4) 非奇次方程求解

- 特征值函数展开法
- 奇次化原理 ODE → PDE

例1. 带有热源的热方程

$$\begin{cases} u_t - k u_{xx} = f(x,t), & 0 < x < \pi, t > 0, \\ u(0,t) = u(\pi,t) = 0, & t \geq 0, \\ u(x,0) = \varphi(x), & 0 \leq x \leq \pi. \end{cases}$$

解 考虑其特征值问题

$$\phi'' + \lambda\phi = 0,\ \ 0 < x < \pi, \quad \phi(0) = \phi(\pi) = 0$$

$$\Rightarrow\ e.v.\ \ \lambda_n = n^2, \quad e.f.\ \ \phi_n(x) = \sin nx,\ \ n = 1, 2, \cdots$$

特征函数系$\{\sin nx\}_1^\infty$构成$L^2[0,\pi]$的一组正交基.

$$u(x,t) = \sum_1^\infty u_n(t)\phi_n(x),\ \ f(x,t) = \sum_1^\infty f_n(t)\phi_n(x),\ \ g(x) = \sum_1^\infty g_n\phi_n(x)$$

将这些展开式代入

$$\Rightarrow\ \textstyle\sum_1^\infty u_n'(t)\phi_n(x) - k\sum_1^\infty u_n(t)\phi_n''(x) = \sum_1^\infty f_n(t)\phi_n(x)$$

$$\textstyle\sum_1^\infty u_n(0)\phi_n(x) = \sum_1^\infty g_n\phi_n(x).$$

注意到$\phi_n''(x)=-\lambda_n\phi_n(x)$，比较$\phi_n(x)$的系数

$$\Rightarrow \begin{cases} u_n'(t)+k\lambda_n u_n(t)=f_n(t), & t>0, \\ u_n(0)=g_n, & n=1,2,\cdots \end{cases}$$

$$\Rightarrow \quad u_n(t)=g_n e^{-k\lambda_n t}+\int_0^t f_n(s)e^{-k\lambda_n(t-s)}\mathrm{d}s$$

$$\Rightarrow \quad u(x,t)=\sum_1^\infty\left(g_n e^{-k\lambda_n t}+\int_0^t f_n(s)e^{-k\lambda_n(t-s)}\mathrm{d}s\right)\sin nx$$

练习1. 求解热方程初边值问题

$$\begin{cases} u_t - ku_{xx} = \cos 2x, & 0 < x < \pi, t > 0, \\ u_x(0,t) = u_x(\pi,t) = 0, & t \geq 0, \\ u(x,0) = 1, & 0 \leq x \leq \pi. \end{cases}$$

参考答案： $u(x,t) = 1 + \dfrac{1}{4k}(1 - e^{-4kt})\cos 2x.$

练习2：求下列问题的级数形式的解，

$$
\begin{cases}
u_{tt} = a^2 u_{xx} - r u_t, 0 < x < l, t > 0; \\
u(0,t) = u(l,t) = 0, t \geq 0; \\
u(x,0) = \varphi(x), u_t(x,0) = \psi(x), 0 \leq x \leq l.
\end{cases}
$$

其中$0 < r < \dfrac{2\pi a}{l}$，$r$为常数。

请思考，如果$r > \dfrac{2\pi a}{l}$，那么求解过程有哪些不同？

$$u(x,t)=\sum_{n=1}^{\infty} e^{-\frac{r}{2}t}\left(\varphi_n \cos\beta_n t+\frac{\psi_n+\frac{r}{2}\varphi_n}{\beta_n}\sin\beta_n t\right)\sin\frac{n\pi x}{l},$$

$\beta_n=\dfrac{\sqrt{(\frac{2n\pi a}{l})^2-r^2}}{2}$, φ_n, ψ_n是$\varphi(x)$, $\psi(x)$的Fourier系数.

例3.用特征函数展开法求解

$$\begin{cases} -\Delta u = 1+\cos\theta, & r<a\ , \\ u=0, & r=a. \end{cases}$$

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

方程化为

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = -r^2(1+\cos\theta)$$

解　先考虑关于变量θ的特征值问题

$$\begin{cases} T''(\theta)+\lambda T(\theta)=0, \\ T(\theta)=T(\theta+2\pi). \end{cases}$$

$e.v.\ \ \lambda_0=0,\ \ \lambda_n=n^2,$

$e.f.\ \ T_0=1,\ \ T_n=A_n\cos n\theta+B_n\sin n\theta,\quad n=1,2,\cdots$

用此特征函数系将$u(r,\theta)$展开

$$u(r,\theta)=A_0(r)+\sum_1^{\infty}A_n(r)\cos n\theta+B_n(r)\sin n\theta$$

代入方程得

$$r^2\left[A_0''(r)+\sum_1^\infty\left(A_n''(r)\cos n\theta+B_n''(r)\sin n\theta\right)\right]$$

$$+r\left[A_0'(r)+\sum_1^\infty\left(A_n'(r)\cos n\theta+B_n'(r)\sin n\theta\right)\right]$$

$$-\sum_1^\infty n^2\left(A_n(r)\cos n\theta+B_n(r)\sin n\theta\right)=-r^2(1+\cos\theta)$$

