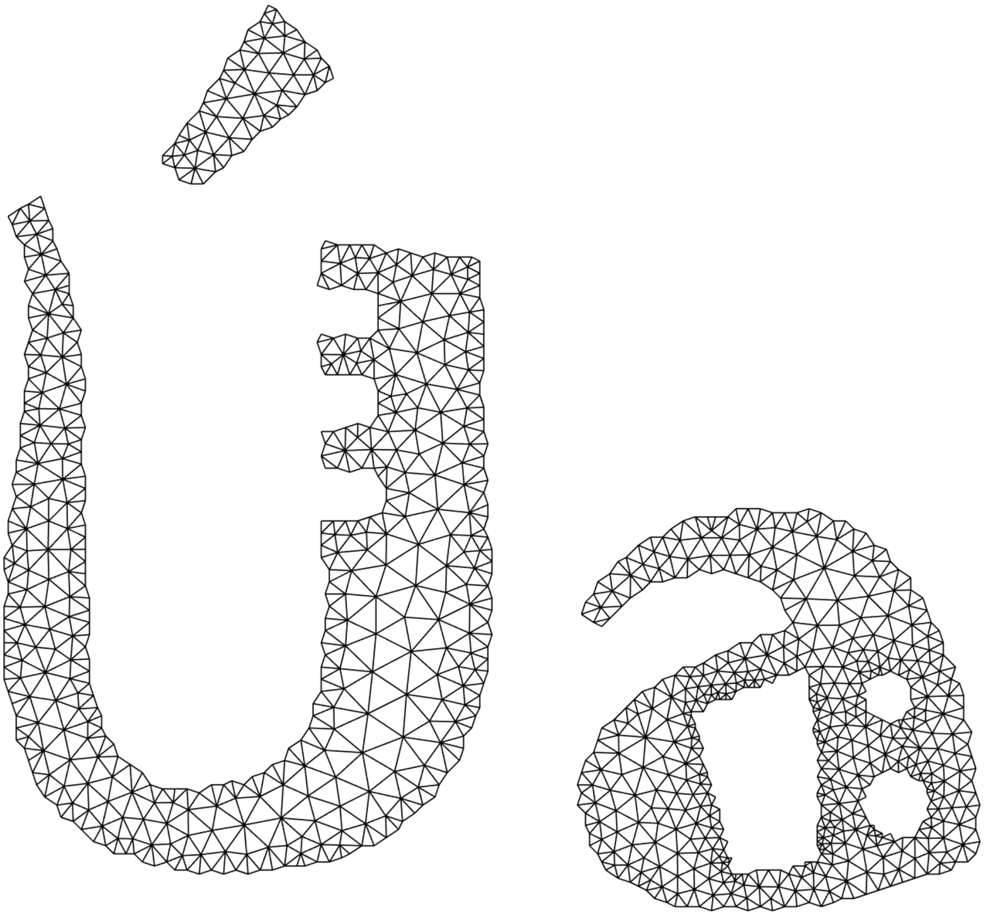

# Úa *Compendium*

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# Introduction

*Úa* is a finite-element ice flow model. This document provides some theoretical background information to *Úa* . It is not a manual. This is a 'life' document, i.e. it is constantly being modified and changed and the current from of the document is in no means final. It contains some general material on glacier mechanics, a bit on the FE method, and lots of some very specific *Úa* related stuff. Most of the material related to glacier mechanics is from a course given at Caltech in 2014.

# Notation and definitions

Typical problem geometry is depicted in Fig. 1 and the main geometrical variables listed in Table 2. Upper and lower glacier surfaces are denoted by $s$ and $b$, respectively, while the ocean surface and the bedrock are denoted by $S$ and $B$, respectively. The ice thickness is $h = s - b$ and is always positive. The distance from bedrock ($B$)) to the ocean surface ($S$) is $H = S - B$, and this quantity can be either positive or negative, depending on location.

The maximal ice thickness possible without grounding is denoted by $h_f$ and is

$$h_f := (S - B)\rho_o/\rho \,,$$

where $\rho$ and $\rho_o$ are the ice and the ocean densities, respectively. The ice grounds if the ice thickness exceeds $h_f$, that is whenever $h > h_f$. Note that $h_f$ becomes negative for $B > S$, i.e. whenever the bedrock ($B$) is located above sea level ($S$). In that case we always have $h > h_f$ for any positive ice thickness $h$ and the ice is always grounded. We also define the positive flotation thickness, $h_f^+$, as

$$h_f^+ = h_f\,\mathcal{H}(h_f) \,, \tag{1}$$

where $\mathcal{H}$ is the Heaviside function, defined as

$$\mathcal{H}(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1/2 & \text{for} \quad x = 0 \\ 1 & \text{for} \quad x > 0 \end{cases}$$

The function $\mathcal{H}(h - h_f)$ (equal to one if grounded, zero if afloat), is needed frequently and we define the grounding/flotation mask $\mathcal{G}$ as

$$\mathcal{G} = \mathcal{H}(h - h_f)$$

Hence

$$\mathcal{G} = \mathcal{H}(h - h_f) = \begin{cases} 0 & \text{over floating areas} \\ 1/2 & \text{at the grounding line} \\ 1 & \text{over grounded areas} \end{cases} \tag{2}$$

The variable $\mathcal{G}$ can the thought of as a ‘grounding’ parameter.

The freeboard is

$$f := s - S,$$

and the draft $d$ is defined as

$$d := \begin{cases} S - b, & \text{if} \;\; S > b \\ 0, & \text{otherwise} \end{cases}$$

which can also be written as

$$d = \mathcal{H}(H)(S - b),$$

The ‘positive’ ocean depth $H^+$ is defined as

$$H^+ := \mathcal{H}(H)H, \tag{3}$$

i.e. $H^+ = H$ if $H > 0$ and zero otherwise.

To distinguish between continuous quantities and discrete quantities we use bold face for the latter. In a finite-element context we might write

$$f(x) = f_q\,\phi_q(x) \,.$$

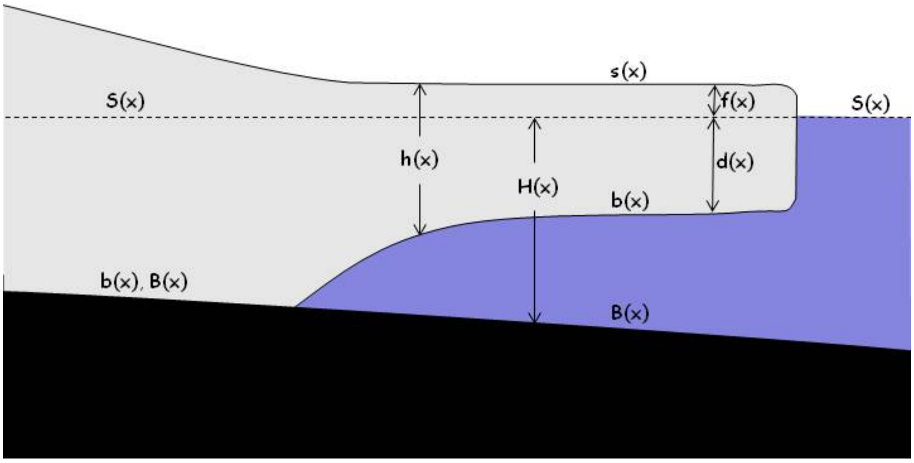



Figure 1: Geometrical variables: Glacier surface ($s$)), glacier bed ($b$), ocean surface ($S$), ocean floor ($B$), glacier thickness ($h = s - b$)), ocean depth ($H = S - B$), glacier draft ($d = S - b$), glacier freeboard ($f = s - S$).

Here $f$ is a continuous function, $f_q$ the nodal variables, and $\phi_q$ the shape/form functions. We sometimes group the nodal variables together into a vector writing

$$\boldsymbol{f} = [f_1, f_1, \ldots, f_N]^T ,$$

and then

$$f(x) = \boldsymbol{f}^T \cdot \boldsymbol{\phi}(x) .$$

Note that $f(x)$ and $f_q$ very are different quantities: $f(x)$ is a function, whereas $f_q$ is a scalar and can be thought of as the $q$th component of the function $f(x)$ in the basis $\{\phi\}$.

If, for a vector $\boldsymbol{f}$, we need to refer to the $q$ element of the vector, we write $[\boldsymbol{f}]_q$, i.e.

$$f_q = [\boldsymbol{f}]_q .$$

The matrix representation of a continuous operator $L : H_1 \to H_2$ is written in bold as $\boldsymbol{L}$. The elements of the matrix are $L_{pq} = [\boldsymbol{L}]_{pq}$.

The $L^2$ inner product is

$$(f, g)_{L^2} = \int f(x)\, g(x)\, dx ,$$

where $f$ and $g$ are square intergrable functions and the $l^2$ inner product is

$$(\boldsymbol{f}, \boldsymbol{g})_{l^2} = \boldsymbol{f}^T \cdot \boldsymbol{g} ,$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are vectors. The inner products define corresponding $L^2$ and $l^2$ norms.

The notation used for inner products is currently a bit inconsistent in this compendium, as I use both the round bracket and the angle bracket (bra/ket) notations. Hence, for inner product from $V \to \mathbb{R}$ all these notations are used

$$(f, g) =< f, g >=< f \mid g > ,$$

and they all have the same meaning.

From the definition of an inner produce we have the induced norm

$$\|f\|^2 =< f \mid f > .$$

Hence

$$\|f\|_{L^2} = \sqrt{< f \mid f >_{L^2}} = \sqrt{\int f(x)\, f(x)\, dx}$$

We often need to linearise various quantities, which we do by calculating directional derivatives. The directional derivative $Df$ of a function $f$ of the variable $x$ in the direction $v$ is defined as

$$Df(x)[v] = \lim_{\epsilon \to 0} \frac{d}{d\epsilon} f(x + \epsilon v) .$$

Table 1: List of variables

| | |
|---|---|
| $s$ | elevation of upper glacier surface |
| $\tilde{s}$ | measurements of the elevation of upper glacier surface |
| $b$ | elevation of lower glacier surface |
| $\tilde{b}$ | measurements of the elevation of lower glacier surface |
| $S$ | ocean surface |
| $B$ | bedrock / ocean floor |
| $h := s - b$ | glacier thickness |
| $H := S - B$ | ocean depth (pos. or neg. depending on location) |
| $H^+ = \mathcal{H}(H)H$ | (positive) ocean depth |
| $d := \mathcal{H}(H)(S - b)$ | glacier draft (positive by definition) |
| $f := s - S$ | freeboard (always positive) |
| $\mathcal{G} := \mathcal{H}(h - h_f)$ | grounding/flotation mask, 1 if ice grounded, 0 otherwise. |
| $h_f := \rho_o H/\rho$ | flotation thickness (maximal ice thickness without grounding) |
| $h_f^+ := \rho_o H^+/\rho$ | positive flotation thickness |
| $\alpha$ | tilt of coordinate system |
| $\rho$ | ice density |
| $\rho_o$ | ocean density |
| $\varrho = \rho\,(1 - \rho/\rho_o)$ | |
| $(u_b, v_b, w_b)$ | $xyz$ components of basal velocity |
| $(u_s, v_s, w_s)$ | $xyz$ components of surface velocity |
| $\tau_{xx}$, $\tau_{yy}$, $\tau_{xy}$ etc. | deviatoric stress components |
| $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{xy}$ etc. | Cauchy stress components |
| $\dot{\epsilon}_{xx}$, $\dot{\epsilon}_{yy}$, $\dot{\epsilon}_{xy}$ etc. | strain rates |
| $g$ | gravitational acceleration |

Often we think of $v$ being a small perturbation to $x$ in which case we write

$$Df(x)[\Delta x] = \lim_{\epsilon \to 0} \frac{d}{d\epsilon} f(x + \epsilon \Delta x) \ ,$$

and may then simply write $Df(x)$ or just $\delta f$. In this document I use either $Df(x)[v]$) or, if the direction $v$ is clear from the context, $\delta f$ to denote the directional derivative.

