	$(q_y,\ 0,\ h)^T$
Sources (**S**)	$(R - f,\ gh\,(S_{0x} - S_{fx}),\ gh\,(S_{0y} - S_{fy}))^T$	
Character	*Hyperbolic*	*Parabolic*

Mathematical formulation of surface flow:
Finite Volume method

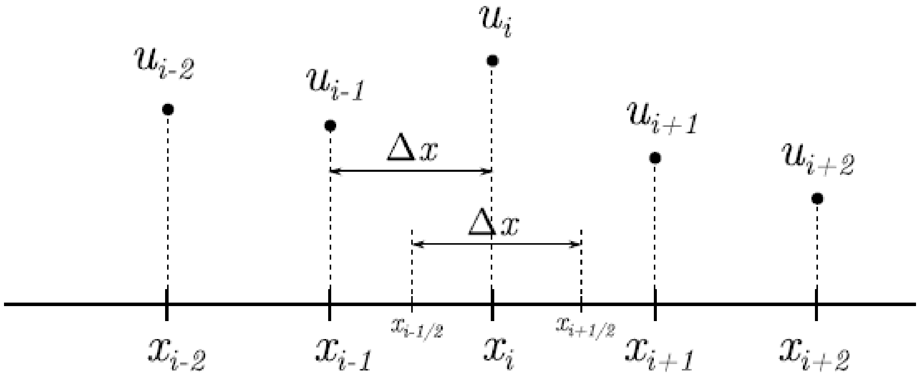


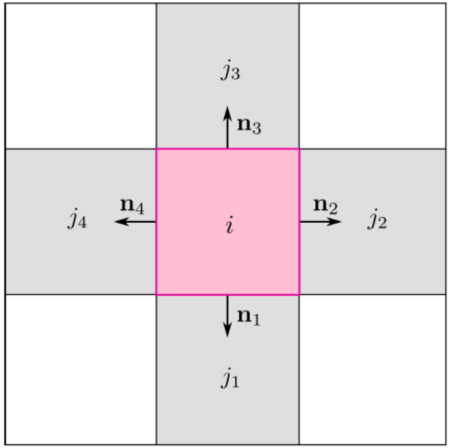
Mathematical formulation of surface flow:
Basic concepts

There are 3 "classic" families of numerical methods: *finite differences*, *finite elements*, and *finite volumes*.

- FINITE DIFFERENCES: It is based on the definition of the derivative and the properties of Taylor series expansions. It is very simple to apply but requires a structured mesh.

$$\frac{\partial u}{\partial x} \approx \frac{u(x+\Delta x) - u(x)}{\Delta x}$$



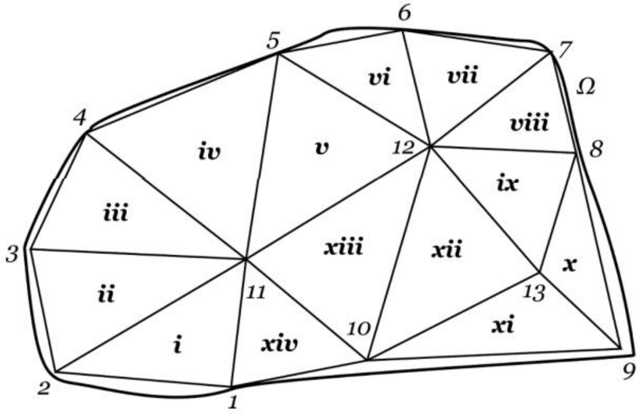


Mathematical formulation of surface flow:
Basic concepts

There are 3 "classic" families of numerical methods: *finite differences*, *finite elements*, and *finite volumes*.

- FINITE ELEMENTS: The method is based on dividing the computational domain into small elements and finding local solutions that satisfy the differential equation of the physical problem. Then, all the individual solutions are combined to find a global solution to the problem.

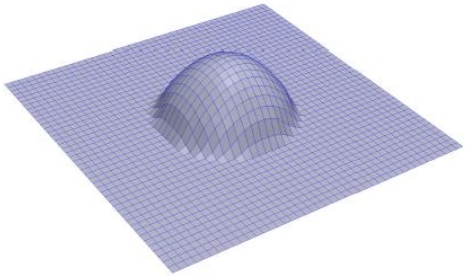
  Normally, the method works at the mesh nodes and interpolates the solution within the cells.

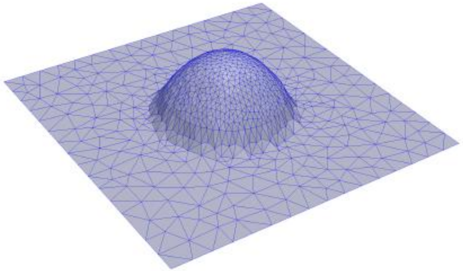


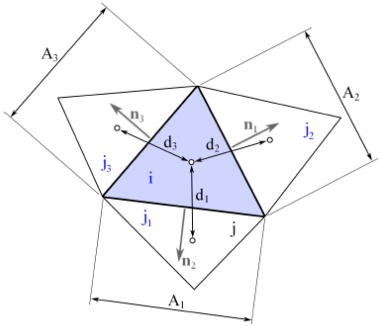
Mathematical formulation of surface flow:
Basic concepts

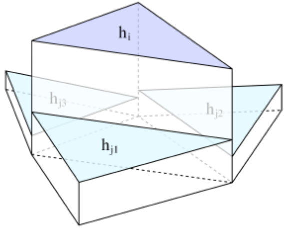
There are 3 "classic" families of numerical methods: *finite differences*, *finite elements*, and *finite volumes*.

- FINITE VOLUMES: They try to combine the geometric flexibility of finite elements with the simplicity of the discrete formulation of variables and fluxes associated with finite differences.









Mathematical formulation of surface flow:
Basic concepts

There are 2 ways to discretize temporal derivatives: *explicit* discretization and *implicit* discretization:

- EXPLICIT DISCRETIZATION: The values of the variables at time n+1 are calculated from the information at time n.

$$\mathbf{x}^{n+1} = f\left(\mathbf{x}^{n}\right)$$

PRO: easy implementation
CON: restricted time step due to numerical stability

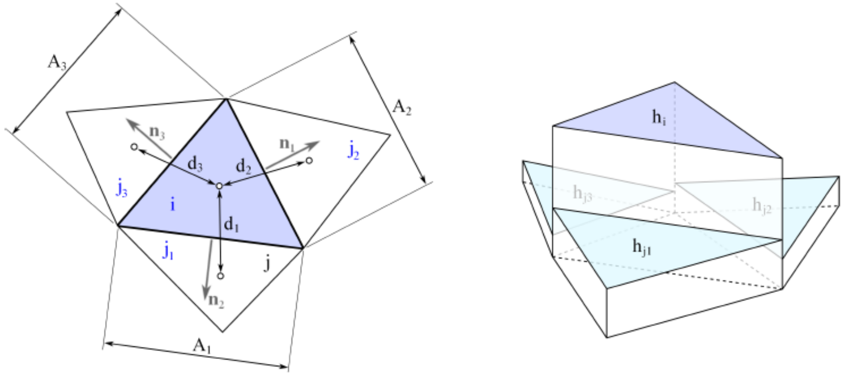
- IMPLICIT DISCRETIZATION: The solution is calculated by solving a system of N equations with the values at times n and n+1 simultaneously, where N is the number of cells.

PRO: unconditionally stable
CON: complex implementation, highly diffusive method

$$\mathbf{A}\mathbf{x}^{n+1} = \mathbf{b}^{n}$$

Mathematical formulation of surface flow:

Finite Volume method



By integrating the system of equations

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{E} = \mathbf{S}$$

In a control volume Ω representing a computational cell of the mesh:

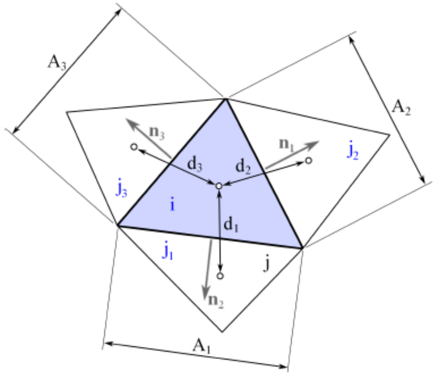
$$\frac{d}{dt}\int_{\Omega} \mathbf{U} \mathbf{d}\Omega + \int_{\Omega} (\nabla \cdot \mathbf{E}) d\Omega = \int_{\Omega} \mathbf{S} d\Omega$$

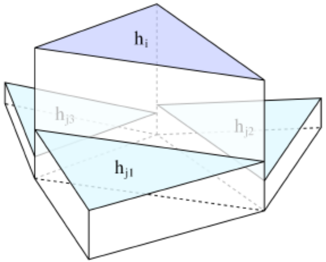
By invoking the Gauss' theorem:

Upwind method

$$\frac{d}{dt}\int_{\Omega} \mathbf{U} d\Omega + \sum_{j,w=1}^{N_w} (\delta \mathbf{E} \cdot \mathbf{n})_w^- l_w = \sum_{j,w=1}^{N_w} \mathbf{S}_w^- l_w$$

Mathematical formulation of surface flow:
Finite Volume method





By discretizing the time derivative:

$$\frac{\Delta \mathbf{U}_i}{\Delta t} S_i + \sum_{j,w=1}^{N_w} (\delta \mathbf{E} \cdot \mathbf{n})_w^- l_w = \sum_{j,w=1}^{N_w} \mathbf{S}_w^- l_w$$

where

$$\Delta \mathbf{U}_i = \mathbf{U}_i^{n+1} - \mathbf{U}_i^n$$

We can evaluate the fluxes in terms of a parameter θ to obtain an hybrid scheme:

$$\frac{\Delta \mathbf{U}_i}{\Delta t} S_i + \theta \sum_{j,w=1}^{N_w} \left[(\delta \mathbf{E} \cdot \mathbf{n})_w^-\right]^{n+1} l_w + (1-\theta) \sum_{j,w=1}^{N_w} \left[(\delta \mathbf{E} \cdot \mathbf{n})_w^-\right]^n l_w = \sum_{j,w=1}^{N_w} \left(\mathbf{S}_w^-\right)^n l_w$$

in where $\theta = 1$ represents an implicit scheme and $\theta = 0$ an explicit scheme.

Application to 2D Shallow Water Equations

The 2D shallow water equations form a nonlinear hyperbolic system of equations that can also be written in conservative form:

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{E} = \mathbf{S} \qquad \mathbf{E} = (\mathbf{F}, \mathbf{G})$$

where

$$\mathbf{U} = (h, q_x, q_y)^T, \qquad q_x = hu, \qquad q_y = hv$$

$$\mathbf{F} = \left(q_x, \frac{q_x^2}{h} + \frac{1}{2}gh^2, \frac{q_x q_y}{h}\right)^T, \qquad \mathbf{G} = \left(q_y, \frac{q_x q_y}{h}, \frac{q_y^2}{h} + \frac{1}{2}gh^2\right)^T$$

$$\mathbf{S} = \left(R - f, gh\left(S_{0x} - S_{fx}\right), gh\left(S_{0y} - S_{fy}\right)\right)^T$$

Application to 2D Shallow Water Equations

The Jacobian matrix of the system can be written as:

$$\mathbf{J}_n = \frac{\partial \mathbf{E}\cdot\mathbf{n}}{\partial \mathbf{U}} = \begin{pmatrix} 0 & n_x & n_y \\ -u(\mathbf{u}\cdot\mathbf{n}) + c^2 n_x & \mathbf{u}\cdot\mathbf{n} + u n_x & u n_y \\ -v(\mathbf{u}\cdot\mathbf{n}) + c^2 n_y & v n_x & \mathbf{u}\cdot\mathbf{n} + v n_y \end{pmatrix}$$

with the associated eigenvalues and eigenvector as follows:

$$\lambda^1 = \mathbf{u}\cdot\mathbf{n} - c, \qquad \lambda^2 = \mathbf{u}\cdot\mathbf{n}, \qquad \lambda^3 = \mathbf{u}\cdot\mathbf{n} + c$$

$$\mathbf{e}^1 = \begin{pmatrix} 1 \\ u - c n_x \\ v - c n_y \end{pmatrix}, \qquad \mathbf{e}^2 = \begin{pmatrix} 0 \\ -c n_y \\ c n_x \end{pmatrix}, \qquad \mathbf{e}^3 = \begin{pmatrix} 1 \\ u + c n_x \\ v + c n_y \end{pmatrix}$$

Application to 2D Shallow Water Equations

We will focus on the explicit temporal discretization, for simplicity (θ=0):

$$\frac{\Delta \mathbf{U}_i}{\Delta t} S_i + \cancel{\theta \sum_{j,w=1}^{N_w} \left[(\delta \mathbf{E} \cdot \mathbf{n})_w^- \right]^{n+1}} l_w + (1-\theta) \sum_{j,w=1}^{N_w} \left[(\delta \mathbf{E} \cdot \mathbf{n})_w^- \right]^{n} l_w = \sum_{j,w=1}^{N_w} \left(\mathbf{S}_w^- \right)^n l_w$$

We can express the fluxes as a function of the conserved variables by means of the Jacobian matrix:

$$\mathbf{J}_n = \frac{\partial \mathbf{E} \cdot \mathbf{n}}{\partial \mathbf{U}} \rightarrow \quad \delta(\mathbf{E} \cdot \mathbf{n})_w = \mathbf{J}_n \delta \mathbf{U}_w = \sum_{m=1}^{3} (\tilde{\lambda} \alpha \tilde{\mathbf{e}})_w^m, \qquad \mathbf{S}_w = \sum_{m=1}^{3} (\beta \tilde{\mathbf{e}})_w^m \rightarrow \quad \mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{A_i} \sum_{k=1}^{N} \sum_{m}^{3} \left[\left(\tilde{\lambda}^- \tilde{\gamma} \tilde{\mathbf{e}} \right)_k^m l_k \right]$$

$$\downarrow$$

$$\mathbf{J}_n = \begin{pmatrix} 0 & n_x & n_y \\ -u(\mathbf{u} \cdot \mathbf{n}) + c^2 n_x & \mathbf{u} \cdot \mathbf{n} + u n_x & u n_y \\ -v(\mathbf{u} \cdot \mathbf{n}) + c^2 n_y & v n_x & \mathbf{u} \cdot \mathbf{n} + v n_y \end{pmatrix}$$

$$\uparrow$$

$$\tilde{\gamma}_k^m = \left(\tilde{\alpha} - \frac{\tilde{\beta}}{\tilde{\lambda}} \right)_k^m$$

Froude number: as in 1D, it represents the relationship between inertial and gravitational forces. However, in 2D, it is associated with the direction of the flow normal vector (**n**), giving it a local character. The same applies to the concepts of normal depth and critical depth.

$$Fr = \frac{|\mathbf{u}|}{c}, \qquad |\mathbf{u}| = \sqrt{u^2 + v^2}$$

Application to 2D Shallow Water Equations

Some additional calculations for the first order Roe's scheme:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{A_i} \sum_{k=1}^{N} \sum_{m}^{3} \left[\left(\tilde{\lambda}^- \tilde{\gamma} \tilde{\mathbf{e}} \right)_k^m l_k \right] \qquad \tilde{\gamma}_k^m = \left(\tilde{\alpha} - \frac{\tilde{\beta}}{\tilde{\lambda}} \right)_k^m$$

Roe averaged variables at the walls:

$$\tilde{u}_k = \frac{\sqrt{h_i}\, u_i + \sqrt{h_j}\, u_j}{\sqrt{h_i} + \sqrt{h_j}}, \quad \tilde{v}_k = \frac{\sqrt{h_i}\, v_i + \sqrt{h_j}\, v_j}{\sqrt{h_i} + \sqrt{h_j}}, \quad \tilde{c}_k = \sqrt{g\, \frac{h_i + h_j}{2}}$$

Wave and source strengths:

$$\tilde{\alpha}_1 = \frac{\delta h}{2} - \frac{1}{2\tilde{c}} \left(\delta \mathbf{q} \cdot \mathbf{n} - \tilde{\mathbf{u}} \cdot \mathbf{n}\, \delta h \right), \quad \tilde{\alpha}_2 = \frac{1}{\tilde{c}} [\delta q_y - \tilde{v}\, \delta h) n_x - (\delta q_x - \tilde{u}\, \delta h) n_y)],$$

$$\tilde{\alpha}_3 = \frac{\delta h}{2} + \frac{1}{2\tilde{c}} (\delta \mathbf{q} \cdot \mathbf{n} - \tilde{\mathbf{u}} \cdot \mathbf{n}\, \delta h),$$

$$\tilde{\beta}_1 = -\frac{1}{2c} (\delta z + S_{f,\mathbf{n}}), \quad \tilde{\beta}_2 = 0, \quad \tilde{\beta}_3 = -\tilde{\beta}_1$$

where

$$\tilde{\mathbf{u}} \cdot \mathbf{n} = \tilde{u}\, n_x + \tilde{v}\, n_y, \; \delta \mathbf{q} \cdot \mathbf{n} = \delta q_x\, n_x + \delta q_y\, n_y$$

Application to 2D Shallow Water Equations

Numerical stability

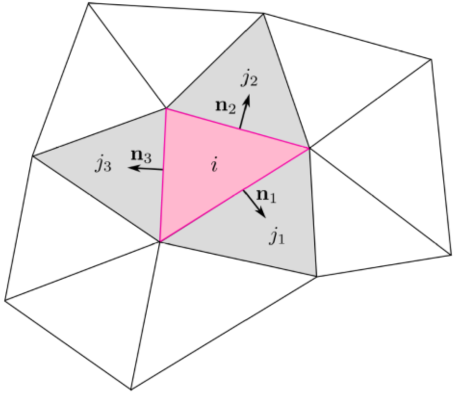
- Implicit schemes are unconditionally stables. Hence, there is no need for any stability condition.

- It is convenient to choose the time step dynamically:

$$\Delta t = \mathrm{CFL}\, \min_{Mesh}(\Delta t_{i,w})$$

where

$$\Delta t_{i,w} = \frac{min(\Delta x'_i, \Delta x'_j)}{max_{m=1,2,3}(|\tilde{\lambda}_w^m|)}, \qquad \Delta x'_i = \frac{S_i}{max_{1,N_w}(l_w)}$$



- In a 2D explicit scheme:
  CFL ≤ 0.5 for rectangular structured meses
  CFL ≤ 0.9 for triangular unstructured meses

- In an implicit scheme, the CFL number represents a multiplicative factor for the máximum time step allowed by the explicit scheme.

GPU massive parallelization

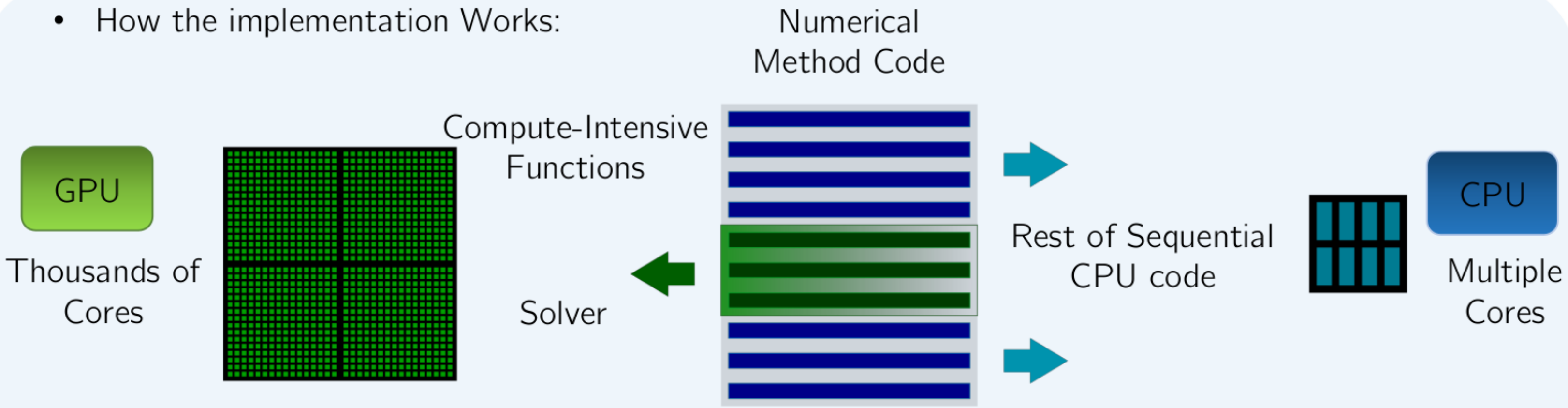
- GPU = Graphic Processing Unit
- Initially designed for videogames
- Requires a re-design of the computer code

GPU massive parallelization

- Numerical method implemented to run on GPU (Graphics Processing Unit)

Parallelized calculation method → From 8/12 cores (CPU) to thousands (GPU) → Acceleration of simulation ~20x-50x

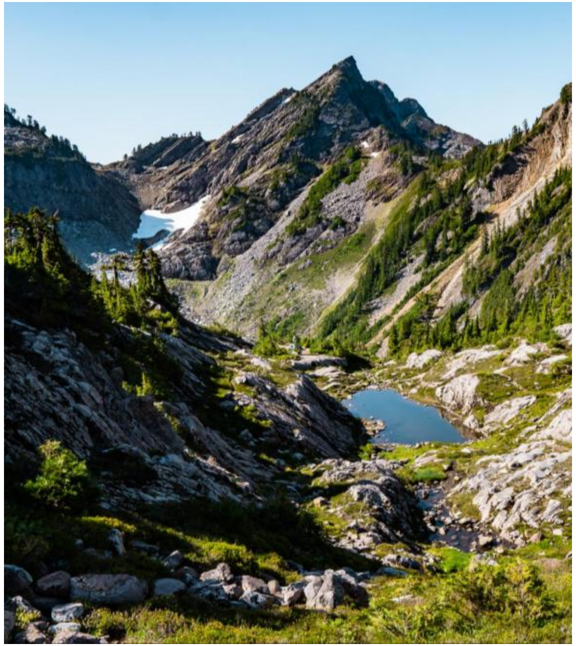
- How the implementation Works:

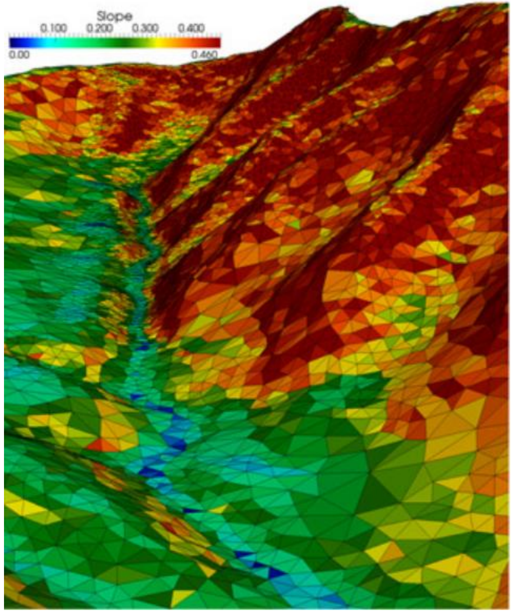


Computational meshes

What is meshing?

Meshing is a fundamental process in physics and engineering that involves discretizing a spatial domain into a set of discrete elements or cells





How important is it to choose an adequate mesh for a specific problem?

Let's find it out with an hydrological example:

How important is it to choose an adequate mesh for a specific problem?

Let's find it out with an hydrological example:

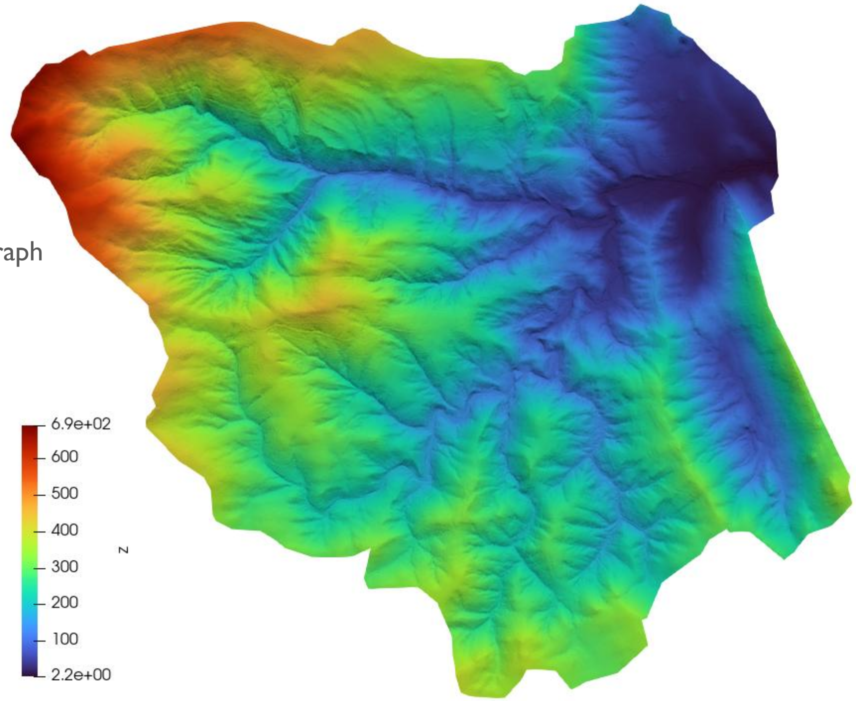
Mape River catchment:
- Located in the Basque Country in the North of Spain
- Basin area: 20,66 Km2
- Very steep slopes

After a rain event over the whole basin, we seek to reproduce the outlet hydrograph with our numerical model.

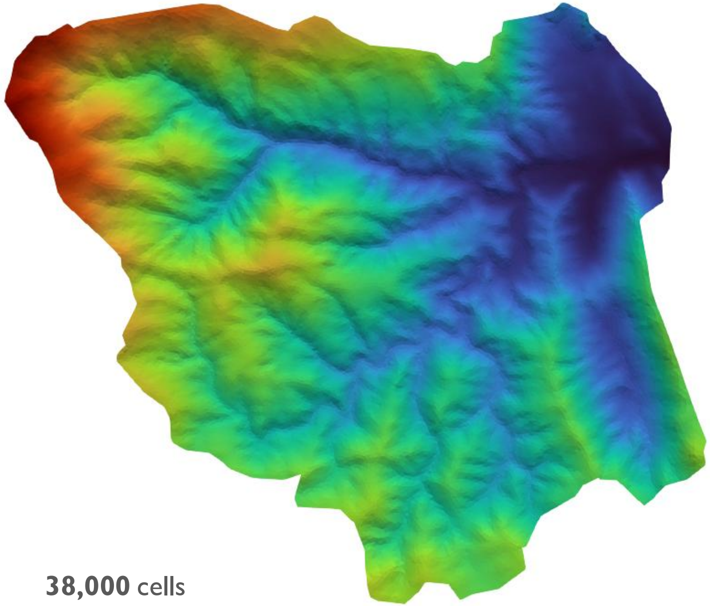
We need to discretize the basin topography (set of elevations)

How do I know what kind of mesh I need and what resolution is appropriate?
[DISCLAIMER: I am going to advance some concepts that we will see later]

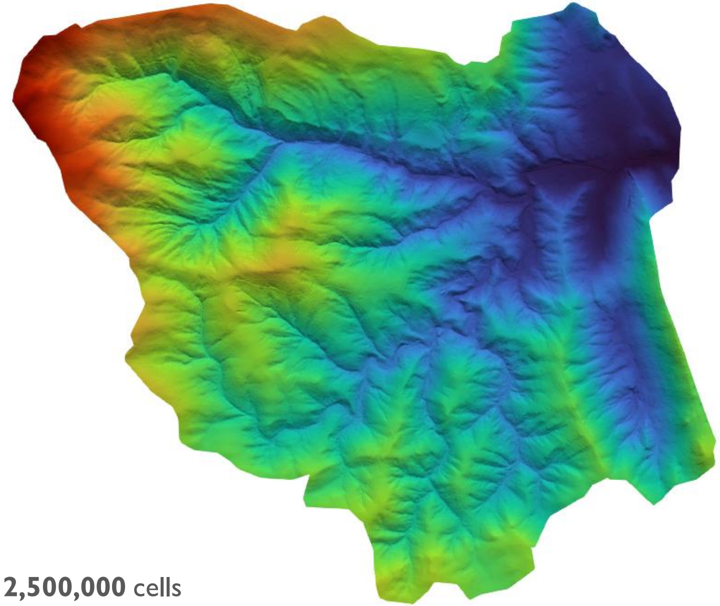
Let's try an unstructured triangular mesh with two different resolution:
38,000 cells vs. **2,500,000** cells



How important is it to choose an adequate mesh for a specific problem?



38,000 cells

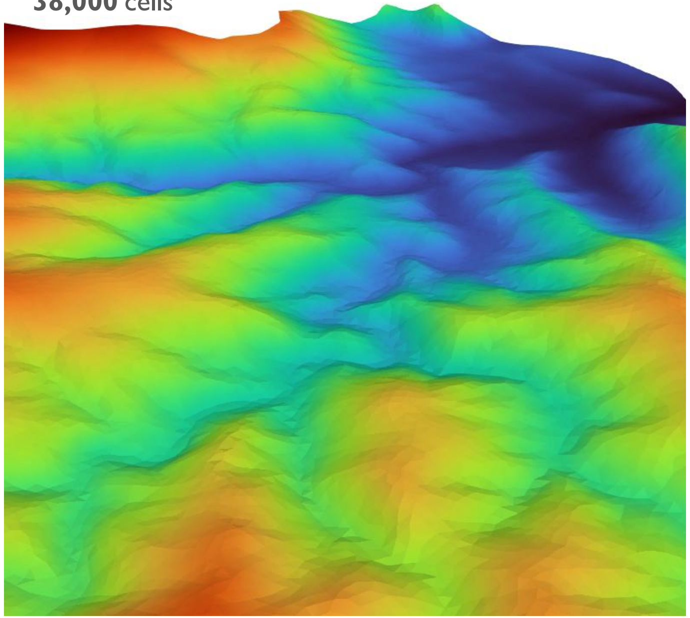


2,500,000 cells

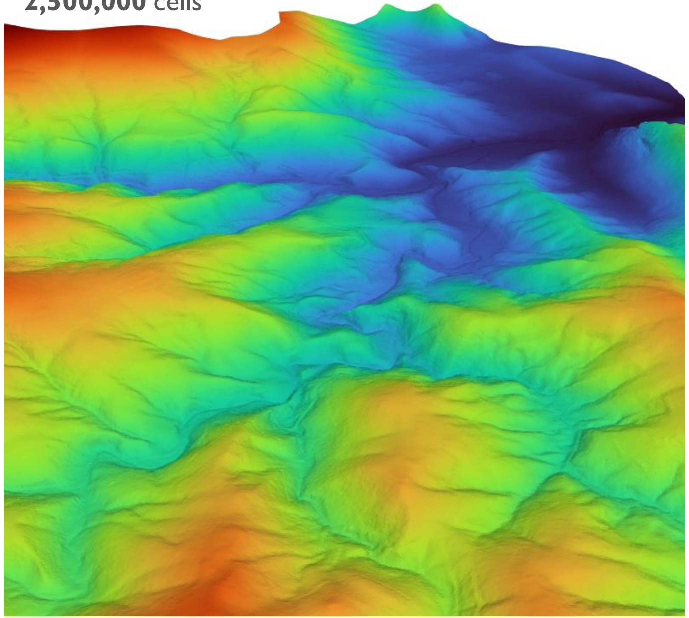
Apparently, not a big difference. Let's zoom in

How important is it to choose an adequate mesh for a specific problem?

38,000 cells

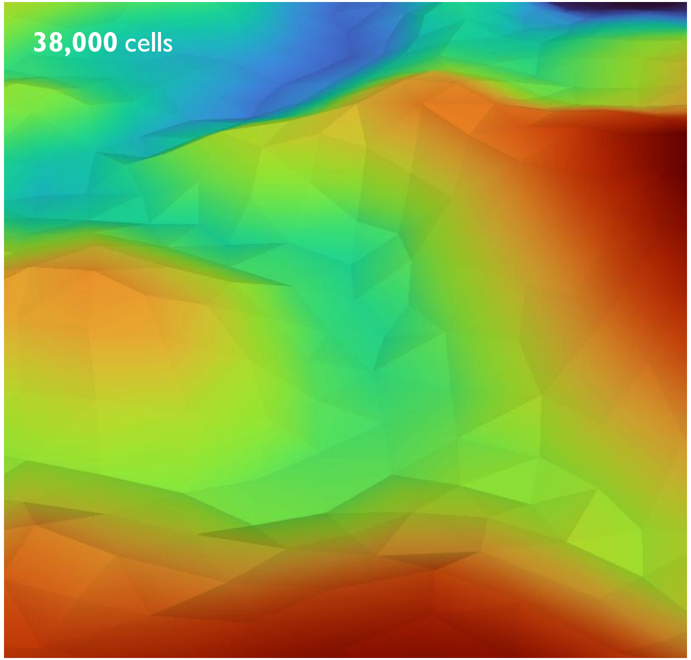


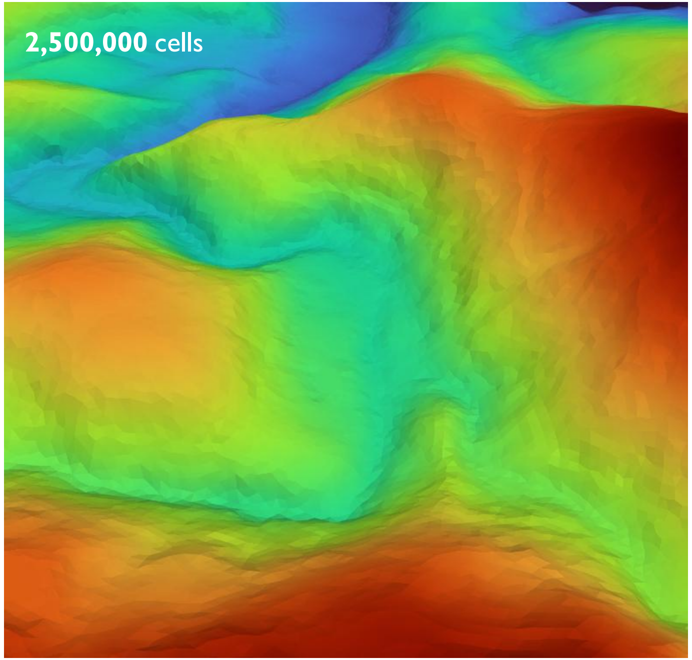
2,500,000 cells



Now, the differences are noticeable. Let's zoom in even more

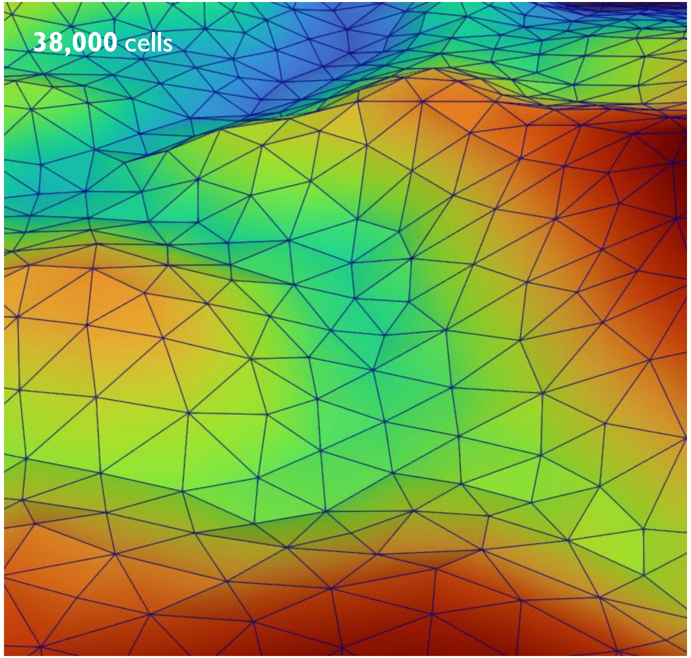
How important is it to choose an adequate mesh for a specific problem?

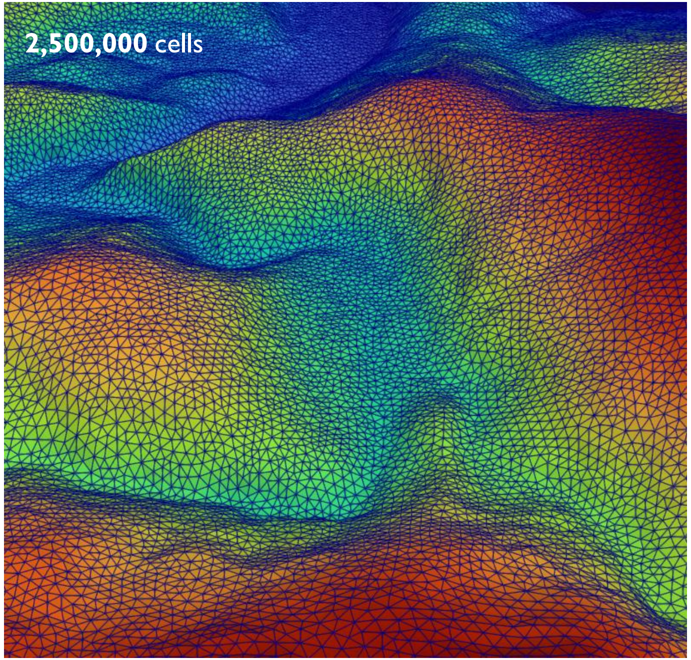




Coarse mesh is not able to faithfully reproduce the topography. Let's draw the mesh

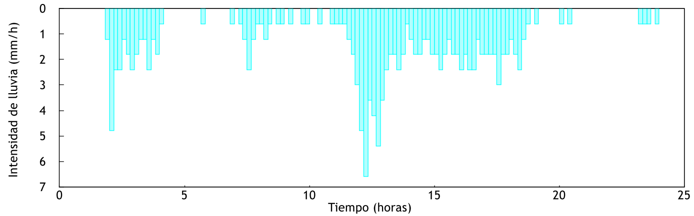
How important is it to choose an adequate mesh for a specific problem?

