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p-Medians and Multi-Medians

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The classical p-median problem is discussed, together with methods for its solution. The multi-median problem, a generalization of the p-median problem in which more than one type of facility is allowed, is introduced and methods of solution developed. Numerical results are presented.

INTRODUCTION

MANY FACILITIES, concerned with the provision of a service to a set of customers J, involve travel (by customers or the provider of the service) between each customer and a 'nearby' facility. Examples include libraries, schools and post offices. Establishing few facilities leads to high transportation costs but low fixed costs (resulting from setting up the facilities). Establishing many facilities leads to the reverse effect. A relevant and much studied problem is the simple, or uncapacitated, location problem (S.L.P.)

$$\text{S.L.P.: minimize } \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} + \sum_{i=1}^{n} f_i y_i$$

subject to

$$\sum_{i=1}^{n} x_{ij} = 1 \quad j = 1, \ldots, m$$

$$y_i - x_{ij} \leqslant 0 \quad i = 1, \ldots, n, \quad j = 1, \ldots, m$$

$$x_{ij}, y_i = 0, 1 \quad i = 1, \ldots, n, \quad j = 1, \ldots, m$$

where

c_{ij} is the cost of meeting customer j's requirements from a facility at i;
f_i is the cost of establishing a facility at site i;

$$x_{ij} = \begin{cases} 1 \text{ if customer } j \text{ is served from } i; \\ 0 \text{ otherwise;} \end{cases}$$

$$y_i = \begin{cases} 1 \text{ if a facility is established at } i; \\ 0 \text{ otherwise.} \end{cases}$$

S.L.P. may be appropriate in private sector location problems in which capacity constraints are not binding. (Rand[1] advocates relaxation of capacity constraints in the first instance anyway, as this can lead to further insight into the effects of these constraints.) However, in the public sector, locational decisions are often made in the political sphere, and budget constraints are frequently represented by a limit on the number of facilities to be established. Addition to the formulation of S.L.P. of the constraint

$$\sum_{i=1}^{n} y_i = p$$

and removal of the fixed cost terms from the objective leads to the p-median problem (pM.P.)

$$p\text{M.P.: minimize } x_0 = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} \tag{1}$$

subject to

$$\sum_{i=1}^{n} x_{ij} = 1 \quad j = 1, \ldots, m \tag{2}$$

$$\sum_{i=1}^{n} y_i = p \tag{3}$$

$$y_i - x_{ij} \geq 0 \quad i = 1, \ldots, n, \quad j = 1, \ldots, m \tag{4}$$

$$x_{ij}, y_i = 0, 1 \quad i = 1, \ldots, n, \quad j = 1, \ldots, m. \tag{5}$$

Warzawski and Peer[2] discussed the location, on a large building site, of such facilities as a concrete-mixing plant, a block-making plant and a centre for cutting, bending and storing reinforcing steel. In their example the blocks of flats constructed were subdivided into 38 consumer areas (customers), each consisting of four dwelling units, and each customer required the service of each facility at one time or another. Warzawski and Peer made the assumption that only one facility can occupy any one site and proposed the following generalization of S.L.P. called the multi-commodity problem (M.L.P.)

$$\text{M.L.P.: minimize} \sum_{r=1}^{w} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}^r x_{ij}^r + \sum_{r=1}^{w} \sum_{i=1}^{n} f_i^r y_i^r \tag{6}$$

subject to

$$\sum_{i=1}^{n} x_{ij}^r = 1 \quad r = 1, \ldots, w, \quad j = 1, \ldots, m \tag{7}$$

$$\sum_{r=1}^{w} y_i^r \leq 1 \quad i = 1, \ldots, n \tag{8}$$

$$y_i^r - x_{ij}^r \geq 0 \quad r = 1, \ldots, w, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m \tag{9}$$

$$x_{ij}^r, y_i^r = 0, 1 \quad r = 1, \ldots, w, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m \tag{10}$$

(The formulation given differs slightly from the original one and is that given in Karkazis and Boffey[3].) The constants and variables are as for S.L.P. except that the superscript r specifies which of the w types of facility is being considered. Clearly M.L.P. would reduce to w S.L.P.s (one for each value of r) if the non-interference constraints (8) were ignored, and solution methods put forward by Warszawski[4] and Karkazis and Boffey[3] are based on this observation.