As explained in more detail in Appendix C the gradient is defined in terms of the directional derivative for a given inner product as

$$Df(p)[\phi] = < \nabla J(p), \phi >_H \ ,$$

where $H$ is a Hilbert space and $f : H \to \mathbb{R}$. In Euclidean inner product the first-order Taylor expansion of a scalar function $f$ is here written as

$$f(\boldsymbol{x} + \boldsymbol{h}) = f(\boldsymbol{x}) + (\nabla f)^T \, \boldsymbol{h} \ ,$$

and similarly for a vector function $\boldsymbol{f}$ as

$$\boldsymbol{f}(\boldsymbol{x} + \boldsymbol{h}) = \boldsymbol{f}(\boldsymbol{x}) + (\nabla \boldsymbol{f})^T \, \boldsymbol{h} \ ,$$

which implies that the gradient of a scalar function $f$ is a column vector and the gradient of a vector function $\boldsymbol{f}$ the matrix [1]

$$\begin{aligned} \nabla \boldsymbol{f} &= \begin{pmatrix} \partial_{x_1} f_1 & \partial_{x_1} f_2 & \cdots & \partial_{x_1} f_m \\ \partial_{x_2} f_1 & \partial_{x_2} f_2 & \cdots & \partial_{x_2} f_m \\ \ldots & \vdots & \ddots & \vdots \\ \partial_{x_n} f_1 & \partial_{x_2} f_2 & \cdots & \partial_{x_n} f_m \end{pmatrix} \\ &= \nabla \otimes \boldsymbol{f} \end{aligned}$$

[1] Sometimes an outer product between two (column) vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as

$$\boldsymbol{a}\boldsymbol{b}^T = \boldsymbol{a} \otimes \boldsymbol{b} \ .$$

If we think of the operator $\nabla$ as a (column) vector, then one could argue that a correct notation for the operation of $\nabla$ on $\boldsymbol{f}$ is, written using the outer product, $\nabla \boldsymbol{f}^T$.

Table 2: List of main geometrical variables.

| symbol | definition |
|---|---|
| $s$ | upper glacier surface |
| $b$ | lower glacier surface |
| $S$ | ocean surface |
| $B$ | bedrock / ocean floor |
| $h := s - b$ | glacier thickness |
| $H := S - B$ | ocean depth (pos. or neg. depending on location) |
| $d := \mathcal{H}(H)(S - b)$ | glacier draft (positive by definition) |
| $f := s - S$ | freeboard (always positive) |
| $h_f := \rho_w H/\rho$ | maximum ice thickness without grounding |

and also

$$\nabla \boldsymbol{f}(\boldsymbol{x}) = \frac{\partial f_i}{\partial x_j} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \ .$$

Using index notation and the summation and comma conventions we can write the Taylor expansion as

$$f_p(\boldsymbol{x} + \boldsymbol{h}) = f_p(\boldsymbol{x}) + f_{p,q}\, h_q$$

and

$$[\nabla \boldsymbol{f}]_{pq} = f_{q,p} \ .$$

The Jakobian matrix $\boldsymbol{K}$ for a vector function $\boldsymbol{f} = (f_1, f_2, \cdots, f_m)^T$ of the vector $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T$ is commonly defined as the $m \times n$ matrix having the elements $[\boldsymbol{K}]_{ij} = \partial f_i / \partial_{x_j}$. Therefore $\boldsymbol{K} = (\nabla \boldsymbol{f})^T$ and the first-order Taylor expansion can be written as

$$\boldsymbol{f}(\boldsymbol{x} + \boldsymbol{h}) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{K}\, \boldsymbol{h} \ .$$

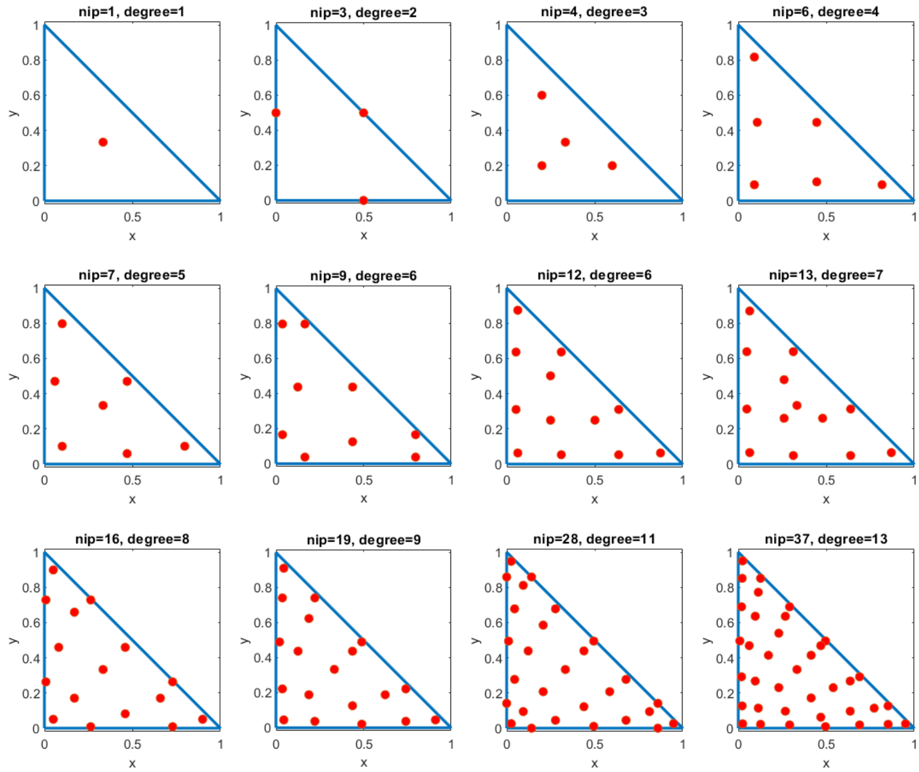



Figure 2: Distribution of integration points with the unit reference triangle and the degree of precision.

# Part I

# General theoretical background for the use of the *Úa* ice-flow model

# Chapter 1

# Equations of ice flow

## 1.1 Shallow Ice Stream Approximation (SSTREAM/SSA)

The shallow-ice stream (SSTREAM/SSA/Shelfy) equations are

$$\partial_x(h(2\tau_{xx}+\tau_{yy}))+\partial_y(h\tau_{xy})-t_{bx}=(\rho g h\partial_x s+\frac{1}{2}h^2 g\,\partial_x\rho)\cos\alpha-\rho g h\sin\alpha \tag{1.1}$$

$$\partial_y(h(2\tau_{yy}+\tau_{xx}))+\partial_x(h\tau_{xy})-t_{by}=(\rho g h\partial_y s+\frac{1}{2}h^2 g\,\partial_y\rho)\cos\alpha \tag{1.2}$$

were $\alpha$ is the tilt of the coordinate system with respect to the gravity vector. Variables are defined in Tables 1 and 2 and in Fig. 1. where

$$\boldsymbol{t}_{bh}=\begin{pmatrix} t_{bx} \\ t_{by} \end{pmatrix}$$

is the horizontal part of the basal stress vector (basal traction)

$$\boldsymbol{t}_b=\boldsymbol{\sigma}\hat{\boldsymbol{n}}-(\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}\hat{\boldsymbol{n}})\hat{\boldsymbol{n}},$$

with $\hat{\boldsymbol{n}}$ being a unit normal vector to the bed pointing into the ice. Note that it is the horizontal component of the basal traction that enters the field equations. We will sometimes just write $\boldsymbol{t}_b$ instead of $\boldsymbol{t}_{bh}$ which is a slight abuse of notation, and strictly speaking incorrect.

Defining the *resistive stress tensor* as

$$\boldsymbol{R}=\begin{pmatrix} 2\tau_{xx}+\tau_{yy} & \tau_{xy} \\ \tau_{xy} & 2\tau_{yy}+\tau_{xx} \end{pmatrix} \tag{1.3}$$

and

$$\nabla_{xy}=(\partial_x,\partial_y)$$

the field equations can be written in a compact form as

$$\nabla_{xy}\cdot(h\,\boldsymbol{R})-\boldsymbol{t}_{bh}=\rho g h\nabla_{xy}\,s+\frac{1}{2}gh^2\nabla_{xy}\,\rho\,, \tag{1.4}$$

for $\alpha=0$.

As shown in section 1.11 we have

$$g\rho h\partial_x s+\frac{1}{2}gh^2\partial_x\rho=\frac{g}{2}\partial_x(\rho h^2-\rho_o d^2)+g(\rho h-\rho_o d)\partial_x b, \tag{1.5}$$

and also

$$g(\rho h-\rho_o d)\partial_x b=\mathcal{G}\,g(\rho h-\rho_o H^+)\partial_x B. \tag{1.6}$$

We can therefore also write the field equations as

$$\nabla_{xy}\cdot(h\,\boldsymbol{R})-\frac{g}{2}\nabla_{xy}(\rho h^2-\rho_o d^2)=\boldsymbol{t}_{bh}+g(\rho h-\rho_o d)\nabla_{xy}b, \tag{1.7}$$

and as

$$\nabla_{xy}\cdot(h\,\boldsymbol{R})-\frac{g}{2}\nabla_{xy}(\rho h^2-\rho_o d^2)=\boldsymbol{t}_{bh}+\mathcal{G}\,g(\rho h-\rho_o H^+)\nabla_{xy}B. \tag{1.8}$$

again for $\alpha = 0$. Note that where the ice is afloat the right hand sides of Eqs. (1.7) and (1.8) become equal to zero.

We will also show (see section 1.10) that the stress boundary condition at both grounded and floating ice edges can be written as

$$h\,\boldsymbol{R}\cdot\hat{\boldsymbol{n}}_{xy} = \frac{g}{2}(\rho h^2 - \rho_o d^2)\hat{\boldsymbol{n}}_{xy}. \tag{1.9}$$

where

$$\hat{\boldsymbol{n}}_{xy} = (n_x, n_y, 0)^T,$$

is a unit normal pointing horizontally outward from the ice front.

## 1.2 Shallow Ice Shelf (SSHELF/SSA)

The shallow ice shelf approximation is simply the shallow ice stream approximation with the drag term dropped.

Since

$$s = S + h\,(1 - \rho/\rho_o)$$

we have over a floating ice shelf

$$\rho g h \nabla_{xy}\, s = \frac{1}{2}\varrho g \nabla_{xy}\, h^2$$

and the momentum equations have the form

$$\nabla_{xy}\cdot(h\,\boldsymbol{R}) = \frac{1}{2}\varrho g \nabla_{xy}\, h^2 + \frac{1}{2} g h^2 \nabla_{xy}\,\rho\,, \tag{1.10}$$

or if we skip the spatial density gradient

$$\nabla_{xy}\cdot\left(\boldsymbol{R} - \frac{1}{2}\varrho g \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}\right) = 0 \tag{1.11}$$

## 1.3 Shallow Ice Sheet (SSHEET/SIA)

The shallow ice sheet equations for the deformational velocities are

$$(u, v) = -E\,\|\nabla_{xy} s\|^{n-1}\,\left(h^{n+1} - (s - z)^{n+1}\right)\,(\partial_x s, \partial_y s)$$

where

$$E = \frac{2A}{n+1}(\rho g)^n$$

The vertically integrated flux is

$$q_x = \int_b^s \rho u\, dz,$$

giving

$$(q_x, q_y) = -\rho D\,\|\nabla_{xy} s\|^{(n-1)}\, h^{n+2}\,(\partial_x s, \partial_y s)$$

where

$$D = \frac{2A}{n+2}(\rho g)^n$$

or

$$(q_x, q_y) = F\,\rho\, h\,(u, v)$$

with

$$F = \frac{n+1}{n+2}$$

where

$$\|\nabla_{xy} s\| = \sqrt{(\partial_x s)^2 + (\partial_y s)^2}$$

## 1.4 Equation of mass conservation

The prognostic equation

$$\rho\,\partial_t h + \nabla_{xy}\cdot \boldsymbol{q}_{xy} = \rho a, \tag{1.12}$$

where

$$a = a_s + a_b,$$

is a vertically integrated expression of mass conservation.