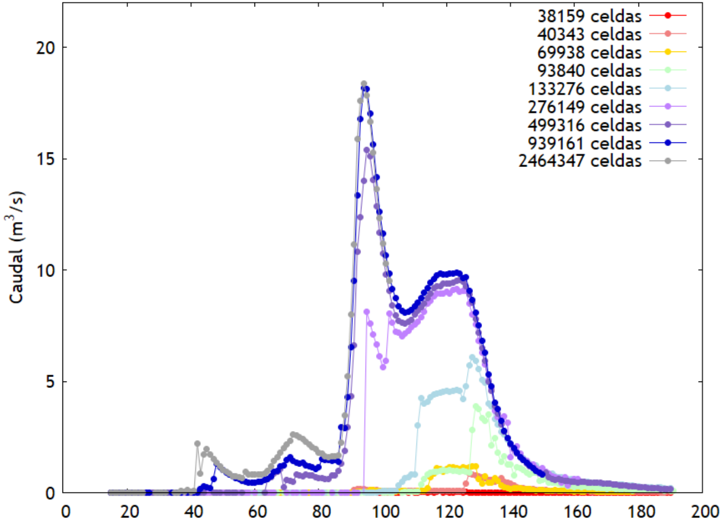


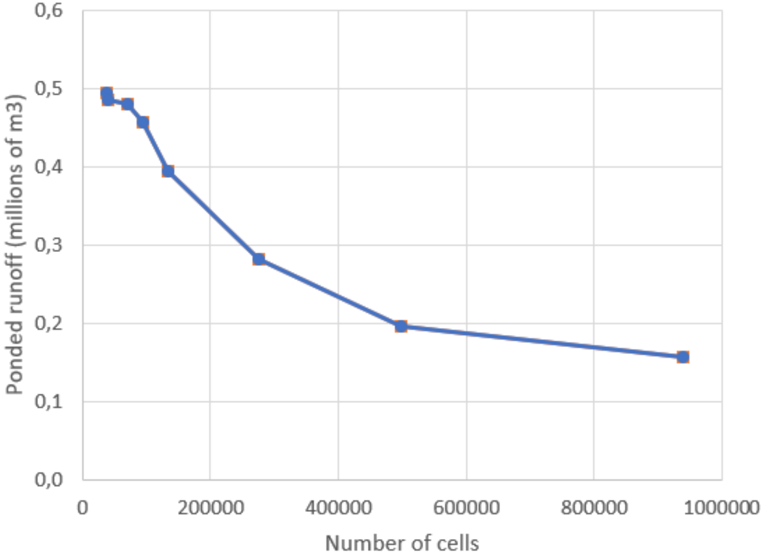


How important is it to choose an adequate mesh for a specific problem?

What about the outlet hydrographs?

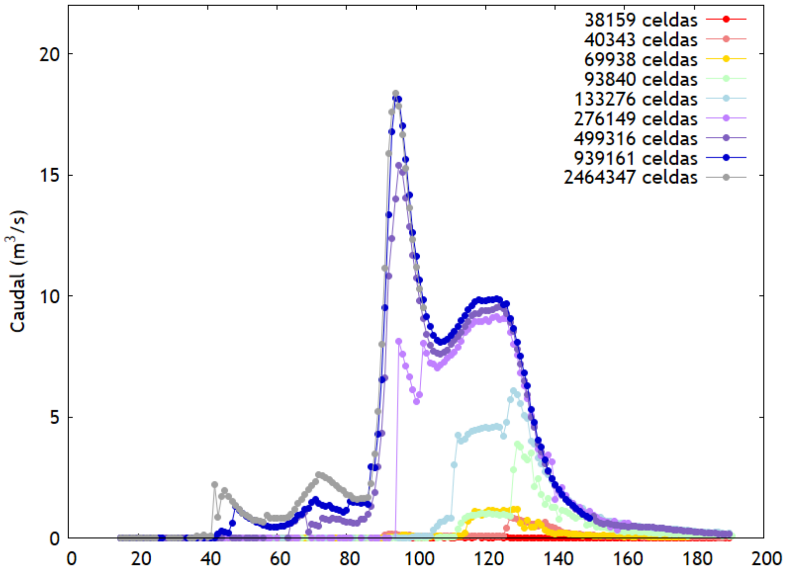


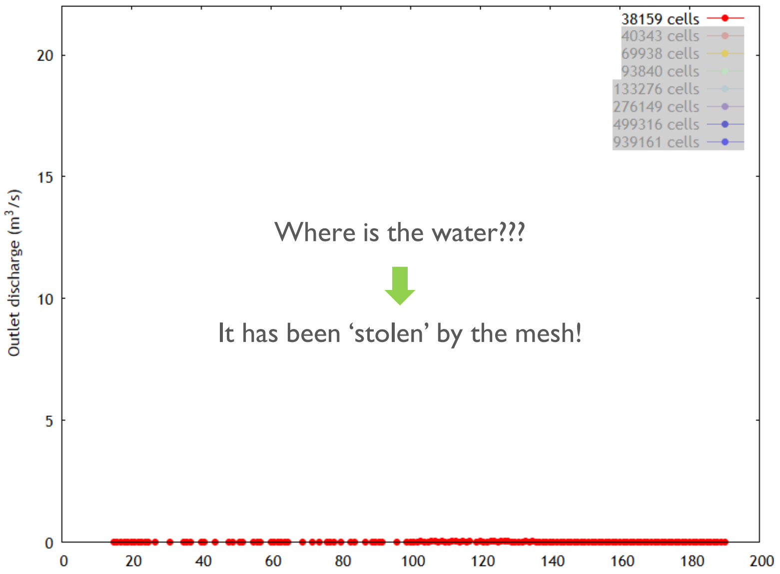


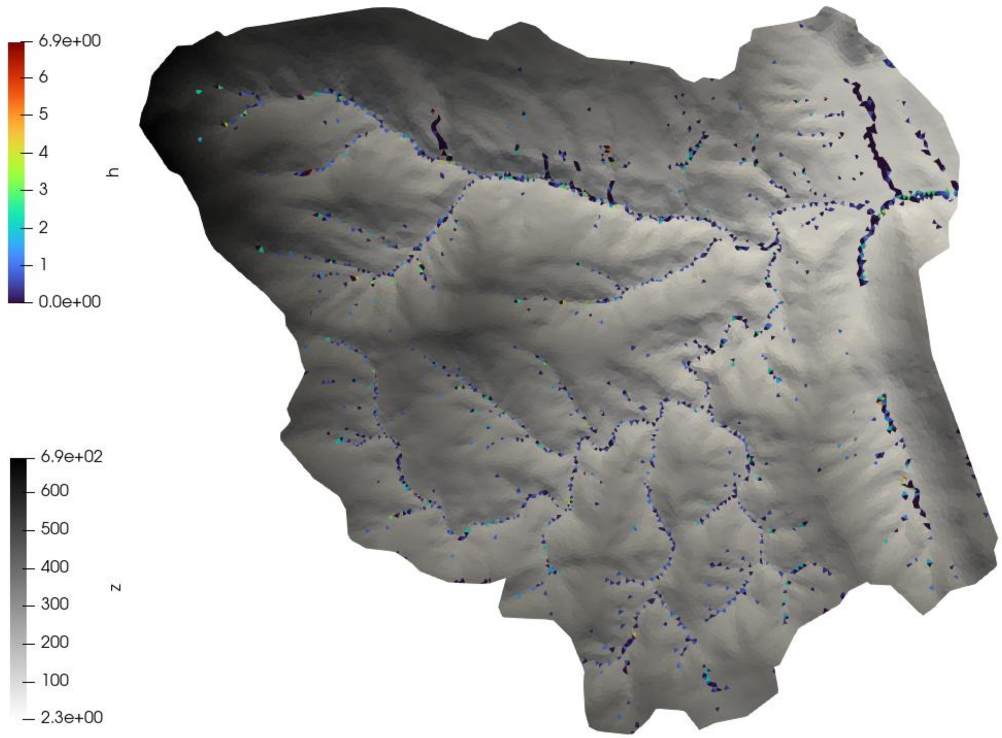


How important is it to choose an adequate mesh for a specific problem?

What about the outlet hydrographs?







Where is the water???

Here it is!

So…

How important is it to choose an adequate mesh for a specific problem?

A LOT!

So…

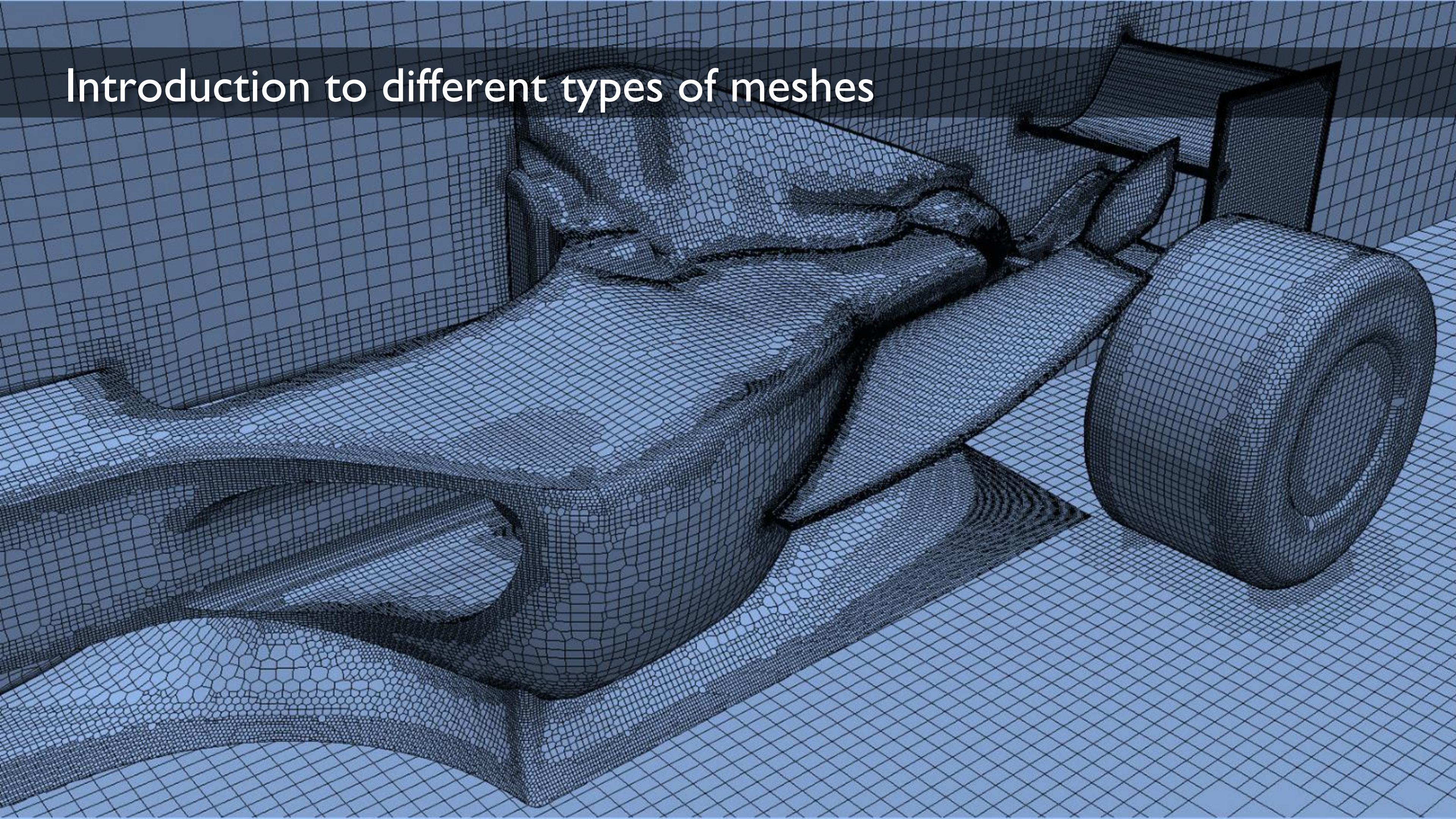
How important is it to choose an adequate mesh for a specific problem?

A LOT!

Sometimes we go crazy looking for the reason why our model generates incorrect results. We check the code, looking for bugs, we check the initial and boundary conditions or we even doubt whether we will be working outside the range of applicability of the model but we forget to…

CHECK THE MESH

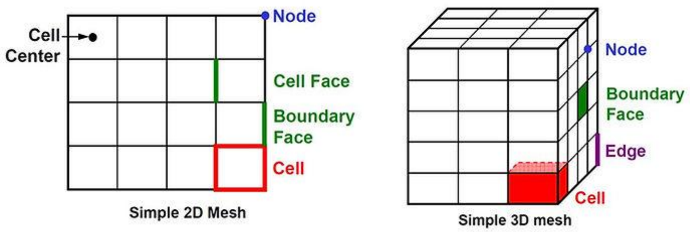
Introduction to different types of meshes



A **mesh** divides a geometry into many **elements**. These are used by the CFD solver **to construct control volumes**.

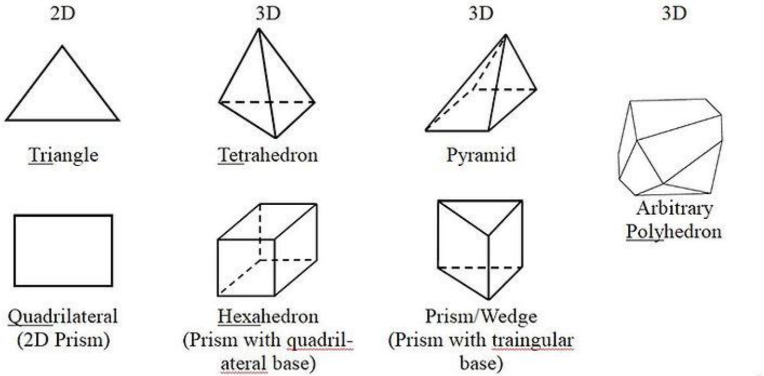
Terminology:

- ***Cell*** = control volume into which domain is broken up
- ***Node*** = grid point.
- ***Cell center*** = center of a cell.
- ***Edge*** = boundary of a face.
- ***Face*** = boundary of a cell.



Points to consider when generating a mesh are:

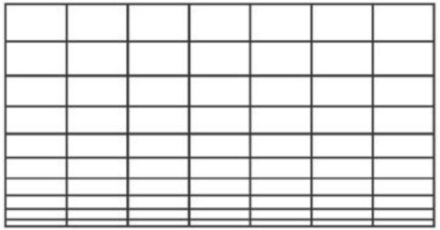
- Mesh resolution
- Type of mesh
- Computer resources



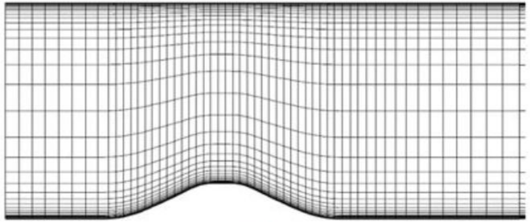
Mesh types:

1) Structured meshes

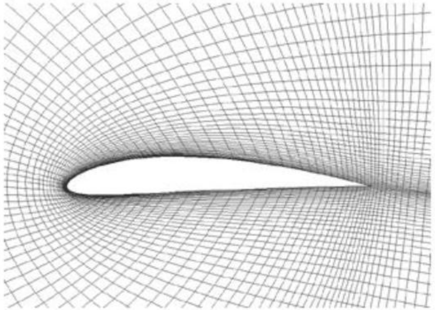
Grids can be **Cartesian** or **curvilinear** (usually body-fitting). In the former, grid lines are always parallel to the coordinate axes. In the latter, coordinate surfaces are curved to fit boundaries. There is an alternative division into orthogonal and non-orthogonal grids. In orthogonal grids (for example, Cartesian or polar meshes) all grid lines cross at 90°.



Cartesian



Curvilinear

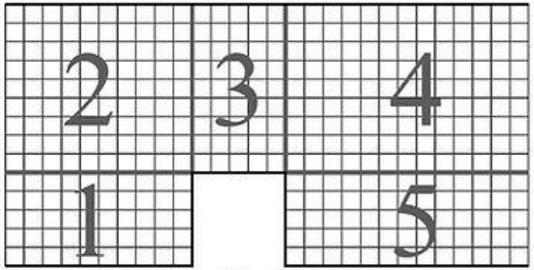


Curvilinear (Body-fitted)

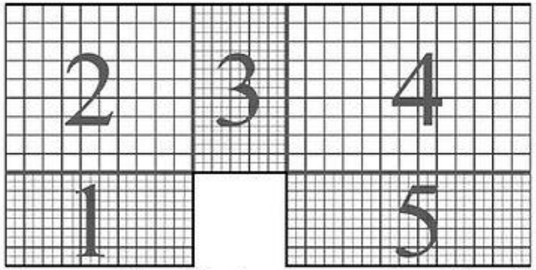
Mesh types:

2) Block-structured meshes

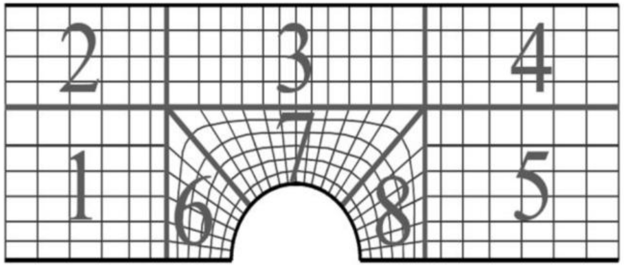
In multi-block structured grids the domain is decomposed into a small number of regions, in each of which the mesh is structured (i.e. cells can be indexed by (i,j,k)). A common arrangement is that grid lines match at the interface between two blocks, so that there are cell vertices that are common to two blocks i.e. matching cells. In some cases, the cell counts do not match at the interface i.e. non-matching cells. Non-matching cells (which could be 2 to 1, 3 to 2, …) should be avoided as much as possible as they tend to increase the computational time.



Matching block-structures



Non-matching block-structures (2 to 1 here)

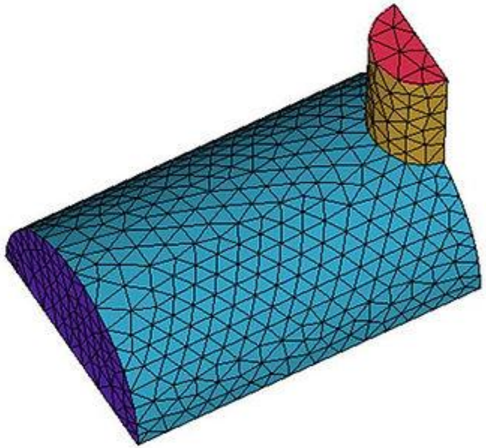


Non-orthogonal block-structured

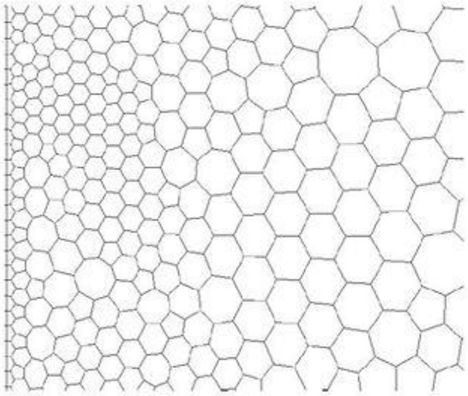
Mesh types:

3) Unstructured meshes

Unstructured meshes can accommodate completely arbitrary geometries. However, there are significant penalties to be paid for this flexibility, both in terms of the connectivity data structures and solution algorithms. Grid generators and plotting routines for such meshes are also very complex.



Triangular/tetrahedral

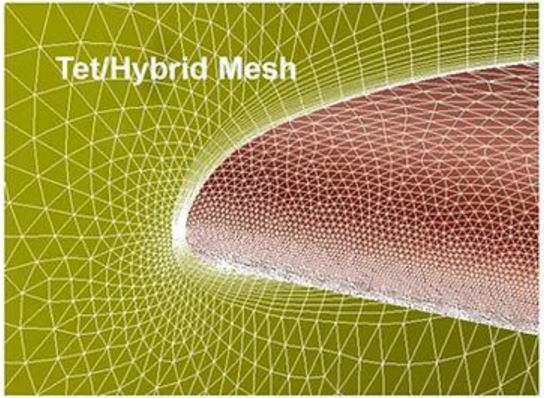


Polygon (polyhedral)

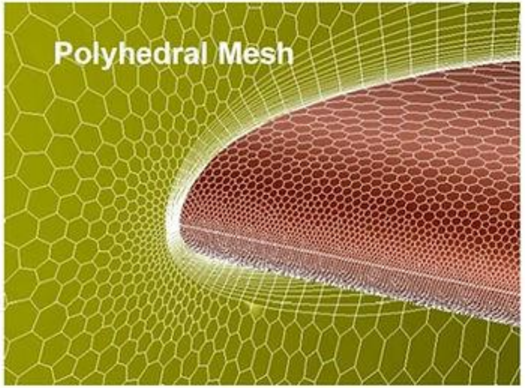
Mesh types:

4) Hybrid meshes

Hybrid meshes typically combine tri/tet elements with other elements in selected regions



Hybrid Tetrahedral



Hybrid Polyhedral

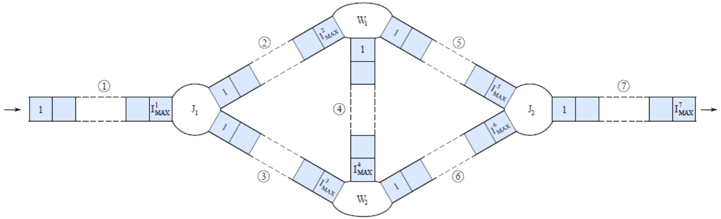

20 min break

Examples of 2D test case simulation

Urban Hidrology

Urban hidrology

- Combination of a 2D model for surface flow and a 1D model for sewer network flow.
- In the 1D model, the existence of confluences and bifurcations must be considered:



$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial H}\frac{\partial H}{\partial t} = A_s \frac{\partial H}{\partial t} = \sum Q$$

where

$V =$ Volume of the confluence
$A_s =$ Area of the confluence
$\sum Q =$ Net flow (inlet-outlet) in the confluence

Model coupling: exchange flow between both domains, acting as a mass source term:

2D Surface flow

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{E} = \mathbf{S}$$

being $\mathbf{E} = (\mathbf{F}, \mathbf{G})$, where

$$\mathbf{U} = (h, q_x, q_y)^T, \qquad q_x = hu, \qquad q_y = hv$$

$$\mathbf{F} = \left(q_x, \frac{q_x^2}{h} + \frac{1}{2}gh^2, \frac{q_x q_y}{h}\right)^T, \qquad \mathbf{G} = \left(q_y, \frac{q_x q_y}{h}, \frac{q_y^2}{h} + \frac{1}{2}gh^2\right)^T$$

$$\mathbf{S} = \left(R - f + q_e, gh\left(S_{0x} - S_{fx}\right), gh\left(S_{0y} - S_{fy}\right)\right)^T$$

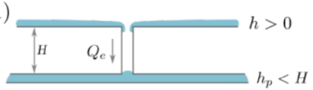
1D pipe flow

$$\frac{\partial A}{\partial t} - \frac{\partial Q}{\partial x} = q_e$$

$$\frac{1}{A}\frac{\partial Q}{\partial t} + \frac{1}{A}\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + g\frac{\partial y}{\partial x} - g\left(S_0 - S_f\right) = 0$$

Exchange flow

1)



$$q_e = -\frac{2}{3} C \pi D_M (2g)^{1/2} h^{3/2}$$

2)

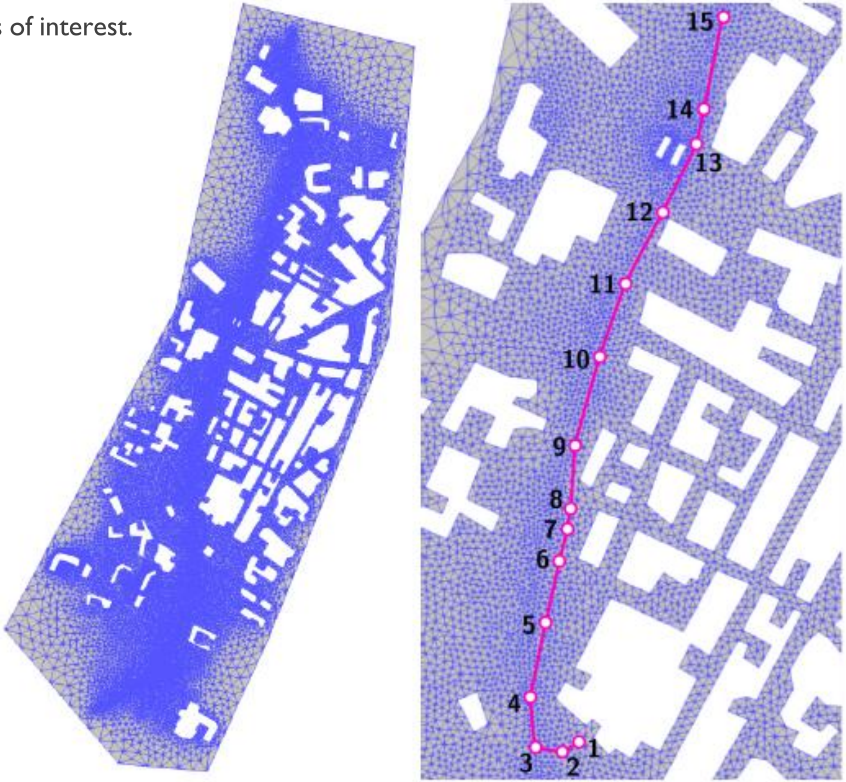
$$q_e = -C A_M (2g)^{1/2} (h + H - h_p)^{1/2}$$

3)

$$q_e = C A_M (2g)^{1/2} (h_p - H - h)^{1/2}$$

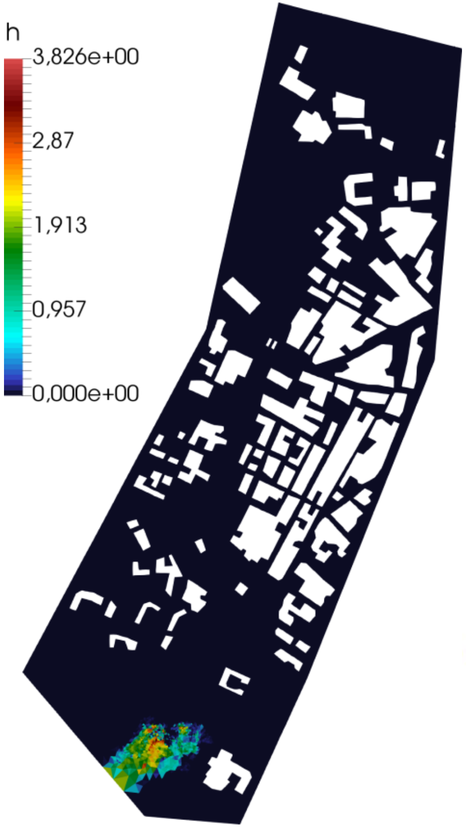
Example 1: Surface flow + drainage pipe in the town of Fuentes de Ebro (Zaragoza, Spain)

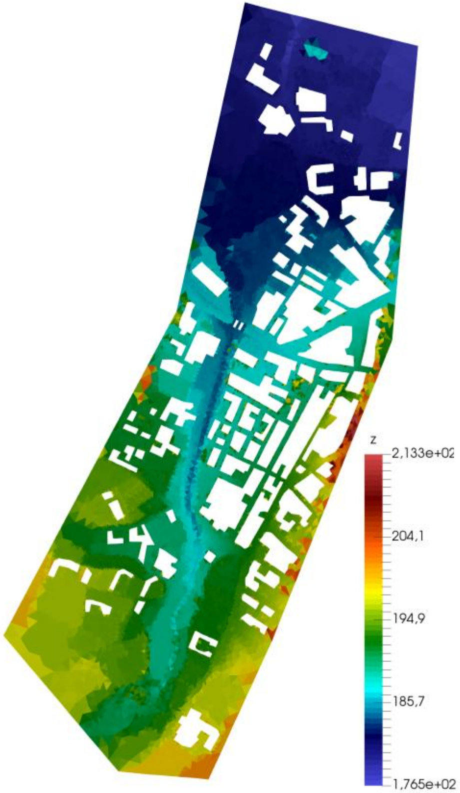
- Topography obtained from a public repository
- 2D computational mesh of 29600 triangular cells, locally refined at the areas of interest.
- The main drainage pipe has 15 connections with the surface (manholes).
- Models → **2DSWE** for the surface, **1DSWE** for the pipe
- Inlet boundary condition: h = 2m to simulate an extraordinary flood (T500)