This paper discusses solution methods for pM.P. and introduces the multi-median problem (M.M.P.), in which the numbers of the various types of facility are pre-specified.

$$\text{M.M.P.: minimize} \sum_{r=1}^{w} \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij}^r x_{ij}^r$$

subject to

$$\sum_{i=1}^{n} x_{ij}^r = 1, \quad r = 1, \ldots w, \quad j = 1, \ldots, m$$

$$\sum_{i=1}^{n} y_i^r = p^r \quad r = 1, \ldots, w$$

$$\sum_{r=1}^{w} y_i^r \leq 1 \quad i = 1, \ldots, n$$

$$y_i^r - x_{ij}^r \geq 0 \quad r = 1, \ldots, w, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m$$

$$x_{ij}^r, y_i^r = 0, 1 \quad r = 1, \ldots, w, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m.$$

M.M.P. bears the same relation to M.L.P. as pM.P. does to S.L.P. Also M.M.P. bears the same relation to pM.P. as M.L.P. does to S.L.P.

This can be represented diagrammatically by

$$\begin{array}{ccc} \text{S.L.P.} & \longrightarrow & p\text{M.P.} \\ \downarrow & & \downarrow \\ \text{M.L.P.} & \longrightarrow & \text{M.M.P.} \end{array}$$

Finally, it might be noted that we are discussing a 'green fields' situation in which no facilities are already present. This is not an important restriction, as the presence of existing facilities is readily built into the branch-and-bound scheme that is developed.

RESUMÉ OF RELEVANT WORK

The relevant background material will be treated here very briefly, and further details may be found in Karkazis and Boffey[3] and references therein. First we note that the dual problem to S.L.P. with integrality constraints relaxed is

$$D\colon \text{maximize} \sum_{j=1}^{m} \lambda_j \tag{11}$$

subject to

$$\sum_{j=1}^{m} \mu_{ij} \le f_i \quad i = 1, \ldots, n \tag{12}$$

$$\lambda_j - \mu_{ij} \le c_{ij} \quad i = 1, \ldots, n, \quad j = 1, \ldots, m \tag{13}$$

$$\mu_{ij} \ge 0, \lambda_j \text{ unrestricted}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m \tag{14}$$

and that μ_{ij} may be taken to be max $(0, \lambda_j - c_{ij})$.

As in Karkazis and Boffey,[3] we designate the dual-based procedure as algorithm B.K.E. The procedure is applied twice, first with the variables always scanned in the order $\lambda_1, \lambda_2, \ldots, \lambda_m$ then with the variables always scanned in reverse order $\lambda_m, \ldots, \lambda_1$. The better of the two solutions is adopted and the values of the dual variables designated by $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_m$. Then

$$\text{dlb}\,(\bar{\boldsymbol{\lambda}}) = \sum_{j=1}^{m} \bar{\lambda}_j$$

is a lower bound to the optimal value $U(D)$ of D and hence a dual lower bound to the optimal value U(S.L.P.) of S.L.P.

If for $i = 1, \ldots, n$ we set

$$\bar{y}_i = \begin{cases} 1 & \text{if} \sum_{j=1}^{n} \mu_{ij} = f_i \\ 0 & \text{otherwise} \end{cases}$$

and assign customers to a nearest open facility, then a feasible solution is obtained for S.L.P. and its value pub($\bar{\mathbf{y}}$) is a primal upper bound to U(S.L.P.). Thus

$$\text{dlb}(\bar{\boldsymbol{\lambda}}) \le U(\text{S.L.P.}) \le \text{pug}\,(\bar{\mathbf{y}}) \tag{15}$$

and if $\varepsilon = \text{pub}\,(\bar{\mathbf{y}}) - \text{dlb}(\bar{\boldsymbol{\lambda}})$ is zero, then S.L.P. has been solved and $\bar{\mathbf{y}}$ defines an optimal solution.

If $\varepsilon > 0$, then an attempt may be made to improve the quality of $\bar{\boldsymbol{\lambda}}$ (and thereby quite likely improve the quality of $\bar{\mathbf{y}}$) by subgradient optimization or some other means. In any case the solution may be completed by branch-and-bound if ε is not reduced to zero.

Karkazis and Boffey[3] extended the above approach to M.L.P.