We will now derive Eq. (1.12) using the local form of the mass-conservation equation and the kinematic boundary conditions at the upper and lower surfaces. The local form of the mass-conservation equation is

$$\nabla\cdot(\rho\boldsymbol{v}) + \partial_t\rho = 0, \tag{1.13}$$

and the kinematic boundary conditions at the upper and lower surface are

$$\partial_t s + u_s\partial_x s + v_s\partial_x s - w_s = a_s \quad \text{at} \quad z = s, \tag{1.14}$$

$$\partial_t b + u_b\partial_x b + v_b\partial_x b - w_b = -a_b \quad \text{at} \quad z = b. \tag{1.15}$$

The mass balance distributions, $a_b$ and $a_s$, are in the units of meters of water equivalent per time. Note the sign convention for $a_s$ and $a_b$ used in (1.14) and (1.15). *Mass flux into the ice is defined positive irrespective over which surface it takes place. Surface accumulation is positive, melting always negative.*

The horizontal ice flux is defined as

$$\boldsymbol{q}_{xy} = \int_b^s \rho\boldsymbol{v}_{xy}.$$

Focusing on the flow-line case for the moment we find that

$$\begin{aligned}
\partial_x q_x &= \partial_x \int_b^s (\rho u)\,dz \\
&= \int_b^s \partial_x(\rho u)\,dz + \rho u_s\partial_x s - \rho u_b\partial_x b \\
&= -\int_b^s (\partial_z(\rho w) + \partial_t\rho)\,dz + \rho u_s\partial_x s - \rho u_b\,\partial_x b \\
&= -\rho w_s + \rho w_b - h\partial_t\rho + \rho u_s\partial_x s - \rho u_b\,\partial_x b \\
&= -h\partial_t\rho - \rho w_s + \rho u_s\partial_x s + \rho w_b - \rho u_b\,\partial_x b \\
&= -h\partial_t\rho + \rho a_s - \rho\partial_t s + \rho a_b + \rho\partial_t b \\
&= -h\partial_t\rho + \rho a - \rho\partial_t h
\end{aligned}$$

where again

$$a = a_s + a_b,$$

and therefore, once the $y$ component has been added

$$\rho\,\partial_t h + \nabla_{xy}\cdot\boldsymbol{q}_{xy} + h\,\partial_t\rho = \rho a. \tag{1.16}$$

In most modelling work of large ice masses it is assumed that the density $\rho$ is uniform in space and does not vary with time. In *Úa* we relax this condition somewhat and only assume that the density at a given location does not change with time, i.e.

$$\partial_t\rho = 0\ ,$$

but allow the density to vary in the horizontal ($\partial_x\rho$ and $\partial_y\rho$ are not assumed to be equal to zero). Hence the mass conservation equations (1.13) becomes

$$\nabla\cdot(\rho\boldsymbol{v}) = 0.$$

and (1.16)

$$\rho\,\partial_t h + \nabla_{xy}\cdot\boldsymbol{q}_{xy} = \rho a \tag{1.17}$$

or

$$\rho\,\partial_t h + \partial_x(\rho h u) + \partial_y(\rho h v) = \rho a \tag{1.18}$$

for a velocity field that does not change with depth.

Eq. (1.17) is the form of the mass continuity equation used in *Úa*. The effect of horizontal gradients in (vertically integrated) density are also included in the momentum equations (1.4).

**Vertical velocities**

Note that

$$w_s = u_s\,\partial_x s + w_b - u_b\,\partial_x b - \frac{1}{\rho}\partial_x q_x - \frac{h}{\rho}\partial_t \rho$$

or

$$w_s = u_s\,\partial_x s + \partial_t b + a_b - \frac{1}{\rho}\partial_x q_x - \frac{h}{\rho}\partial_t \rho$$

which can be used to calculate vertical velocities.

If the bed elevation does not change with time and if furthermore $\partial_t \rho = 0$, we have the special case

$$w_s = u_s\,\partial_x s + a_b - \frac{1}{\rho}\partial_x q_x$$

## 1.5 Sliding law

Basal drag is often assumed to be a function of velocity, i.e. Robin type boundary condition, and written as

$$\boldsymbol{t}_b = f(u, v, h)\frac{\boldsymbol{v}_b}{\|\boldsymbol{v}_b\|}\ ,$$

where $f$ is some unknown function, $\boldsymbol{v}_b$ is the tangential basal velocity, i.e. the basal sliding velocity

$$\begin{aligned}\boldsymbol{v}_b &= \boldsymbol{v} - (\hat{\boldsymbol{n}}^T \cdot \boldsymbol{v})\hat{\boldsymbol{n}},\\ &= \boldsymbol{T}\boldsymbol{v},\end{aligned}$$

and $\boldsymbol{t}_b$ is the tangential component of the basal traction

$$\boldsymbol{t}_b = -\boldsymbol{T}(\boldsymbol{\sigma}\hat{\boldsymbol{n}}).$$

where

$$\boldsymbol{T} = \mathbf{1} - \hat{\boldsymbol{n}} \otimes \hat{\boldsymbol{n}},$$

is the tangential operator[1]. We refer to $\boldsymbol{t}_b$ as *basal shear traction.* Note that the shear traction is, in general, not equal to basal shear stress. The basal drag caused by ice flow only acts over the grounded parts, a fact which we can express as

$$\boldsymbol{t}_b = \mathcal{G}\boldsymbol{f}(\boldsymbol{v}_b)\ ,$$

where $\mathcal{G}$ is the grounding/flotation mask defined as

$$\mathcal{G} := \mathcal{H}(h - h_f)\ .$$

### 1.5.1 Weertman sliding law

The power-law type Weertman sliding law is

$$\boldsymbol{T}\boldsymbol{\sigma}\hat{\boldsymbol{n}} + C^{-1/m}\|\boldsymbol{T}\boldsymbol{v}\|^{1/m-1}\,\boldsymbol{T}\boldsymbol{v} = 0,$$

[1] This can be seen as follows:

$$\begin{aligned}\boldsymbol{v}_b &= \boldsymbol{v} - (\boldsymbol{v}^T \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}}\\ &= (\mathbf{1} - \hat{\boldsymbol{n}} \otimes \hat{\boldsymbol{n}})\boldsymbol{v}\\ &= \boldsymbol{T}\boldsymbol{v}\end{aligned}$$

where $\boldsymbol{v}_b$ it the bed tangential component of $\boldsymbol{v}$ and we have used the identity $(\hat{\boldsymbol{n}} \otimes \hat{\boldsymbol{n}})\boldsymbol{v} = (\boldsymbol{v}^T \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}}$. One can also define the normal operator $\boldsymbol{N}$ as

$$\boldsymbol{N} = \hat{\boldsymbol{n}} \otimes \hat{\boldsymbol{n}},$$

and we have

$$\boldsymbol{\sigma} = \boldsymbol{N}\boldsymbol{\sigma} + \boldsymbol{T}\boldsymbol{\sigma}.$$

In general, given a unit vector $\hat{\boldsymbol{n}}$, the orthogonal projection onto the subspace spanned by $\hat{\boldsymbol{n}}$ is $\hat{\boldsymbol{n}} \otimes \hat{\boldsymbol{n}}$.

Table 1.1: Sliding laws and definitions.

| Name | Abbreviation | Definition |
|---|---|---|
| Weertman | W | $\boldsymbol{t}_b = \mathcal{G}\, \beta^2\, \boldsymbol{v}_b$ |
| Coulomb | C | $\boldsymbol{t}_b = \mu_k N \, \frac{\boldsymbol{v}_b}{\lVert \boldsymbol{v}_b \rVert}$ |
| Budd | W | $\boldsymbol{t}_b = \mathcal{G}\, N^{q/m}\, \beta^2\, \boldsymbol{v}_b$ |
| Tsai | minCW | $\boldsymbol{t}_b = \min(\mathcal{G}\, \beta^2 \lVert \boldsymbol{v}_b \rVert\, \boldsymbol{v}_b, \mu_k N)\, \frac{\boldsymbol{v}_b}{\lVert \boldsymbol{v}_b \rVert}$ |
| Cornford | rpCW | $\boldsymbol{t}_b = \frac{\mathcal{G}\, \beta^2\, \lVert \boldsymbol{v}_b \rVert\, \mu_k N}{((\mu_k N)^m + (\mathcal{G}\, \beta^2 \lVert \boldsymbol{v}_b \rVert)^m)^{1/m}}\, \frac{\boldsymbol{v}_b}{\lVert \boldsymbol{v}_b \rVert}$ |
| Umbi | rCW | $\boldsymbol{t}_b = \frac{\mathcal{G}\, \beta^2\, \lVert \boldsymbol{v}_b \rVert\, \mu_k N}{\mu_k N + \mathcal{G}\, \beta^2 \lVert \boldsymbol{v}_b \rVert}\, \frac{\boldsymbol{v}_b}{\lVert \boldsymbol{v}_b \rVert}$ |
| Zeroth-order hydrology | N0 | $N = \mathcal{G} g \rho (h - h_f^+)$ |
| Note: When combined with the zeroth-order hydrology the abbreviations are: W, C, W-N0, minCW-N0, rpCW-N0 , and rCW-N0. | | |

Different formulations are

$$\boldsymbol{t}_b = \mathcal{G}\, C^{-1/m} \lVert \boldsymbol{v}_b \rVert^{1/m-1}\, \boldsymbol{v}_b, \tag{1.19}$$

$$\boldsymbol{t}_b = \mathcal{G}\, \beta^2\, \boldsymbol{v}_b, \tag{1.20}$$

$$\boldsymbol{t}_b = \lVert \boldsymbol{t}_b \rVert \frac{\boldsymbol{v}_b}{\lVert \boldsymbol{v}_b \rVert}, \tag{1.21}$$

$$\lVert \boldsymbol{t}_b \rVert = \mathcal{G}\, \beta^2\, \lVert \boldsymbol{v}_b \rVert, \tag{1.22}$$

where $\beta^2$ is defined as,

$$\beta^2 = C^{-1/m} \lVert \boldsymbol{v}_b \rVert^{1/m-1}\ , \tag{1.23}$$

and[2]

$$\boldsymbol{v}_b = \mathcal{G}^m C \lVert \boldsymbol{t}_b \rVert^{m-1}\, \boldsymbol{t}_b, \tag{1.24}$$

$$\lVert \boldsymbol{v}_b \rVert = \mathcal{G}^m\, C \lVert \boldsymbol{t}_b \rVert^m, \tag{1.25}$$

In *Úa* the power-law sliding law is given as

$$\begin{pmatrix} t_{bx} \\ t_{by} \end{pmatrix} = \mathcal{G}\, \beta^2 \begin{pmatrix} u_b \\ v_b \end{pmatrix}$$

with

$$\beta^2 = (C + C_0)^{-1/m}\, (u_b^2 + v_b^2 + u_o^2)^{(1-m)/2m}\ ,$$

and where $C_0$ and $u_0$ are some (small) regularisation parameters.

The basal slipperiness $C$ has the physical dimensions of velocity divided by stress to the power $m$. It can be useful to think of $C$ as the ratio

$$C = \frac{C_v}{C_\tau^m}$$

where $C_v$ has the dimensions of velocity and $C_\tau$ the dimensions of stress. We can then write (1.19) and (1.24) as

$$\boldsymbol{t}_b = \mathcal{G}\, C_\tau \left( \frac{\lVert \boldsymbol{v}_b \rVert}{C_v} \right)^{1/m} \frac{\boldsymbol{v}_b}{\lVert \boldsymbol{v}_b \rVert} \tag{1.26}$$

$$\boldsymbol{v}_b = C_v \left( \frac{\lVert \boldsymbol{t}_b \rVert}{\mathcal{G} C_\tau} \right)^m \frac{\boldsymbol{t}_b}{\lVert \boldsymbol{t}_b \rVert} \tag{1.27}$$

[2]Since

$$\mathcal{G} = \mathcal{G}^m\ ,$$

for any power $m$ we do not strictly need to include the stress exponent $m$ in any equations involving $\mathcal{G}$. However if we use an approximation to Heaviside function $\mathcal{H}$ in the definition of $\mathcal{G}$ given by Eq. (2), then this stress exponent should be included.