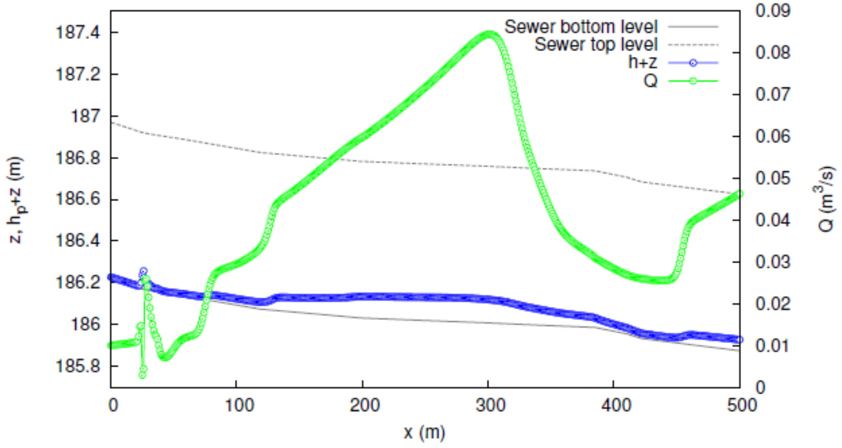


Workshop 2: Numerical models for hydraulic simulation

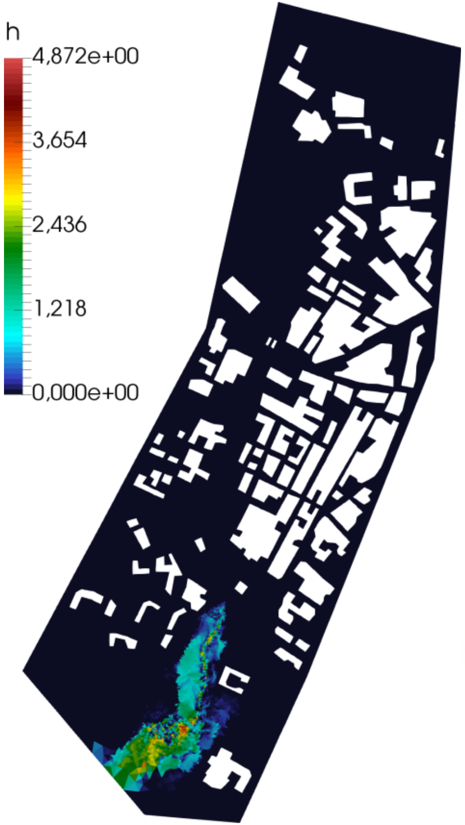
Examples of 2D test case simulation

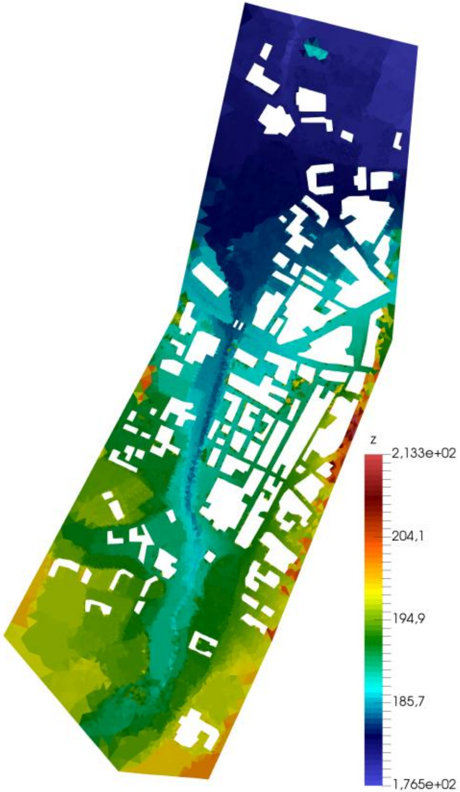


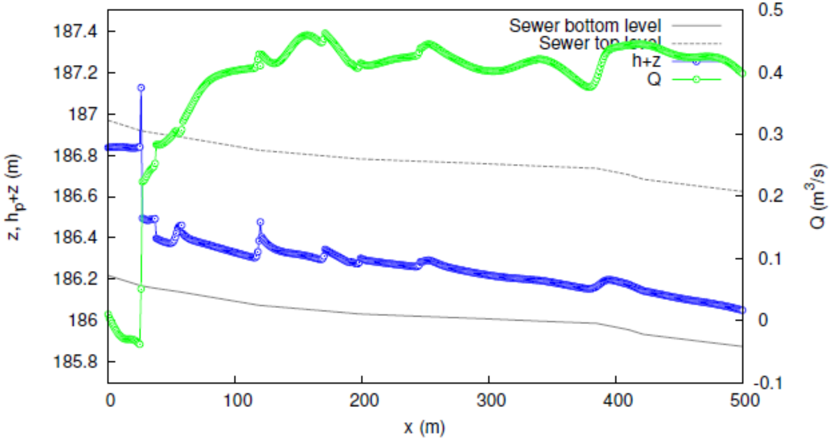




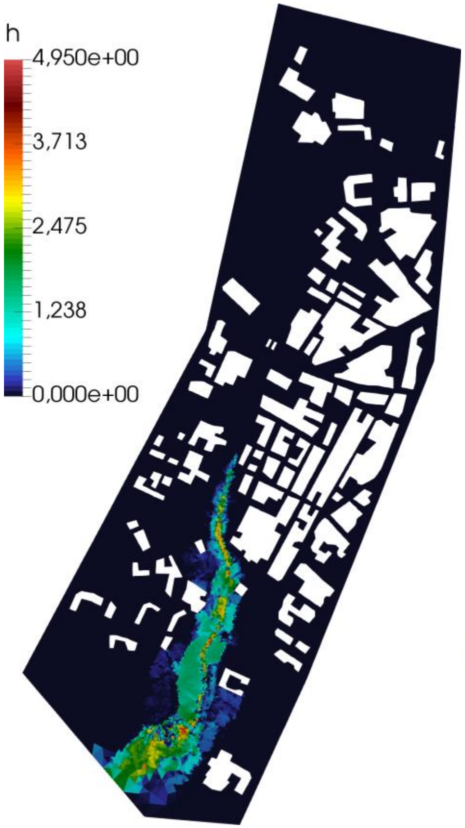
Examples of 2D test case simulation

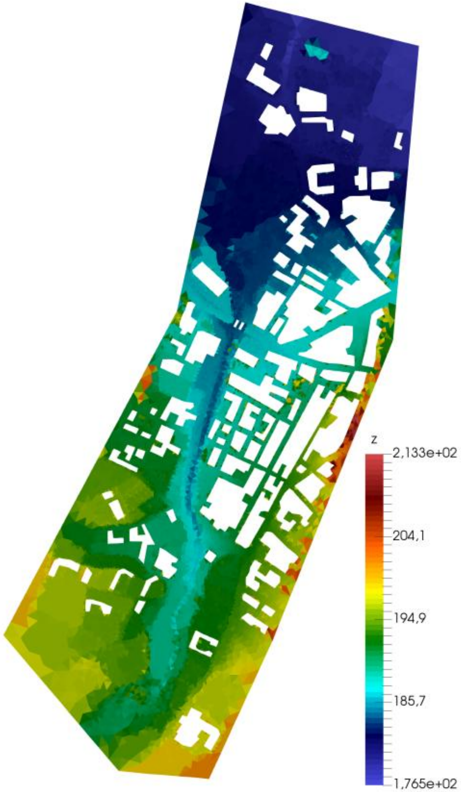


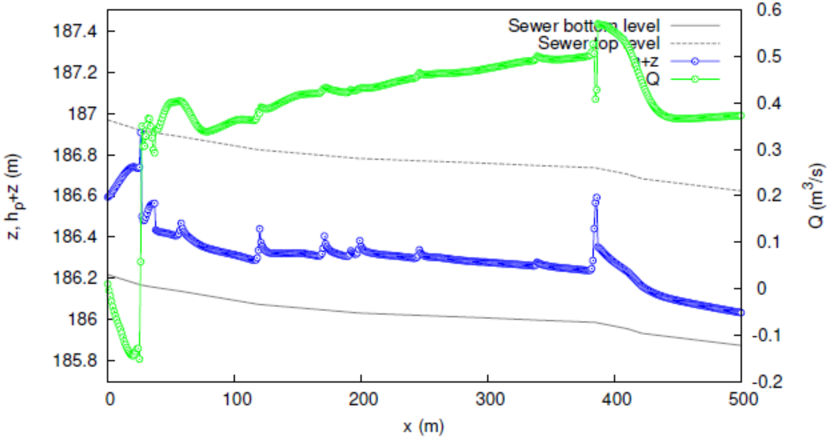




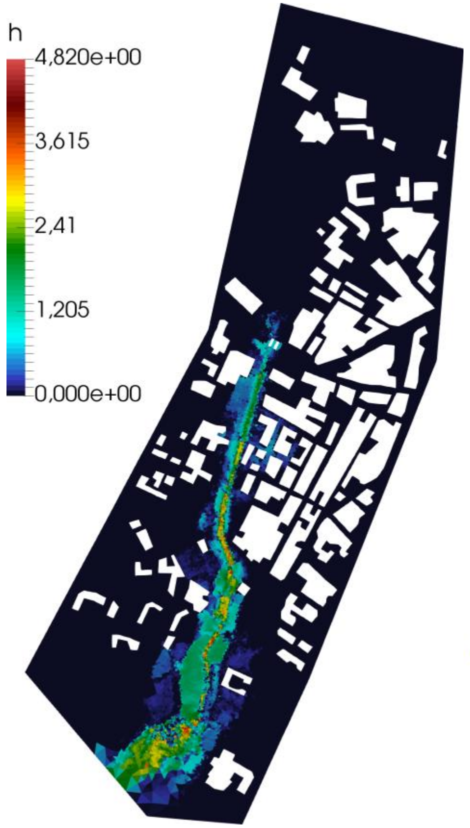
Examples of 2D test case simulation

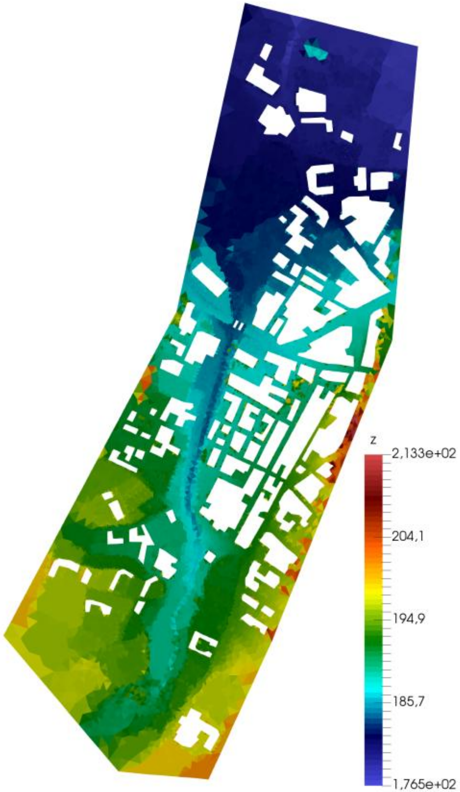


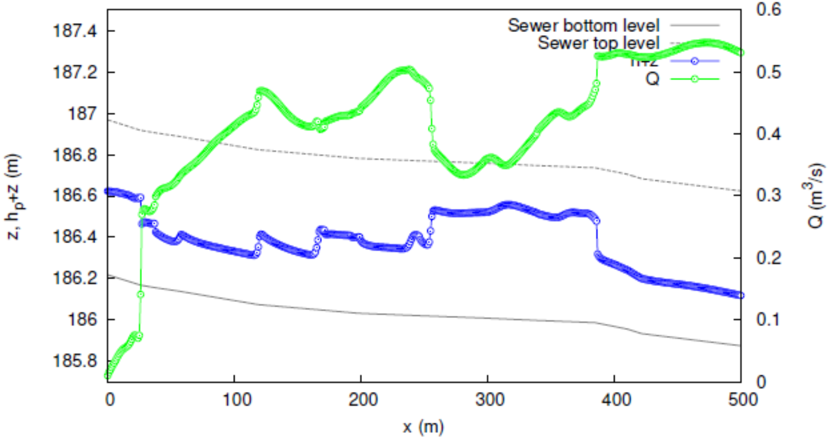




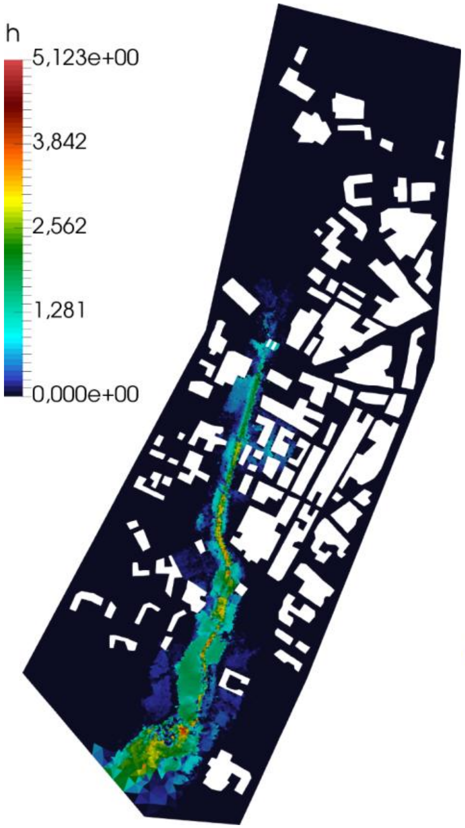
Examples of 2D test case simulation

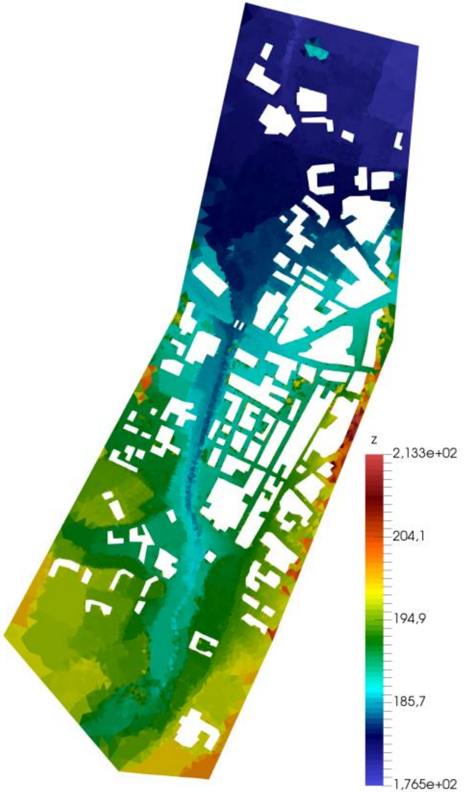


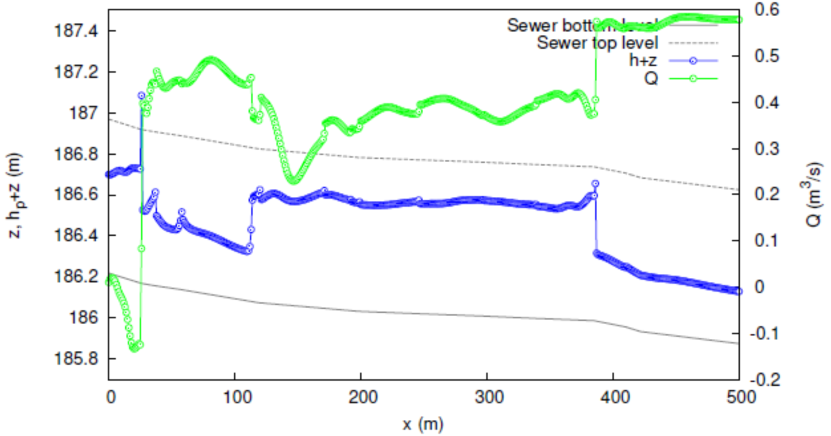




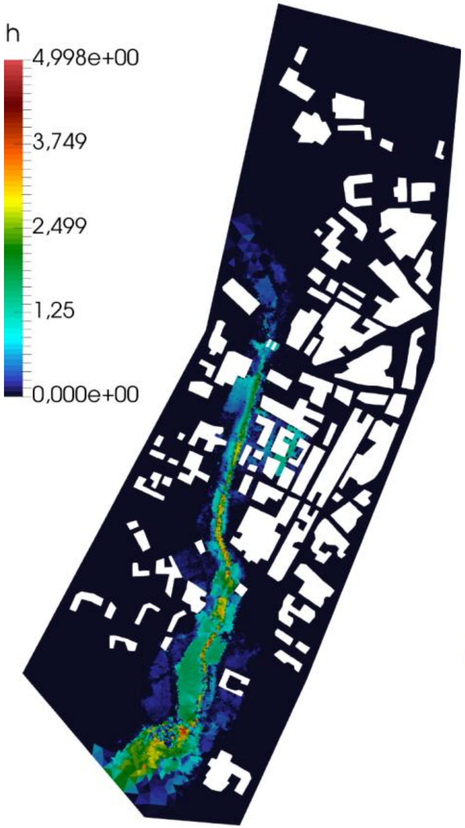
Examples of 2D test case simulation

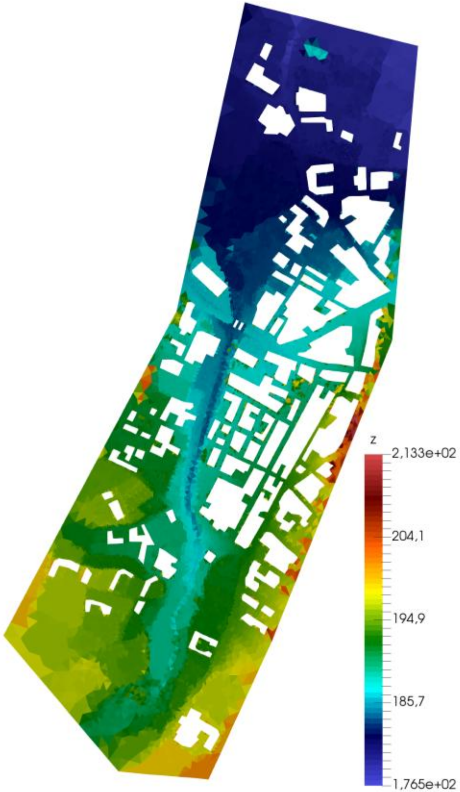


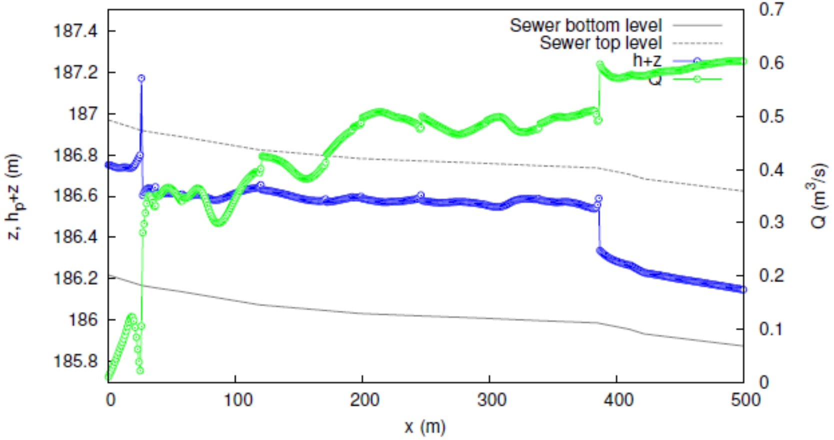




Examples of 2D test case simulation







Contaminant transport

Solute/contaminant transport

We add the Advection-Diffusion-Reaction equation to the system of equations.

$$\frac{\partial\left(h\phi_i\right)}{\partial t}+\frac{\partial\left(hu\phi_i\right)}{\partial x}+\frac{\partial\left(hv\phi_i\right)}{\partial y}=E\frac{\partial}{\partial x}\left(h\frac{\partial\phi_i}{\partial x}\right)+E\frac{\partial}{\partial y}\left(h\frac{\partial\phi_i}{\partial y}\right)\pm hr_i\pm s_i$$

where

ϕ_i = Average concentration of each solute/contaminant

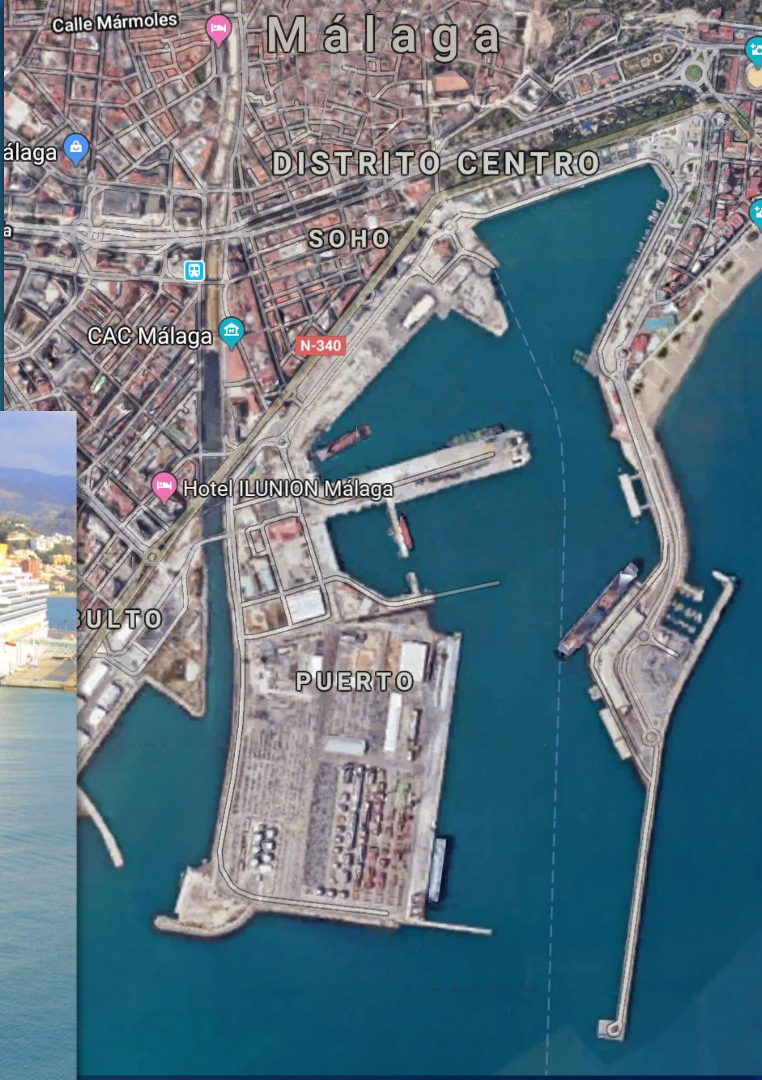
E = Longitudinal diffusion coefficient

r_i = Generation/consumption term of each solute (20-process x 10-variable matrix).

s_i = Source term

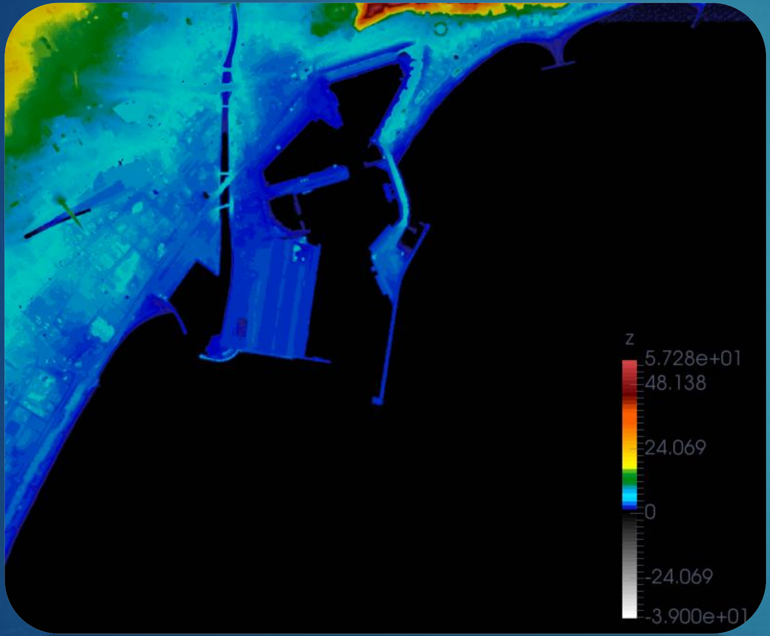
Seaport of Málaga: Case setup

- The oldest international seaport operating in Spain
- The second most important port in Spain for cruise passengers
- Located in a natural bay, well protected by the shape of coastline



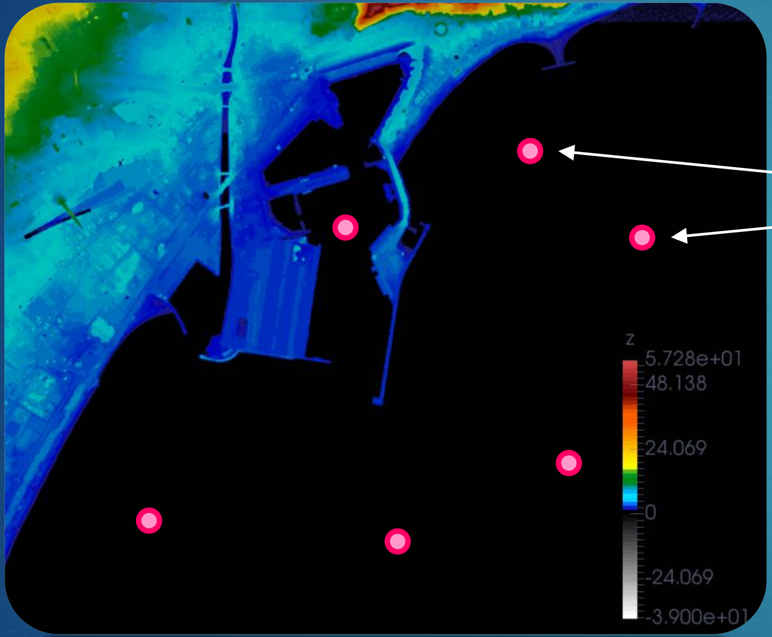
Seaport of Málaga: Case setup

Starting point: 5m x 5m Digital Elevation Model



Seaport of Málaga: Case setup

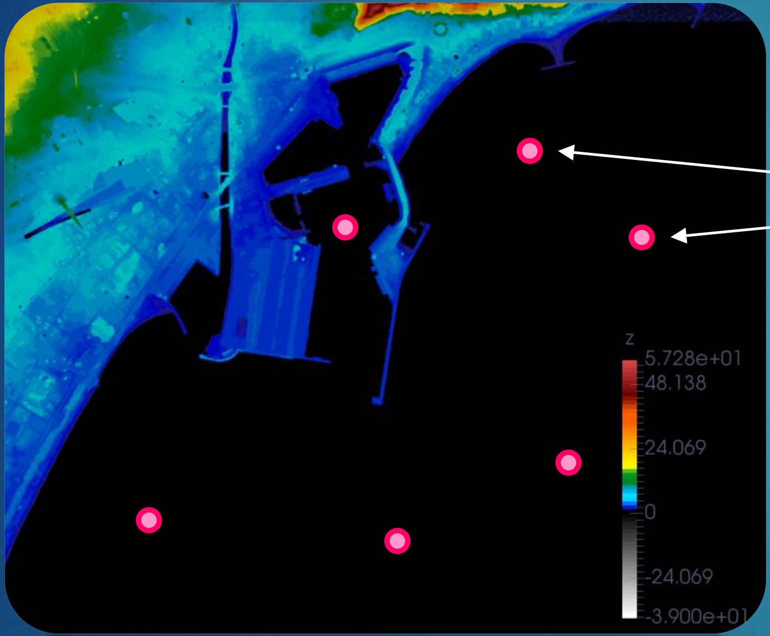
Starting point: 5m x 5m Digital Elevation Model



Limited bathymetric data at certain points

Seaport of Málaga: Case setup

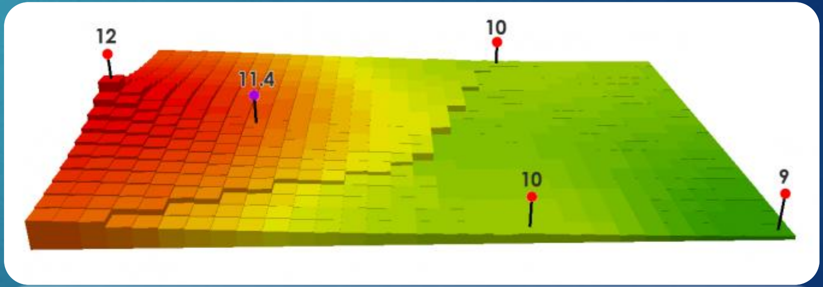
Starting point: 5m x 5m Digital Elevation Model



Limited bathymetric data at certain points

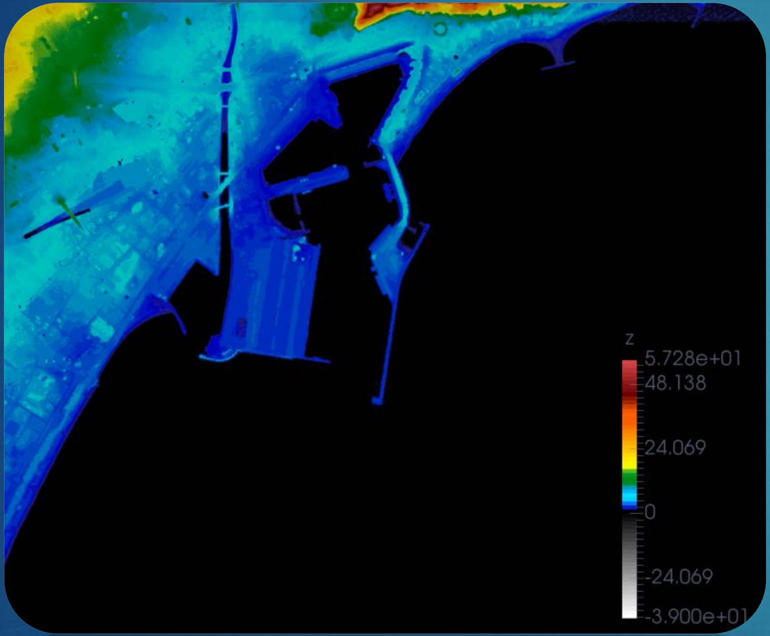
↓

Inverse Distance Weighting interpolation

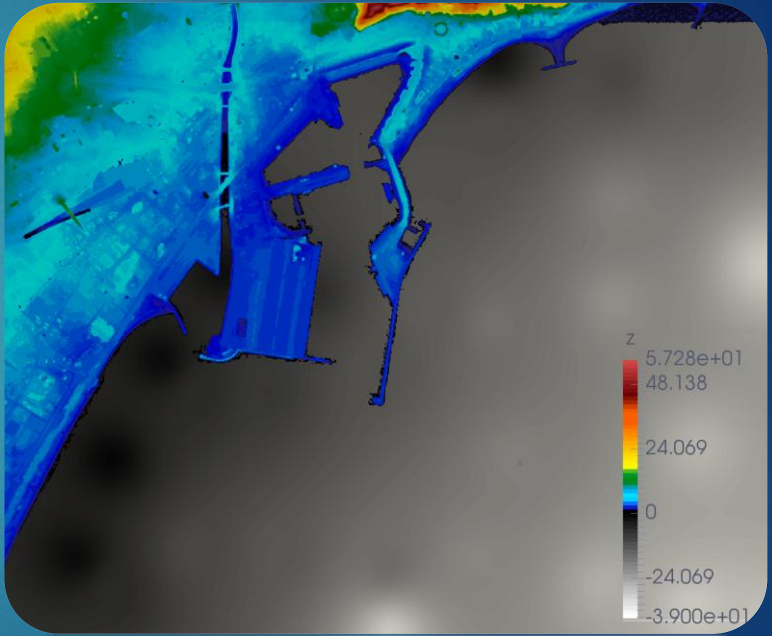


Seaport of Málaga: Case setup

Starting point: 5m x 5m Digital Elevation Model

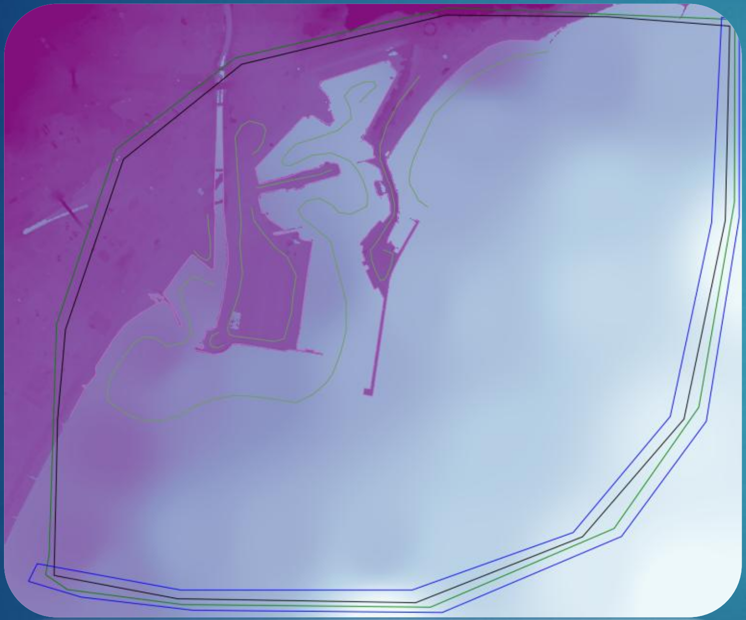


Bathymetry (Inverse Distance Weighting interp.)

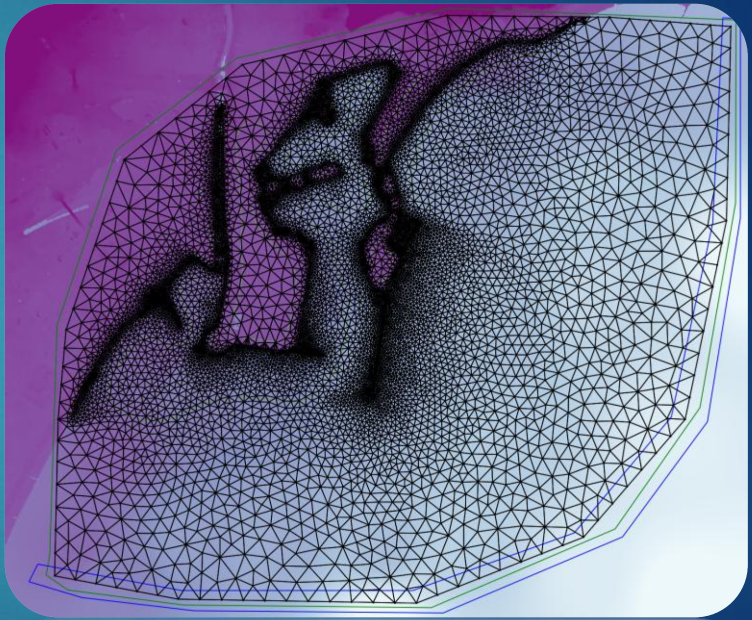


Seaport of Málaga: Case setup

Domain and boundary conditions setup

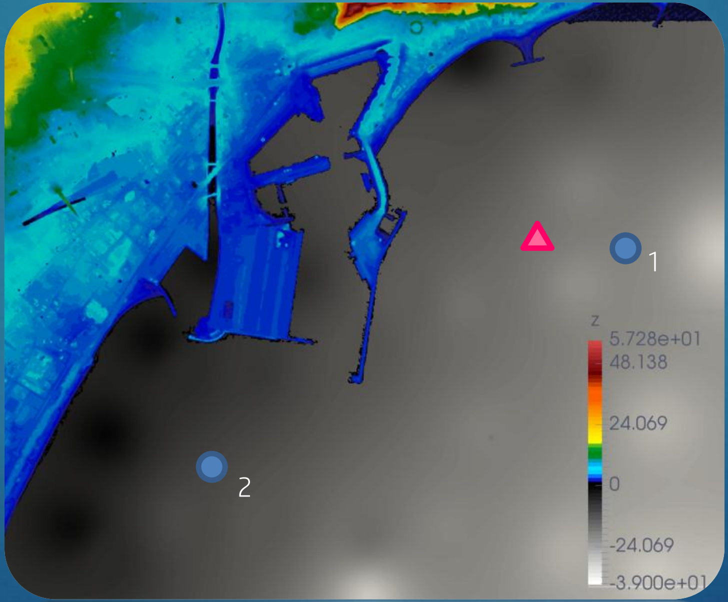


Local mesh refinement at the coastline



Seaport of Málaga: Case setup

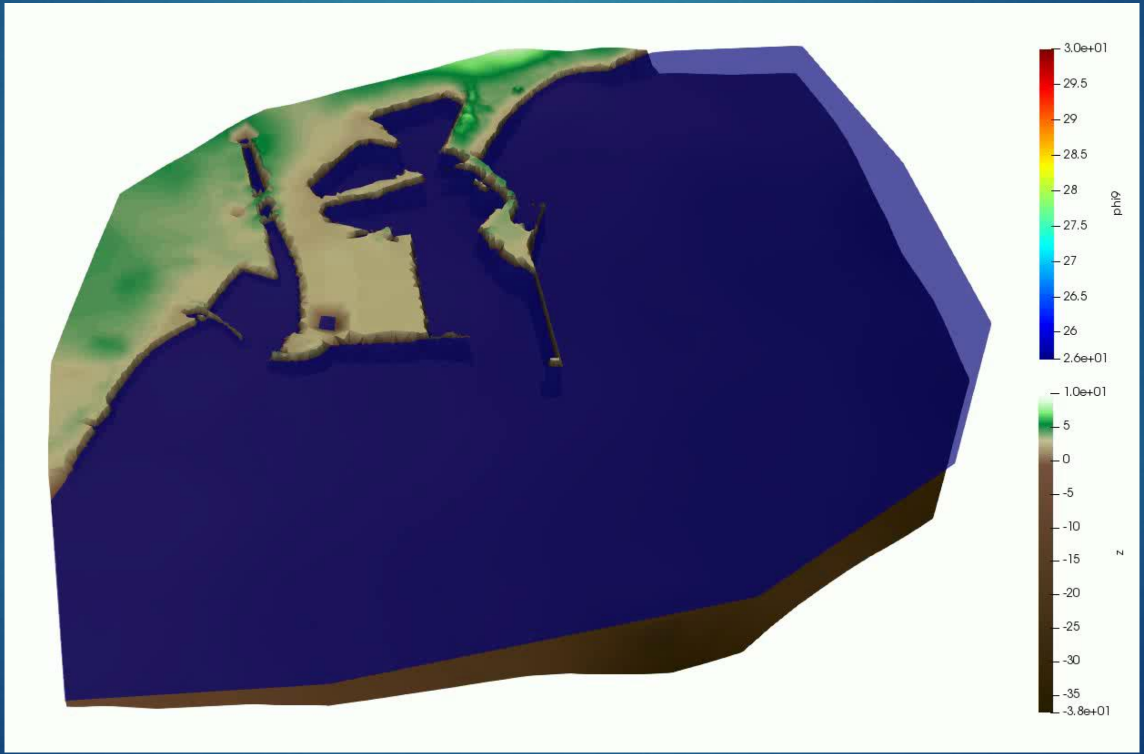
A contaminant source is assumed with Q=10m^3/s and 10 solutes
Only temperature, dissolved oxygen and total coliform bacteria are monitored
A nominal tidal wave is imposed at the offshore boundary
Simulation time = 100 hours



Pollutant source
Gauges

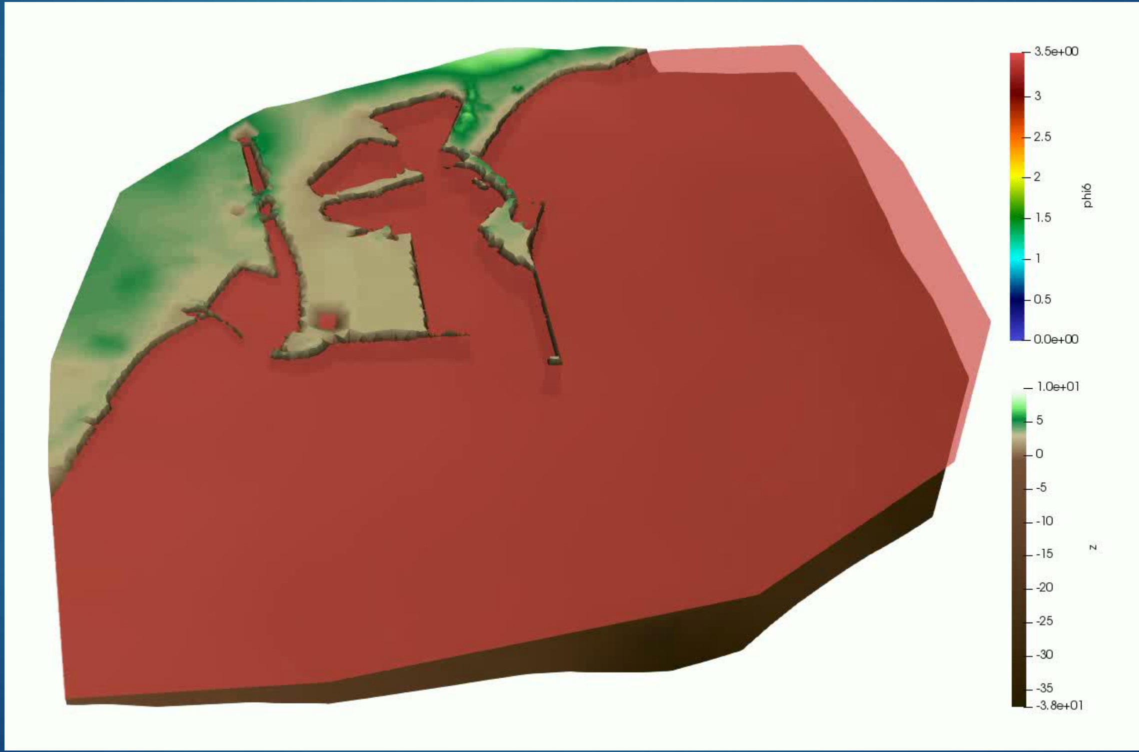
Seaport of Málaga: Numerical results

Numerical results for the Temperature:



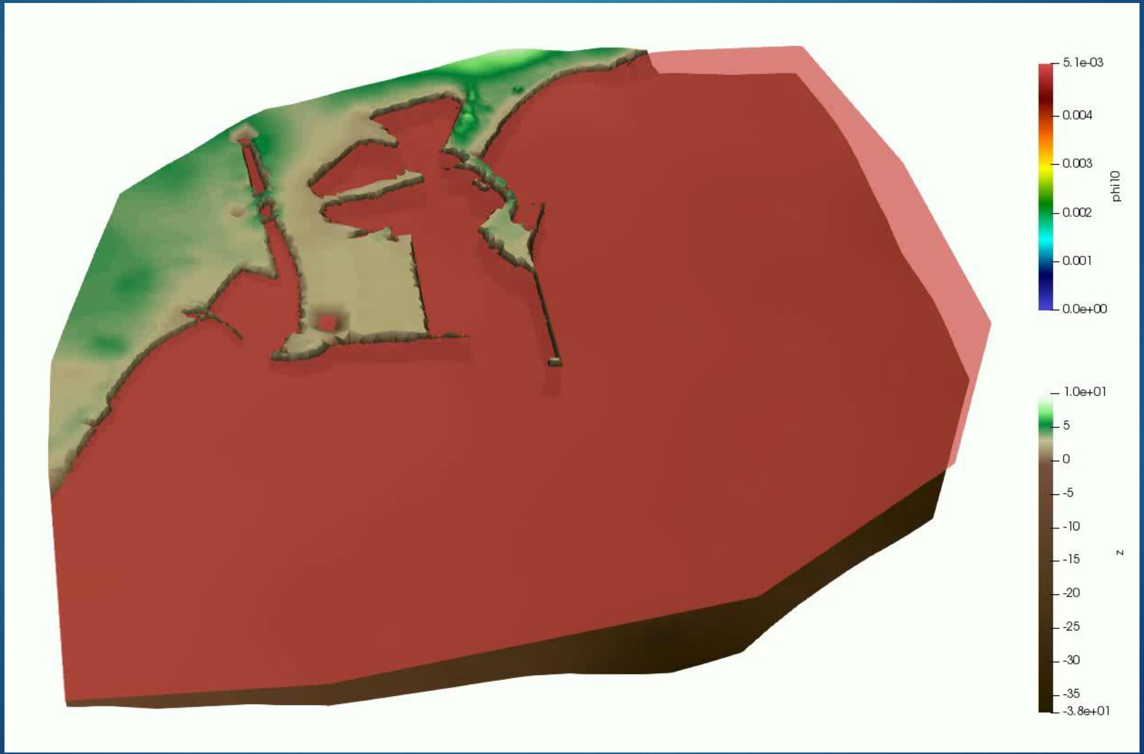
Seaport of Málaga: Numerical results

Numerical results for the Dissolved Oxigen:



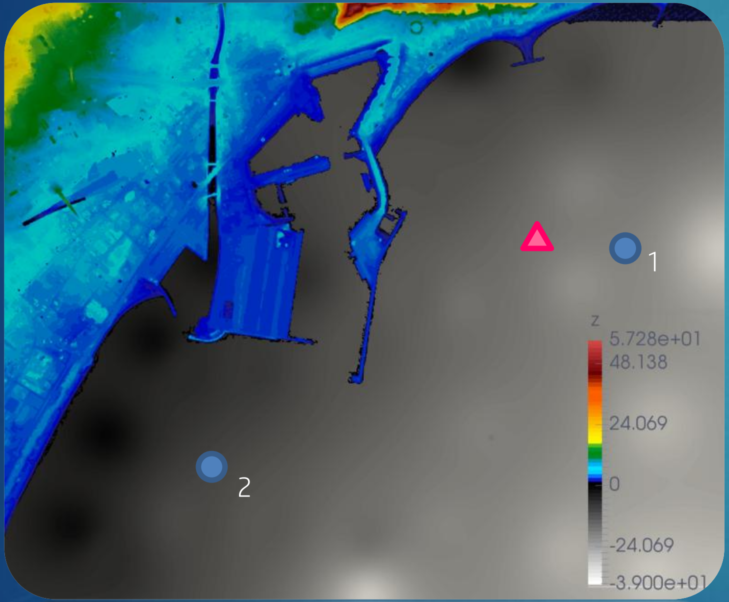
Seaport of Málaga: Numerical results

Numerical results for the Total Coliform Bacteria:



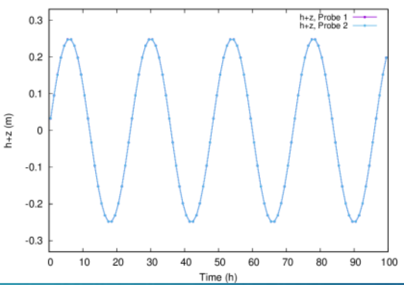
Seaport of Málaga: Numerical results

- Temporal evolution for Probes 1 and 2:

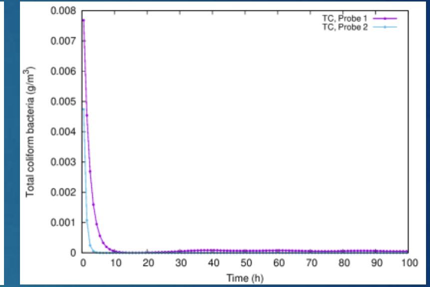


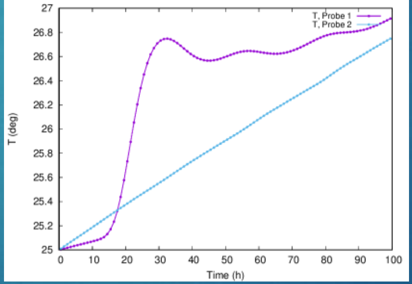
Pollutant source
Gauges

Water level

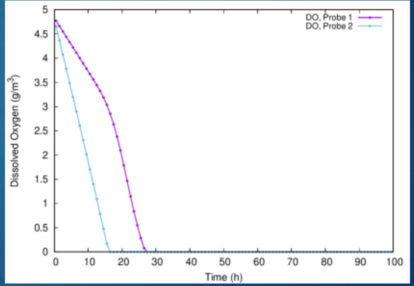


Total coliform bacteria





Temperature

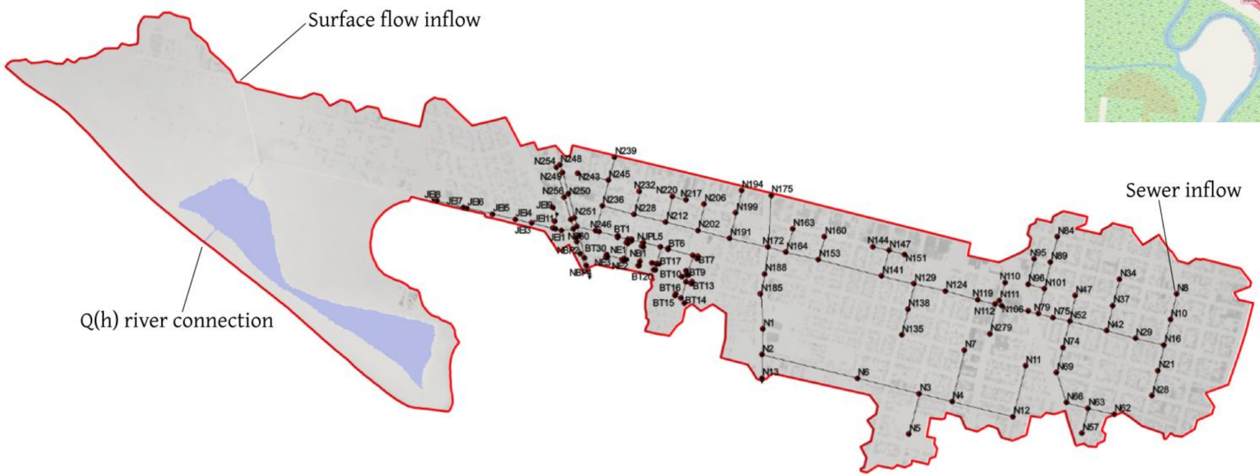


Dissolved Oxygen

Solute transport in a mixed environment in the city of Santa Fé (Argentina).