PRESENT APPROACH

p-*Median problem*

For the p-median problem the number of facilities is fixed but no account is taken of the magnitudes of the fixed costs. This is equivalent to assigning sites a constant fixed cost $f_i = g$ and using the objective

$$\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij}x_{ij} + \sum_{i=1}^{n} f_i y_i$$

since the latter summation merely adds a constant

$$pg \; (\Sigma \, y_i = p).$$

Ignoring the constraint

$$\sum_{i=1}^{n} y_i = p$$

for the moment, we return to the formulation for S.L.P. If this S.L.P. is solved and it happens that

$$\sum_{i=1}^{n} y_i = p,$$

then we also have a feasible and hence optimal solution to pM.P., and the value of g gives a facility fixed cost which is equivalent to imposing the constraint that there be precisely p facilities opened.

If S.L.P. solves to give

$$\sum_{i=1}^{n} y_i < p,$$

then the fixed cost g is too high, having deterred sufficient facilities from being set up. Conversely, if

$$\sum_{i=1}^{n} y_i > p,$$

then g is too low. By varying g we may hope to find a value leading to an optimal solution to pM.P. Moreover, since the objective

$$\min \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij}x_{ij} + pg$$

is concave with respect to g, we may use a Fibonacci search. However it may happen that $g = f_i$ leads to

$$\sum_{i=1}^{n} y_i > p,$$

whereas $g = f_i + \delta$ leads to

$$\sum_{i=1}^{n} y_i < p$$

for any $\delta > 0$. This situation can be resolved by branch-and-bound.

In practice the S.L.P.s are only solved approximately, with a solution $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\mu}})$ obtained for D giving a lower bound dlb $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\mu}})$ to S.L.P. and hence, after subtracting pg, to pM.P. [This latter assertion may be justified formally by considering the Lagrangean relaxation of pM.P. with respect to constraint (3):

$$\text{Minimize} \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij}x_{ij} + g\left(\sum_{i=1}^{n} y_i - p\right)$$

subject to (2), (4) and (5).

The optimal value, which is easily seen to be the optimal value of S.L.P. (with all $f_i = g$) minus pg, must give a lower bound to pM.P. The assertion follows immediately.]

An upper bound pub $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ to pM is found by taking the value of solutions obtained from

$$\bar{y}_i = \begin{cases} 1 \text{ if } \sum_{j=1} \bar{\mu}_{ij} = g \\ 0 \text{ otherwise} \end{cases}$$

if feasible. Otherwise, if there are too many nonzero $\bar{y}_i$, sufficient are set to zero to obtain feasibility. This is done using a routine SETPRIMAL, which repeatedly eliminates a site that leads to a minimal increase in objective.

A test for optimality is made by checking whether $\varepsilon = \text{pub}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) - \text{dlb}(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\mu}})$ is zero or not. As before, the solution may be completed if necessary by using subgradient optimization and branch-and-bound. It might be noted here that the bounds can sometimes be improved at 'exclusion' nodes of the branch-and-bound tree as follows. Suppose

$$\sum_{i=1}^{n} \bar{y}_i = p,$$

that is SETPRIMAL is not required; then if $(\bar{\mathbf{y}}, \bar{\mathbf{x}})$ is not optimal, there must be some $i_0, \bar{y}_{i_0} \neq 1$ such that $y'_{i_0} = 1$ in an optimal solution $(\mathbf{y}', \mathbf{x}')$. Adding the constraint $y_{i_0} = 1$ and associating with it dual variable k, the objective of D is modified to

$$\max \sum_{j=1}^{m} \lambda_j + k$$

and constraint (12), for $i = i_0$, modified to

$$\sum_{j=1}^{m} \mu_{ij} + k \leq g.$$

It is thus clear that the value of the objective may be increased by

$$g - \sum_{j=1}^{m} \mu_{i_0 j}.$$

However, since i_0 is not known in general, the increase is limited to

$$\min\left(g - \sum_{j=1}^{m} \mu_{ij}\right) = k > 0,$$

the minimization being over non-zero terms only. It follows that either $(\bar{\mathbf{y}}, \bar{\mathbf{x}})$ is optimal or dlb $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\mu}})$ may be increased by k. Hence $(\bar{\mathbf{y}}, \bar{\mathbf{x}})$ must be optimal if

$$\text{dlb}(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\mu}}) + k > \text{pub}(\bar{\mathbf{y}}, \bar{\mathbf{x}}),$$

a stronger condition than given by (15).