**Weertman sliding law limits**

If we now consider the limit $m \to +\infty$ we see from Eq. (1.24) that $\|\boldsymbol{t}_b\| \to C_\tau$ over the grounded areas (where $\mathcal{G} = 1$) for the velocity to remain finite. With increasing $m$, the basal velocity $\boldsymbol{v}_b$ becomes increasingly sensitive to basal shear traction, and in the limit $m \to +\infty$ the velocity can be considered to become infinitely sensitive to basal shear traction. For $m \to +\infty$, $1/m \to 0$ and inserting $1/m = 0$ into Eq. (1.26) gives

$$\boldsymbol{t}_b = C_\tau \frac{\boldsymbol{v}_b}{\|\boldsymbol{v}_b\|} \quad \text{for} \quad m \to +\infty.$$

Hence, the basal shear traction is in this limit equal to $C_\tau$, and $C_\tau$ can be considered to be a yield stress. In this particular limit it therefore arguably better to recast the 'sliding' law as

$$\boldsymbol{t}_b = \tau^* \frac{\boldsymbol{v}_b}{\|\boldsymbol{v}_b\|} \quad \text{for} \quad m \to +\infty,$$

where $\tau^* = C_\tau$ is a yield stress, and the 'sliding law' is now simply a stress condition for the basal shear traction and $\tau^*$ a property of the bed (e.g. till) that is determined by some other physical principle. Using this viewpoint, in the limit $m \to +\infty$ the parameter $C_v$ has no effect on the solution, but can be calculated afterwards from the velocity. The basal sliding law does not impose any direct constraints on the basal sliding velocity, i.e. for given basal shear traction, the basal velocity can have any value. The basal velocity can still be calculated by solving the SSTREAM/SSA equations provided the velocity is set to a value somewhere along the boundary by the boundary conditions (In this limit the SSTREAM field equations only provide constraints on the gradients of velocities). In the SIA equations the velocity can not be determined using the momentum equation (as by definition there is no direct functional relationship between velocity and basal shear traction), and must be determined from other consideration (such as mass conservation).

Considering the opposite limit where $m \to 0$, we see from Eq. (1.26) that now it is the basal shear traction that becomes infinitely sensitive to basal velocity, or conversely, the basal velocity becomes in this limit independent of basal shear traction. Inserting $m = 0$ into Eq. (1.24) gives

$$\boldsymbol{v}_b = C_v \frac{\boldsymbol{t}_b}{\|\boldsymbol{t}_b\|} \quad \text{for} \quad m \to 0.$$

This is a limit which is (I find) difficult to understand in physical terms. The basal velocity is now no longer a function of the stress state and must be determined by some other physical principle. What physical conditions at the bed would force the basal sliding velocity to obtain some particular value independently of the state of stress?

Note that the limits $m \to +\infty$ and $m \to 0$ are fundamentally different. In the former the basal shear traction is fixed, in the latter the basal velocity.

### 1.5.2 Budd sliding law (Generalised Weertman sliding law)

The Budd sliding law is

$$\boldsymbol{t}_b = \mathcal{G}\, N^{q/m}\, C^{-1/m} \|\boldsymbol{v}_b\|^{1/m-1}\, \boldsymbol{v}_b, \tag{1.28}$$

$$= \mathcal{G}\, N^{q/m}\, \beta^2\, \boldsymbol{v}_b, \tag{1.29}$$

$$\boldsymbol{v}_b = \frac{C}{\mathcal{G}^m N^q} \|\boldsymbol{t}_b\|^{m-1}\, \boldsymbol{t}_b, \tag{1.30}$$

$$\|\boldsymbol{v}_b\| = \frac{C}{\mathcal{G}^m N^q}\, \|\boldsymbol{t}_b\|^m, \tag{1.31}$$

$$\|\boldsymbol{t}_b\| = \mathcal{G} \left(\frac{N^q}{C}\right)^{1/m} \|\boldsymbol{v}_b\|^m, \tag{1.32}$$

where $N$ is the effective pressure.

If one assumes perfect hydrological connection with the ocean (see section 1.7.1), Budd sliding law takes the form

$$\boldsymbol{t}_b = \mathcal{G}\, \left(\rho g (h - h_f^+)\right)^{q/m}\, C^{-1/m}\, \|\boldsymbol{v}_b\|^{1/m-1}\, \boldsymbol{v}_b, \tag{1.33}$$

$$= \mathcal{G}\, \left(\rho g (h - h_f^+)\right)^{q/m}\, \beta^2\, \boldsymbol{v}_b, \tag{1.34}$$

It follows that $N = 0$ where $h = h_f = H\rho_o/\rho$ and therefore $\boldsymbol{t}_b = \boldsymbol{0}$ at the grounding line.

### 1.5.3 Coulomb friction law

The Coulomb friction law is

$$\boldsymbol{t}_b = \mu_k N \, \frac{\boldsymbol{v}_b}{\|\boldsymbol{v}_b\|} \,, \qquad \text{(Coulomb)}$$

where $N$ is the effective pressure the and coefficient of kinetic friction, $\mu_k$, an empirical property. [3]

### 1.5.4 Combining Weertman with Coulomb

Budd sliding law ensures that basal drag goes to zero as the grounding line is approached. But there are many other ways of ensuring that $\|\boldsymbol{t}_b\| \to 0$ as $N \to 0$, for example by combining Weertman sliding and the Coulomb friction laws:

$$\boldsymbol{t}_b = \mathcal{G}\, \beta^2\, \boldsymbol{v}_b \,, \qquad \text{(Weertman)} \tag{1.20}$$

$$\boldsymbol{t}_b = \mu_k N \, \frac{\boldsymbol{v}_b}{\|\boldsymbol{v}_b\|} \,, \qquad \text{(Coulomb)} \tag{1.5.3}$$

where again

$$\beta^2 = C^{-1/m} \|\boldsymbol{v}_b\|^{1/m-1} \,. \tag{1.23}$$

#### Minimum Value

Here the idea is that basal drag is limited by these two processes acting independently, and the process giving the lower value at a given location is the one to use at that location (Tsai et al., 2015). So calculate both

$$\|\boldsymbol{t}_b\| = \mathcal{G}\, \beta^2\, \|\boldsymbol{v}_b\| \qquad \text{(Weertman)} \tag{1.35}$$

$$\|\boldsymbol{t}_b\| = \mu_k N \qquad \text{(Coulomb)} \tag{1.36}$$

and at each location use the law producing the lower value of the two. Again assuming perfect hydrological connection, the effective pressure can be approximated close to the grounding line using Eq. (1.45).

#### Reciprocal sum

Another option of combining Weertman and Coulomb drag laws is to use a reciprocal sum of the two, i.e.

$$\frac{1}{\|\boldsymbol{t}_b\|} = \frac{1}{\|\boldsymbol{t}_b^W\|} + \frac{1}{\|\boldsymbol{t}_b^C\|} \,,$$

or

$$\|\boldsymbol{t}_b\| = \frac{1}{\frac{1}{\|\boldsymbol{t}_b^W\|} + \frac{1}{\|\boldsymbol{t}_b^C\|}} \,,$$

where again

$$\|\boldsymbol{t}_b^W\| = \mathcal{G}\, \beta^2\, \|\boldsymbol{v}_b\|, \tag{1.22}$$

$$\|\boldsymbol{t}_b^C\| = \mu_k N \tag{1.36}$$

[3] Weertman sliding law is

$$\boldsymbol{t}_b = \mathcal{G}\, C^{-1/m} \|\boldsymbol{v}_b\|^{1/m} \, \frac{\boldsymbol{v}_b}{\|\boldsymbol{v}_p\|}$$

and therefore

$$\mu_k N = \mathcal{G}\, C^{-1/m} \|\boldsymbol{v}_b\|^{1/m} \,,$$

and, if $C$ has been determined, it is tempting to calculate $N$ using

$$N = \mathcal{G}\, \mu_k^{-1} C^{-1/m} \|\boldsymbol{v}_b\|^{1/m} \,.$$

However, there is no guarantee that this will produce a physically plausible effective pressure distribution. For example, for any finite-valued inverted $C$ distribution, $N$ calculated in this manner will not go to zero as the grounding line is approached from above.

giving[4],

$$
\begin{aligned}
\|\boldsymbol{t}_b\| &= \frac{1}{1/\|\boldsymbol{t}_b^W\| + 1/\|\boldsymbol{t}_b^C\|} \\
&= \frac{\|\boldsymbol{t}_b^C\|\,\|\boldsymbol{t}_b^W\|}{\|\boldsymbol{t}_b^W\| + \|\boldsymbol{t}_b^C\|} \\
&= \mathcal{G}\,\frac{\mu_k N\,\beta^2\|\boldsymbol{v}_b\|}{\mu_k N + \mathcal{G}\beta^2\|\boldsymbol{v}_b\|}\;. \qquad (1.37)
\end{aligned}
$$

By looking at the limits where either the Coulomb or Weertman drag becomes small compared to the other, we see that this idea is similar to the *minimum value* idea above, i.e. that these two processes act independently, and the basal drag is at each location effectively determined by the process giving the lower drag of the two.

Note that the grounding/floating mask $\mathcal{G}$ in the numerator is arguably redundant as $\mathcal{G}N = N$, but we keep it here to get the right limits as $N \to +\infty$. The basal drag vector is then calculated as

$$
\begin{aligned}
\boldsymbol{t}_b &= \|\boldsymbol{t}_b\|\frac{\boldsymbol{v}_b}{\|\boldsymbol{v}_b\|} \\
&= \frac{\mu_k N\,\mathcal{G}\beta^2\|\boldsymbol{v}_b\|}{\mu_k N + \mathcal{G}\beta^2\|\boldsymbol{v}_b\|}\,\frac{\boldsymbol{v}_b}{\|\boldsymbol{v}_b\|} \qquad (1.38)\\
&= \mathcal{G}\,\frac{\mu_k N}{\mu_k N + \mathcal{G}\,\beta^2\|\boldsymbol{v}_b\|}\,\beta^2\,\boldsymbol{v}_b\;. \qquad (1.39)
\end{aligned}
$$

This gives the limits

$$
\begin{aligned}
&\|\boldsymbol{t}_b\| = \mathcal{G}\,\beta^2\|\boldsymbol{v}_b\| \quad \text{for} \quad N \to +\infty && \text{(Weertman)}\\
&\|\boldsymbol{t}_b\| = \mu_k N \quad \text{for} \quad N \to 0 && \text{(Coulomb)}\\
&\|\boldsymbol{t}_b\| = \mu_k N \quad \text{for} \quad \|\boldsymbol{v}_b\| \to +\infty && \text{(Coulomb)}\\
&\|\boldsymbol{t}_b\| = \mathcal{G}\beta^2\|\boldsymbol{v}_b\| \quad \text{for} \quad \|\boldsymbol{v}_b\| \to 0 && \text{(Weertman)}
\end{aligned}
$$

### Cornford's reciprocal power weighting

In Asay-Davis et al. (2016), Cornford suggests combining of Weertman sliding and Coulomb friction as [5]

$$
\frac{1}{\|\boldsymbol{t}_b\|^m} = \frac{1}{\|\boldsymbol{t}_b^W\|^m} + \frac{1}{\|\boldsymbol{t}_b^C\|^m}\;,
$$

[4] This is not dissimilar from Schoof's suggestion which, apart from some stress exponents, is

$$
\|\boldsymbol{t}_b\| = \frac{\mu_k N\|\boldsymbol{v}_b\|}{\lambda A N + \|\boldsymbol{v}_b\|}
$$

and we get Eq. (1.37) if we set $\lambda A = \mu_k/\beta^2$.