A very complex case that includes many simultaneous phenomena:

- Coupled 2D flow (surface) + 1D (pipes)
- Rainfall
- Infiltration
- Inflow of water + contaminants
- Connection to a river
- >100 connections with an urban drainage network.



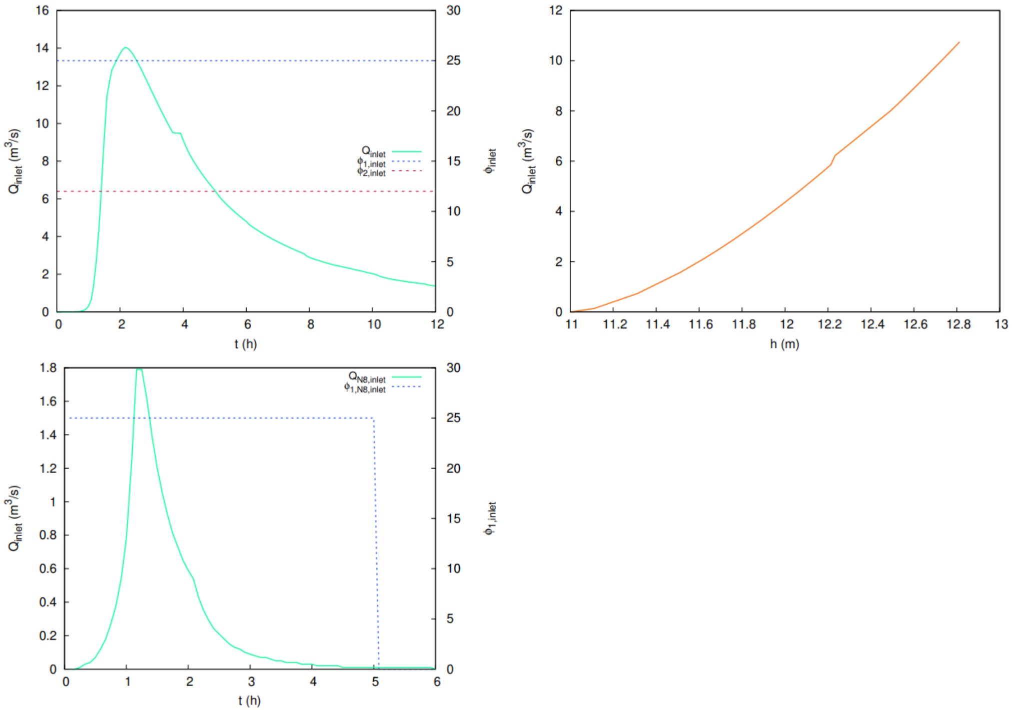


Figure 11. Case 3: Water discharge and concentration for both pollutants at the surface boundary inlet point (**left**). Stage discharge rating curve used for the river connection (**right**). Water discharge and TSS concentration at the sewer node N8 (**lower**).

Examples of 2D test case simulation

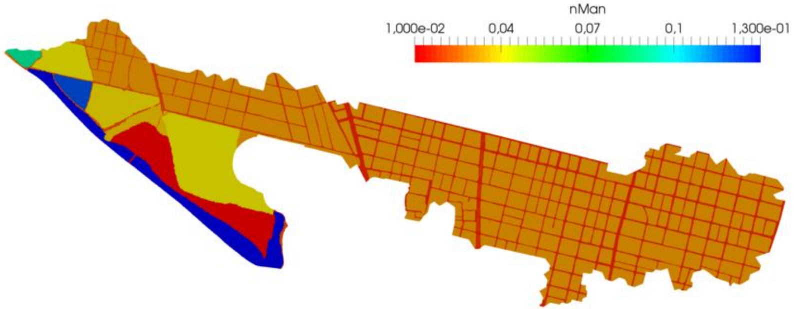


Figure 12. Case 3: Manning's roughness map.

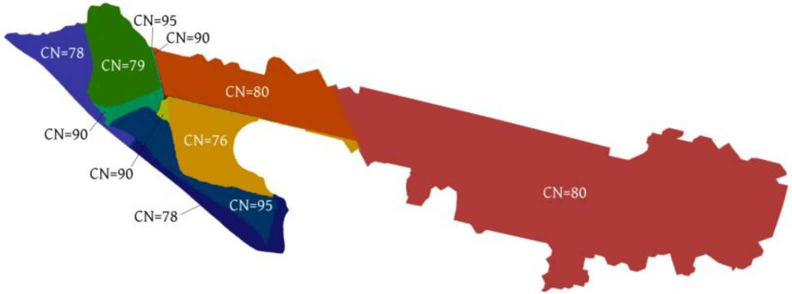


Figure 14. Case 3: SCS Curve Number values (CN).

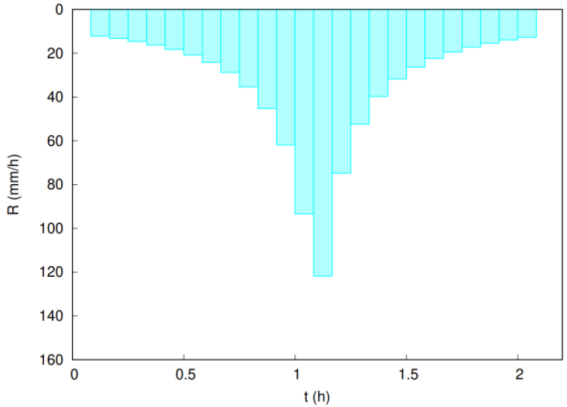


Figure 13. Case 3: Temporal evolution of the incoming rainfall intensity.

Examples of 2D test case simulation

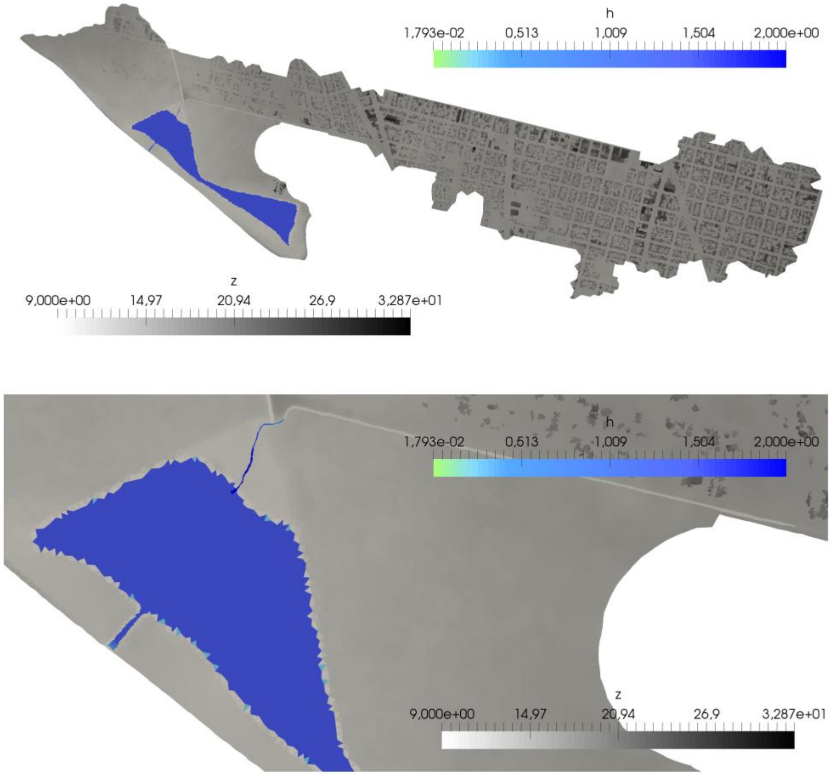


Figure 15. Case 3: Bed level (gray scale) and initial water depth (color scale) of the full domain (**upper**) and detail of the lake (**lower**).

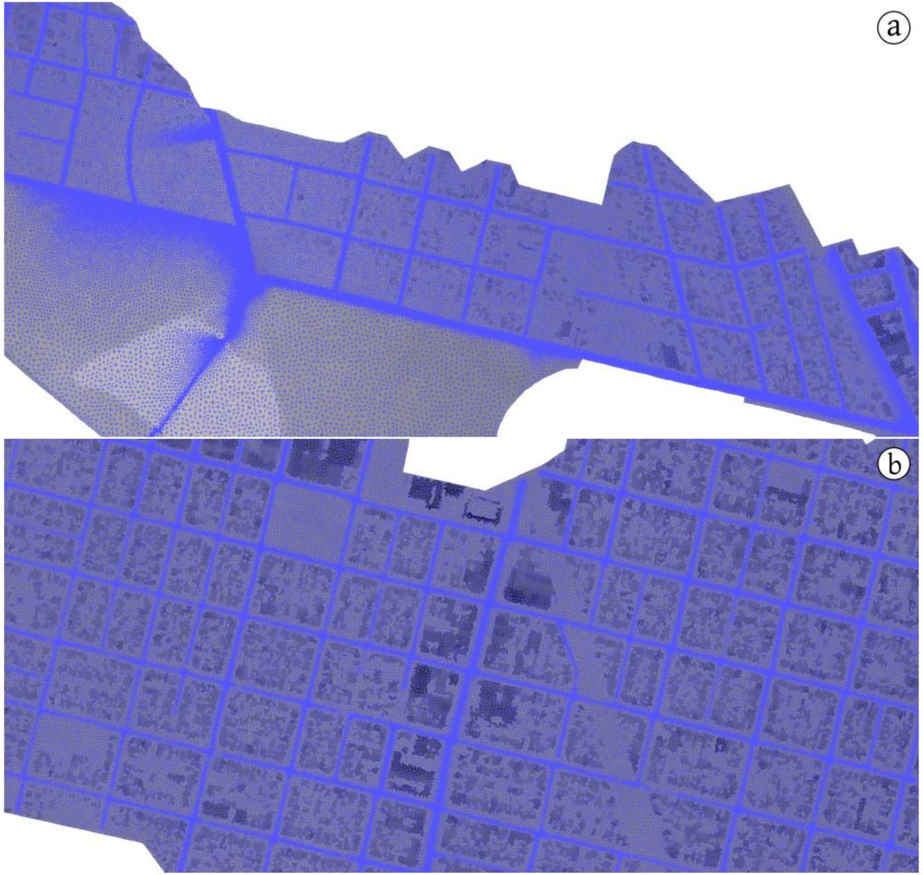


Figure 16. Case 3: Detail of the unstructured triangular and locally refined computational mesh (**a**,**b**).

Examples of 2D test case simulation

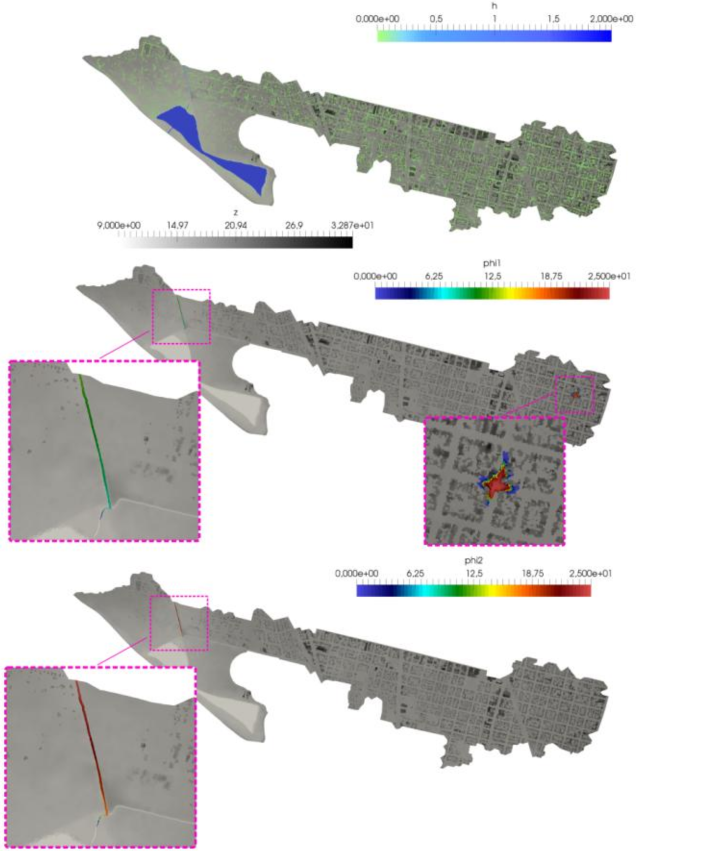


Figure 17. Case 3: Surface water depth h (**upper**) and pollutant concentration for both solutes ϕ_1 (**center**) and ϕ_2 (**lower**) at $t = 1.1$ h.

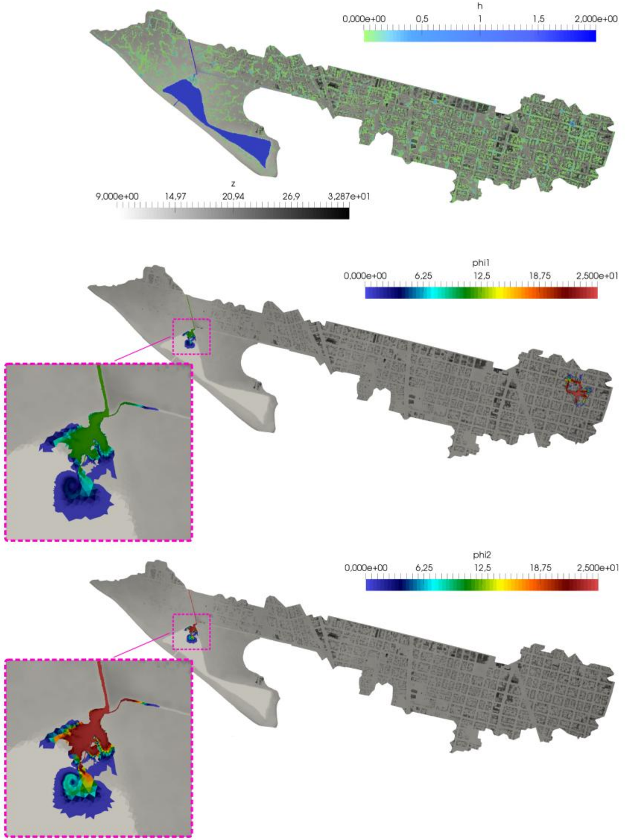


Figure 18. Case 3: Surface water depth h (**upper**) and pollutant concentration for both solutes ϕ_1 (**center**) and ϕ_2 (**lower**) at $t = 1.5$ h.

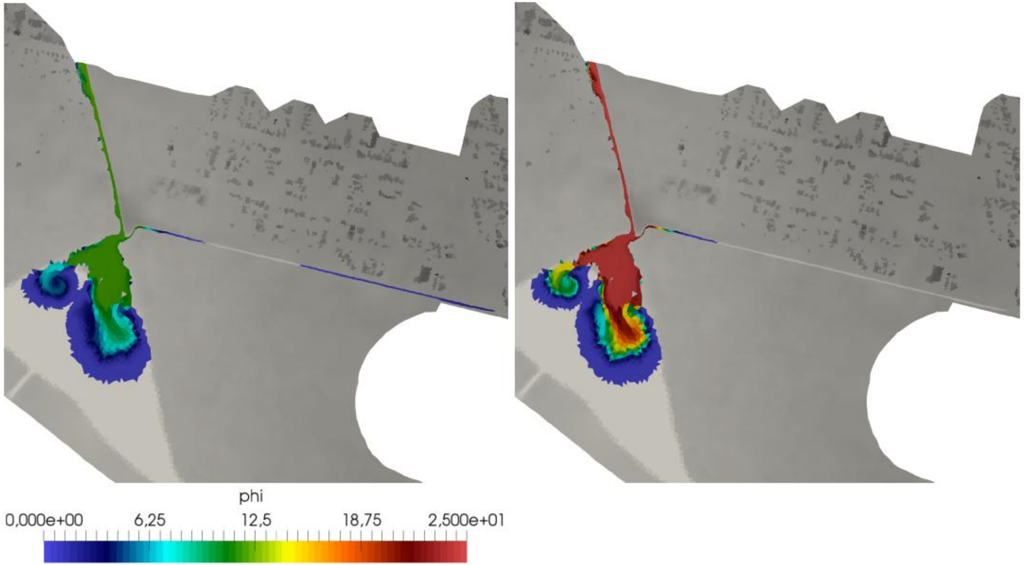


Figure 19. Case 3: Detail of the surface-sewer connecting channel showing the pollutant concentration for both solutes ϕ_1 (**left**) and ϕ_2 (**right**) at $t = 1.8$ h.

Workshop 2: Numerical models for hydraulic simulation

Examples of 2D test case simulation

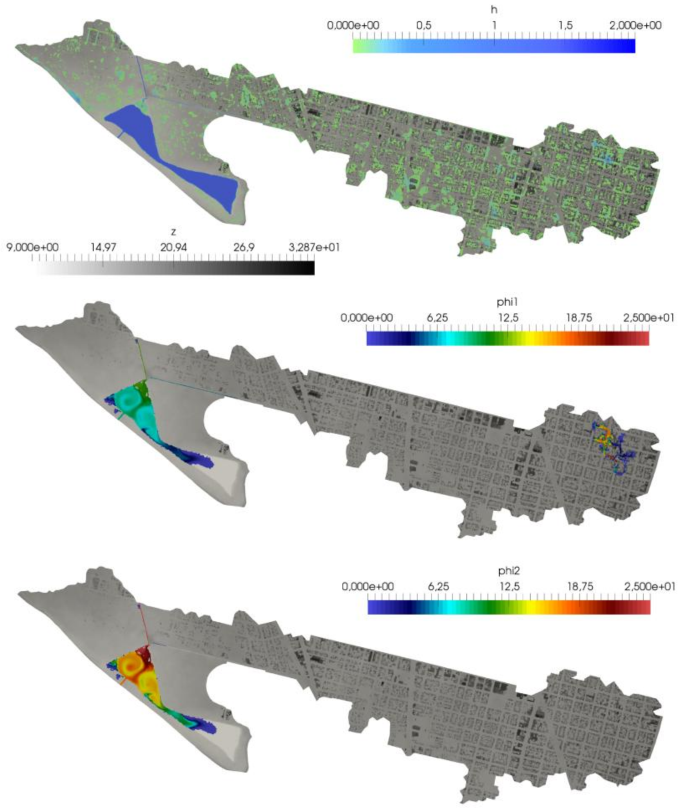


Figure 20. Case 3: Surface water depth h (**upper**) and pollutant concentration for both solutes ϕ_1 (**center**) and ϕ_2 (**lower**) at $t = 5.2$ h.

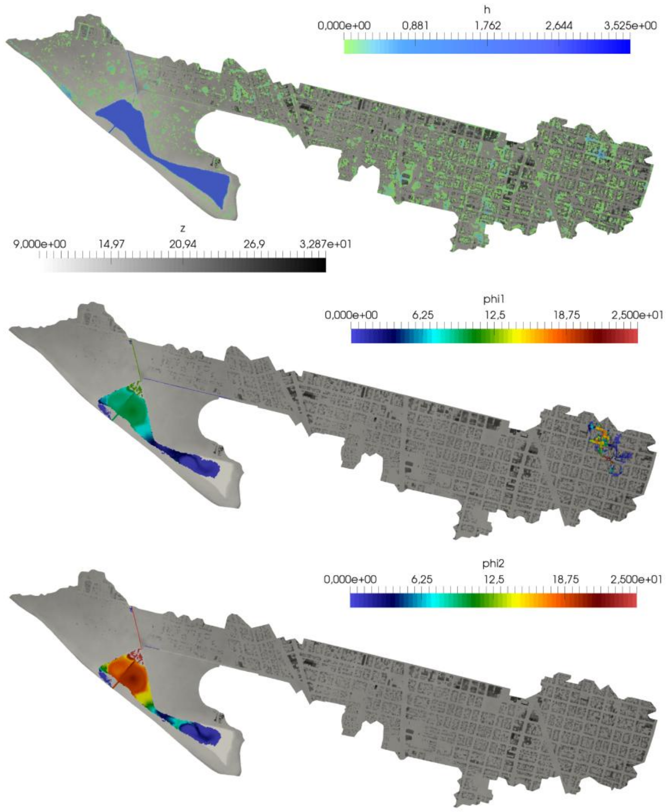


Figure 21. Case 3: Surface water depth h (**upper**) and pollutant concentration for both solutes ϕ_1 (**center**) and ϕ_2 (**lower**) at $t = 12$ h.

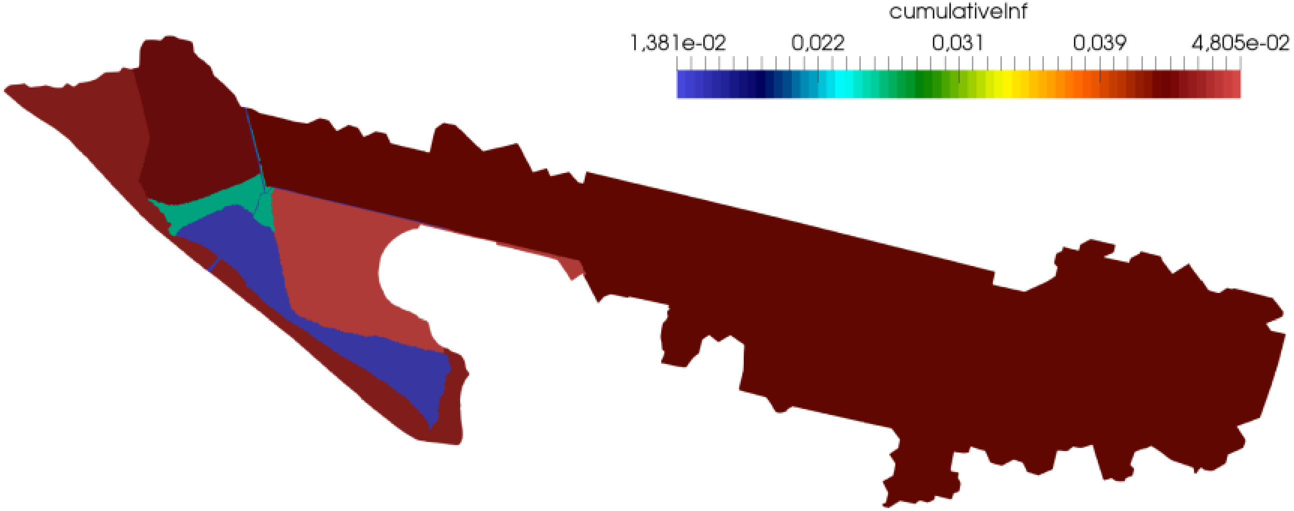


Figure 22. Case 3: Cumulative water infiltration F at $t = 12$ h.

Groundwater flow

Mathematical model: Among the multiple options to formulate groundwater flow, one of the simplest is the combination of Darcy's law for flows in porous media with the continuity equation.

2D Darcy's law: $\mathbf{v} = -K_s \nabla h_s \Rightarrow \mathbf{q} = (h_s - z_s)\mathbf{v} = -K_s (h_s - z_s) \nabla h_s$

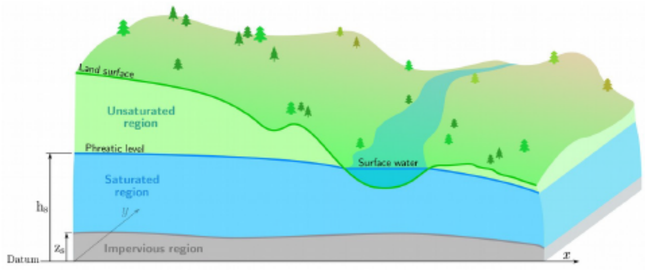
Continuity equation: $\eta \frac{\partial (h_s - z_s)}{\partial t} + \nabla \mathbf{q}_s = S_s$

$$\eta \frac{\partial h_s}{\partial t} - K_s \nabla \left[(h_s - z_s) \nabla h_s \right] = S_s$$

Porosity

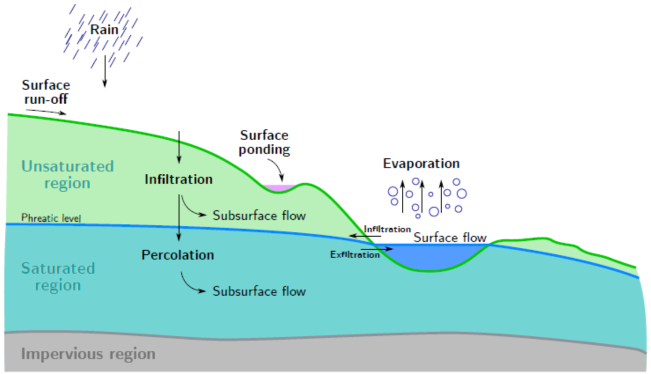
Saturated hydraulic conductivity

Source terms



Coupling mechanisms with the surface flow

1) Soil infiltration: water infiltration through the soil is one of the connection mechanisms between surface and groundwater flow models:



- Horton model suggests an exponential equation for modeling infiltration capacity:

$$f_p = f_c + (f_0 - f_c)\, e^{-kt}$$

Parameters: $f_0,\ f_c,\ k.$

- Green-Ampt model is a simple approach with a theoretical basis on Darcy's law:

$$f_p = K_s + \frac{K_s\,(\theta_s - \theta_i)\,\Psi}{F}$$

Parameters: $K_s,\ \Psi,\ \Delta\theta = \theta_s - \theta_i.$

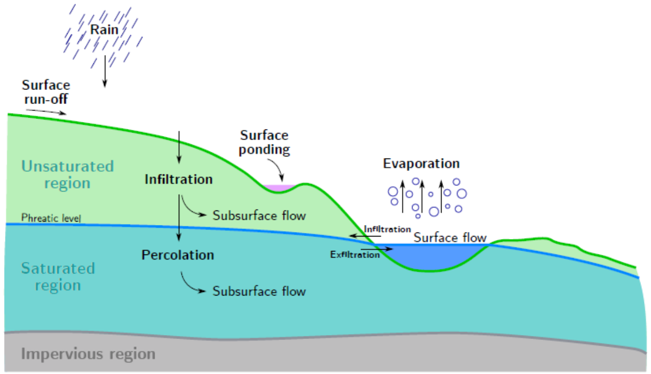
- SCS-CN is a rainfall/runoff conversion model:

$$R_e = \begin{cases} \frac{(RV-\alpha S)^2}{RV-\alpha S+S} & (RV > \alpha S) \\ 0 & (RV \leq \alpha S) \end{cases}, \quad S = \frac{25400}{CN} - 254$$

Parameters: $CN,\ \alpha$

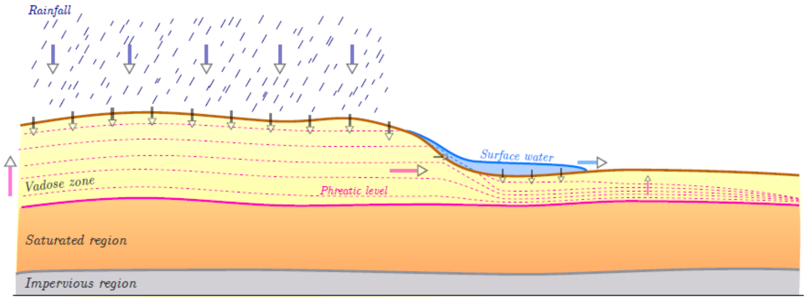
Phreatic level updating

1) Soil infiltration: water infiltration through the soil is one of the connection mechanisms between surface and groundwater flow models:



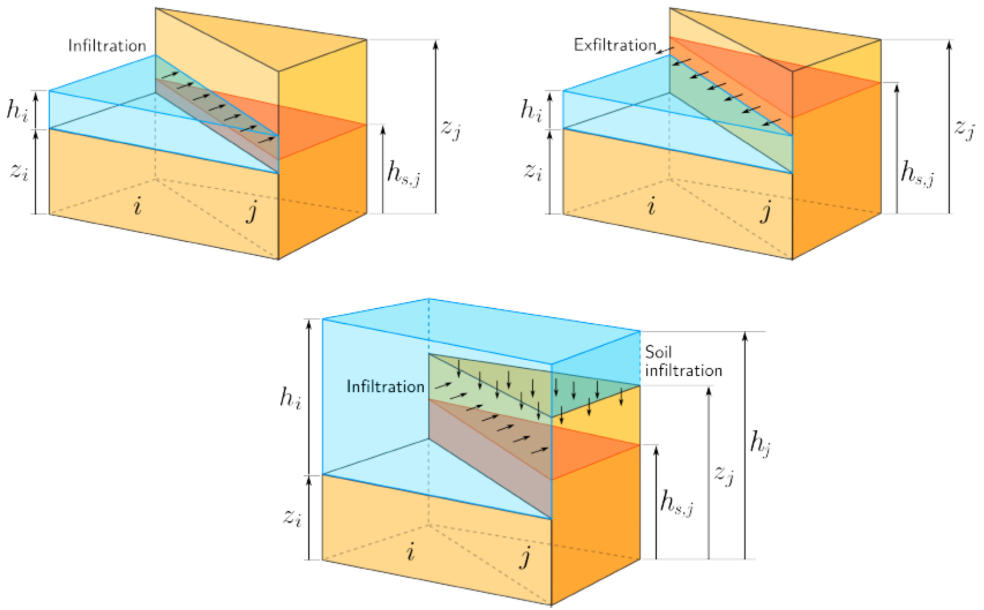
Water from rainfall of surface flow infiltrates throught the soil and may recharge the phreatic level (PL). Several options to model this process:

1. Instant updating of the PL (not realistic).
2. Delayed updating in terms of the non-saturated hydraulic conductivity and the distance between the surface and position of the PL.
3. Modeling of the groundwater flow in the vadose zone.



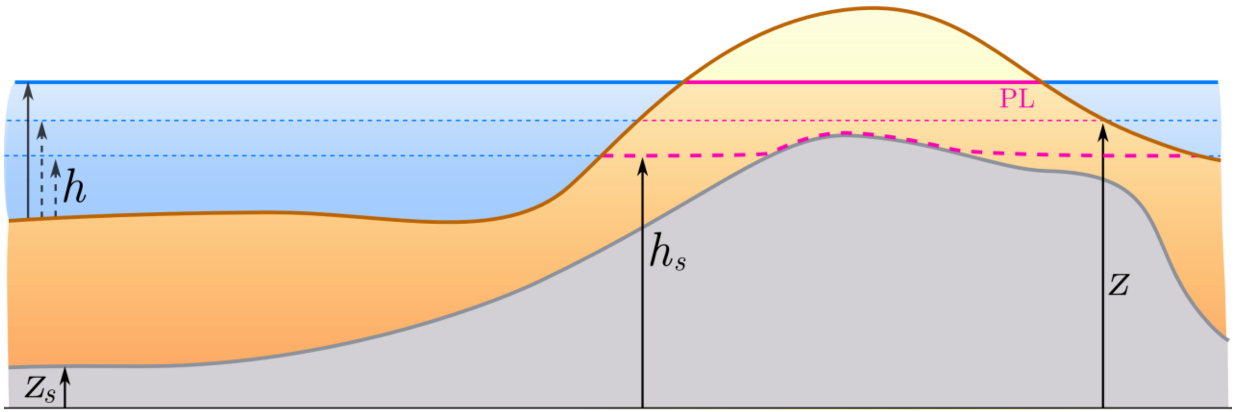
Coupling mechanisms with the surface flow

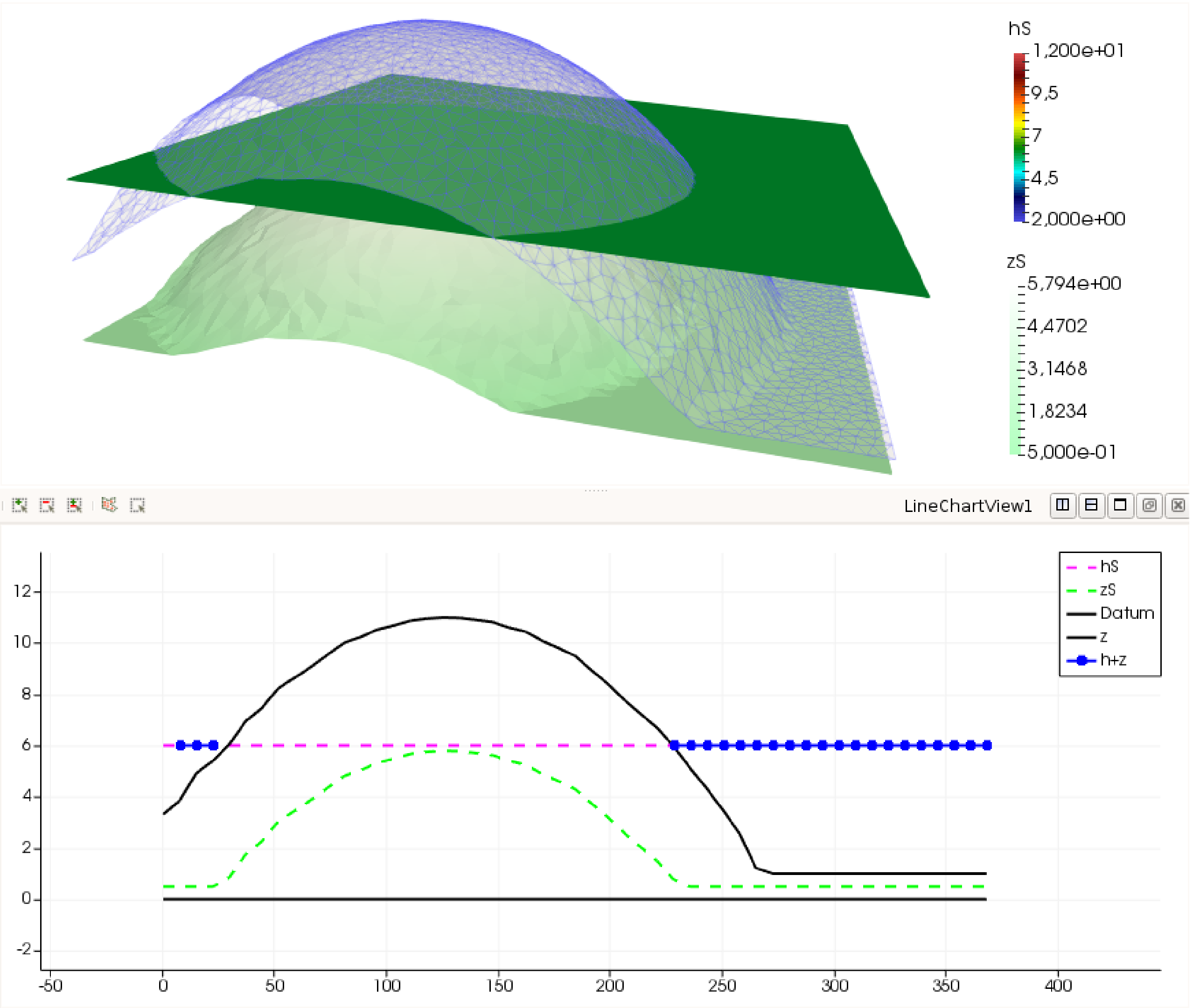
2) Surface-subsurface levels coupling: infiltration and exfiltration processes



Interaction between surface and groundwater flow (Example 1):

- Rise of the surface water level
- Update of the groundwater table
- Interaction with the impermeable part of the soil