代入边界条件得

$$A_0(a)+\sum_1^\infty A_n(a)\cos n\theta+B_n(a)\sin n\theta=0.$$

$$\begin{cases} r^2A_0''(r) + rA_0'(r) = -r^2, & 0 < r < a, \\ |A_0(0)| < \infty, \quad A_0(a) = 0, \end{cases}$$

解得 $\quad A_0(r) = \dfrac{1}{4}(a^2 - r^2)$

$$\begin{cases} r^2A_1''(r) + rA_1'(r) - A_1(r) = -r^2, & 0 < r < a, \\ |A_1(0)| < \infty, \quad A_1(a) = 0, \end{cases}$$

解得 $\; A_1(r) = \dfrac{r}{3}(a - r)$

其余系数 $A_n(r) = 0 \;\; (n \neq 0, 1), \qquad B_n(r) = 0.$

$$\Rightarrow \quad u(r,\theta) = \frac{1}{4}(a^2 - r^2) + \frac{r}{3}(a - r)\cos\theta.$$

特征函数展开法小结

仔细分析可以看到特征函数展开法对分离变量法进行了改进，先求解特征值问题，然后在每个特征子空间内分离变量. 利用此法可将非齐次PDE初边值问题转化为非齐次ODE初值问题，求出ODE初值问题的解即可得到原问题的解.

复习:二阶常系数非齐次ODE特解的求法

$$ay''+by'+cy=f(x)$$

(1)[待定函数法]f(x)具有特殊形式时,上述方程特解的求法.这里的特殊形式是指:f(x)是指数函数、正弦函数、余弦函数、多项式,或这些函数的某种组合.

(2)[常数变易法]将齐次方程通解中的常数变为函数代入。

积分号下求导:

(1) 设$I(t)=\int_a^b f(x,t)dx, f, f_t \in C$,则$I'(t)=\int_a^b f_t(x,t)dx$.

(2)设$H(t)=\int_{a(t)}^{b(t)} f(x,t)dx, f, f_t, a', b' \in C$,则

$$H'(t)=\int_{a(t)}^{b(t)} f_t(x,t)dx + f(b(t),t)b'(t) - f(a(t),t)a'(t).$$

证明:(1) 根据导数定义可得.

(2)设$g(t,a,b)=\int_a^b f(x,t)dx$,则

$$H(t)=g(t,a(t),b(t)) \Rightarrow H'(t)=g_t+g_a a'(t)+g_b b'(t).$$

齐次化原理的思路.

$$\boxed{\begin{cases} y' + p(t)y = q(t) \\ y(0) = y_0 \end{cases}.}$$

$$\begin{cases} v' + p(t)v = 0 \\ v(0) = y_0 \end{cases} \quad \text{(I)} \qquad \begin{cases} u' + p(t)u = q(t) \\ u(0) = 0 \end{cases} \quad \text{(II)}$$

$$\begin{cases} w' + p(t+s)w = 0, \quad t > 0 \\ w(0;s) = q(s) \end{cases} \Rightarrow w(t;s) = q(s) \cdot e^{-\int_0^t p(\tau+s)d\tau}$$

利用已有结论猜测 $\boxed{u(t) = \int_0^t w(t-s;s)ds}$.

- 证明上述猜测的结论(一阶ODE)
- 推广到二阶ODE初值问题

- 推广到热方程
- 推广到波动方程

二阶ODE齐次化原理

$$\begin{cases} u''(t)+b(t)u'(t)+c(t)u(t)=f(t), \quad t>0, \\ u(0)=0, \quad u'(0)=0. \end{cases}$$

考虑其辅助问题

$$\begin{cases} w''(t)+b(t+s)w'(t)+c(t+s)w(t)=0, \quad t>0, \\ w(0;s)=0, \qquad w'(0;s)=f(s). \end{cases}$$

求出该问题的解$w(t;s)$，则原问题的解

$$u(t)=\int_0^t w(t-s;s)\mathrm{d}s.$$

考虑非齐次热方程

$$\begin{cases} u_t - k\Delta u = f(x,t), & x \in D, \ \ t > 0, \\ \text{齐次 } B.C. & x \in \partial D, \ \ t \geq 0, \\ u(x,0) = 0, & x \in D, \end{cases}$$

引入辅助问题

$$\begin{cases} w_t - k\Delta w = 0, & x \in D, \ \ t > 0, \\ \text{齐次 } B.C. & x \in \partial D, \ \ t \geq 0, \\ w(x,0;s) = f(x,s), & x \in D, \end{cases}$$