[5] In Asay-Davis et al. (2016), the notation is

$$
\boldsymbol{t}_b = \frac{\beta_C^2\|\boldsymbol{v}_b\|^{1/m-1}\alpha_C^2 N}{\left(\beta_C^{2m}\|\boldsymbol{v}_b\| + (\alpha_C^2 N)^m\right)^{1/m}}\,\boldsymbol{v}_b\;,
$$

where

$$
\begin{aligned}
\alpha_C^2 &= \mu_k \\
\beta_C^2\,\|\boldsymbol{v}_b\|^{1/m-1} &= \mathcal{G}\,\beta^2
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\boldsymbol{t}_b &= \frac{\beta_C^2\|\boldsymbol{v}_b\|^{1/m-1}\alpha_C^2 N}{\left(\beta_C^{2m}\|\boldsymbol{v}_b\| + (\alpha_C^2 N)^m\right)^{1/m}}\,\boldsymbol{v}_b \\
&= \frac{\mu_k N\,\mathcal{G}\,\beta^2\|\boldsymbol{v}_b\|}{((\mu_k N)^m + (\mathcal{G}\,\beta^2\|\boldsymbol{v}_b\|)^m)^{1/m}}\,\frac{\boldsymbol{v}_b}{\|\boldsymbol{v}_b\|} \qquad (1.40)
\end{aligned}
$$

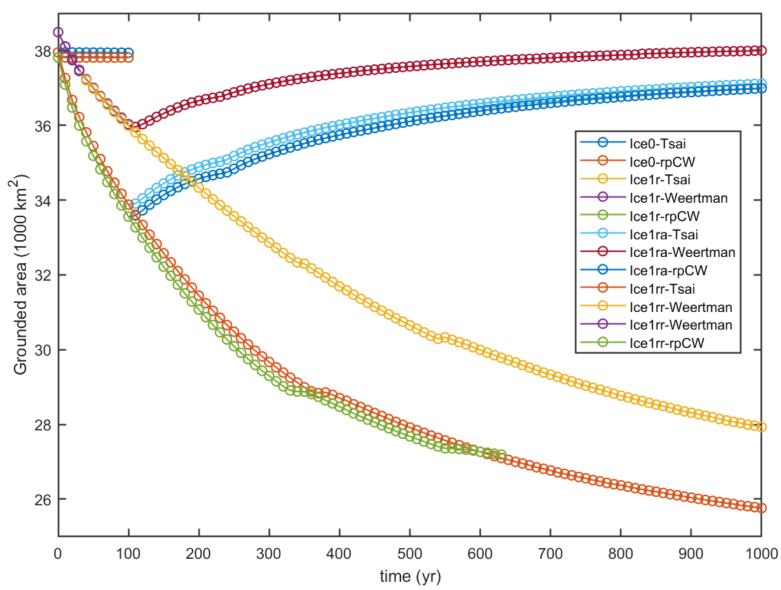



Figure 1.1: MismipPlus: Grounded area as function of time for different sliding laws. Description of the experimental setup is given in Asay-Davis et al. (2015).

giving

$$\begin{aligned}
\|\boldsymbol{t}_b\| &= \frac{1}{\left(1/\|\boldsymbol{t}_b^W\|^m + 1/\|\boldsymbol{t}_b^C\|^m\right)^{1/m}} \\
&= \frac{\|\boldsymbol{t}_b^C\|\,\|\boldsymbol{t}_b^W\|}{\left(\|\boldsymbol{t}_b^C\|^m + \|\boldsymbol{t}_b^W\|^m\right)^{1/m}} \\
&= \frac{\mu_k N\,\mathcal{G}\,\beta^2\|\boldsymbol{v}_b\|}{\left((\mu_k N)^m + (\mathcal{G}\,\beta^2\|\boldsymbol{v}_b\|)^m\right)^{1/m}}\,.
\end{aligned} \tag{1.41}$$

Thus

$$\boldsymbol{t}_b = \frac{\mu_k N\,\mathcal{G}\,\beta^2\|\boldsymbol{v}_b\|}{\left((\mu_k N)^m + (\mathcal{G}\,\beta^2\|\boldsymbol{v}_b\|)^m\right)^{1/m}}\,\frac{\boldsymbol{v}_b}{\|\boldsymbol{v}_b\|}\,. \tag{1.42}$$

While expression (1.42) is not identical to Eq. (1.38) it clearly reflects the same idea, i.e. to combing Weertman and Coulomb laws in a gradual and continuous manner, and it gives the same limits, i.e.

$$\begin{aligned}
&\|\boldsymbol{t}_b\| = \mathcal{G}\,\beta^2\|\boldsymbol{v}_b\| \quad \text{for} \quad N \to +\infty && \text{(Weertman)} \\
&\|\boldsymbol{t}_b\| = \mu_k N \quad \text{for} \quad N \to 0 && \text{(Coulomb)} \\
&\|\boldsymbol{t}_b\| = \mu_k N \quad \text{for} \quad \|\boldsymbol{v}_b\| \to +\infty && \text{(Coulomb)} \\
&\|\boldsymbol{t}_b\| = \mathcal{G}\beta^2\|\boldsymbol{v}_b\| \quad \text{for} \quad \|\boldsymbol{v}_b\| \to 0 && \text{(Weertman)}
\end{aligned}$$

## 1.6 Ocean drag term

To simulate drag exerted on the ice by the ocean we add an ocean drag term over the floating section of the form

$$\boldsymbol{t}_b^o = \mathcal{H}(h_f - h)C_o^{-1/m_o}\|\boldsymbol{v}_b - \boldsymbol{v}_o\|^{1/m_o - 1}(\boldsymbol{v}_b - \boldsymbol{v}_o)$$

where $\boldsymbol{v}_0$ is the velocity of the ocean current.

The sea-ice literature suggest $m_0 = 1/2$, i.e.

$$\boldsymbol{t}_b^o = \mathcal{H}(h_f - h) C_o^{-2} \|\boldsymbol{v}_b - \boldsymbol{v}_o\| \, (\boldsymbol{v}_b - \boldsymbol{v}_o)$$

and defines

$$D_o = C_o^{-2}$$

where

$$D_o = \rho_o c_o$$

and typically $c_0 = 0.0055$. Hence

$$C_0 = \frac{1}{\sqrt{D_o}} = \frac{1}{\sqrt{\rho_o c_o}} \approx 0.4 \quad [\sqrt{(m/s)/Pa}]$$

The total drag is a sum of that due to basal sliding and ocean currents.

$$\boldsymbol{t}_b = \mathcal{G} \, \beta^2 \, \boldsymbol{v} + \mathcal{H}(h_f - h) \, \beta_o^2 \, (\boldsymbol{v} - \boldsymbol{v}_o)$$

## 1.7 Effective basal water pressure

### 1.7.1 Hydrology

The first law of observational subglacial hydrology is that if you drill a hole in a warm-based glacier and measure the subglacial water pressure, that pressure will tend to be at or close to the ice overburden pressure. This observation can be expressed as either relative or absolute terms as (relative)

$$\frac{N}{\rho g h} < \gamma_w \tag{1.43}$$

where $\gamma_w$ is small compared to unity, e.g.

$$\gamma_w \lesssim 0.1$$

or (absolute)

$$N < \Gamma_w$$

where $\Gamma_w$ is a dimensional number based on observations, e.g. $\Gamma_w \lesssim 1$ kPa.

#### Perfect hydrological connection

It is sometimes assumed that the sub-glacial pressure in the vicinity of the grounding line equals the ocean pressure, in which case

$$N = g(\rho h - \rho_o H^+) \tag{1.44}$$

$$= g\rho(h - h_f^+) \, . \tag{1.45}$$

where $H^+$ is the positive ocean depth and $h_f^+$ the positive flotation thickness defined by Eqs. (3) and (1), respectively. It follows that

$$\frac{dN}{dh} = \begin{cases} \rho g & \text{if} \quad h \geq h_f \\ 0 & \text{if} \quad h < h_f \end{cases}$$

when $h \geq h_f$, and this might seem rather unrealistic if $h \gg h_f$.

Note that the assumption of perfect hydrological implies

$$\begin{aligned} \gamma_w &= \frac{\rho g (h - h_f^+)}{\rho g h} \\ &= 1 - h_f^+/h \end{aligned}$$

clearly violating the relative formulation of the first law of observational subglacial hydrology (Eq. 1.43) whenever $h_f^+ < (1 - \gamma_w) h$.

### Rosier hydrology

The Rosier hydrology, defined as

$$N = \min\left(\rho g(h - h_f^+), \gamma_w \rho g h\right)$$

where

$$\gamma_w = (e\pi)^{-1} \approx 0.117$$

is one approach to satisfy the first law in its relative from.[6] For the Rosier hydrology

$$\frac{dN}{dh} = \begin{cases} \gamma_w \rho g & \text{if} \quad h > h_f/(1-\gamma_w) \\ \rho g & \text{if} \quad h_f \leq h \leq h_f/(1-\gamma_w) \\ 0 & \text{if} \quad h < h_f \end{cases}$$

### Passive hydrology

The *passive hydrology* model is motivated by the absolute formulation of the first law. It does not predict the exact value of $N$ at all, but states that the effective pressure is independent of ice thickness, i.e.

$$\frac{\partial N}{\partial h} = 0 \, .$$

It can be argued that currently many (most?) ice-sheet models implicitly assume the hydrology to be passive in this sense.

## 1.8 Flow law

Glen's flow law is

$$\dot{\epsilon}_{ij} = A \tau^{n-1} \tau_{ij},$$

where

$$\tau = \sqrt{\tau_{ij}\tau_{ij}/2}$$

The flow law can also be written as

$$\tau_{ij} = A^{-1/n} \, \dot{\epsilon}^{(1-n)/n} \, \dot{\epsilon}_{ij}, \tag{1.46}$$

where

$$\dot{\epsilon} = \sqrt{\dot{\epsilon}_{ij}\dot{\epsilon}_{ij}/2}$$

which in the Shallow Ice Stream Approximation takes the form

$$\begin{aligned} \dot{\epsilon} &= \sqrt{(\dot{\epsilon}_{xx})^2 + (\dot{\epsilon}_{yy})^2 + \dot{\epsilon}_{xx}\,\dot{\epsilon}_{yy} + (\dot{\epsilon}_{xy})^2} && (1.47) \\ &= ((\partial_x u)^2 + (\partial_y v)^2 + \partial_x u \, \partial_y v + (\partial_x v + \partial_y u)^2/4)^{1/2} && (1.48) \end{aligned}$$

If we write

$$\tau_{ij} = 2\eta \dot{\epsilon}_{ij}$$

then $\eta$ is the effective viscosity given by

$$\eta = \frac{1}{2} A^{-1/n} \, \dot{\epsilon}^{(1-n)/n}$$

or

$$\eta = \frac{1}{2} A^{-1/n} \, ((\partial_x u)^2 + (\partial_y v)^2 + \partial_x u \, \partial_y v + (\partial_x v + \partial_y u)^2/4)^{(1-n)/2n} \tag{1.49}$$

[6] The exact mathematical expression for $\gamma_w$ is based on unpublished work by S. Rosier but appears to be in parts motivated by aesthetic principles.

## 1.9 Floating relationships

For a given bedrock geometry $B$ and ocean surface $S$ the ice is afloat for $h < h_f$ where

$$h_f := \rho_o H/\rho \tag{1.50}$$

For $h \geq h_f$ the glacier is grounded, and at the grounding line we have

$$h = h_f,$$
$$d = H,$$

where $h_f$ is defined by Eq. (1.50).

Where the ice is afloat, we have the floating condition

$$\rho g\,(s-b) = \rho_o g\,(S-b) \tag{1.51}$$

or equivalently with $h = s - b$ and $d = S - b$,

$$\rho g h = \rho_o g d\;.$$

Various other relations are obtained by rearranging the floating condition (1.51), for example

$$h = \rho_o d/\rho = \frac{s-S}{1-\rho/\rho_o} = \frac{\rho_o}{\rho}(S-b), \tag{1.52}$$
$$b = \frac{\rho_o S - \rho s}{\rho_o - \rho} = S - \frac{\rho}{\rho_o}h, \tag{1.53}$$
$$s = S + (1-\rho/\rho_o)h = (1-\rho_o/\rho)b + \frac{\rho_o}{\rho}S, \tag{1.54}$$
$$f := s - S = (1-\rho/\rho_o)h. \tag{1.55}$$

Furthermore, if $\partial_x S = 0$ the slopes of the upper and the lower boundary are related through

$$b\,\partial_x s - s\,\partial_x b = S\,\partial_x h, \tag{1.56}$$

and also

$$\partial_x s = (1-\rho/\rho_o)\partial_x h.$$

### 1.9.1 Expressing geometrical variables in terms of ice thickness

For a fully implicit treatment, i.e. implicit with respect both velocity and geometry, is advantageous to be able to express geometrical variables such as $s$, $b$, and $d$ in terms of ice thickness $h$.

It is easy to see that

$$s = \mathcal{H}(h-h_f)\,(h+B) + \mathcal{H}(h_f-h)\,(S + (1-\rho/\rho_o)\,h), \tag{1.57}$$
$$b = \mathcal{H}(h-h_f)\,B + \mathcal{H}(h_f-h)\,(S - \rho h/\rho_o), \tag{1.58}$$

and that

$$d := \mathcal{H}(H)(S-b) \tag{1.59}$$
$$= \mathcal{H}(H)\,[\mathcal{H}(h_f-h)\rho h/\rho_o + \mathcal{H}(h-h_f)H], \tag{1.60}$$

i.e.

$$d = \begin{cases} H, & \text{if } h > h_f \text{ and } H > 0 \\ \rho h/\rho_o, & \text{if } h < h_f \text{ and } H > 0 \\ 0, & \text{if } H < 0 \end{cases}$$

The draft is always $0 \leq d \leq \rho h/\rho_o$.