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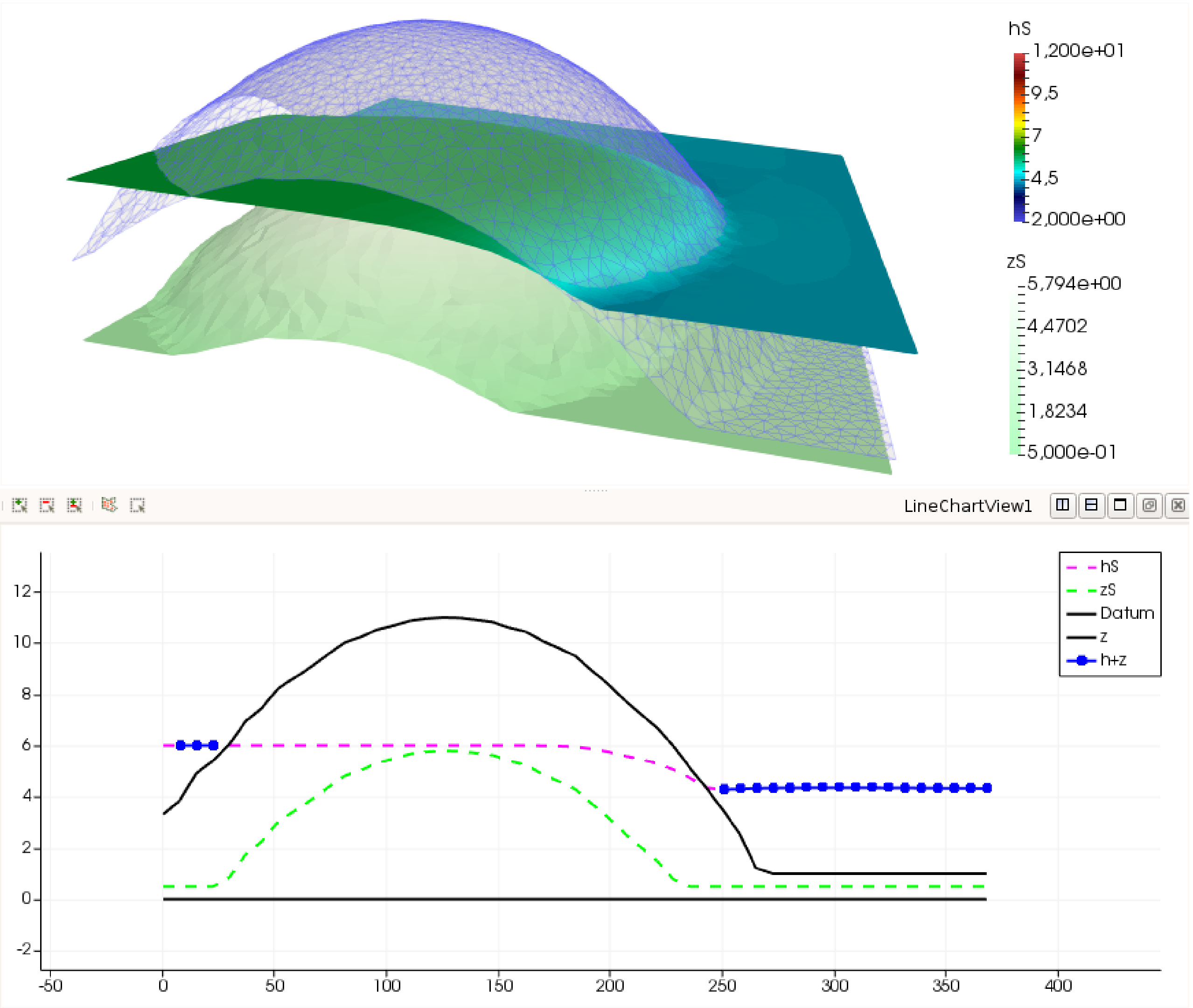
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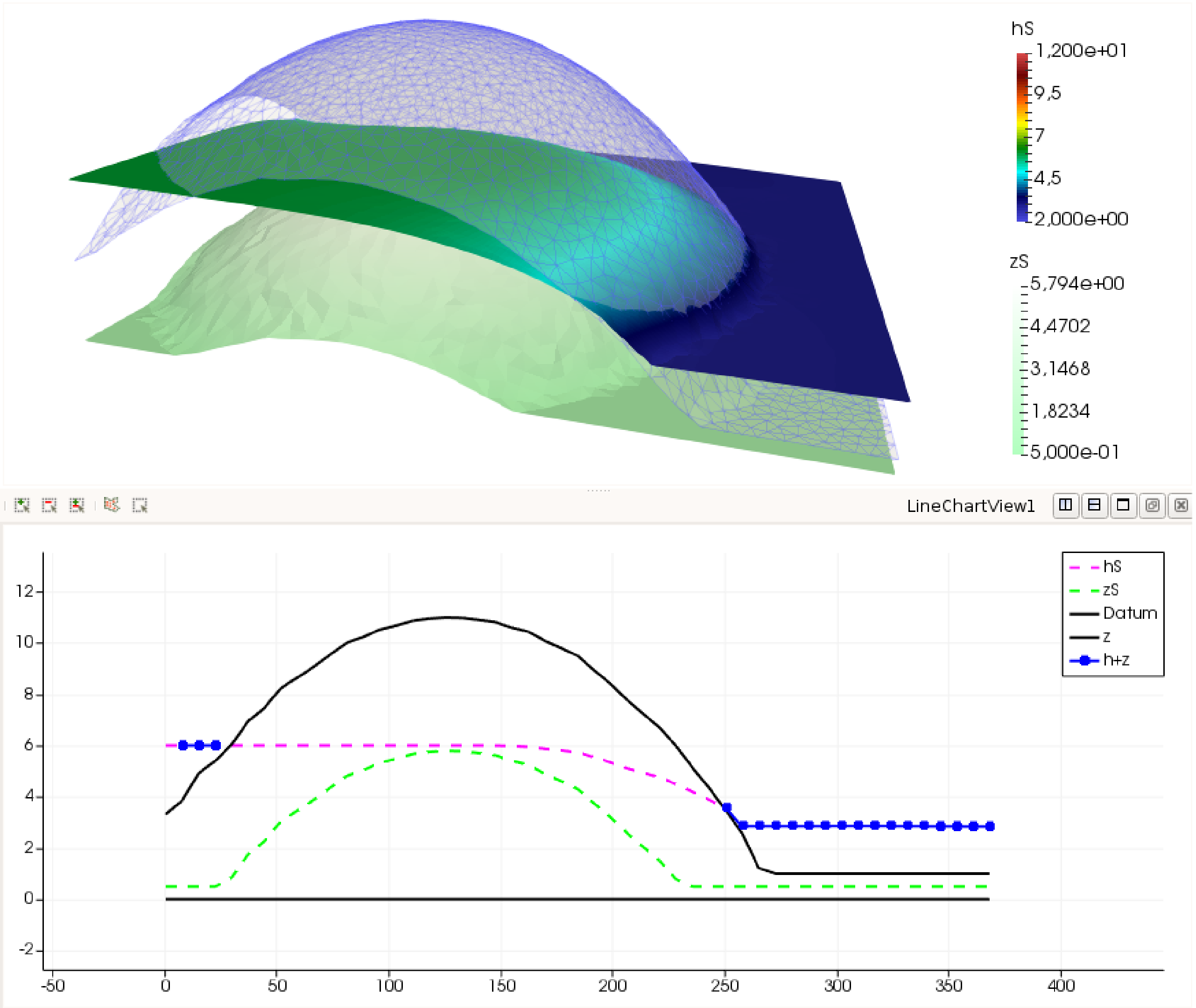
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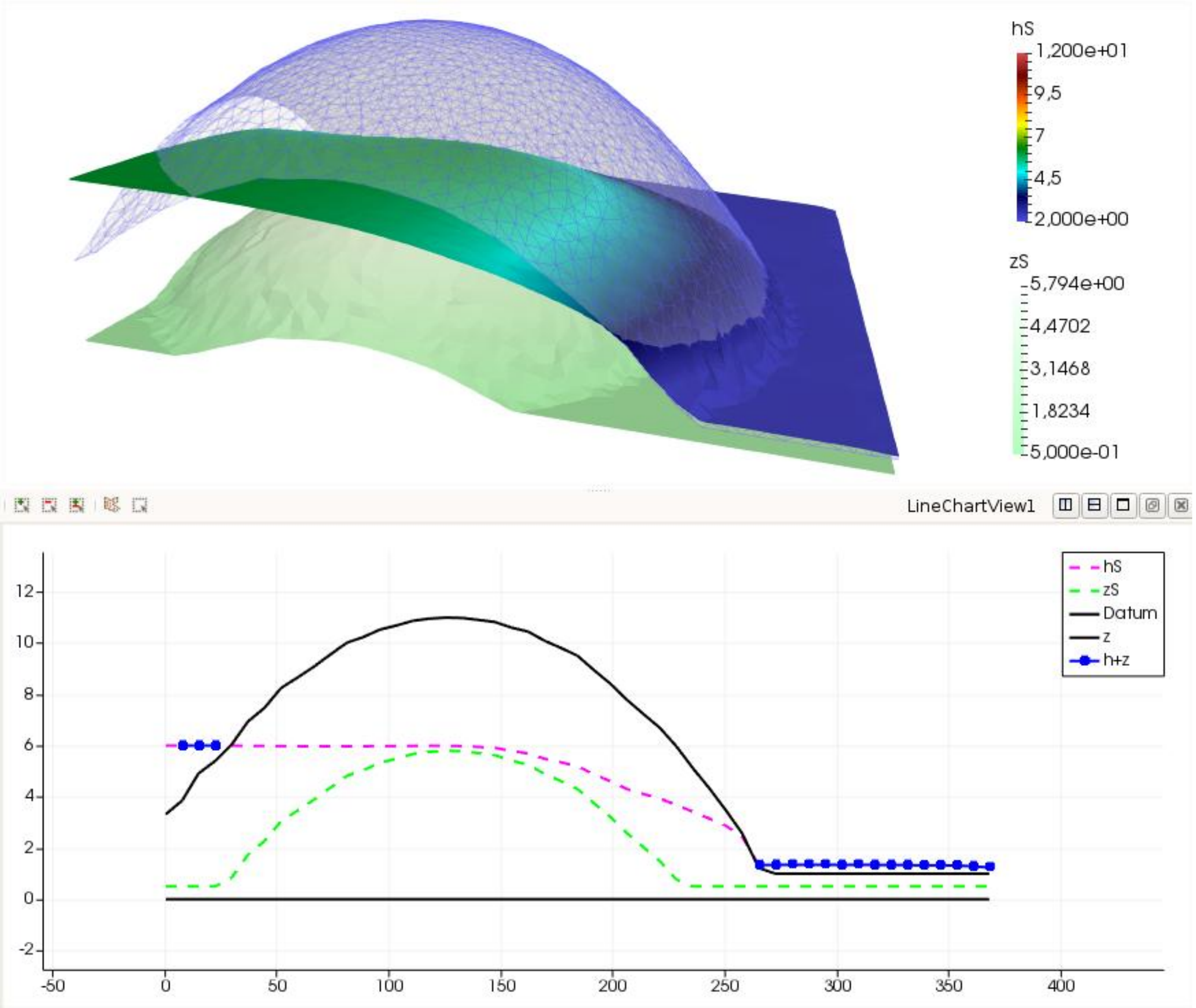

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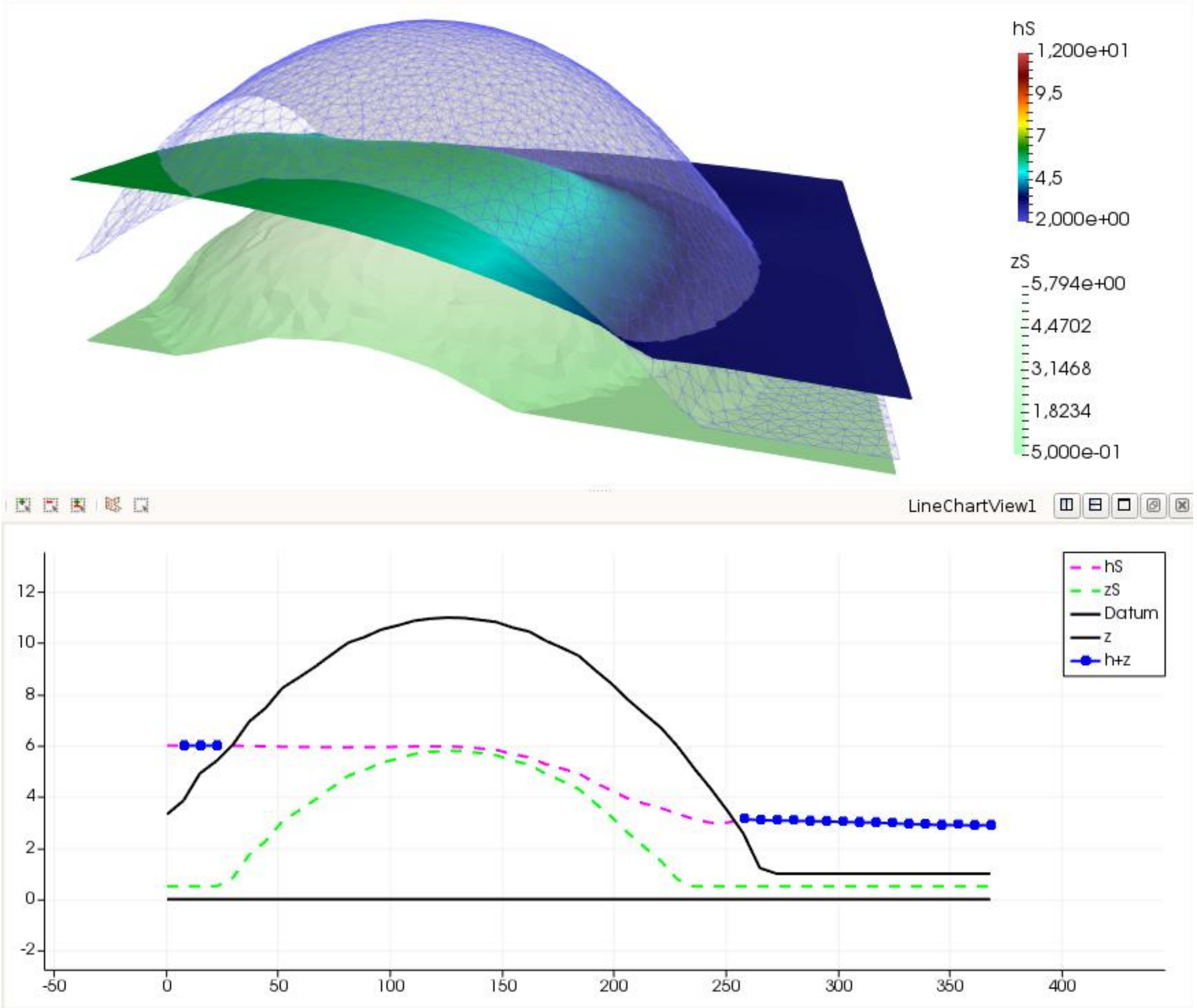
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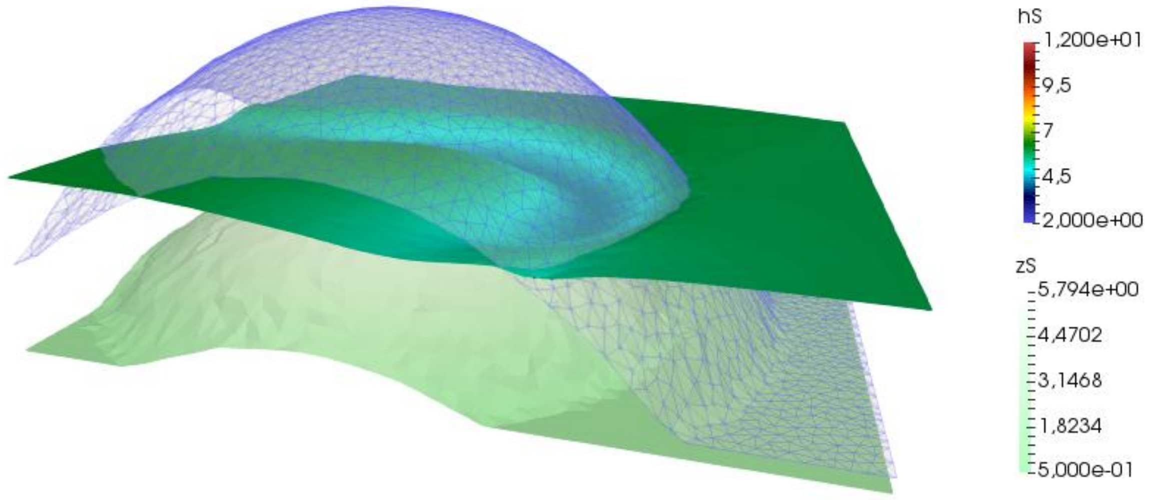
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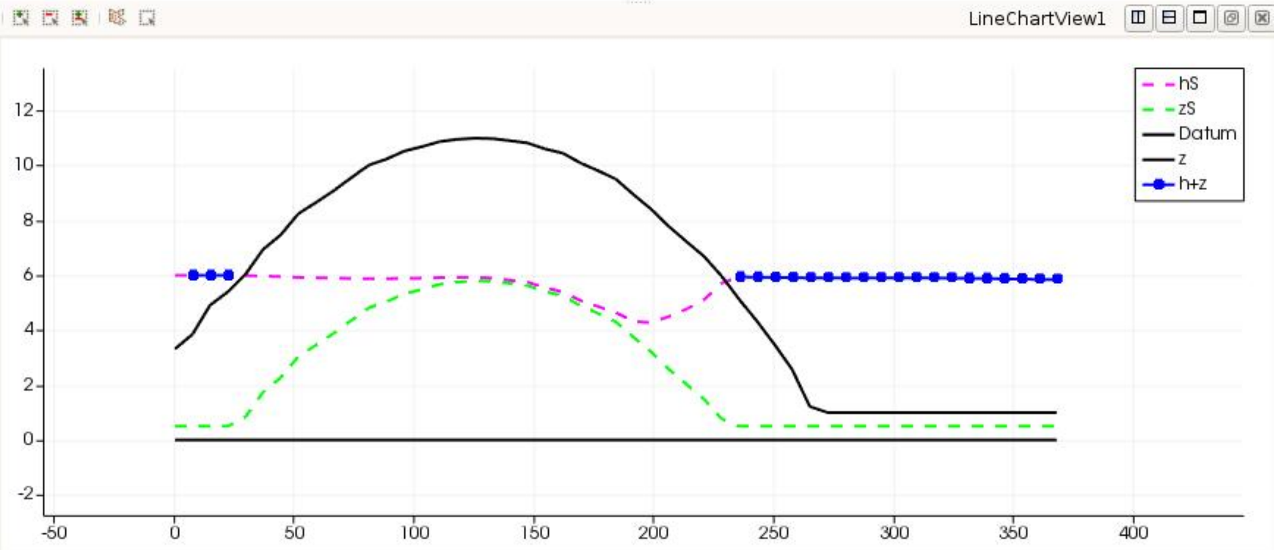
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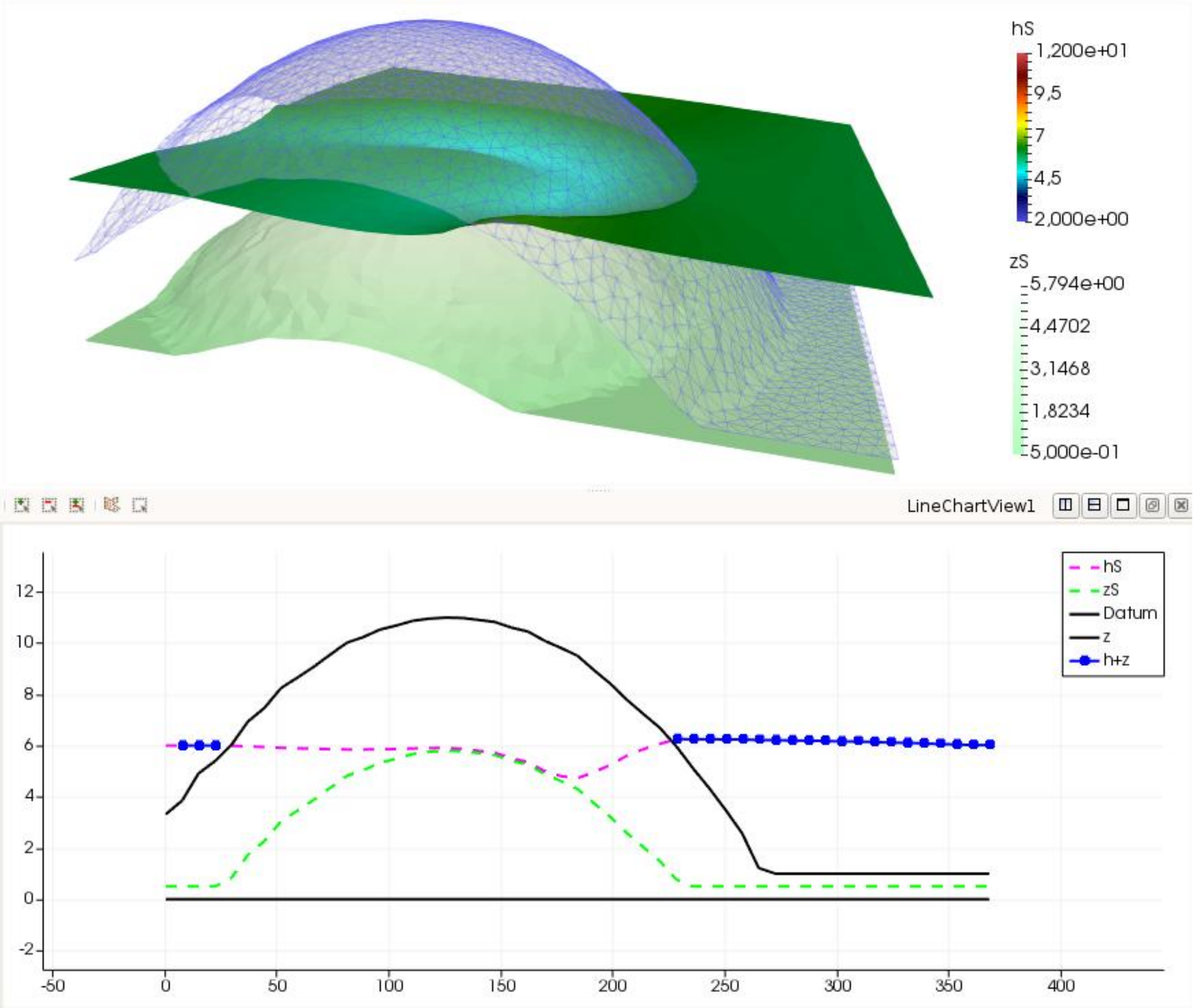
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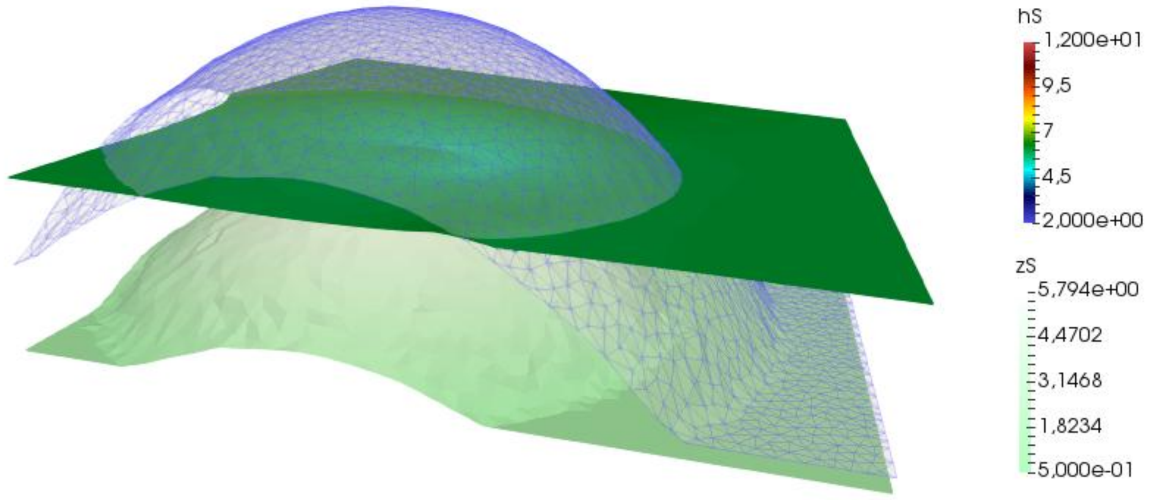
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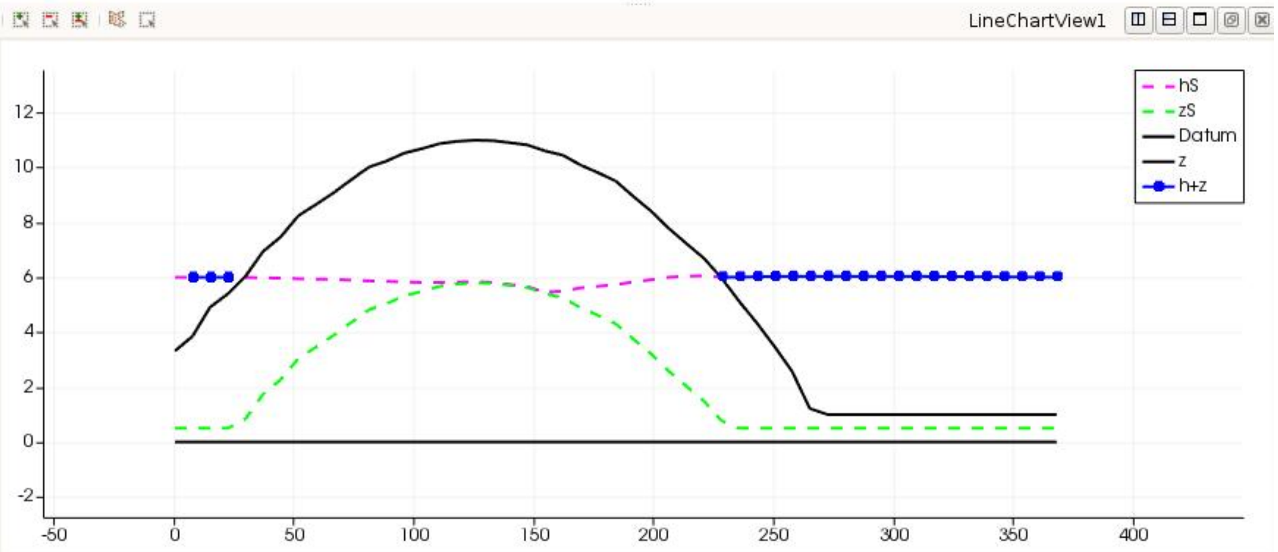
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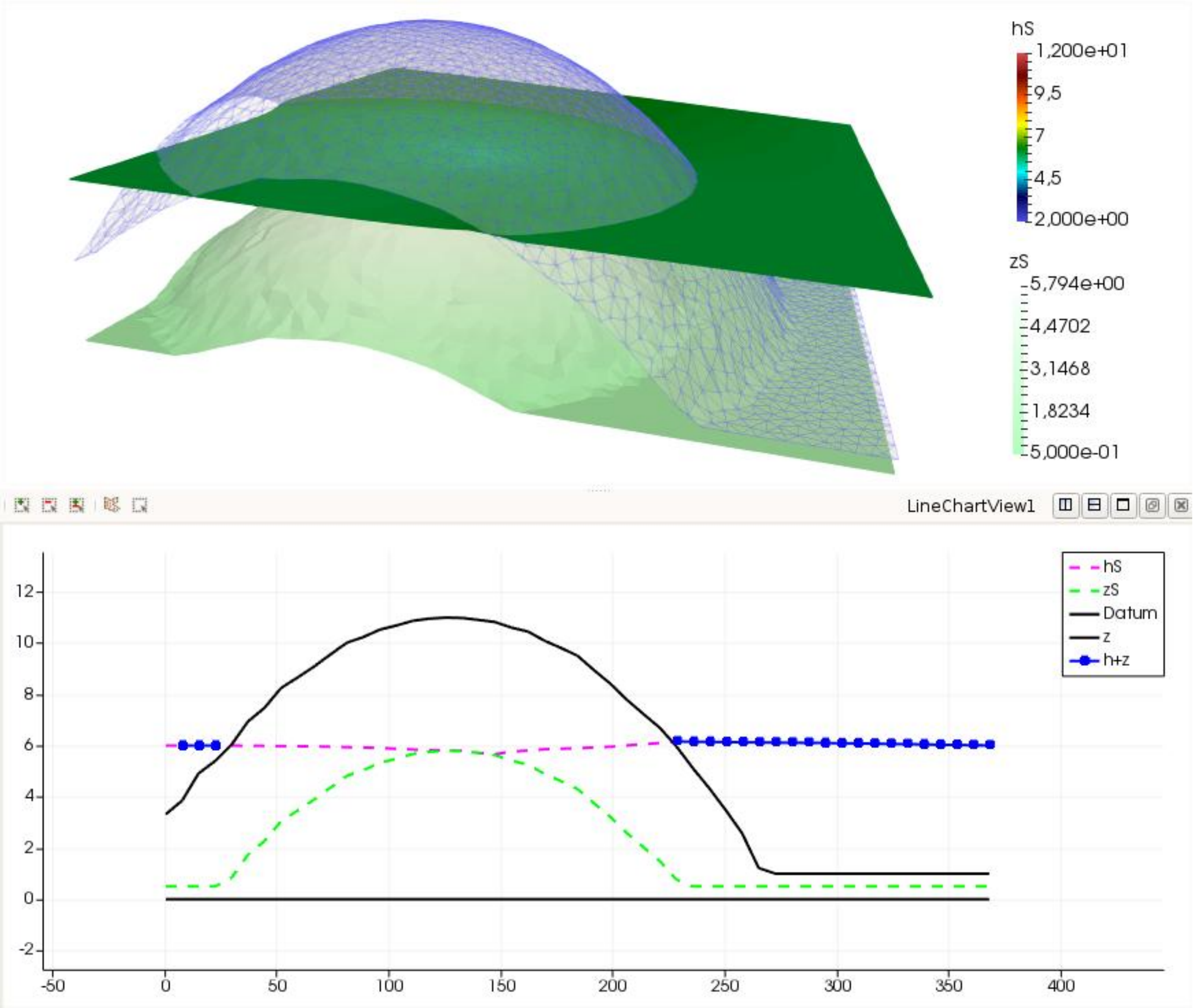
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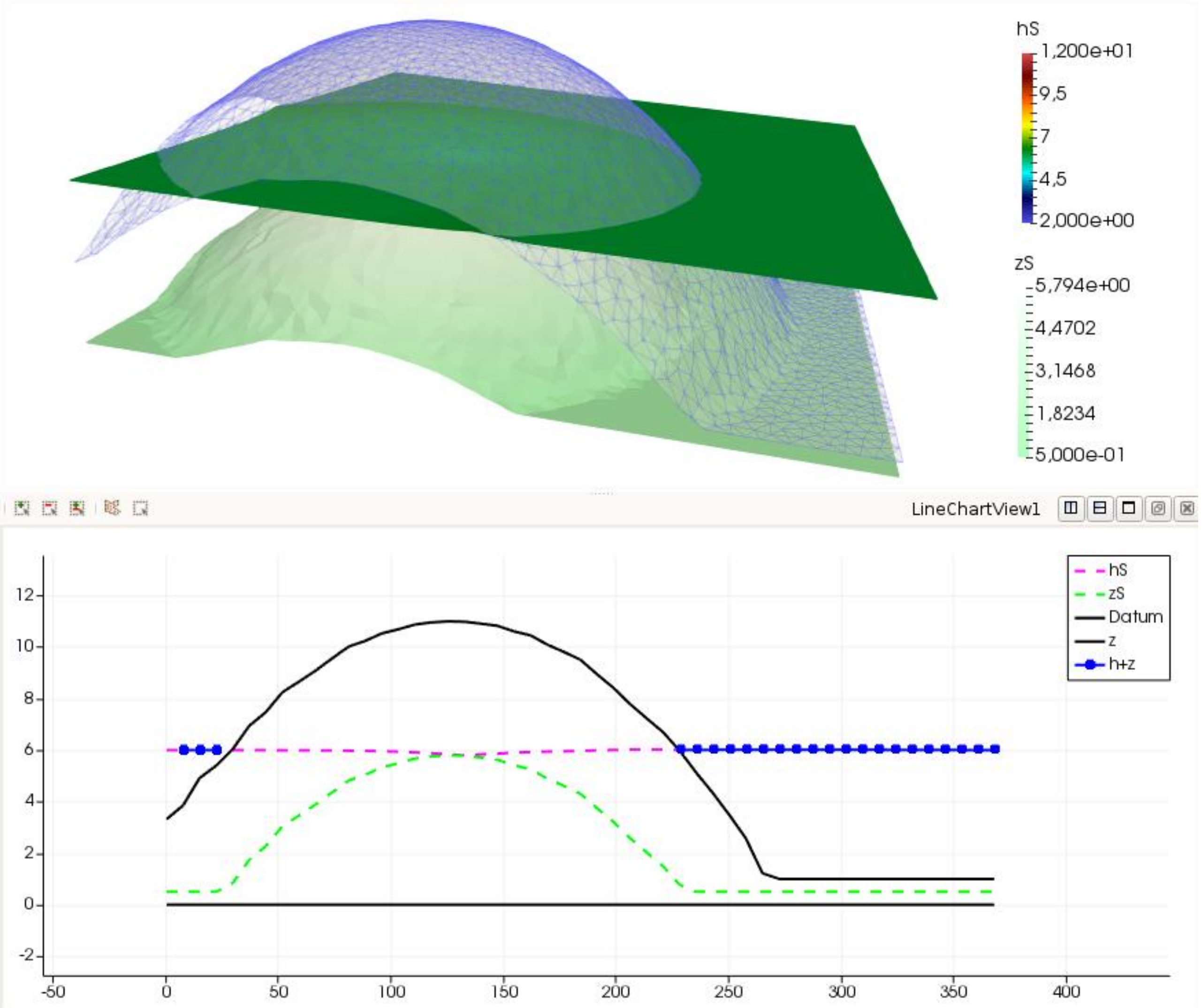

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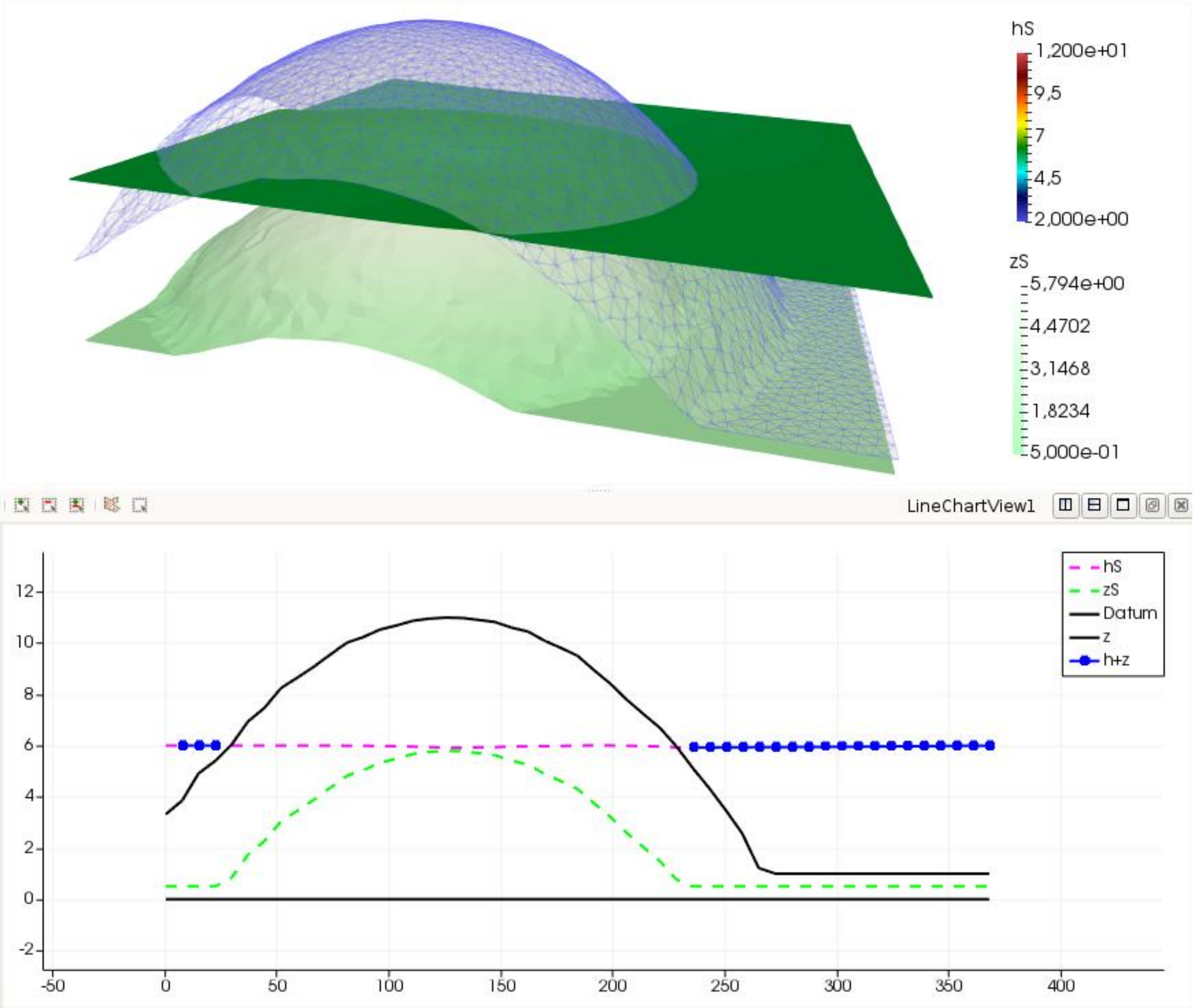
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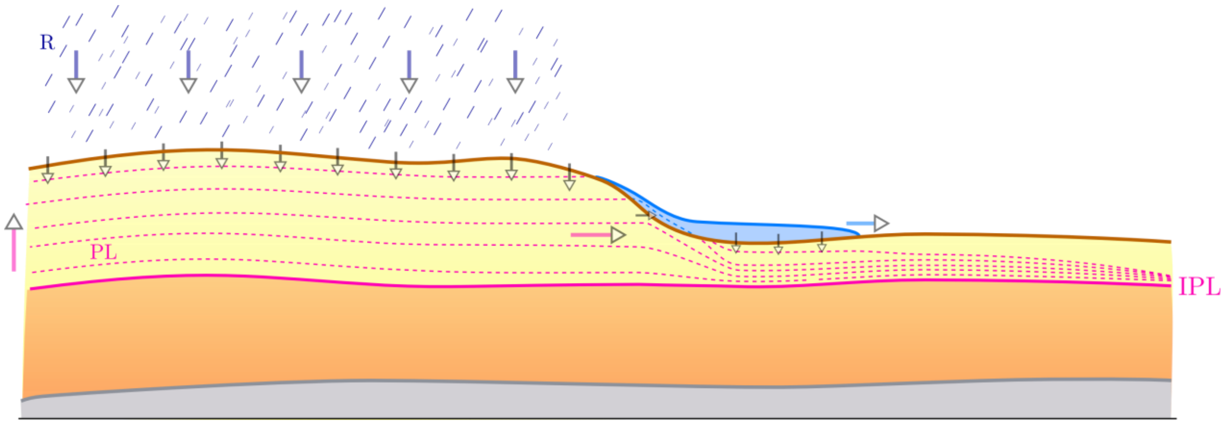
Interaction between surface and groundwater flow (Example 2):