Multi-median problem

For M.M.P. the non-interference constraints

$$\sum_{r=1}^{w} y_i^r \leq 1$$

may be relaxed to obtain w ordinary pM.P.s in which the rth has the facility limiting constraint (see constraint(3))

$$\sum_{i=1}^{n} y_i^r = p^r.$$

These w pM.P.s may then be solved, and if the noninterference constraints are satisfied, M.M.P. has been solved. If not, there is a *conflict* at some site k; that is

$$\sum_{r=1}^{w} y_k^r \leq 1 \quad \text{(dual variable } \theta\text{)}$$

is violated. Adding this constraint to the set of wpM.P.s and dualizing (the relaxed problems) results in corresponding S.L.P.s

$$D(r)\text{: maximize } \sum_{j=1}^{m} \lambda_j^r$$

subject to

$$\lambda_j^r - \mu_{ij}^r \leq c_{ij}^r$$

$$\sum_{j=1}^{m} \mu_{ij}^r \leq g^r + \theta_i \quad \theta_i = \begin{cases} 0 \text{ if } i \neq k \\ \theta_k \text{ if } i = k \end{cases}$$

$$\mu_{ij}^r, \theta_i \geq 0, \lambda_j^r \text{ unrestricted.}$$

Thus if a value of θ_i is specified, we again have wpM.P.s to solve. If dlb $(\boldsymbol{\theta}, \boldsymbol{\lambda}^r, \boldsymbol{\mu}^r)$ is an optimal value for $D(r)$, then

$$\text{dlb} = \sum_{r=1}^{w} \text{dlb}(\boldsymbol{\theta}, \boldsymbol{\lambda}^r, \boldsymbol{\mu}^r) - \sum_{i=1}^{n} \theta_i$$

is a lower bound for M.M.P. This may be seen by considering the Lagrangean relaxation of M.M.P. formed by adding

$$\sum_{i=1}^{n} \theta_i \left(\sum_{r=1}^{w} y_i^r - 1 \right)$$

to the objective and relaxing the constraints

$$\sum_{r=1}^{w} y_i^r \leqslant 1.]$$

A good value for dlb can be obtained by 'hill-climbing' in some way in $\boldsymbol{\lambda}$-space.

In practice the $D(r)$ would not necessarily be solved exactly but dual variables $\bar{\lambda}_j^r, \bar{\mu}_{ij}^r$ obtained by, for example, algorithm B.K.E.[3], which leads to a good value for $D(r)$.

Suppose now that all conflicts have been removed, then a primal solution $(\bar{\mathbf{y}}, \bar{\mathbf{x}})$ is obtained by setting for all i and r

$$\bar{y}_i^r = \begin{cases} 1 \text{ if } \sum_{j=1}^{m} \mu_{ij}^r = g^r + \theta_i \\ 0 \text{ otherwise} \end{cases}$$

and assigning customers to a nearest facility. Of course this solution may fail to be feasible. First the SETPRIMAL routine is applied for each r for which

$$\sum_{i=1}^{n} y_i^r > p^r.$$

After this has been done, there may still be conflicts at some sites

$$\left(\text{i.e. } \sum_{r=1}^{w} y_i^r > 1 \right).$$

These are removed, one at a time, so as to lead at each step to minimal increase in the value of the objective. The new solution, which is feasible, will also be denoted by $(\bar{\mathbf{y}}, \bar{\mathbf{x}})$, its value being pub$(\bar{\mathbf{y}}, \bar{\mathbf{x}})$.

If it should happen that

$$\text{dlb}(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\mu}}) = \text{pub}(\bar{\mathbf{y}}, \mathbf{x}),$$

then M.M.P. has been solved. Indeed by a straightforward extension of the result for pM.P., we may assert that the optimal solution has been found if

$$\mathrm{dlb}(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\mu}}) + k \geq \mathrm{pub}(\bar{\mathbf{y}}, \bar{\mathbf{x}}),$$

where

$$k = \underset{\substack{i,r \\ y_i^r = 0}}{\text{minimum}} \left(g^r - \theta_i - \sum_{j=1}^{m} \mu_{ij}^r\right).$$

RESULTS

The algorithm for solving pM.P., described above, was tested using data set ES30, which relates to the thirty largest U.S. cities, and the subset ES20 corresponding to the 20 largest of these. The city populations were as in El-Shaieb,[5] but intercity distances were road distances rather than 'crowfly' (airline) distances. The program, written in Algol 68R and run on an I.C.L. 1906S computer, was used to solve pM.P. for several values of p, and times, in seconds and exclusive of data input, are given in Table 1. Distance data are sorted for application of algorithm B.K.E., but this need be done only once for each data set and the sorting times are given separately.