Eq. (1.60) can be simplified a bit further by noticing that if $H > 0$ then $\mathcal{H}(H)\mathcal{H}(h_f-h) = \mathcal{H}(h_f-h)$. On the other hand if $H < 0$ then $\mathcal{H}(H) = 0$ but so is $\mathcal{H}(h_f-h)$ because if $H = S - B < 0$ then $h_f = \frac{\rho_o}{\rho}(S-B) < 0$ and since $h$ is always positive we have $\mathcal{H}(h_f-h) = 0$, i.e.

$$\mathcal{H}(H)\mathcal{H}(h_f-h) = \mathcal{H}(h_f-h),$$

and $d$ can therefore be written as

$$d = \mathcal{H}(h_f - h)\rho h/\rho_o + \mathcal{H}(H)\mathcal{H}(h - h_f)H. \tag{1.61}$$

or as

$$d = \mathcal{H}(h_f - h)\rho h/\rho_o + \mathcal{H}(h - h_f)H^+. \tag{1.62}$$

using

$$H^+ := \mathcal{H}(H)\,H.$$

### 1.9.2 Calculating $b$ and $s$ given $h$, $S$ and $B$

If we think of $S$, $B$, and $h$ as independent variables, i.e.

$$s = s(h, S, B, \rho, \rho_o)$$
$$b = b(h, S, B, \rho, \rho_o)$$

then $s$ and $b$ are given by

$$\begin{aligned} s &= \mathcal{G}(B + h) + (1 - \mathcal{G})\left((1 - \rho_o/\rho)b + \rho_o S/\rho\right), \\ &= \mathcal{G}(B + h) + (1 - \mathcal{G})\left(b + \frac{\rho_o}{\rho}(S - b)\right), \\ b &= \mathcal{G}B + (1 - \mathcal{G})(S - \rho h/\rho_o), \end{aligned}$$

valid both over the grounded and the floating sections. where again the grounding/flotation mask $\mathcal{G}$ is defined as

$$\mathcal{G} := \mathcal{H}(h - h_f),$$

with

$$h_f := \rho_o(S - B)/\rho.$$

Here $s$ and $b$ are explicit functions of $h$ because for $h$ and $S$ and $B$ given, the flotation/grounding mask $\mathcal{G}$ is already determined.

In numerical calculations, it may sometimes be desired to use a smooth and continuous approximation of the Heaviside step function.[7] However, any smooth approximation of the Heaviside step function will inescapably imply that the upper and lower surfaces are not in a perfect point-wise agreement with flotation/grounding conditions. In particular, we can have the situation where the lower ice surface $b$ is lower than the bedrock $B$. To see this note that the condition $b \geq B$ implies

$$\begin{aligned} b - B &= B\,(\mathcal{G} - 1) + (1 - \mathcal{G})(S - \rho h/\rho_o) \\ &= (S - B)(1 - \mathcal{G}) - (1 - \mathcal{G})\rho h/\rho_o \\ &= H(1 - \mathcal{G}) - (1 - \mathcal{G})\rho h/\rho_o \\ &= (H - \rho h/\rho_o)\,(1 - \mathcal{G}) \\ &\geq 0 \end{aligned}$$

or

$$(\rho_0 H/\rho - h)\,\mathcal{H}(\rho_0 H/\rho - h) \geq 0$$

or simply

$$x\,\mathcal{H}(x) \geq 0\ .$$

Whenever $\mathcal{H}(x)$ is an exact step function, this inequality is clearly fulfilled for any $x$. If, however, $\mathcal{H}(x)$ is approximated by some smooth function this is no longer the case. For example, if $\mathcal{H}(-\epsilon) = \delta > 0$ where $\epsilon$ and $\delta$ are some arbitrarily small positive numbers, then $x\mathcal{H}(x) < 0$ for $x = -\epsilon$ and therefore $b < B$. One can argue that for any sensible approximation of the Heaviside step function, we must have $\mathcal{H}(x) \geq 1/2$ for $x \geq 0$. Assuming this, then at the grounding line where $h = h_f$, we have $\mathcal{G} = 1/2$ and therefore

$$\begin{aligned} b &= B/2 + (S - \rho h_f/\rho_o)/2 \\ &= B/2 + (S - (S - B))/2 \\ &= B \end{aligned}$$

[7]This is, for example, needed to achieve second-order convergence in the NR iteration.

and downstream of the grounding line $b > B$. However, upstream of the grounding line any approximation of the Heaviside step function unavoidably implies that the we may encounter a situation where numerically $b < B$. This can be seen to be a consequence of $b$ being partly determined by flotation conditions upstream of the grounding line despite $h$ there being greater than flotation thickness $h_f$.

### 1.9.3 Calculating $b$ and $h$ given $s$, $S$ and $B$

The lower glacier surface ($b$) and the ice thickness ($h$) can be calculated from the upper glacier surface ($s$), the ocean surface ($S$) and the bedrock ($B$) as

$$b = \max\left\{B, \frac{\rho s - \rho_o S}{\rho - \rho_o}\right\} , \tag{1.63}$$

$$h = s - b\ . \tag{1.64}$$

Hence, $b$ should never be lower than the bedrock, $B$, or the flotation limit for the lower surface. Again as discussed above, this is potentially violated if flotation mask is calculated using a (slightly) smoothed step function (as is done in Úa). For that reason this problem is a bit more complicated, and in Úa it is solved implicitly as follows. We write

$$b = b(s, S, B, \rho, \rho_o)\ ,$$
$$h = h(s, S, B, \rho, \rho_o)\ ,$$

as

$$b = \mathcal{G}\ B + (1-\mathcal{G})\frac{\rho s - \rho_o S}{\rho - \rho_o}, \tag{1.65}$$

$$h = \mathcal{G}\ (s - B) + (1-\mathcal{G})\frac{s - S}{1 - \rho/\rho_o}, \tag{1.66}$$

where

$$\mathcal{G} = \mathcal{H}(h - \rho_o(S - B)/\rho)\ . \tag{1.67}$$

This is a non-linear system because $\mathcal{G}$ depends on $h$. Finding $b$ given $s$, $S$ and $B$ is equivalent to solving the equation

$$F_b(b) = 0\ ,$$

where the function $F_b$ is defined as

$$F_b(b) := b - \mathcal{G}\,B - (1-\mathcal{G})(\rho s - \rho_o S)/(\rho - \rho_o)\ ,$$

and $\mathcal{G}$ and $h$ are defined in terms of $b$ through Eqs. (1.65) and (1.66)

We can calculate $b$ from $s$, $S$ and $B$ given $\rho$ and $\rho_o$ by minimising the cost function $J$ defined as

$$J = \frac{1}{2}\|F_b\|^2_{l^2} = \frac{1}{2}\boldsymbol{F}_b \cdot \boldsymbol{F}_b$$

with respect to $b$ using the NR method as follows: For given $s$, $S$, $B$, $\rho$ and $\rho_o$, and an initial guess for $b$ and $h$ repeat:

$$\begin{aligned}
\mathcal{G} &= \mathcal{H}(h - \rho_o(S - B)/\rho) \\
F_b &= b - \mathcal{G}\,B - (1-\mathcal{G})(\rho s - \rho_o S)/(\rho - \rho_o) \\
\partial F_b/\partial b &= 1 + \delta(h - h_f)\ (B - (\rho s - \rho_o S)/(\rho - \rho_o)) \\
(\partial F_b/\partial b)\Delta b &= -F_b \\
b &= b + \Delta b \\
h &= s - b
\end{aligned}$$

until $\|J\| < \epsilon$. An initial guess for $b$ and $h$ is provided by Eqs. (1.63) and (1.64).

## 1.10 Stress boundary conditions at an ice front

We consider the case of an ice front in contact with water of a given depth. The treatment is general and includes the cases of zero water depth, i.e. glacier terminating on land, and a floating ice front, i.e. a glacier terminating in an ocean.

At the calving front the jump condition is

$$\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}_{xy} = -p_o \hat{\boldsymbol{n}}_n.$$

where $p_o$ is the hydrostatic ocean pressure, and

$$\hat{\boldsymbol{n}}_{xy} = (n_x, n_y, 0)^T,$$

is a unit normal pointing horizontally outward from the ice front. The vertically integrated form of this stress condition is

$$\int_b^s (\sigma_{xx} n_x + \sigma_{xy} n_y)\, dz = -\int_b^S p_o n_x \, dz \quad \text{on} \quad \Gamma_2 \tag{1.68}$$

$$\int_b^s (\sigma_{xy} n_x + \sigma_{yy} n_y)\, dz = -\int_b^S p_o n_y \, dz \quad \text{on} \quad \Gamma_2 \tag{1.69}$$

If the draft $d$ at the ice front is zero, i.e. if the ice front is fully grounded, then $S < b$, the right hand sides of (1.68) and (1.69) are to be set to zero.

Using

$$\sigma_{xx} = 2\tau_{xx} + \tau_{yy} + \sigma_{zz},$$

and with

$$\sigma_{zz} = -\rho g (s - z),$$

(where we have set $\alpha = 0$), if follows that

$$\begin{aligned} \int_b^s \sigma_{xx}\, dz &= \int_b^s (2\tau_{xx} + \tau_{yy})\, dz - \int_b^s \rho g (s - z)\, dx \\ &= h(2\tau_{xx} + \tau_{yy}) - \frac{\rho g}{2} h^2. \end{aligned}$$

The $x$ component of the vertically integrated ocean pressure acting on the calving front is

$$\begin{aligned} -\int_b^S p_o n_x \, dz &= -\int_b^S \rho_o g (S - z) n_x \, dz \\ &= -\frac{1}{2} \rho_o g (S - b)^2 \\ &= -\frac{1}{2} \rho_o g d^2. \end{aligned}$$

Boundary conditions (1.68) and (1.69) can therefore be written as

$$h(2\tau_{xx} + \tau_{yy}) n_x + h \tau_{xy} n_y = \frac{g}{2} (\rho h^2 - \rho_o d^2)\, n_x, \tag{1.70}$$

$$h(2\tau_{yy} + \tau_{xx}) n_y + h \tau_{xy} n_x = \frac{g}{2} (\rho h^2 - \rho_o d^2)\, n_y, \tag{1.71}$$

or more compactly as

$$\boldsymbol{R} \cdot \hat{\boldsymbol{n}}_{xy} = \frac{g}{2h} (\rho h^2 - \rho_o d^2) \hat{\boldsymbol{n}}_{xy}. \tag{1.72}$$

The boundary condition (1.72) is valid for both grounded and floating ice edges.

### 1.10.1 Floating

In the particular case where the calving front is afloat, $\rho h = \rho_o d$ boundary conditions (1.70) and (1.71) simplify to

$$h(2\tau_{xx} + \tau_{yy})n_x + h\tau_{xy}\,n_y = \frac{1}{2}\varrho g h^2 n_x \tag{1.73}$$
$$h(2\tau_{yy} + \tau_{xx})n_y + h\tau_{xy}\,n_x = \frac{1}{2}\varrho g h^2 n_y \tag{1.74}$$

where

$$\varrho := \rho(1 - \rho/\rho_o),$$

Written in terms of the velocity components the boundary conditions along a floating ice front are:

$$\eta h(4\partial_x u + 2\partial_y v)n_x + \eta h(\partial_x v + \partial_y u)n_y = \frac{\varrho g h^2}{2} n_x, \tag{1.75}$$
$$\eta h(\partial_x v + \partial_y u)n_x + \eta h(4\partial_y v + 2\partial_x u)n_y = \frac{\varrho g h^2}{2} n_y. \tag{1.76}$$

### 1.10.2 Grounded

On the other hand if the ice terminates on land then $d = 0$ and

$$h(2\tau_{xx} + \tau_{yy})n_x + h\tau_{xy}\,n_y = \frac{g}{2}\rho h^2\, n_x, \tag{1.77}$$
$$h(2\tau_{yy} + \tau_{xx})n_y + h\tau_{xy}\,n_x = \frac{g}{2}\rho h^2\, n_y. \tag{1.78}$$

## 1.11 Boundary condition at a glacier terminus as a natural boundary condition

For solving (1.1) and (1.2) it is advantageous to modify the equations in such a way that the boundary conditions (1.70) and (1.71) become the 'natural' boundary conditions. Furthermore, for an implicit time integration with respect to both velocities, grounding-line position, and ice thickness, it is of advantage to write all evolving geometrical variables ($s$, $b$) in terms of the ice thickness $h$.