- Localized storm in a specific area of the domain
- Update of the groundwater table
- Exfiltration in another area of the domain



Interaction between surface and groundwater flow (Example 2):

- Localized storm in a specific area of the domain
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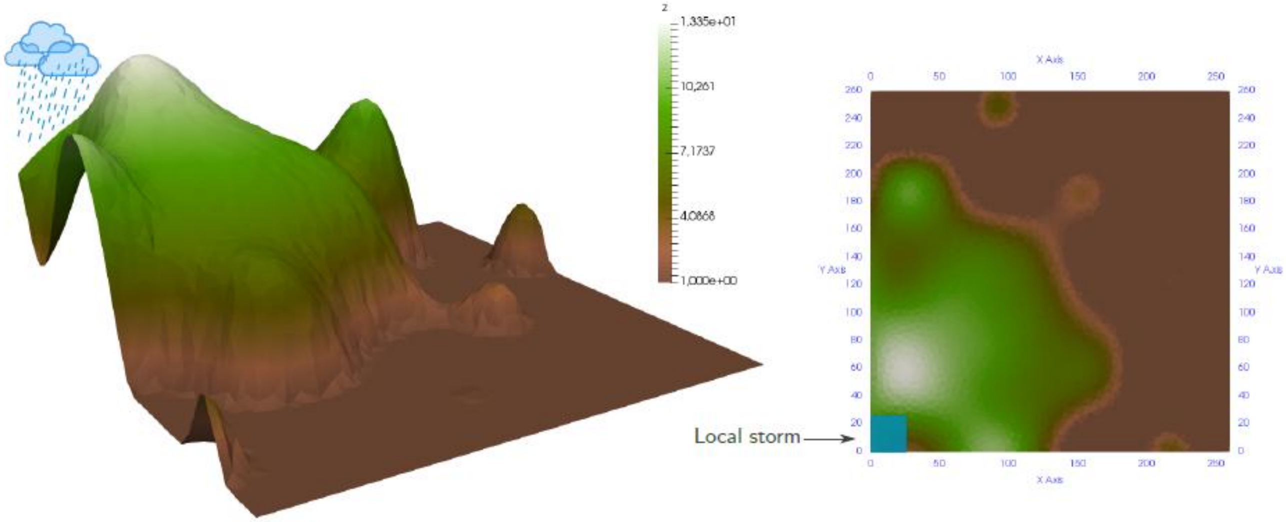

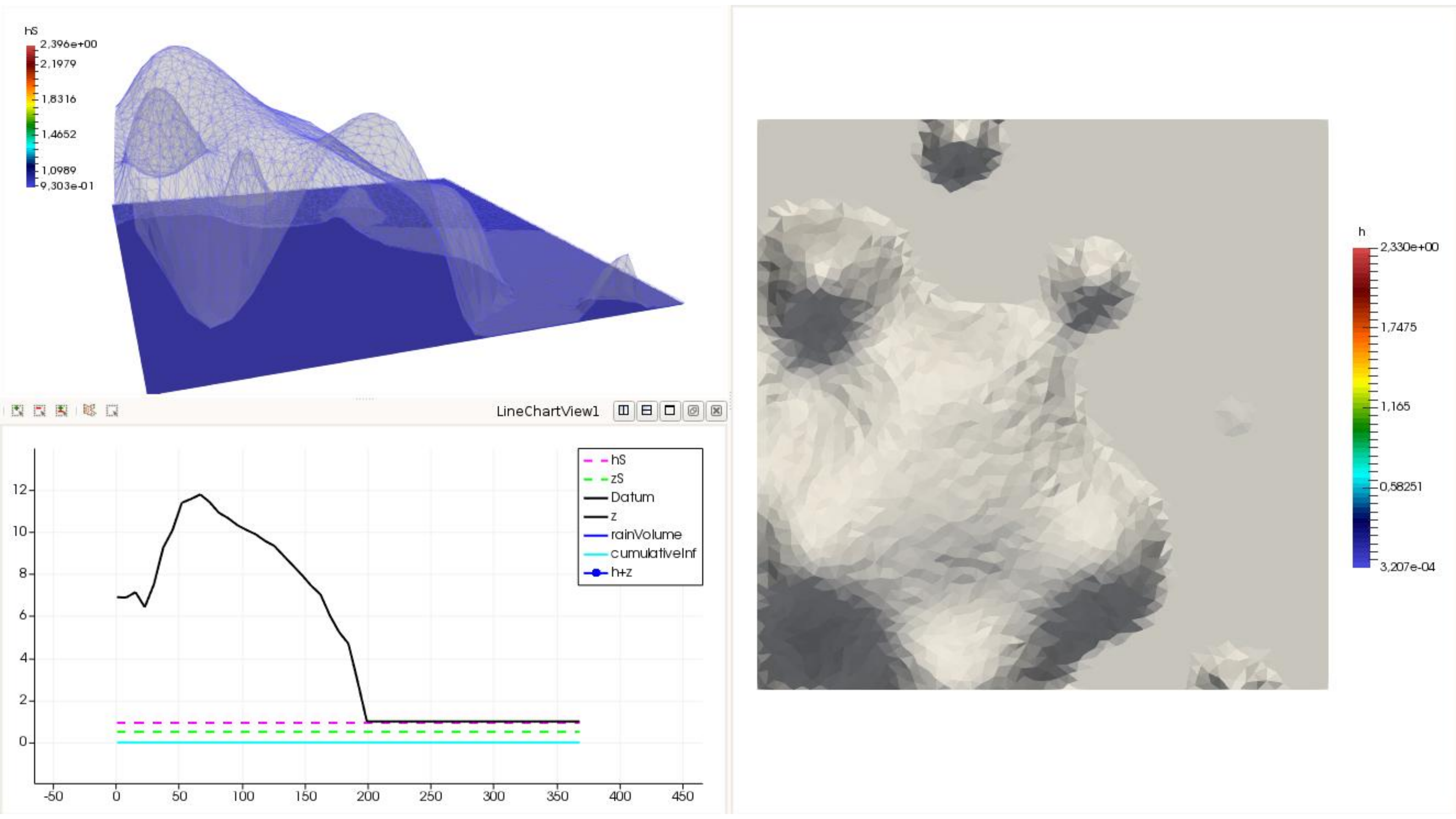

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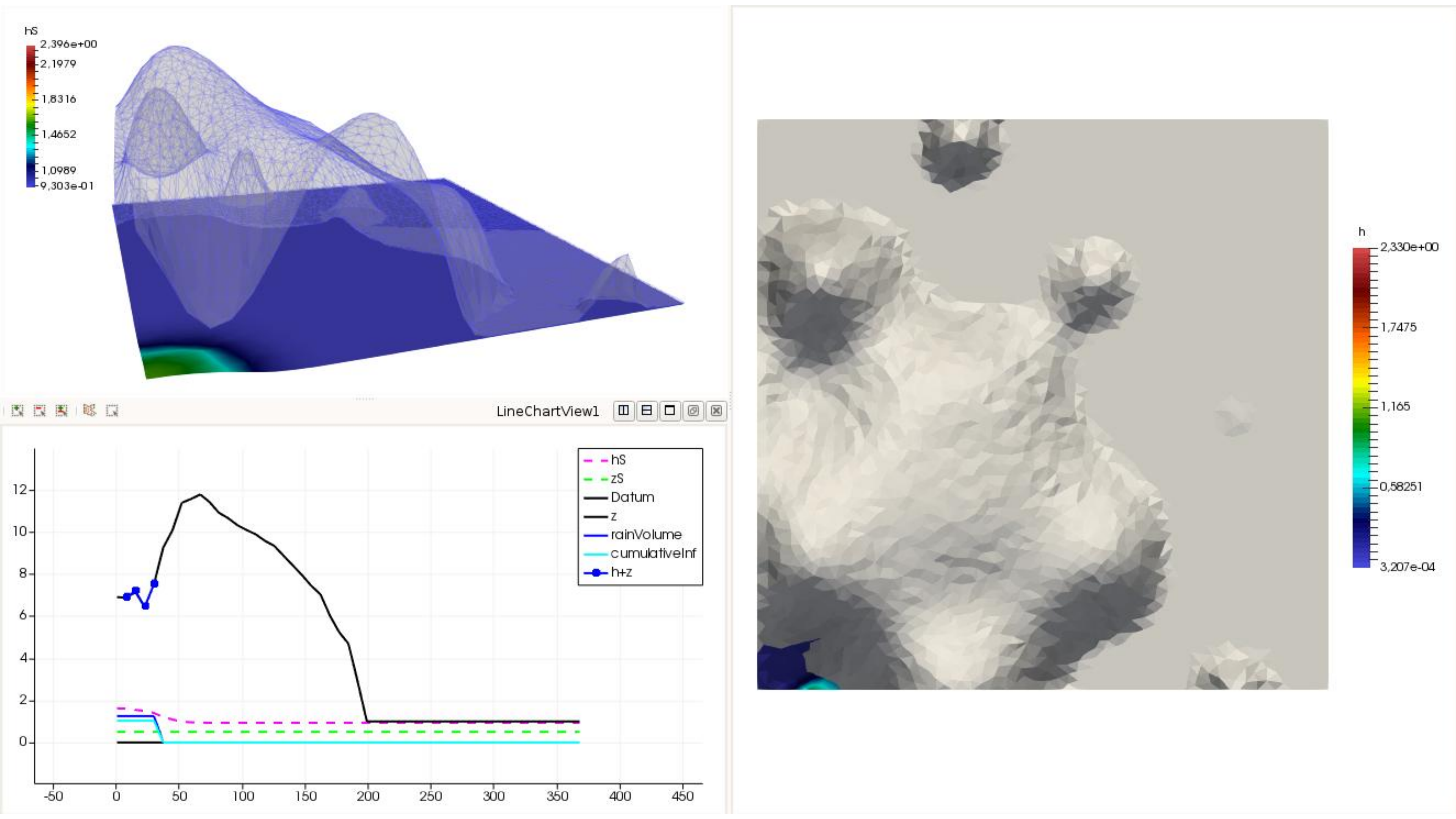
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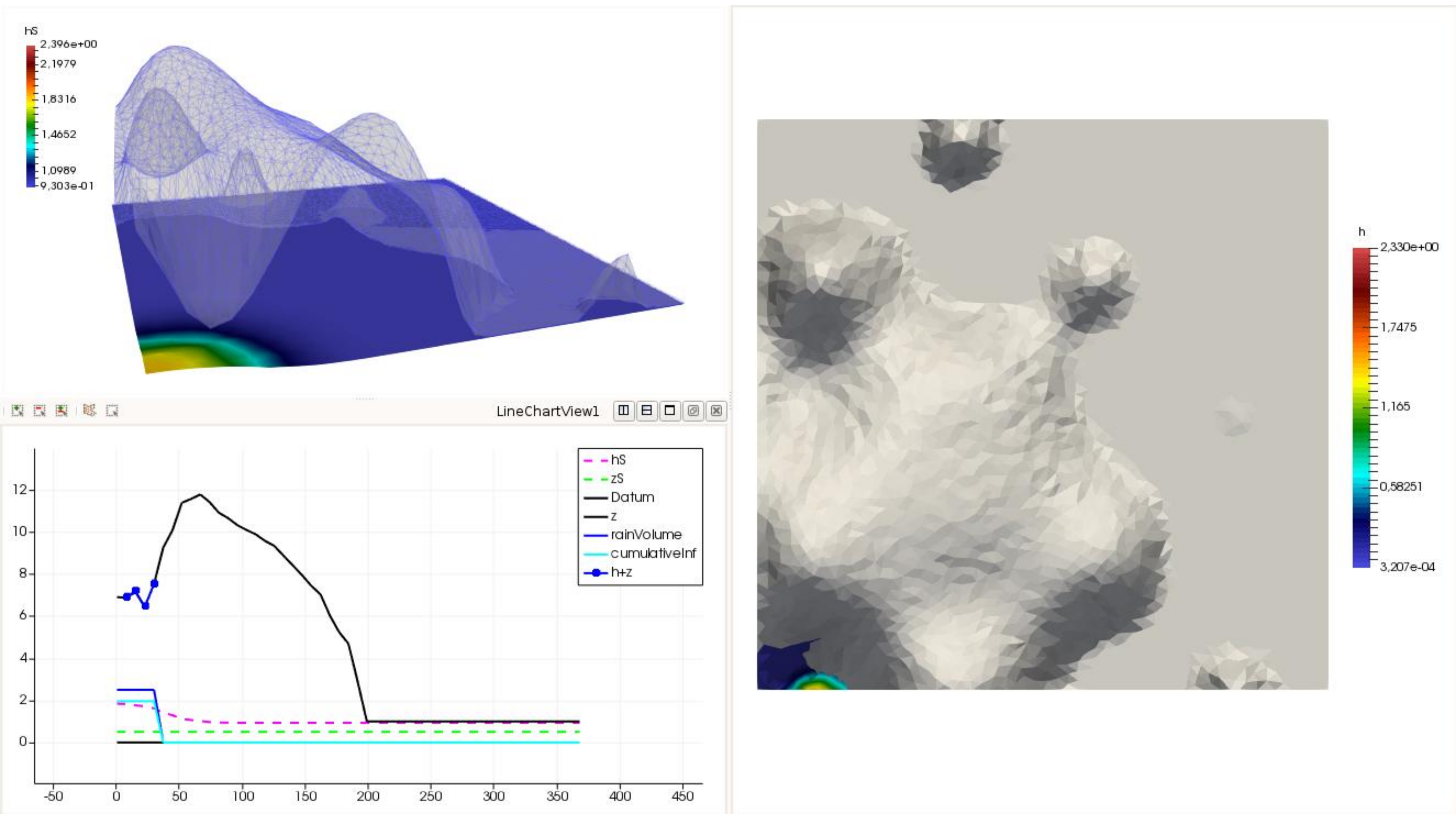
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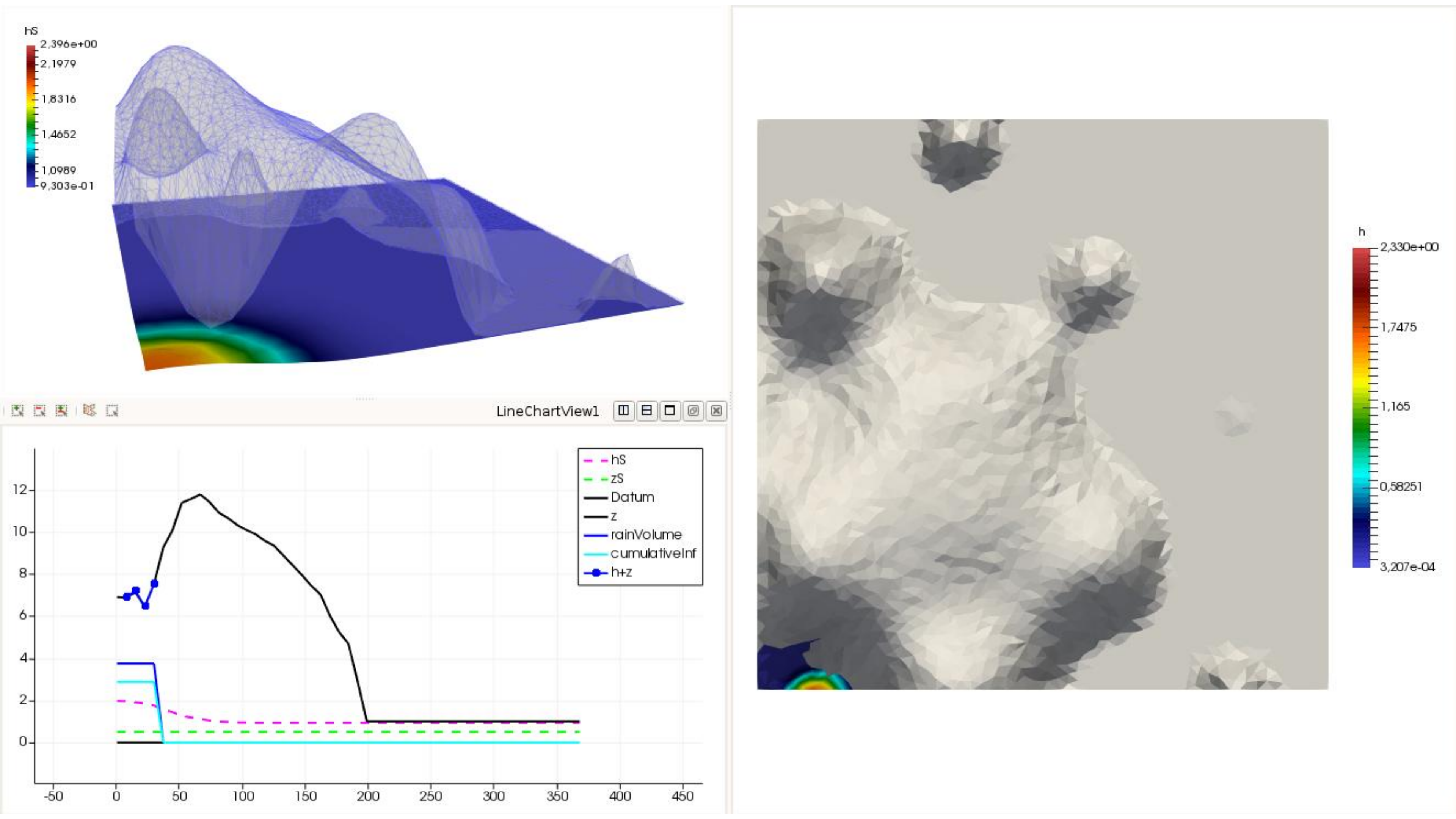

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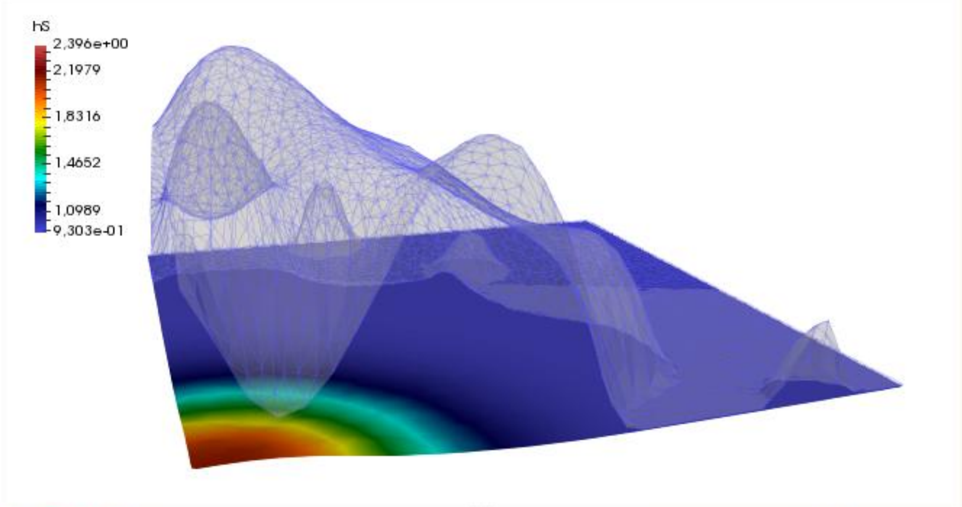
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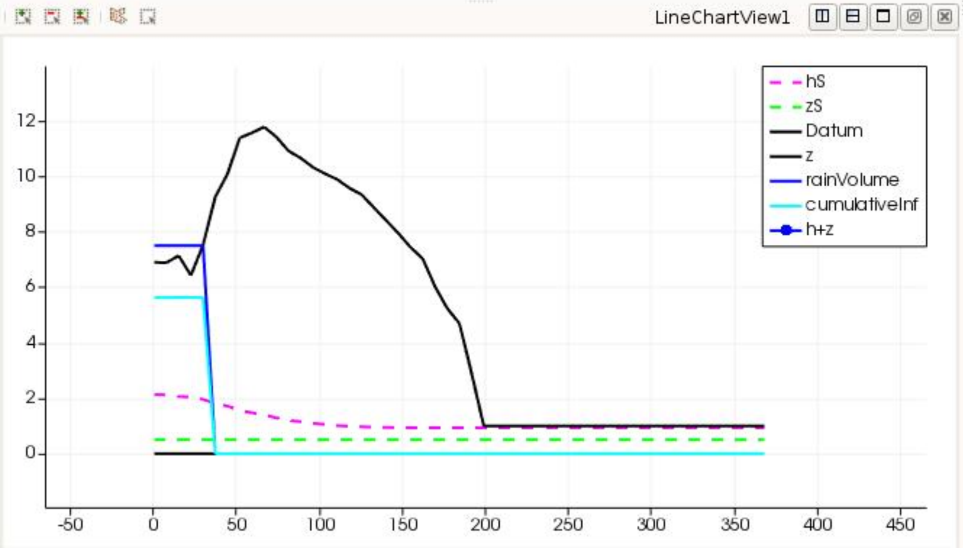
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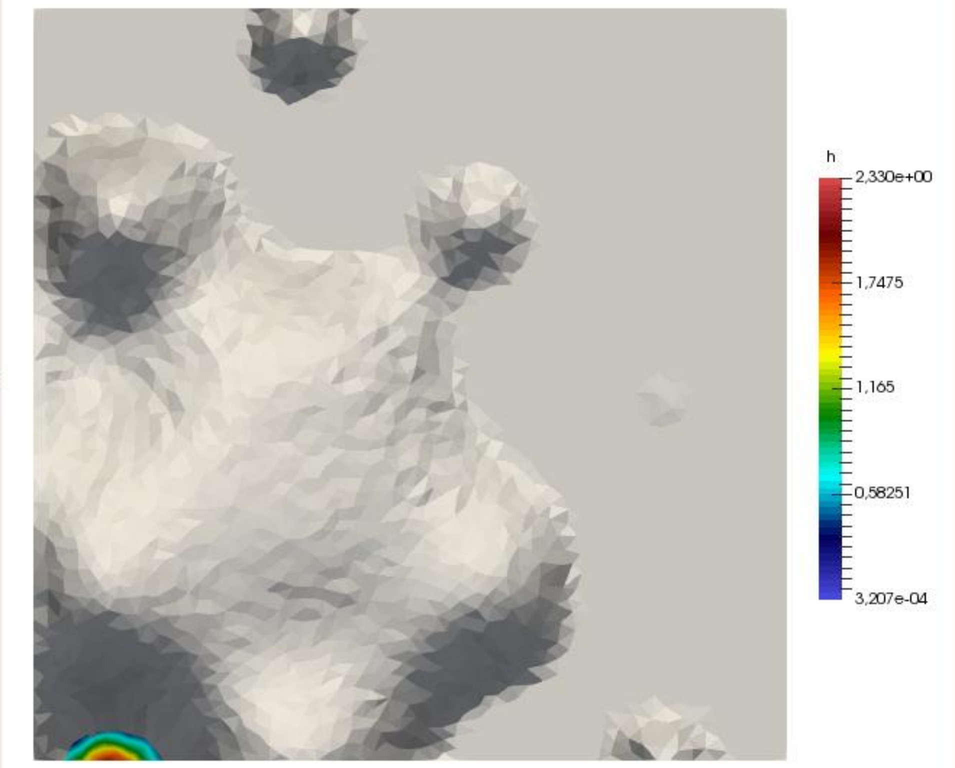
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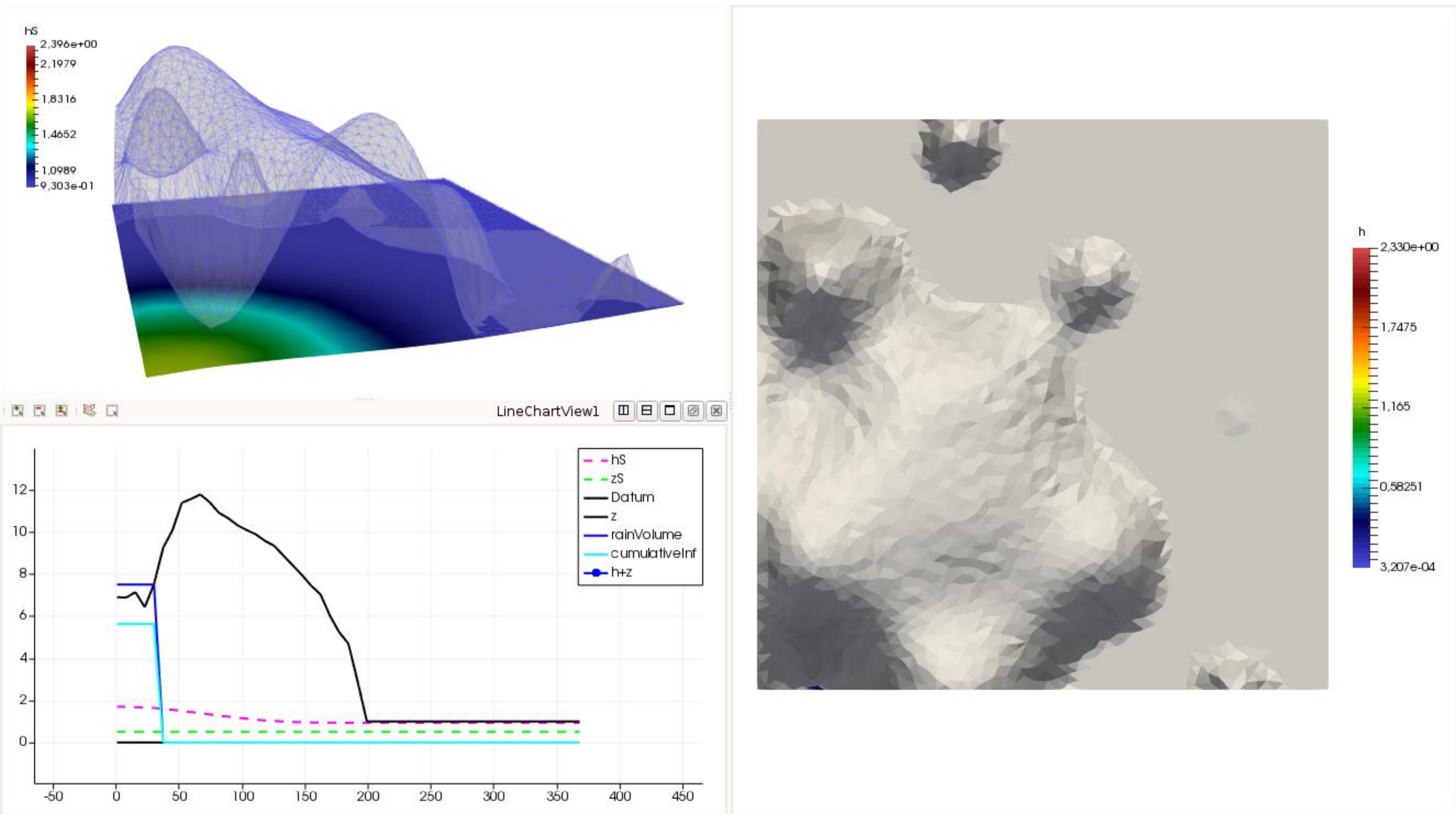
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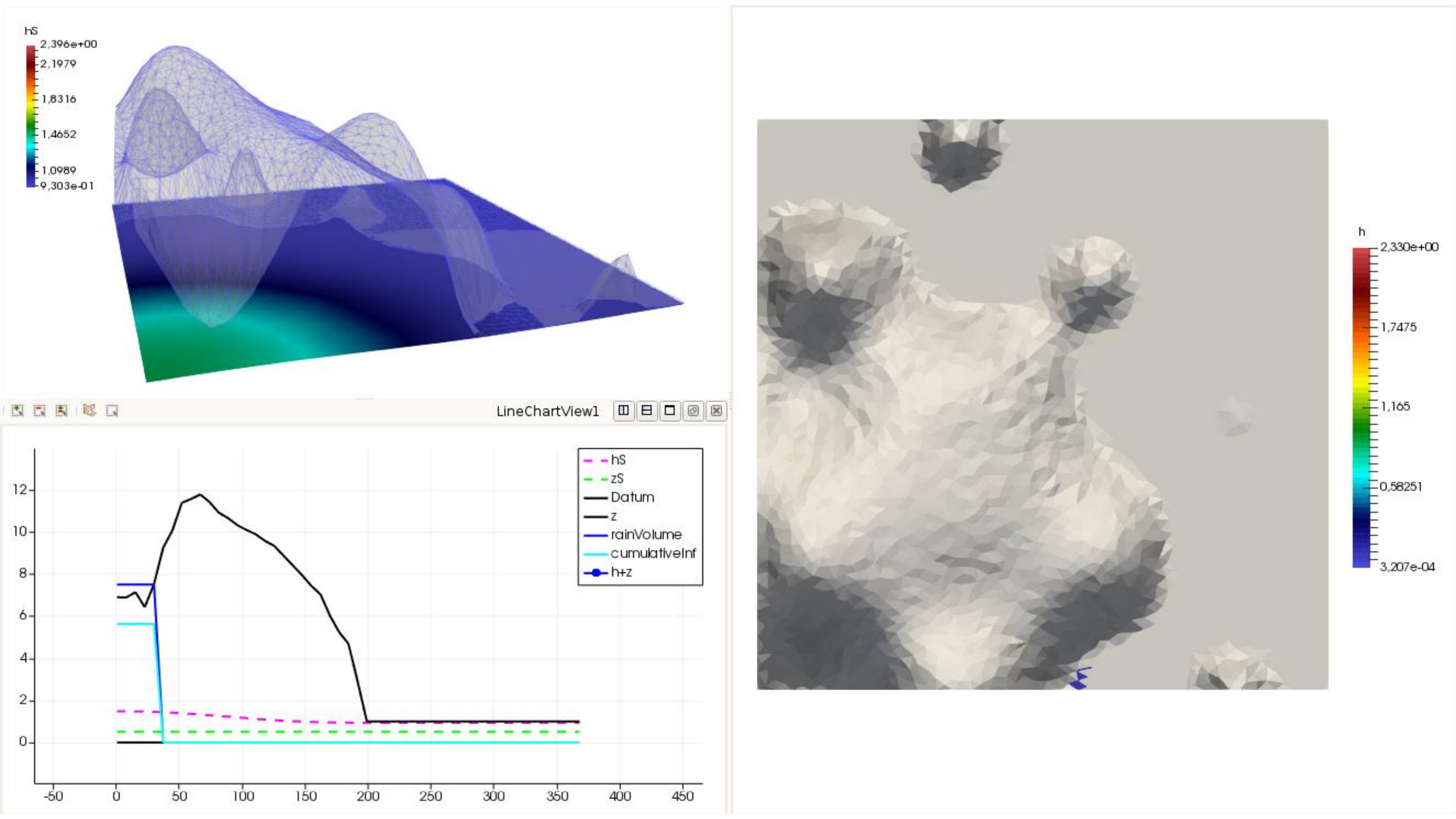
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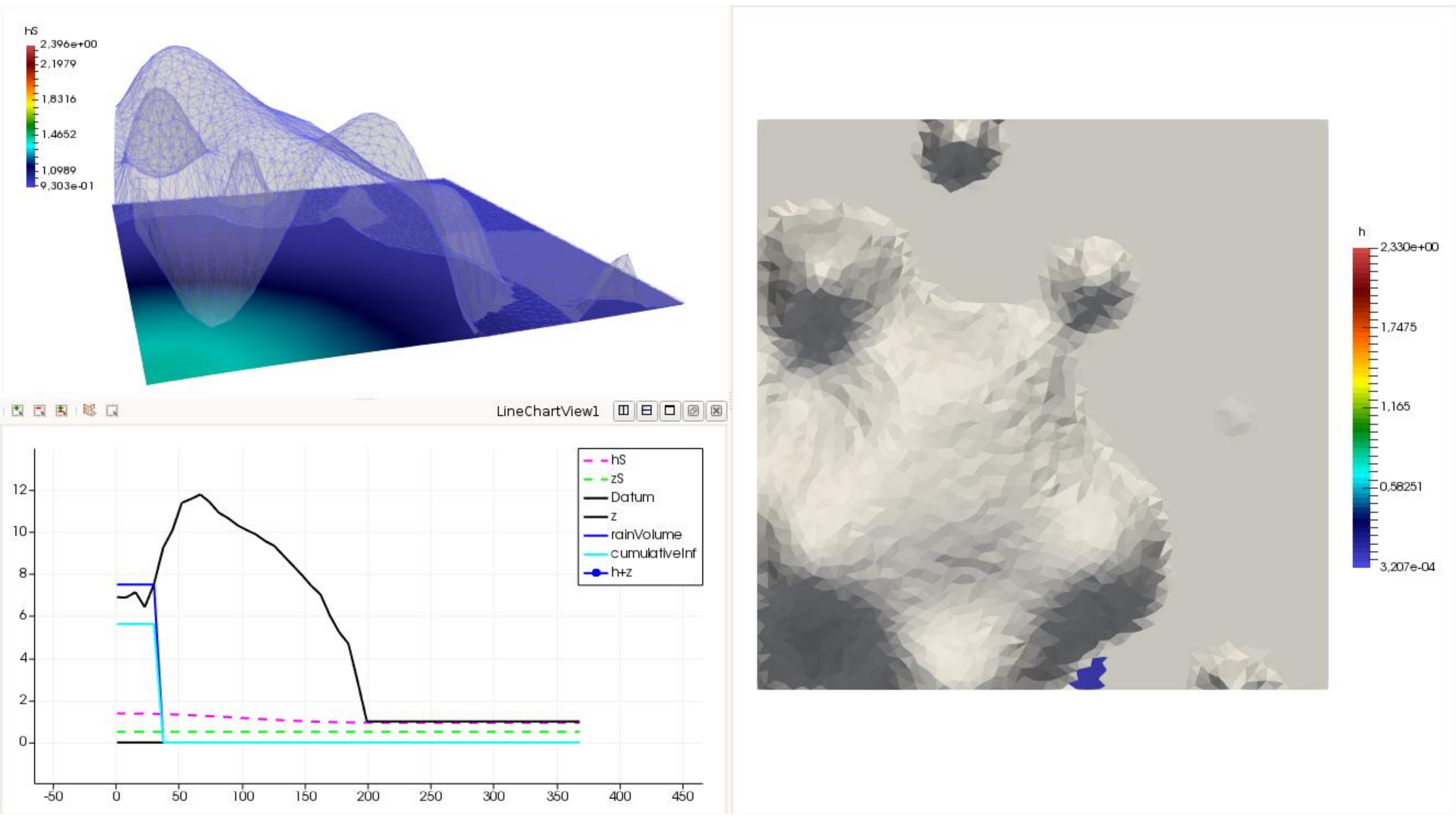
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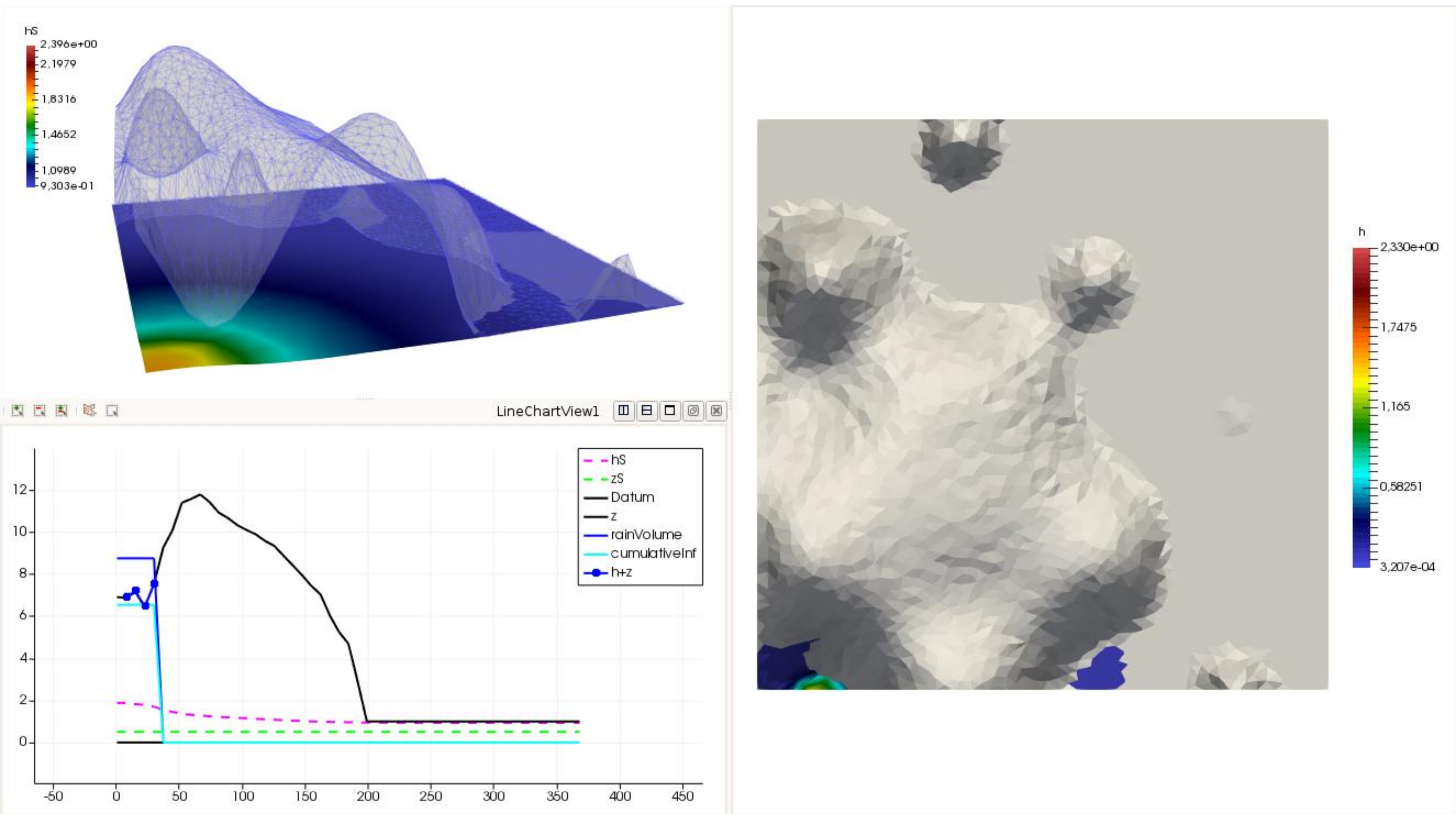
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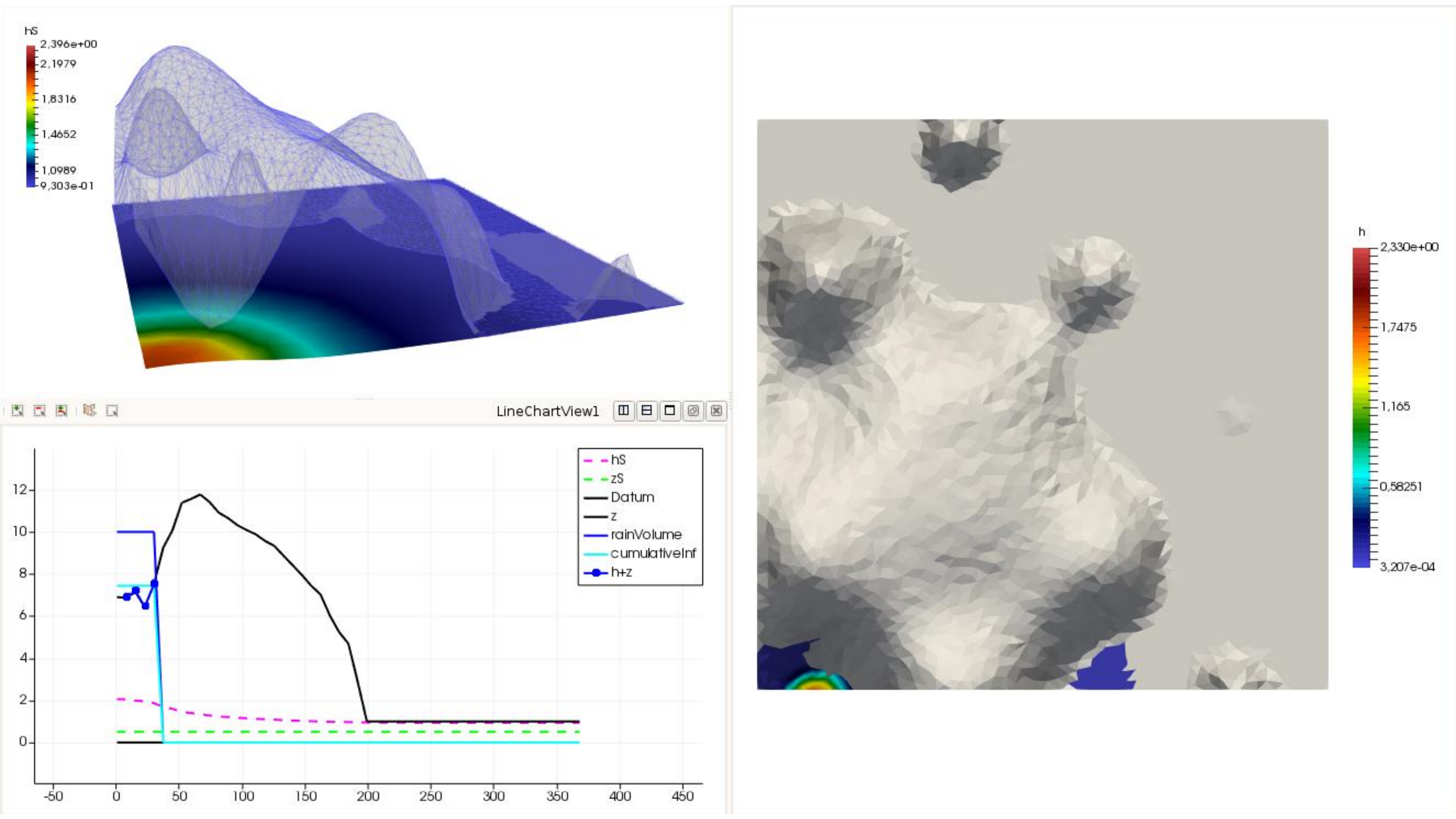