Also given in Table 1 are corresponding times obtained using a Lagrangean relaxation—subgradient method, as described in Narula *et al.*[6] It seems that the proposed algorithm is rather quicker. Of the times quoted for the method of Narula *et al.*, about 10% is taken up by sorting. For both programs a relatively unsophisticated sort routine was used, but improving this will not affect the general conclusion.

The present algorithm was also tested on a very large problem supplied by Dr U. Vaubel. The data concerned the placement of stations in the Ruhr district of Germany; it was required to locate optimally centres at 45 out of 206 possible sites. This problem was solved in 73 seconds (of which 41 were taken in sorting the data!) and, with $n = 206$ and $p = 45$, is perhaps the largest pM.P. so far solved exactly.

For M.M.P., ten data sets $P_1, P_2, \ldots, P_{10}$ were used for testing purposes, their values of n, w and $\mathbf{p} = (p^1, \ldots, p^w)$ being given in Table 2. For each problem the elements of the 'weighted' distance matrix were obtained be generating integers δ from a uniform distribution with $0 < \delta \leq 1000$. Solution times, in seconds and exclusive of data input, obtained using an Algol 68R program and an I.C.L. 1906S computer, are given in Table 2.

TABLE 1. COMPUTATIONAL TIMES FOR pM.P.

(a) ES20 DATA

$p =$	2	4	6	8	10	
Present method	0.436	0.295	0.276	0.261	0.228	
Narula *et al.*	1.192	0.451	0.432	0.429	0.424	
$p =$	12	14	16	18	Sort	Total
Present method	0.214	0.268	0.131	0.090	0.080	2.279
Narula *et al.*	0.231	0.206	0.413	0.196		3.974

(b) ES30 DATA

$p =$	2	4	6	8	10	12	14	16
Present method	0.614	0.678	0.888	0.930	0.817	0.578	0.967	1.090
Narula *et al.*	2.648	1.068	3.446	1.851	2.019	2.528	1.238	1.623
$p =$	18	20	22	24	26	28	Sort	Total
Present method	0.438	0.362	0.336	0.225	0.183	0.100	0.209	8.445
Narula *et al.*	0.453	0.274	0.476	0.424	0.460	0.075		18.583

Both methods were coded in Algol 68R and run on an I.C.L. 1906S computer. Times are in seconds and exclusive of input. The use of road distances led to about the same number of interactions for ES20 as quoted in Narula *et al.*[6] but over 50% more for ES30.

TABLE 2. COMPUTATIONAL RESULTS FOR M.M.P.

Problem	n	w	$\mathbf{p}$	Nodes on branch-and-bound tree	Total time (seconds)
P_1	10	2	(1, 1)	Solved at root node	0.130
P_2	15	2	(1, 1)	2	0.380
P_3	20	2	(1, 1)	40	2.500
P_4	10	3	(1, 1, 1)	8	0.390
P_5	15	3	(1, 1, 1)	3	0.580
P_6	20	3	(1, 1, 1)	142	8.090
P_7	10	2	(2, 1)	4	0.250
P_8	15	2	(2, 1)	28	1.070
P_9	20	2	(2, 1)	628	19.100
P_{10}	10	2	(3, 2)	15	0.360

SUMMARY

It was shown how the p-median problem may be solved by solving S.L.P.s (for which the very efficient algorithm B.K.E. is available) for a sequence of values of the fixed cost g, with completion by branch-and-bound if necessary. Table 1 shows this yields a very competitive method for pM.P.

The multi-median problem (M.M.P.) was introduced in the opening section, and a similar 'parametric' approach was adopted relating M.M.P. to w pM.P.s and vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ of dual variables. Results given in Table 2 are promising, though one problem, P_9, proved to be considerably more difficult than the others.

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