The key idea is to rewrite (assuming $\alpha = 0$) the Eqs. (1.1) and (1.2) as

$$\partial_x(h(2\tau_{xx} + \tau_{yy})) + \partial_y(h\tau_{xy}) - t_{bx} = \frac{1}{2} g\partial_x(\rho h^2 - \rho_o d^2) + g(\rho h - \rho_o d)\partial_x b, \tag{1.79}$$
$$\partial_y(h(2\tau_{yy} + \tau_{xx})) + \partial_x(h\tau_{xy}) - t_{by} = \frac{1}{2} g\partial_y(\rho h^2 - \rho_o d^2) + g(\rho h - \rho_o d)\partial_y b, \tag{1.80}$$

(note the $d$ term is not missing a $\mathcal{H}(H)$ because $d$ is automatically zero whenever $\mathcal{H}(H) = 0$.) where it has been used that $\partial_x \rho_o = \partial_y \rho_o = 0$ and $\partial_x S = \partial_y S = 0$. Note that in Eqs. (1.79) and (1.80) the second terms on the right hand sides are automatically zero where the ice is afloat and that this formulation can also be used if the ice density varies in the horizontal.

The equality of the right-hand terms in (1.1) and (1.79) (for $\alpha = 0$) follows from

$$\begin{aligned}
\frac{1}{2} g\partial_x(\rho h^2 - \rho_o d^2) + g(\rho h - \rho_o d)\partial_x b &= \frac{1}{2} g h^2 \partial_x \rho + g(\rho h \partial_x h - \rho_o d \partial_x d) + g(\rho h - \rho_o d)\partial_x b \\
&= \frac{1}{2} g h^2 \partial_x \rho + g(\rho h \partial_x (s - b) - \rho_o d \partial_x d) + g(\rho h - \rho_o d)\partial_x b \\
&= \frac{1}{2} g h^2 \partial_x \rho + g(\rho h \partial_x s - \rho_o d \partial_x d) - g\rho_o d \partial_x b \\
&= \frac{1}{2} g h^2 \partial_x \rho + g(\rho h \partial_x s - \rho_o d \partial_x (\mathcal{H}(H)(S - b))) - g\rho_o d \partial_x b \\
&= \frac{1}{2} g h^2 \partial_x \rho + g\rho h \partial_x s - g\rho_o d (S - b)\partial_x \mathcal{H}(H)
\end{aligned}$$

and the last term is (in an integrated sense) zero

$$
\begin{aligned}
\int g\rho_o d(S-b)\partial_x \mathcal{H}(H)\,dx &= \int g\rho_o d(S-b)\partial_H \mathcal{H}(H)\,\partial_x H\,dx \\
&= \int g\rho_o d(S-b)\delta(H)\,\partial_x H\,dx \\
&= g\rho_o d\,(S-b) \qquad \text{(for } x \text{ where } H=0\text{)} \\
&= 0
\end{aligned}
$$

where the last step follows from the fact that where $H=0$, we have $S=b$, because if $H=0$, then $h_f = \rho H/\rho_o = 0$, and hence $h \geq h_f$ because $h$ is never negative. Where $H=0$ the ice is therefore grounded, and $B=b$ and therefore $S-b=S-B=H=0$, so $S=b$.

Hence

$$
g\rho h\partial_x s + \frac{1}{2}gh^2\partial_x\rho = \frac{1}{2}g\partial_x(\rho h^2 - \rho_o d^2) + g(\rho h - \rho_o d)\partial_x b. \tag{1.81}
$$

Because $\rho_o d \leq \rho h$, with the equality sign fulfilled where the ice is afloat, the second terms on the right-hand sides of Eqs. (1.79) and (1.80) are positive where the ice is both partly and fully grounded, and zero where it is afloat. Therefore

$$
\begin{aligned}
g(\rho h - \rho_o d)\partial_x b &= \mathcal{H}(h-h_f)g(\rho h - \rho_o d)\partial_x b \\
&= \mathcal{H}(h-h_f)g(\rho h - \rho_o d)\partial_x B \\
&= \mathcal{H}(h-h_f)g(\rho h - \rho_o H^+)\partial_x B
\end{aligned}
$$

where we used (1.62) and

$$
d = \mathcal{H}(h_f - h)\rho h/\rho_o + \mathcal{H}(h-h_f)H^+,
$$

and hence

$$
\mathcal{H}(h-h_f)d = \mathcal{H}(h-h_f)H^+,
$$

again in an integrated sense (i.e. when evaluated under an integral).

The basal drag terms are also zero where the ice is afloat and the system can therefore be written as

$$
\partial_x(h(2\tau_{xx}+\tau_{yy})) + \partial_y(h\tau_{xy}) - t_{bx} = \frac{1}{2}g\partial_x(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h-h_f)(\rho h - \rho_o H^+)\partial_x B \tag{1.82}
$$

$$
\partial_y(h(2\tau_{yy}+\tau_{xx})) + \partial_x(h\tau_{xy}) - t_{by} = \frac{1}{2}g\partial_y(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h-h_f)(\rho h - \rho_o H^+)\partial_y B \tag{1.83}
$$

This suggests how the boundary condition (1.72) can form the natural boundary condition of our FE formulation. This can be achieved by including in the corresponding boundary integral the first terms on the left and right-hand sides of (1.82) and (1.83). The details are given in Section 2.1.

Written in terms of the velocity components:

$$
\begin{aligned}
&\partial_x(h\eta(4\partial_x u + 2\partial_y v)) + \partial_y(h\eta(\partial_y u + \partial_x v)) - t_{bx} = \\
&\qquad \frac{1}{2}g\partial_x(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h-h_f)(\rho h - \rho_o H^+)\partial_x B
\end{aligned} \tag{1.84}
$$

$$
\begin{aligned}
&\partial_y(h\eta(4\partial_y v + 2\partial_x u)) + \partial_x(h\eta(\partial_x v + \partial_y u) - t_{by} = \\
&\qquad \frac{1}{2}g\partial_y(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h-h_f)(\rho h - \rho_o H^+)\partial_y B
\end{aligned} \tag{1.85}
$$

## 1.12 SSTREAM in 1HD

In one horizontal dimension (1HD), i.e. in the flow-line case, the SSTREAM equation becomes

$$
4\partial_x(h\eta\partial_x u) - t_{bx} = \frac{1}{2}g\partial_x(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h-h_f)(\rho h - \rho_o H^+)\partial_x B
$$

with

$$
\eta = \frac{1}{2}A^{-1/n}\dot{\epsilon}^{(1-n)/n} = \frac{1}{2}A^{-1/n}\|\partial_x u\|^{(1-n)/n}
$$

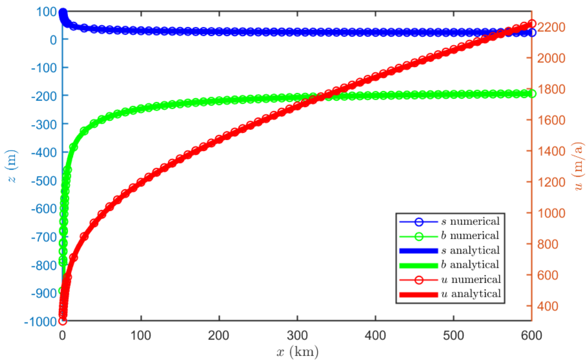



Figure 1.2: Comparison between analytical and numerical solutions for a one-dimensional ice shelf. Parameters: $A = 1.14 \times 10^{-9}\,\mathrm{kPa}^{-3}\mathrm{a}^{-1}$, $n = 3$, $h_{gl} = 1000\,\mathrm{m}$, $u_{gl} = 300\,\mathrm{m\,a}^{-1}$, $\rho = 910\,\mathrm{kg\,m}^{-3}$, $\rho_o = 1030\,\mathrm{kg\,m}^{-3}$. The value for $A$ corresponds to an ice temperature of about -10 degrees Celsius.

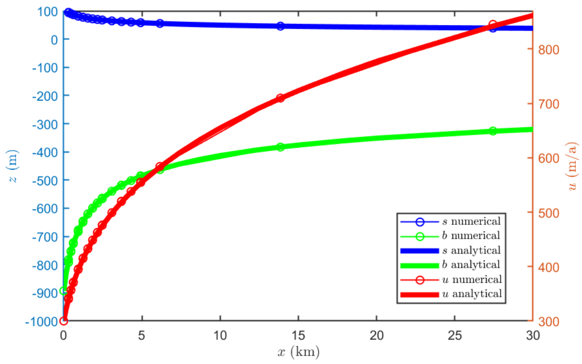



Figure 1.3: Close up of velocity and surface profiles. The analytical solutions are derived in section 9.5, see for example Eq. (9.43). This numerical solution was obtained using linear elements and automated mesh refinement based on the gradient of the effective strain rate.

if we use Glen's flow law, and with

$$t_{bx} = \mathcal{H}(h - h_f)\, C^{-1/m} |u|^{1/m-1}\, u$$

if we use Weertman sliding law.

Inserting Glen's flow law we get

$$2\partial_x (A^{-1/n}\, h\, |\partial_x u|^{(1-n)/n} \partial_x u) - t_{bx} = \frac{1}{2} g \partial_x (\rho h^2 - \rho_o d^2) + g\, \mathcal{H}(h - h_f)(\rho h - \rho_o H^+) \partial_x B \tag{1.86}$$

and if we can assume that $u > 0$ and $\partial_x u > 0$ then the SSTREAM equation is

$$2\partial_x \left( A^{-1/n}\, h\, (\partial_x u)^{1/n} \right) - \mathcal{H}(h - h_f) C^{-1/m} u^{1/m} = \rho g h \partial_x s + \frac{1}{2} g h^2 \partial_x \rho \tag{1.87}$$

Eq. (1.87) is a fairly common way of writing down the SSTREAM/SSA equation in 1HD.

# Chapter 2

# Finite-element implementation

## 2.1 FE formulation of the diagnostic equations

In the FE method the inner product of the field equations with a test function is formed. The inner product is

$$< \phi \mid \theta >= \iint_\Omega \phi\, \theta \, dx\, dy$$

where $\phi$ and $\theta$ are some functions. One form of Green's theorem states that

$$\iint_\Omega \phi \, \partial_x \theta \, dx\, dy = -\iint_\Omega \partial_x \phi\, \theta \, dx\, dy + \oint_\Gamma \phi\, \theta\, n_x \, d\Gamma$$

Applying the Green's theorem on the stress terms and the first term on the right-hand side of that of Eq. (1.82), i.e. $x$ direction, leads to

$$\begin{aligned} 0 = &\iint_\Omega \Big( h(2\tau_{xx} + \tau_{yy})\partial_x \phi + h\tau_{xy}\, \partial_y \phi - \frac{1}{2} g(\rho h^2 - \rho_o d^2)\, \partial_x \phi + t_{bx}\, N \\ &+ \phi\, g\, \mathcal{H}(h - h_f)(\rho h - \rho_o H^+)\partial_x B \Big)\, dx\, dy \\ &- \oint_\Gamma (h(2\tau_{xx} + \tau_{yy})\, \phi\, n_x + h\tau_{xy}\, \phi\, n_y - \frac{1}{2} g(\rho h^2 - \rho_o d^2)\, \phi n_x)\, , d\Gamma \end{aligned}$$

Performing the same calculation of the $y$ direction results in boundary terms that are identically equal to zero for the boundary condition (1.72). The natural boundary condition is therefore exactly (1.72) and covers not only the case of a fully floating ice front, but that of a grounded and partially grounded ice fronts as well. Using Eq. (1.60) the draft ($d$) appearing the equations above can be written in terms of the ice thickness ($h$). This formulation is therefore well suited as a starting point for a linearisation around $h$ required for a fully implicit solution of transient flow.