则

$$u(x,t) = \int_0^t w(x,t-s;s)\,\mathrm{d}s.$$

考虑非齐次波动方程

$$
\begin{cases}
u_{tt} - a^2 \Delta u = f(x,t), & x \in D,\ \ t > 0, \\
\text{齐次 } B.C. & x \in \partial D,\ \ t \geq 0, \\
u(x,0) = 0,\ \ u_t(x,0) = 0, & x \in D,
\end{cases}
$$

引入辅助问题

$$
\begin{cases}
w_{tt} - a^2 \Delta w = 0, & x \in D,\ \ t > 0, \\
\text{齐次 } B.C. & x \in \partial D,\ \ t \geq 0, \\
w(x,0;s) = 0,\ \ w_t(x,0;s) = f(x,s), & x \in D,
\end{cases}
$$

则 $$u(x,t) = \int_0^t w(x,t-s;s)\,\mathrm{d}s.$$

例. 利用齐次化原理求解

$$\begin{cases} u_t - k u_{xx} = t\sin x, & 0 < x < \pi,\ \ t > 0, \\ u(0,t) = u(\pi,t) = 0, & t \geq 0, \\ u(x,0) = 0, & 0 \leq x \leq \pi. \end{cases}$$

解 构造关于含参数s的函数$w(x,t;s)$的辅助问题

$$\begin{cases} w_t - kw_{xx} = 0, & 0 < x < \pi, \ \ t > 0, \\ w(0,t;s) = w(\pi,t;s) = 0, & t \geq 0, \\ w(x,0;s) = s\sin x, & 0 \leq x \leq \pi. \end{cases}$$

解得$w(x,t;s) = se^{-kt}\sin x$.

由热方程齐次化原理知

$$u(x,t) = \int_0^t w(x,t-s;s)\,\mathrm{d}s = \int_0^t se^{-k(t-s)}\sin x\,\mathrm{d}s$$

$$= \left(\frac{t}{k} + \frac{e^{-kt}-1}{k^2}\right)\sin x.$$

课堂练习

求解非齐次波动方程初边值问题

$$\begin{cases} u_{tt} - a^2 u_{xx} = f(x)\sin\omega t, & 0 < x < l,\ \ t > 0, \\ u(0,t) = u(l,t) = 0, & t \geq 0, \\ u(x,0) = u_t(x,0) = 0, & 0 \leq x \leq l, \end{cases}$$

其中$\omega \neq \omega_n$, $n = 1, 2, \cdots$,

这里的$\omega_n = an\pi/l$称为固有频率.

解　考虑特征值问题

$$\begin{cases} \phi''(x)+\lambda\phi(x)=0, \quad 0<x<l, \\ \phi(0)=\phi(l)=0. \end{cases}$$

$$e.v.\ \ \lambda_n=\left(\frac{n\pi}{l}\right)^2, \quad e.f.\ \ \phi_n(x)=\sin\frac{n\pi x}{l},$$

$$u(x,t)=\sum_{n=1}^{\infty}u_n(t)\phi_n(x), \quad f(x)=\sum_{n=1}^{\infty}f_n\phi_n(x)$$

其中　$f_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x.$

将展开式代入方程得

$$\sum_{n=1}^{\infty} u_n''(t)\phi_n(x) - a^2 \sum_{n=1}^{\infty} u_n(t)\phi_n''(x) = \sin\omega t \sum_{n=1}^{\infty} f_n\phi_n(x)$$

$$\begin{cases} u_n''(t) + a^2\lambda_n u_n(t) = f_n \sin\omega t, & t > 0, \\ u_n(0) = 0, \quad u_n'(0) = 0, & n = 1, 2, \cdots \end{cases}$$

该ODE的特征方程是 $\xi^2 + \omega_n^2 = 0 \quad \Rightarrow \quad \xi = \pm i\,\omega_n.$

因为$\omega \neq \omega_n$，故可求得该ODE的特解是

$$u_n^*(t) = \frac{f_n}{\omega_n^2 - \omega^2}\sin\omega t$$

可设ODE的通解为

$$u_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{f_n}{\omega_n^2 - \omega^2} \sin \omega t$$

代入初始条件$u_n(0) = 0,\ u_n'(0) = 0$解得

$$A_n = 0, \qquad B_n = \frac{\omega}{\omega_n} \cdot \frac{f_n}{\omega^2 - \omega_n^2},$$

所以 $$u_n(t) = \frac{f_n}{\omega^2 - \omega_n^2} \left(\frac{\omega}{\omega_n} \sin \omega_n t - \sin \omega t \right)$$

综上所述

$$u(x,t) = \sum_1^{\infty} \frac{f_n}{\omega^2 - \omega_n^2} \left(\frac{\omega}{\omega_n} \sin \omega_n t - \sin \omega t \right) \sin \frac{n\pi x}{l}.$$