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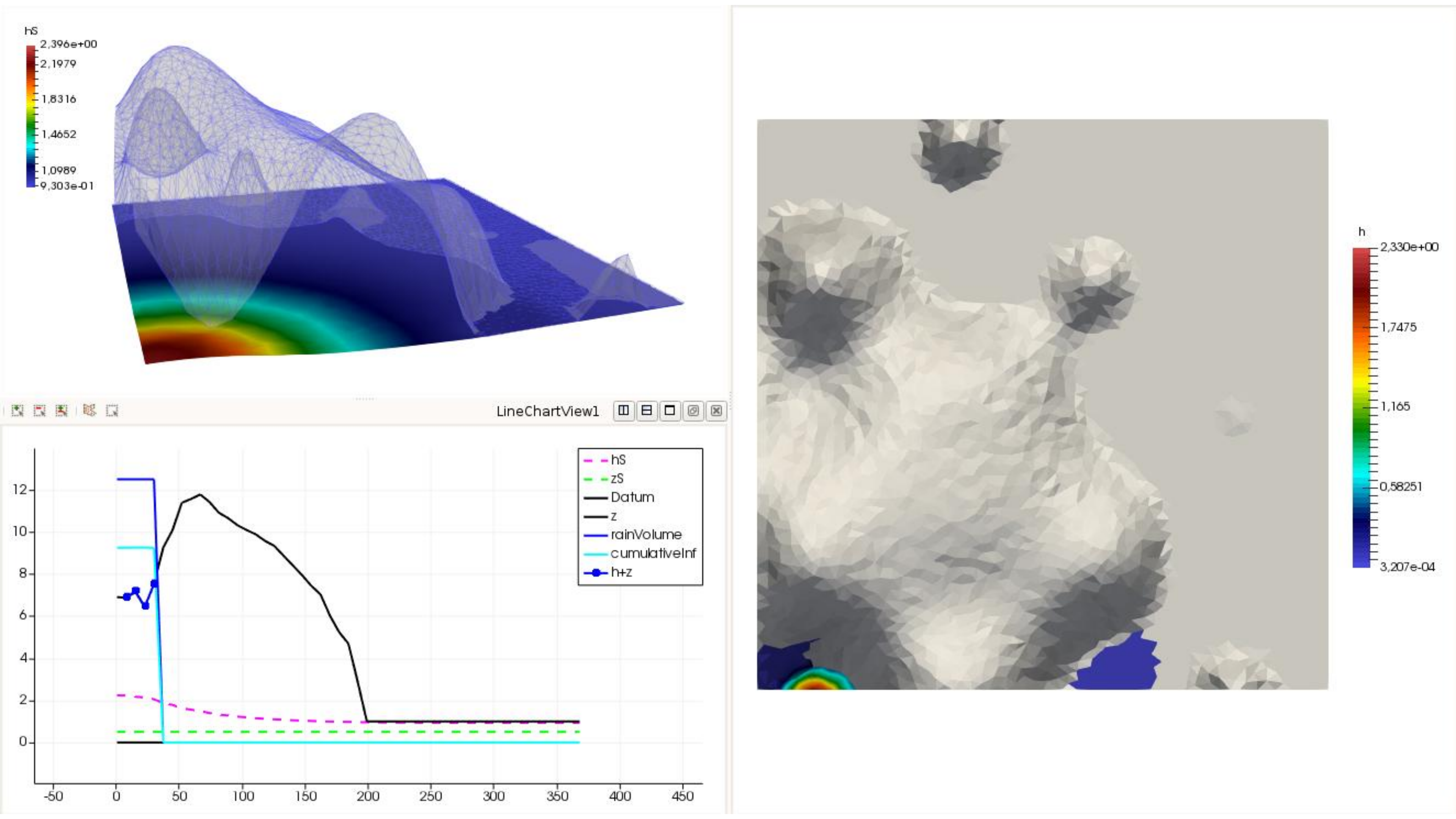
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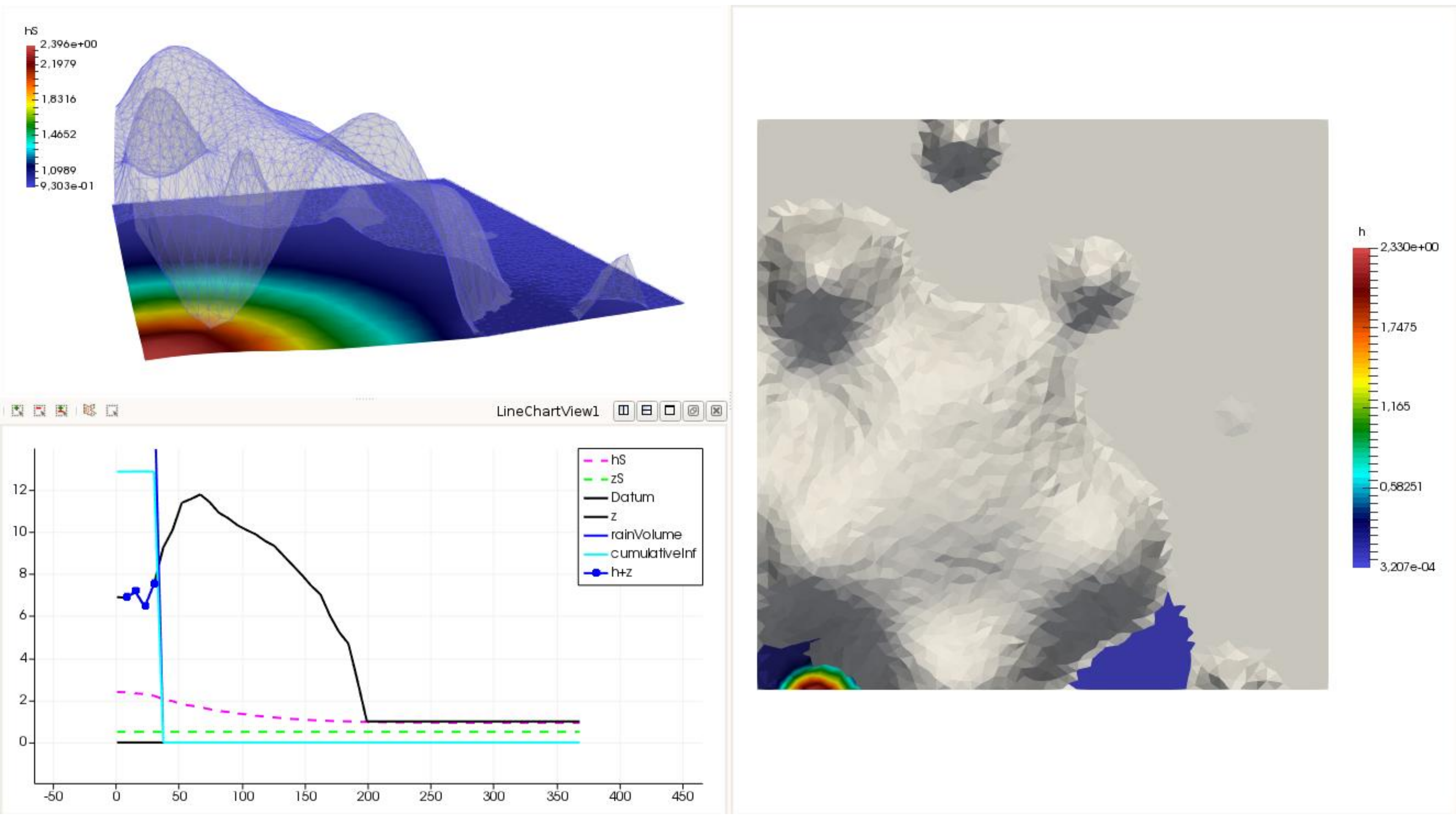

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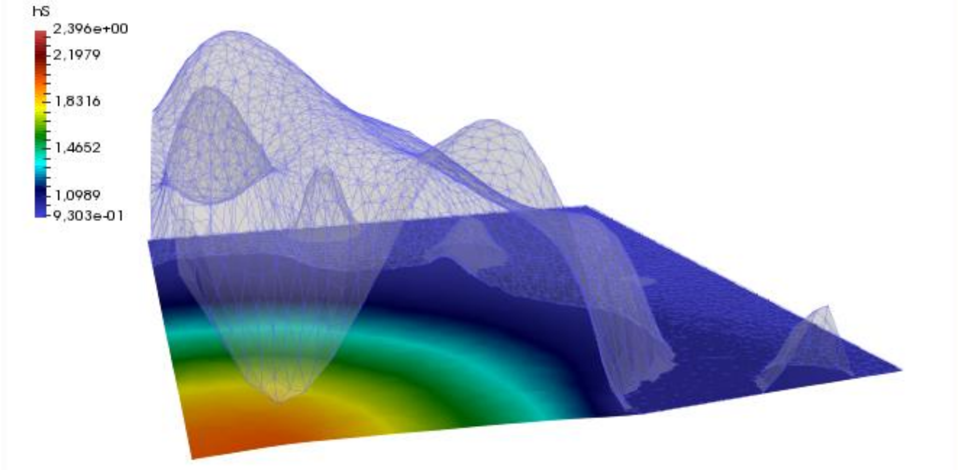
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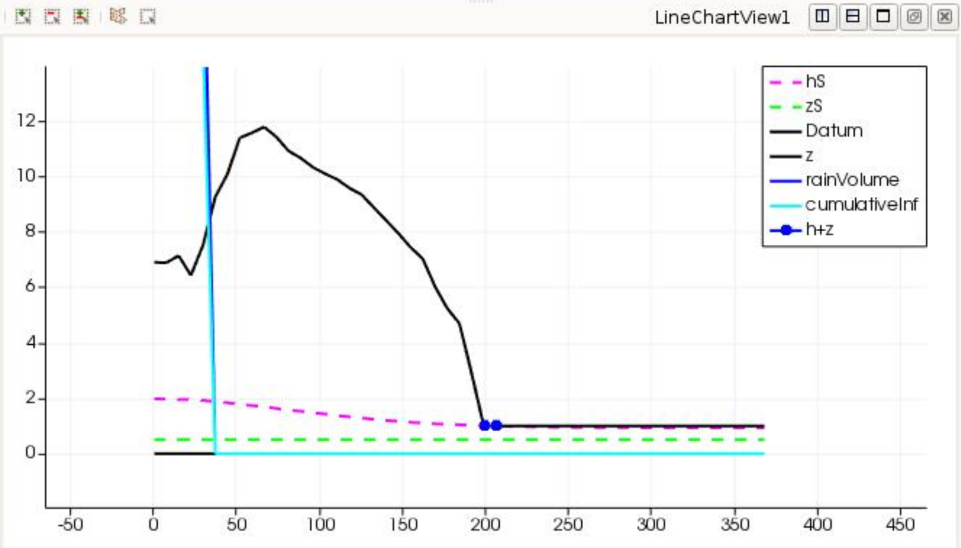
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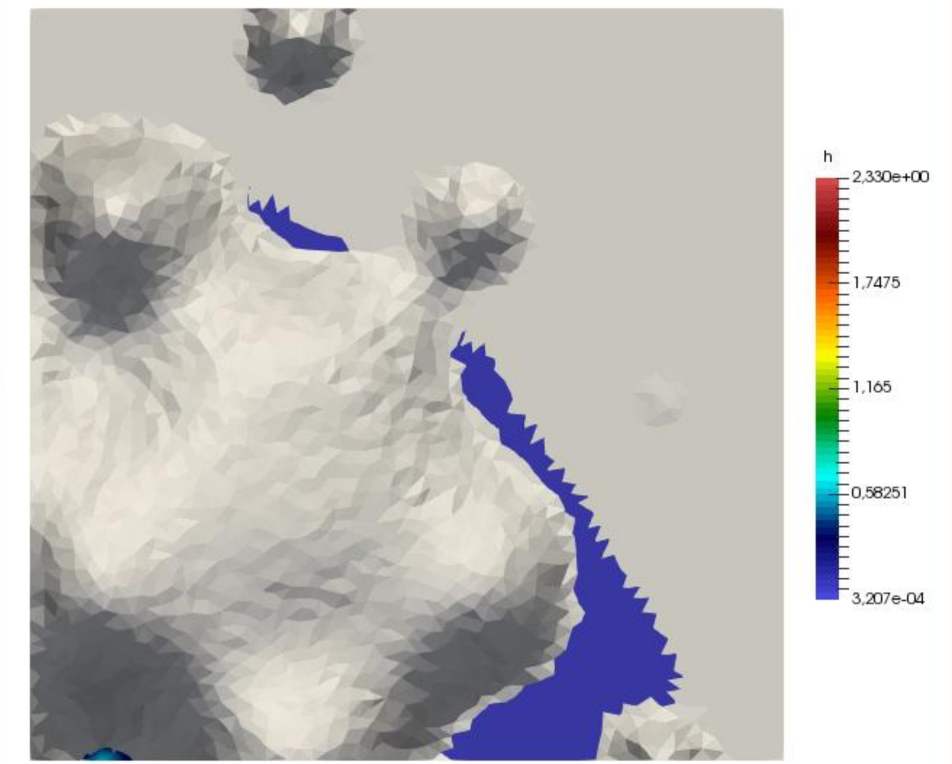
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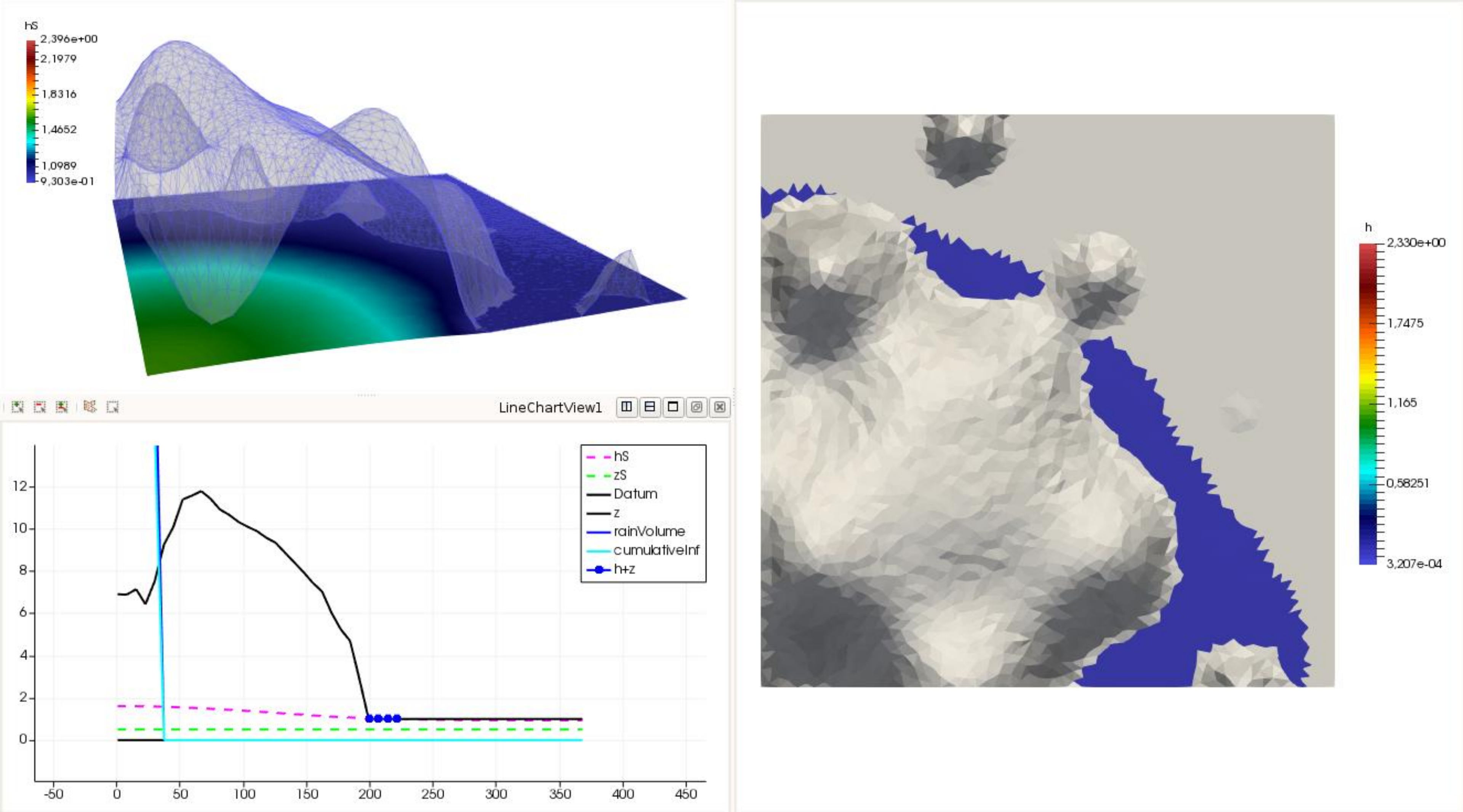
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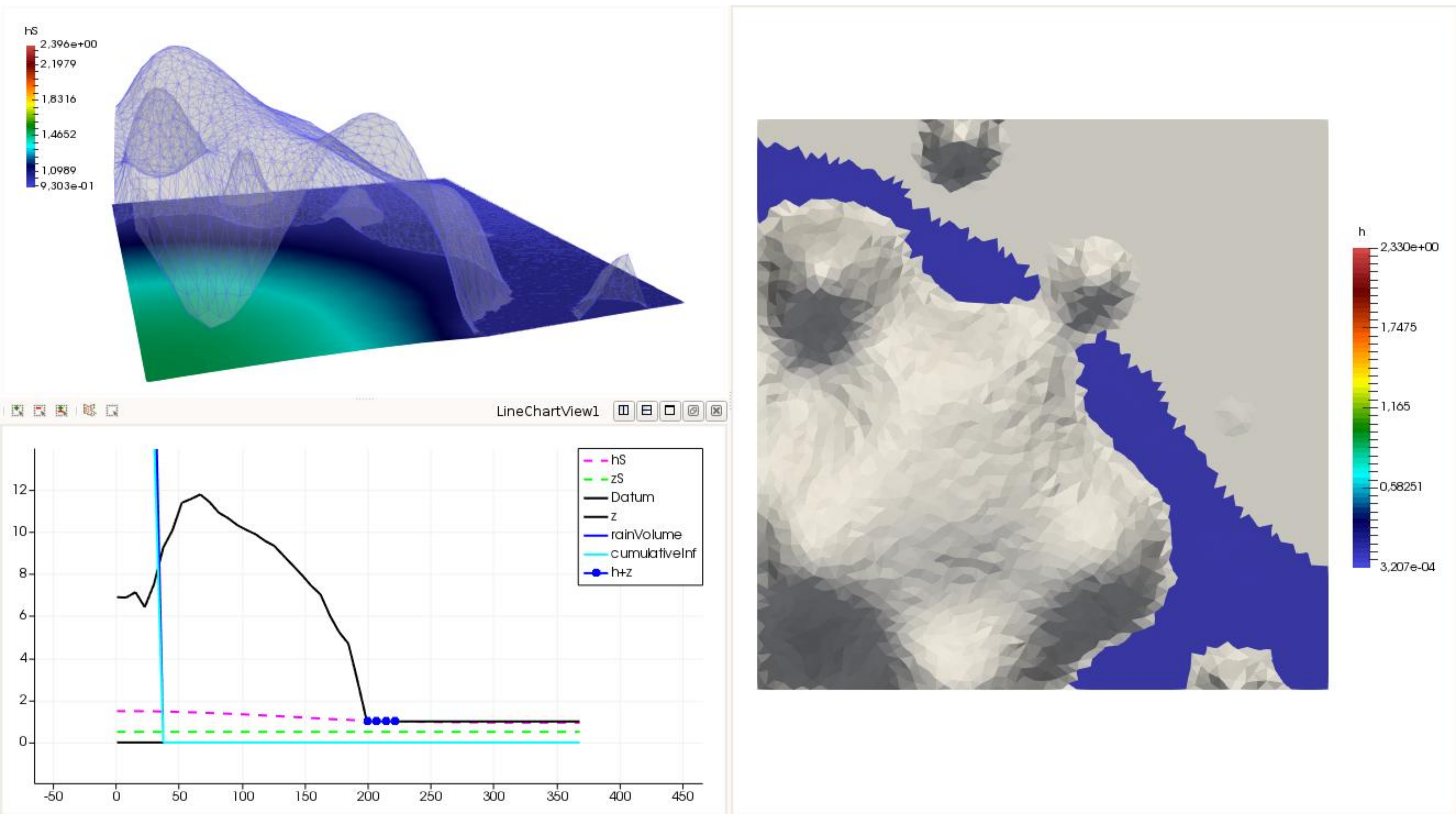
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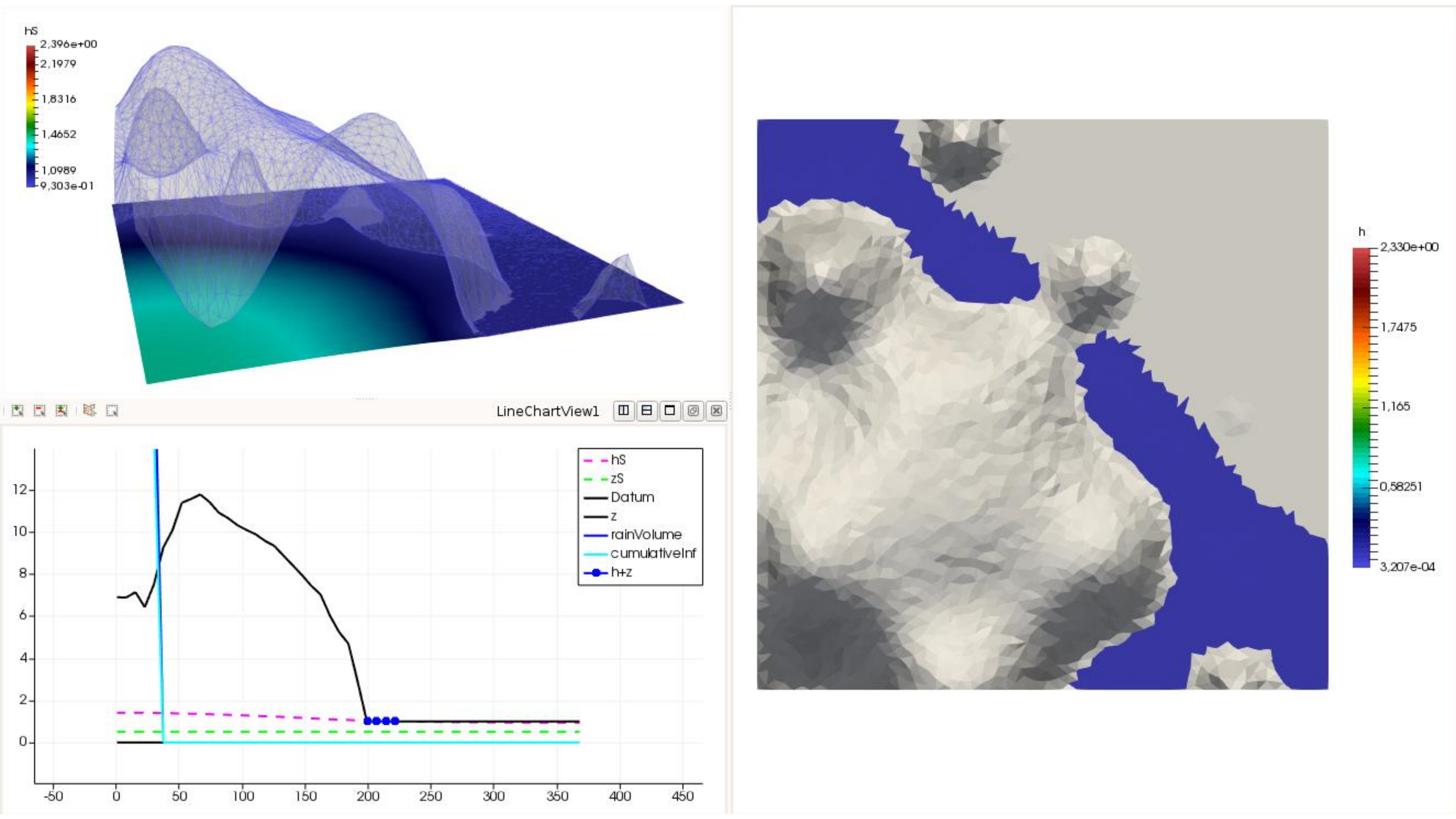

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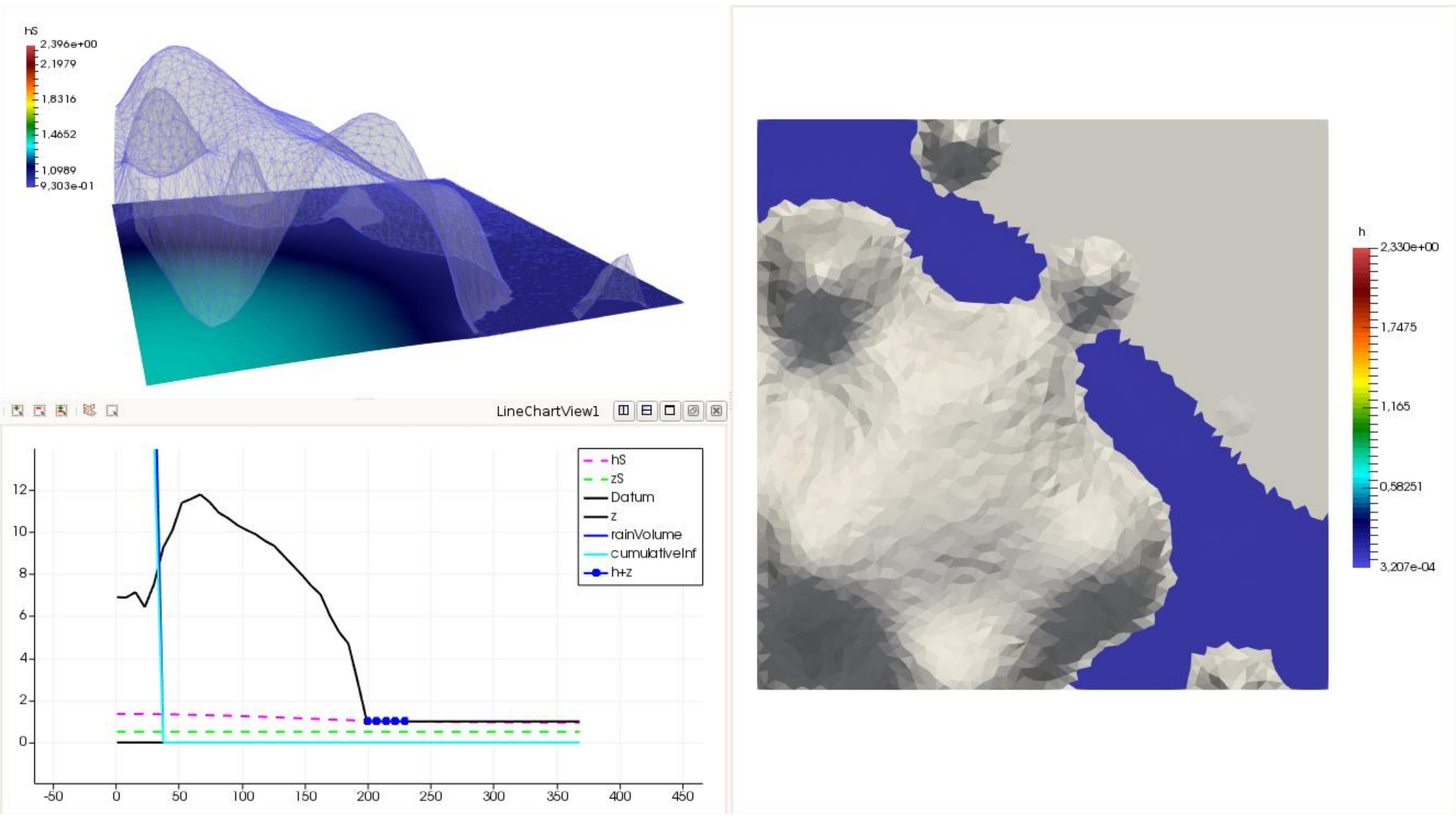
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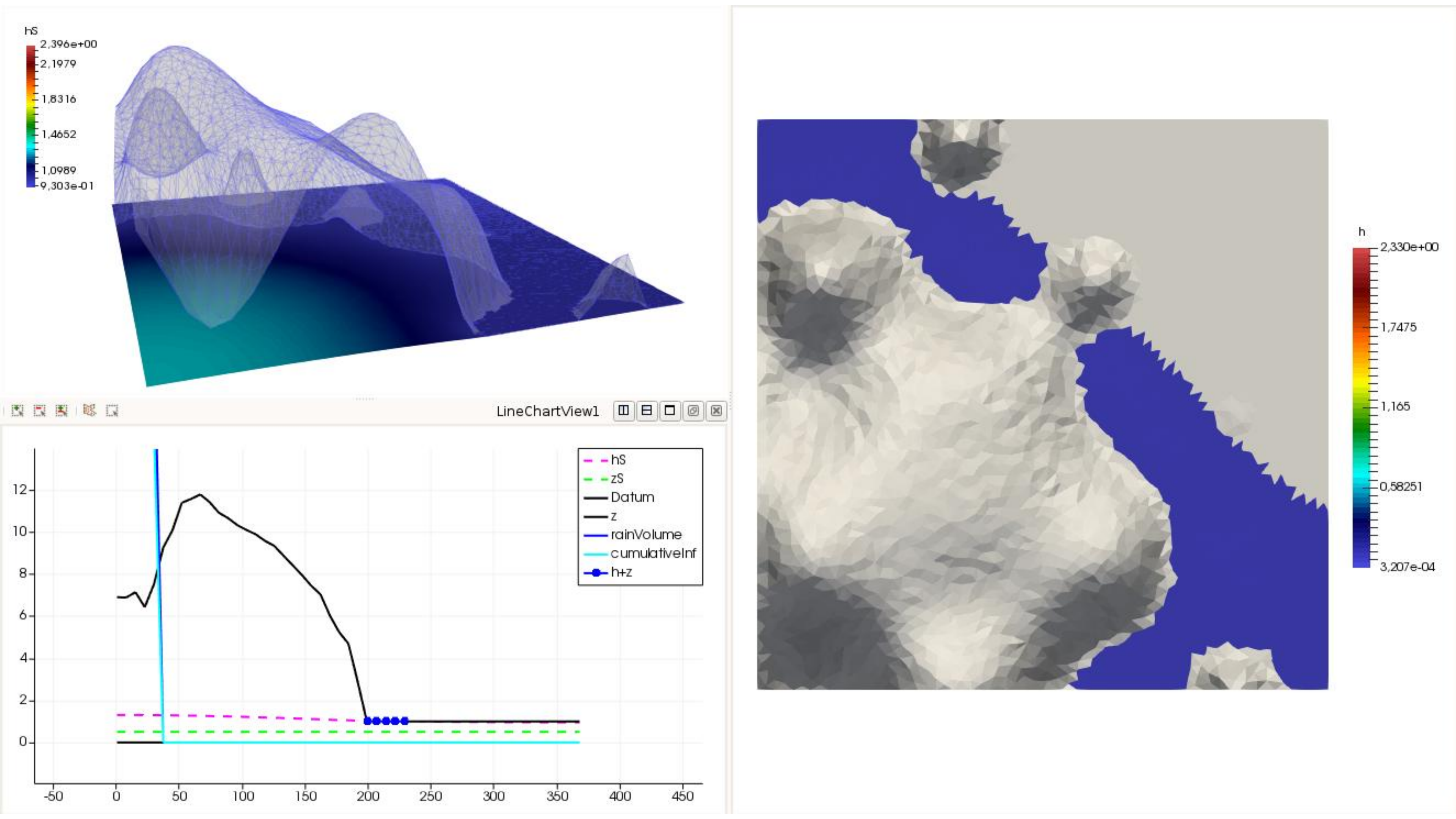

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Volcanic lava flow

2D Model of Lava Flow with Variable Density and Heat Exchange:
Application to the Cumbre Vieja Volcano in La Palma

Motivation

Before and (shortly) after.

19 September 2021

25 September 2021

How is lava flow?

Very complex flow and difficult to model:

- Physical properties highly dependent on the temperature of the lava (density, viscosity, yield, etc.)
- Variable topography: lava solidification, formation of lava tubes, etc.

How is lava flow?

Very complex flow and difficult to model:

- Physical properties highly dependent on the temperature of the lava (density, viscosity, yield, etc.)
- Variable topography: lava solidification, formation of lava tubes, etc.
- Effusion rate and temperature are very difficult to estimate accurately: $Q(t) = ?$ $T(t) = ?$
- Contributions from the explosive phase?
- Creeping

Mathematical model

Mathematical model

Conservation of mass:

$$\frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x}(\rho h u) + \frac{\partial}{\partial y}(\rho h v) = 0$$

We can no longer consider a constant density for the fluid.

Conservation of linear momentum in the x and y directions:

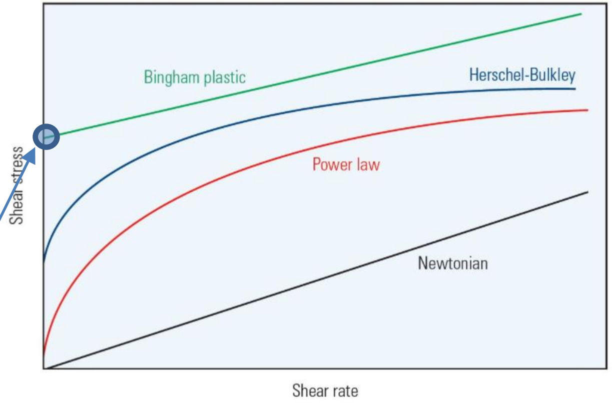
$$\frac{\partial(\rho h u)}{\partial t} + \frac{\partial}{\partial x}\left(\rho h u^2 + \frac{1}{2} g \rho h^2\right) + \frac{\partial}{\partial y}(\rho h u v) = -g \rho h \frac{\partial z_b}{\partial x} - \tau_{bx}$$

$$\frac{\partial(\rho h v)}{\partial t} + \frac{\partial}{\partial x}(\rho h u v) + \frac{\partial}{\partial y}\left(\rho h v^2 + \frac{1}{2} g \rho h^2\right) = -g \rho h \frac{\partial z_b}{\partial x} - \tau_{by}$$

For the shear stress, the Bingham model is used:

$$2\tau_b^3 - 3\left(\tau_y + 2\mu \frac{|\mathbf{u}|}{h}\right)\tau_b^2 + \tau_y^3 = 0$$

Yield stress (threshold value to overcome for the fluid to move)



Temperature equation

$$\frac{\partial(\rho h T)}{\partial t} + \boxed{\frac{\partial(\rho h u T)}{\partial x}} + \boxed{\frac{\partial(\rho h v T)}{\partial y}} = \boxed{-\frac{Q}{c_p}}$$

Heat exchange [J]

Specific heat for lava [J/kg/K]

The source term governs the evolution of the lava temperature, and the convective terms transport it according to the velocity field.

Mathematical model

Temperature equation:

$$\frac{\partial(\rho h T)}{\partial t} + \frac{\partial(\rho h u T)}{\partial x} + \frac{\partial(\rho h v T)}{\partial y} = -\frac{Q}{c_p} \longrightarrow$$ Radiation + Convection: $Q = \boxed{Q_{rad}} + \boxed{Q_{conv}}$

$$\boxed{Q_{rad} = \epsilon\sigma\left(T^4 - T_{air}^4\right)}$$ Stefan-Boltzmann law

$\epsilon \longrightarrow$ *Lava emissivity (we assume 0.55)*

$\sigma{=}5.6710^{-8}\; W/(m^2K^4) \longrightarrow$ *Stefan-Boltzmann constant*

$T_{air} \longrightarrow$ *Air temperature*

$$\boxed{Q_{conv} = h_c\left(T - T_{air}\right)}$$

$h_c[W/(m^2K)] \longrightarrow$ *Heat transfer convective coefficient (we asume a value of 25)*

$Nu = \frac{h_c L}{k}\left(T - T_{air}\right) \longrightarrow$ *Related to Nusselt number*

$L \longrightarrow$ *Characteristic length*

$k \longrightarrow$ *Thermal condutivity*

Mathematical model

System of differential equations:

$$\frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x}(\rho h u) + \frac{\partial}{\partial y}(\rho h v) = 0$$

$$\frac{\partial(\rho h u)}{\partial t} + \frac{\partial}{\partial x}(\rho h u^2 + \frac{1}{2} g \rho h^2) + \frac{\partial}{\partial y}(\rho h u v) = -g\rho h \frac{\partial z_b}{\partial x} - \boxed{\tau_{bx}}$$

$$\frac{\partial(\rho h v)}{\partial t} + \frac{\partial}{\partial x}(\rho h u v) + \frac{\partial}{\partial y}(\rho h v^2 + \frac{1}{2} g \rho h^2) = -g\rho h \frac{\partial z_b}{\partial x} - \boxed{\tau_{by}}$$

$$\boxed{2\tau_b^3 - 3\left(\tau_y + 2\mu\frac{|\mathbf{u}|}{h}\right)\tau_b^2 + \tau_y^3 = 0}$$

$$\frac{\partial(\rho h T)}{\partial t} + \frac{\partial(\rho h u T)}{\partial x} + \frac{\partial(\rho h v T)}{\partial y} = -\frac{\boxed{Q}}{c_p}$$

$$\boxed{Q_{rad} = \epsilon\sigma\left(T^4 - T_{air}^4\right) \qquad Q_{conv} = h_c\left(T - T_{air}\right)}$$

Closure equation to relate density and temperature:

$$\rho(T) = \rho_0 + K(T - T_0)$$

Reference density

Fitting parameter

Temperature also affects viscosity and yield stress:

$$\mu(T) = A_\mu e^{B_\mu T} \qquad \tau_y(T) = A_y e^{B_y T}$$

Mathematical model

System of differential equations:

$$\frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x}(\rho h u) + \frac{\partial}{\partial y}(\rho h v) = 0$$

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$$\boxed{2\tau_b^3 - 3\left(\tau_y + 2\mu \frac{|\mathbf{u}|}{h}\right)\tau_b^2 + \tau_y^3 = 0}$$

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$$\boxed{Q_{rad} = \epsilon\sigma\left(T^4 - T_{air}^4\right) \qquad Q_{conv} = h_c\left(T - T_{air}\right)}$$

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$$\mu(T) = \boxed{A_\mu} e^{\boxed{B_\mu} T} \qquad \tau_y(T) = \boxed{A_y} e^{\boxed{B_y} T}$$

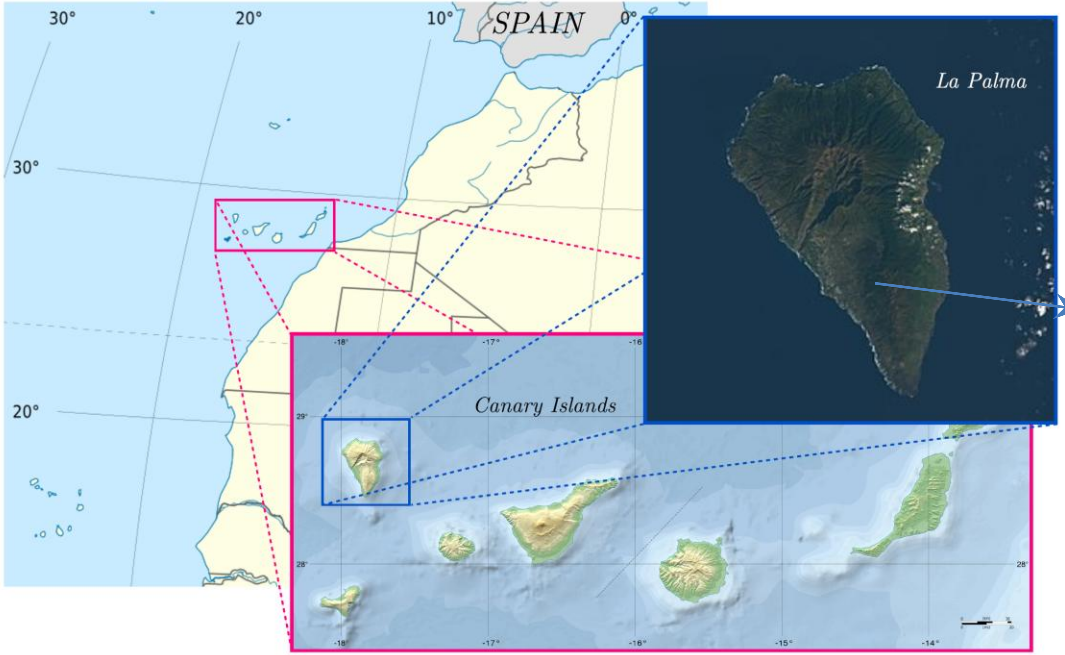
5 degrees of freedom to calibrate the model

Numerical results

Numerical results

Case description

Location of La Palma island



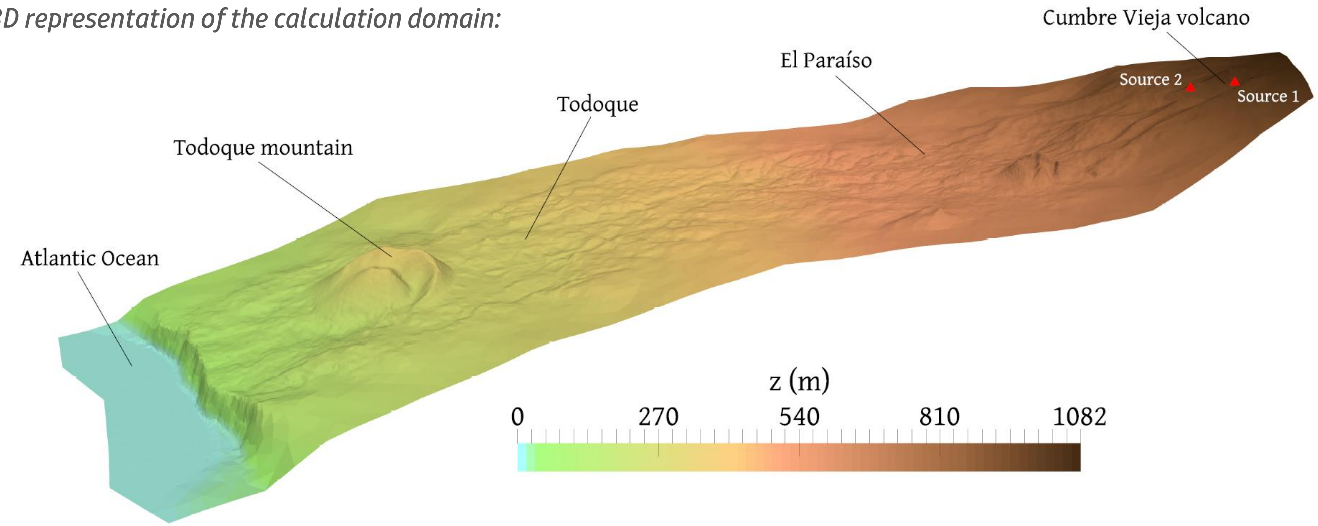
State of the lava flow as of September 30, 2021:

Aim of the first phase of the study

Numerical results

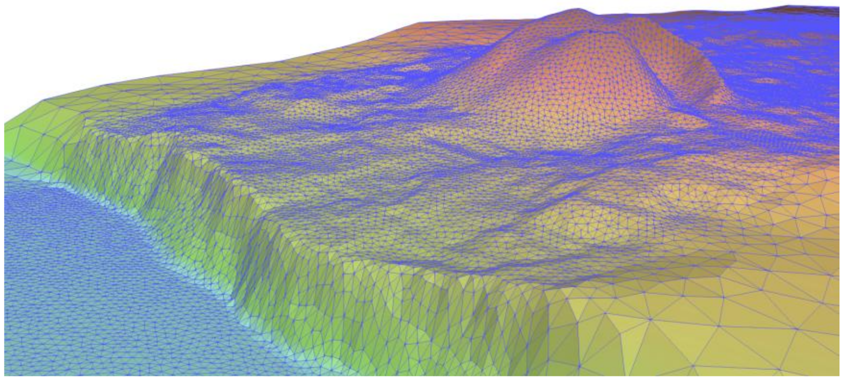
Case description

3D representation of the calculation domain:



Spatial discretization:

- *108.167 triangular cells mesh*
- *Local refinement along observed lava flow path*



Numerical results

Case description

Estimation of the volcano's effusive flow rate:

- *It is estimated that by September 27, the volcano had expelled a lava volume of **V=46.3 million m³**.*
- *With this data and the area of the lava flow corresponding to that day, an average depth is estimated.*

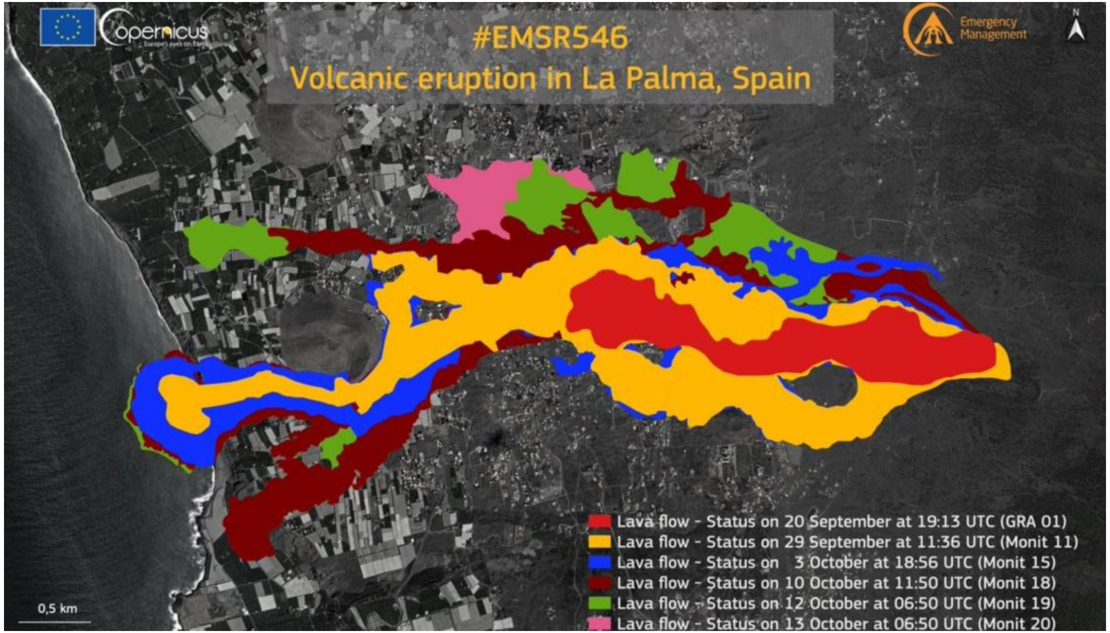
$$\bar{h} = \frac{V}{A}$$

- *With the average depth, the approximate volume of the remaining lava flows is calculated:*

$$V_i = \bar{h} A_i$$

- *By dividing the difference in volumes between two consecutive spots by the time elapsed between them, an estimate of the effusive flow rate is obtained*

$$Q(t) = \frac{V_{i+1} - V_i}{t_{i+1} - t_i}$$



Numerical results

Case description

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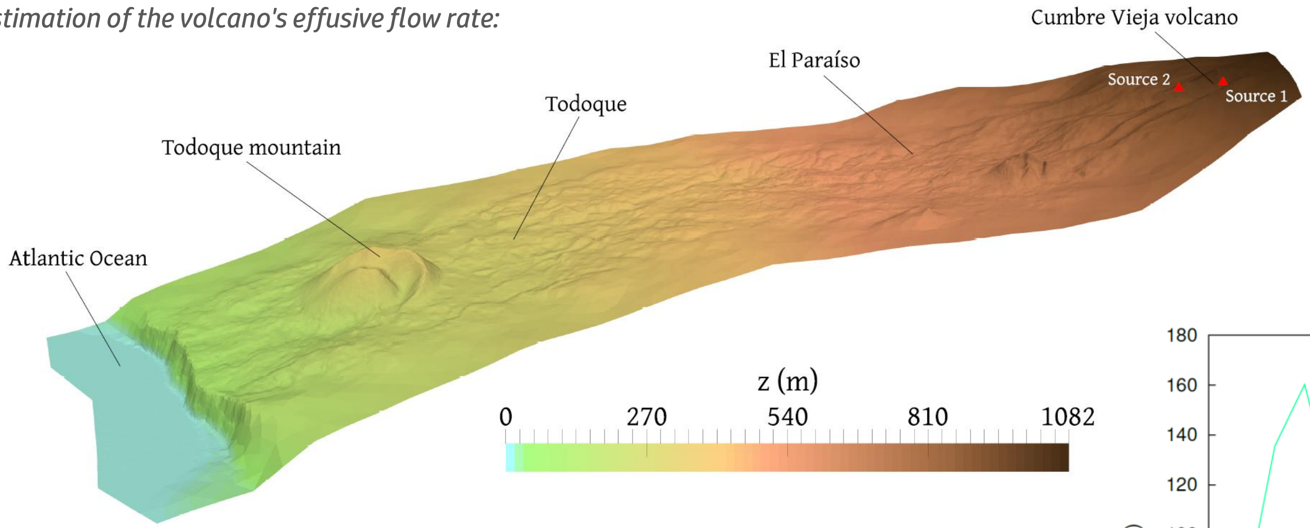
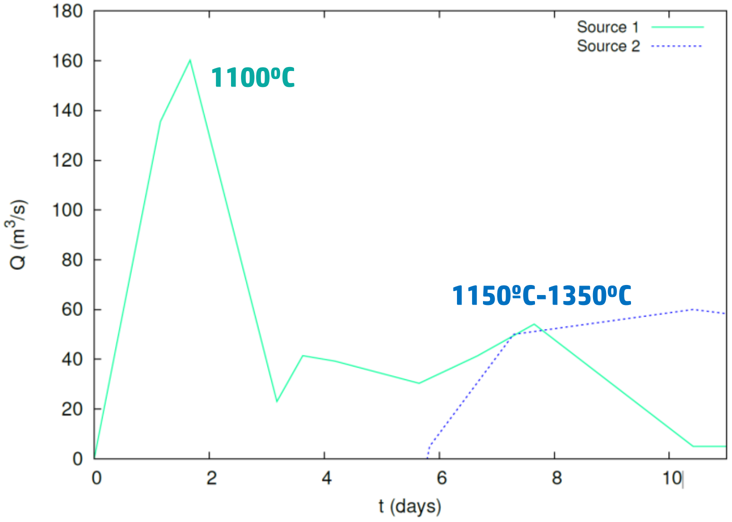
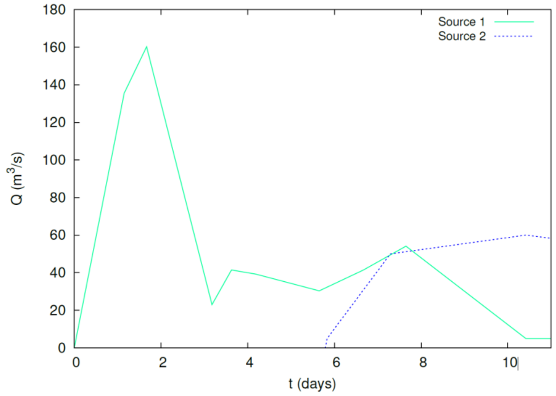


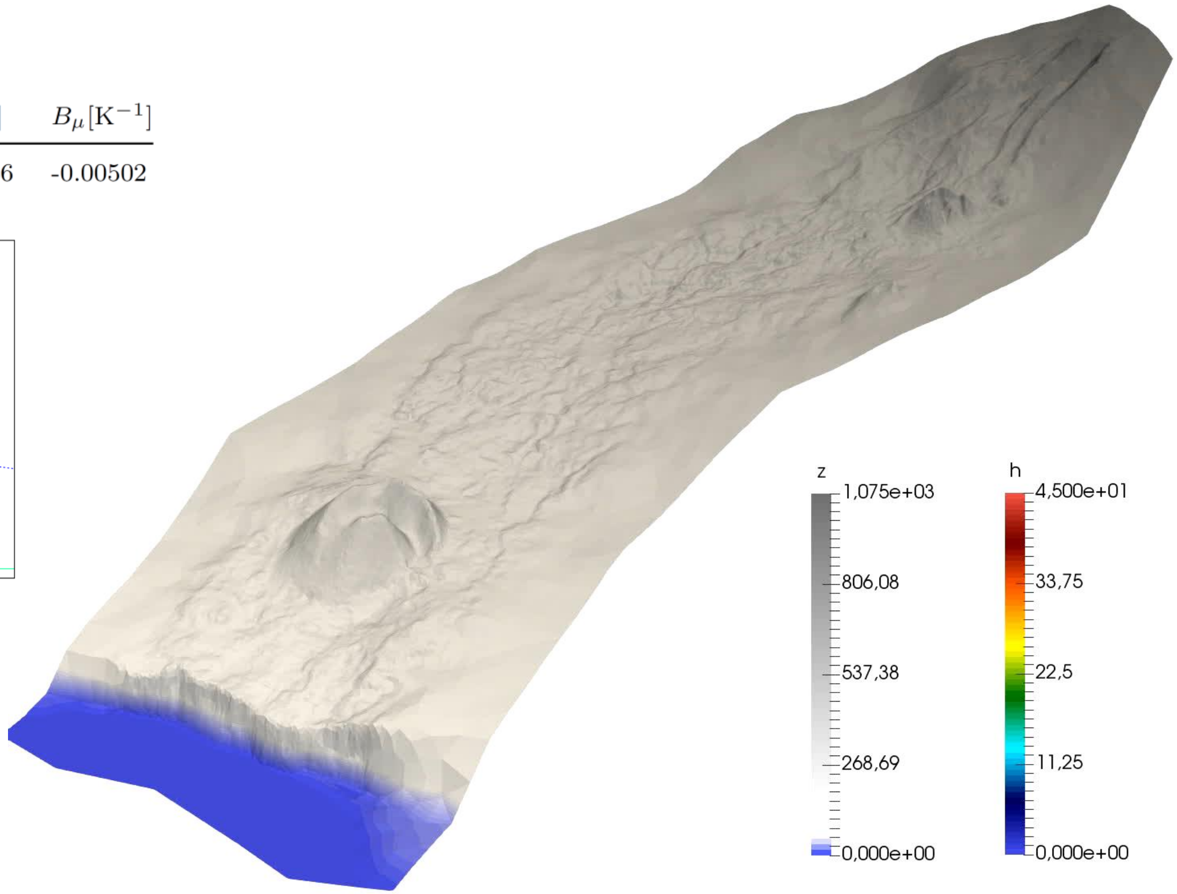


Numerical results

Calibration of the physical properties of lava

$K\,[\mathrm{kg/m^3\ K}]$	$A_y\,[\mathrm{Pa}]$	$B_y\,[\mathrm{K^{-1}}]$	$A_\mu\,[\mathrm{Pa\,s}]$	$B_\mu\,[\mathrm{K^{-1}}]$
-0.1	$2.5\cdot 10^6$	-0.00429	20000336	-0.00502





Numerical results

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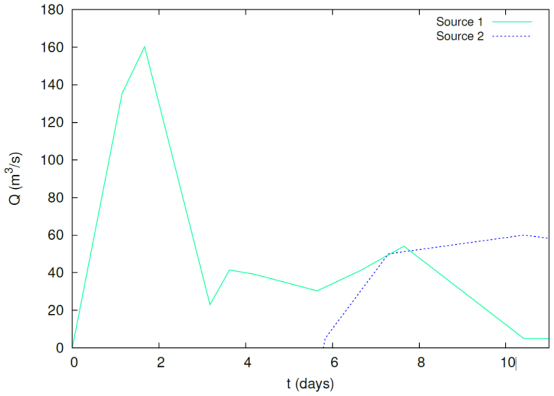

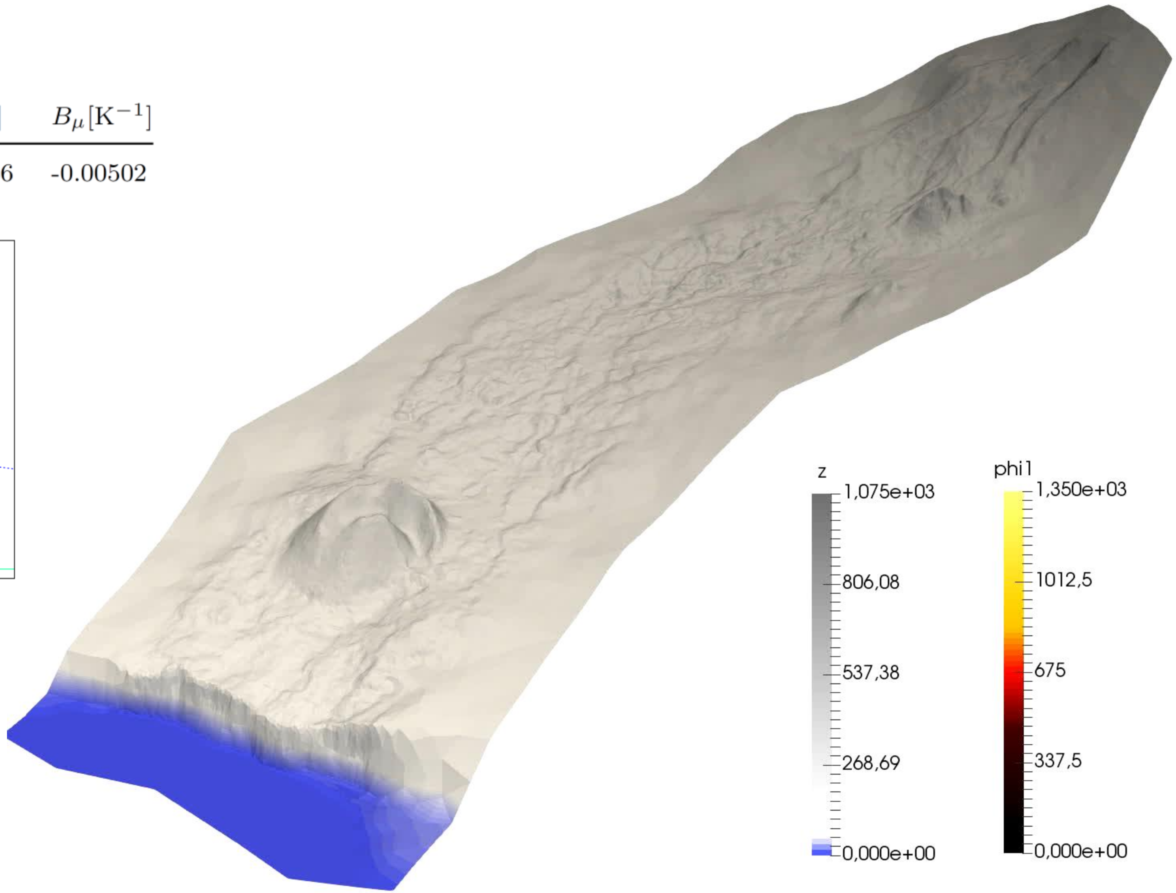

Numerical results

Calibration of the physical properties of lava

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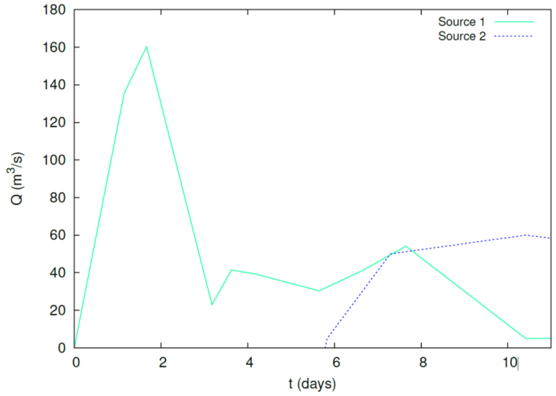
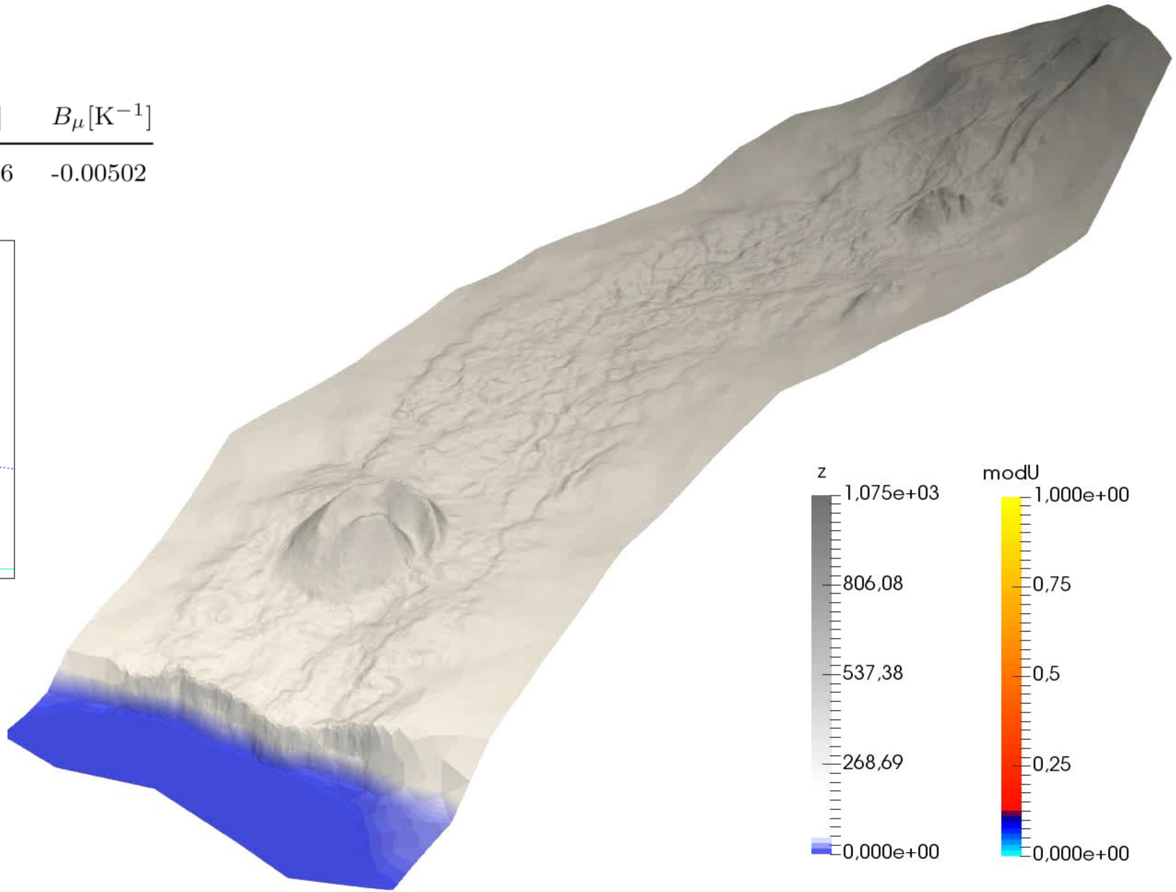
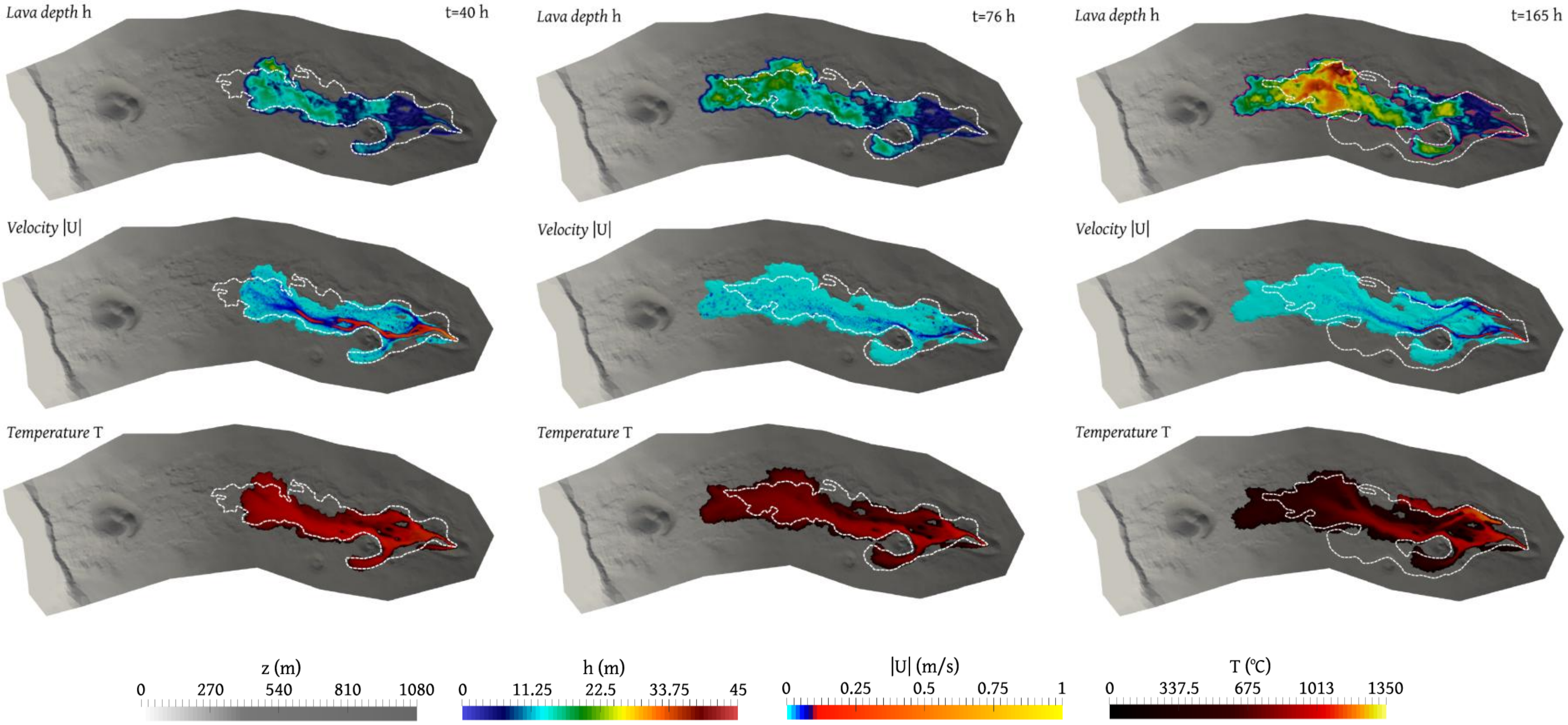




Numerical results

Comparison with observations from the Copernicus satellite



Numerical results

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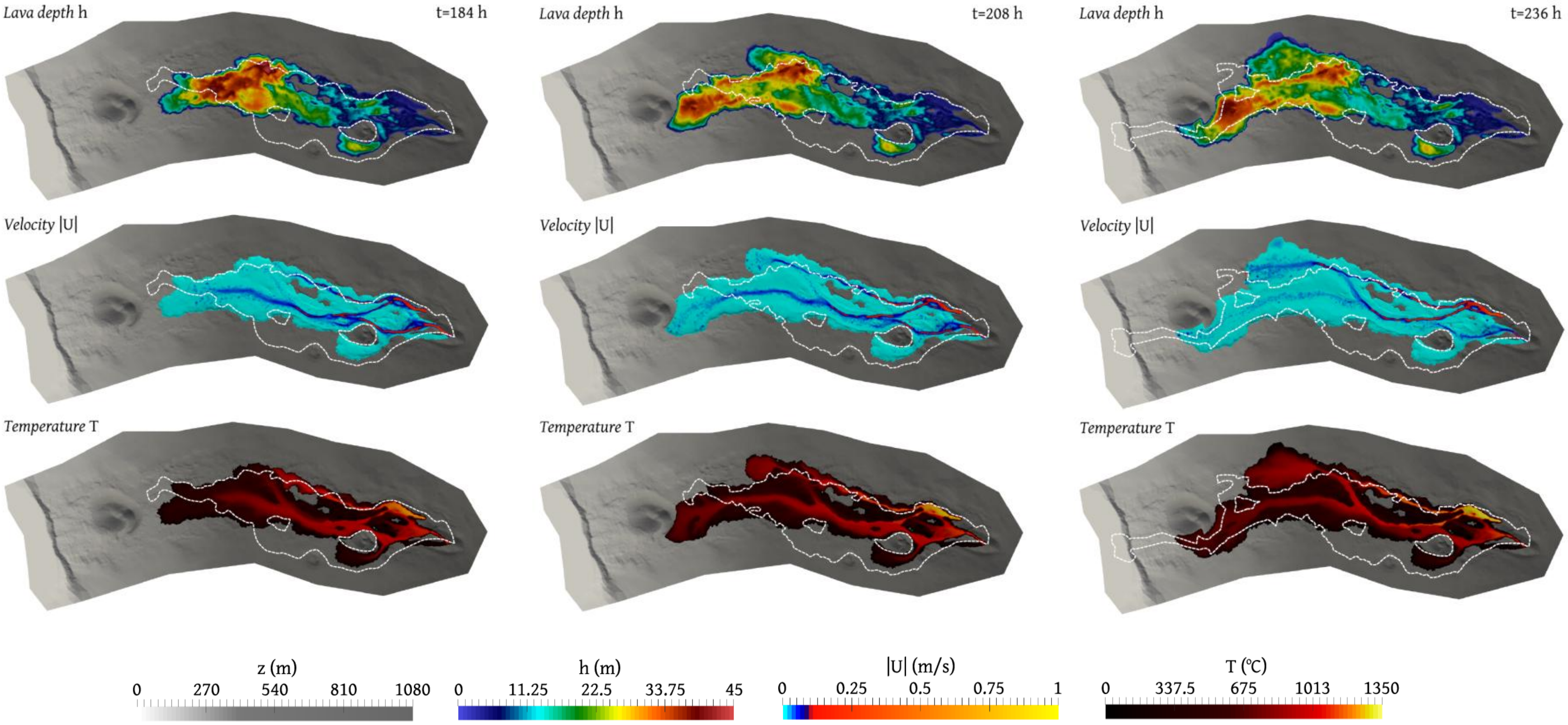

Numerical results

GPU efficiency

CPU: Intel Core i7-10700F
16 cores 2.90Ghz

GPU-1: NVIDIA GeForce GTX Titan Black
2880 cores 980Mhz

GPU-2: NVIDIA A100 Tensor Core
6912 cores 1410Mhz
54.000 millions of transistors
~15000 €

	CPU (1 core)	CPU (16 cores)	GPU-1	GPU-2
Comp. time (h)	477.19	33.37	6.88	1.65
Speed-up	-	14.3	69.4	289.2

Thank you for the attention!

For further questions, please contact me:
jfernandez@eead.csic.es