Expressing this equation in terms of the velocity components $u$ and $v$

$$\begin{aligned} 0 = &\iint_\Omega (h\eta(4\partial_x u + 2\partial_y v)\partial_x \phi + h\eta(\partial_y u + \partial_x v)\partial_y \phi + \mathcal{H}(h - h_f)\beta^2 u\, \phi \\ &- \frac{1}{2} g(\rho h^2 - \rho_o d^2)\partial_x \phi + \phi\, g\, \mathcal{H}(h - h_f)(\rho h - \rho_o H^+)\partial_x B)\, dx\, dy \\ &- \oint_\Gamma (h\eta(4\partial_x u + 2\partial_y v)\, \phi\, n_x + h\eta(\partial_y u + \partial_x v)\, \phi\, n_y - \frac{1}{2} g(\rho h^2 - \rho_o d^2)\, N)\, n_x\, d\Gamma \end{aligned} \tag{2.1}$$

$$\begin{aligned} 0 = &\iint_\Omega (h\eta(4\partial_y v + 2\partial_x u)\partial_y \phi + h\eta(\partial_x v + \partial_y u)\partial_x \phi + \mathcal{H}(h - h_f)\beta^2 v\, \phi \\ &- \frac{1}{2} g(\rho h^2 - \rho_o d^2)\partial_y \phi + \phi\, g\, \mathcal{H}(h - h_f)(\rho h - \rho_o H^+)\partial_x B)\, dx\, dy \\ &- \oint_\Gamma (h\eta(4\partial_y v + 2\partial_x u)\, \phi\, n_y + h\eta(\partial_x v + \partial_y u)\, \phi\, n_x - \frac{1}{2} g(\rho h^2 - \rho_o d^2)\, N)\, n_y\, d\Gamma \end{aligned} \tag{2.2}$$

where the corresponding expression in $y$ direction has been added.

## 2.2 FE formulation of the prognostic equations

*Úa* allows for fully implicit time integration with respect to both geometry, grounding-line migration, and velocity. This approach is not limited by the CFL condition and is unconditionally stable allowing for arbitrarily large times steps irrespective of spatial discretisation. The time step is only limited by the convergence radius of the Newton-Raphson method[1].

*The recommended option in a transient run is to use a fully implicit $\Theta$ method combined with the consistent streamline-upwind Petrov-Galerkin method (SUPG). This is the default option.*

In *Úa* a semi-implicit approach (implicit with respect to geometry, explicit with respect to velocity) is also implemented. Unless memory is a limiting factor, the fully implicit approach is always preferable to the semi-implicit (staggered) approach.

Experience shows the $\Theta$ method to give good results when used in a combination with a fully implicit forward time integration. For a semi-implicit approach a third-order Taylor Galerkin (TG3) is a better approach.

In 2HD both $\Theta$ and TG3 have been implemented for both staggered and implicit approach. (The 1HD fully implicit was only done using the $\Theta$ method and not using TG3.) Using TG3 in 1HD staggered approach resulted in a great improvement over the $\Theta$ method. It appears that in the implicit approach there is no great advantage of using TG3.

There is no separate diffusion term added to the prognostic equations in *Úa* , and no shock-stabilisation term either. Even just using the fully implicit approach without SUPG generally gives good results. But using SUPG is nevertheless recommended, especially for problems involving grounding line migration.

### 2.2.1 Mass flux equation

The vertically integrated form of the mass conservation equation used in *Úa* is Eq. (1.17).

### 2.2.2 $\Theta$ method or the 'generalised trapezoidal rule'

In the $\Theta$ method the left-hand side is approximated by the discrete first-order derivative $\Delta h/\Delta t = (h_1 - h_0)/(t_1 - t_0)$ and the right-hand side is replace by an weighted average of the values at time step $t = t_1$ and $t = t_0$, i.e.

$$\frac{\Delta h}{\Delta t} = \Theta \partial_t h_1 + (1 - \Theta)\partial_t h_0$$

where

$$\rho \partial_t h_0 = \rho a_0 - \partial_x q_{x0} - \partial_y q_{y0}$$
$$\rho \partial_t h_1 = \rho a_1 - \partial_x q_{x1} - \partial_y q_{y1}$$

and $0 \le \Theta \le 1$. For $\Theta > 0$ the resulting system is implicit with respect to both thickness ($h$) and velocity ($u$ and $v$).

### 2.2.3 Third order implicit Taylor Galerkin (TG3)

This method is also referred to as fourth-order Crank-Nicolson time-stepping.

First expand $h$ at time step 1 and 0 using third order Taylor expansion as

$$h_1 = h_0 + \Delta t\, \partial_t h_0 + \frac{(\Delta t)^2}{2} \partial^2_{tt} h_0 + \frac{(\Delta t)^3}{6} \partial^3_{ttt} h_0,$$

$$h_0 = h_1 - \Delta t\, \partial_t h_1 + \frac{(\Delta t)^2}{2} \partial^2_{tt} h_1 - \frac{(\Delta t)^3}{6} \partial^3_{ttt} h_1,$$

adding and simplifying gives

$$\frac{1}{\Delta t}(h_1 - h_0) = \frac{1}{2}(\partial_t h_0 + \partial_t h_1) + \frac{\Delta t}{4}(\partial^2_{tt} h_0 - \partial^2_{tt} h_1) + \frac{(\Delta t)^2}{12}(\partial^3_{ttt} h_0 + \partial^3_{ttt} h_1) \quad (2.3)$$

Note that including only the first term of the Taylor expansion is equal to using the $\Theta$ method with $\Theta = 1/2$.)

[1] Joseph Raphson published his method 50 years earlier than Newton in his book *Analysis Aequationum Universalis*, Raphson (2011)

Then replace the third-order derivative is expressed through finite differences giving

$$\begin{aligned}\frac{1}{\Delta t}(h_1 - h_0) &= \frac{1}{2}(\partial_t h_0 + \partial_t h_1) + \frac{\Delta t}{4}(\partial^2_{tt} h_0 - \partial^2_{tt} h_1) + \frac{(\Delta t)^2}{12\Delta t}(\partial^2_{tt}(h_1 - h_0) + \partial^2_{tt}(h_1 - h_0)) \\ &= \frac{1}{2}(\partial_t h_0 + \partial_t h_1) + \frac{\Delta t}{4}(\partial^2_{tt} h_0 - \partial^2_{tt} h_1) + \frac{\Delta t}{6}\,\partial^2_{tt}(h_1 - h_0)\end{aligned}$$

i.e.

$$h_1 - h_0 = \frac{\Delta t}{2}(\partial_t h_0 + \partial_t h_1) + \frac{(\Delta t)^2}{12}(\partial^2_{tt} h_0 - \partial^2_{tt} h_1) \tag{2.4}$$

The final step is to replace time derivatives with spatial derivatives through repeated use of the prognostic equation.

The flux $\boldsymbol{q}$ is a time dependent function of both $h$ and $\boldsymbol{v}$, using the prognostic equation the second time derivative of $h$ can be written as

$$\begin{aligned}\rho\partial^2_{tt} h &= \rho\partial_t a - \nabla_{xy} \cdot \partial_t \boldsymbol{q} \\ &= \rho\partial_t a - \nabla_{xy} \cdot (\partial_h \boldsymbol{q}\,\partial_t h + (\nabla_{uv}\boldsymbol{q}) \cdot \partial_t \boldsymbol{v}) \\ &= \rho\partial_t a - \nabla_{xy} \cdot (\partial_h \boldsymbol{q}(\rho a - \nabla_{xy} \cdot \boldsymbol{q})/\rho + (\nabla_{uv}\boldsymbol{q}) \cdot \partial_t \boldsymbol{v})\end{aligned}$$

or

$$\rho\partial^2_{tt} h = \rho\partial_t a - \nabla_{xy} \cdot (\partial_h \boldsymbol{q}(\rho a - \nabla_{xy} \cdot \boldsymbol{q})/\rho + (\nabla_{uv}\boldsymbol{q}) \cdot \partial_t \boldsymbol{v}) \tag{2.5}$$

where $\nabla_{uv} := (\partial_u, \partial_v)$ is the *horizontal velocity gradient operator.*

### Third-order Taylor-Galerkin (TG3) for SSHEET/SSA

Using (2.5) in the SSHEET/SIA approximation where

$$\boldsymbol{q} = \boldsymbol{v}(h),$$

we find that

$$\rho\partial^2_{tt} h = \rho\partial_t a - \nabla_{xy} \cdot (\partial_h \boldsymbol{q}(\rho a - \nabla_{xy} \cdot \boldsymbol{q})).$$

### TG3 for SSTREAM/SSA

Using (2.5) in the SSTREAM/SSA approximation where $\boldsymbol{q} = \rho h \boldsymbol{v}$ we find[2]

$$\rho\partial^2_{tt} h = \rho\partial_t a - \nabla_{xy} \cdot (\boldsymbol{v}(\rho a - \nabla_{xy} \cdot \boldsymbol{q}) + \rho h\,\partial_t \boldsymbol{v}) \tag{2.7}$$

The Third-Order-Taylor-Galerkin (TG3) method is obtained by inserting Eqs. (1.18) and (2.7) into Eq. (2.4) leading to

$$\begin{aligned}< \rho(h_1 - h_0), N > = {}& \frac{\Delta t}{2}(< \rho a_0 - \nabla_{xy} \cdot \boldsymbol{q}_0 \mid N > + < \rho a_1 - \nabla_{xy} \cdot \boldsymbol{q}_1 \mid N >) \\ &+ \frac{1}{2}\frac{\Delta t^2}{6}(< \rho a_0 - \nabla_{xy} \cdot \boldsymbol{q}_0 \mid \boldsymbol{v}_0 \cdot \nabla_{xy} N > - < \rho a_1 - \nabla_{xy} \cdot \boldsymbol{q}_1 \mid \boldsymbol{v}_1 \cdot \nabla_{xy} N >)\end{aligned} \tag{2.8}$$

(where a few terms involving $\partial_t u$ and $\partial_t a$ have been omitted as well as the boundary terms, see below). This from is suitable as a starting point of a fully implicit approach, i.e. where both thickness and velocity is solved for implicitly. Note that the higher-order terms (i.e. those of second and third order) in the implicit TG3 method for $t = t_0$ and $t_1 = t_0 + \Delta t$ have opposite signs. In steady-state they will therefore cancel each other out.

[2] This expression can also be derived operating on each component as follows

$$\begin{aligned}\partial^2_{tt} h &= \partial_t a - \partial^2_{tx}(hu) - \partial^2_{ty}(hv) \\ &= \partial_t a - \partial_x(h\partial_t u + u\partial_t h) - \partial_y(h\partial_t v + v\partial_t h)\end{aligned}$$

leading to

$$\partial^2_{tt} h = \partial_t a - \partial_x(h\partial_t u + u(a - \partial_x(hu) - \partial_y(hv))) - \partial_y(h\partial_t v + v(a - \partial_x(hu) - \partial_y(hv))) \tag{2.6}$$

which is identical to Eq. (2.7).

In more detail the TG3 system is as follows (missing $\rho$ in a number of places):

$$
\begin{aligned}
0 &= \frac{1}{\Delta t}(h_1 - h_0) \\
&- \frac{1}{2}(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y}) + a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})) \\
&- \frac{\Delta t}{12}(\partial_t a_0 - \partial_x(h_0\partial_t u_0 + u_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y}))) - \partial_y(h_0\partial_t v_0 + v_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))) \\
&+ \frac{\Delta t}{12}(\partial_t a_1 - \partial_x(h_1\partial_t u_1 + u_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1}))) - \partial_y(h_1\partial_t v_1 + v_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1}))))
\end{aligned}
\tag{2.9}
$$

For (2.9) corresponding Galerkin system is

$$
\begin{aligned}
0 = &\int (h_1 - h_0 - \frac{\Delta t}{2}(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y}) + a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))\, N\, dA \\
&- \frac{\Delta t^2}{12}\int (\partial_t a_0 N + (h_0\partial_t u_0 + u_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))\partial_x N \\
&\qquad + (h_0\partial_t v_0 + v_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))\partial_y N)\, dA \\
&+ \frac{\Delta t^2}{12}\int (\partial_t a_1 N + (h_1\partial_t u_1 + u_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))\partial_x N \\
&\qquad + (h_1\partial_t v_1 + v_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))\partial_y N)\, dA \\
&+ \frac{\Delta t^2}{12}\oint ((h_0\partial_t u_0 + u_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))n_x \\
&\qquad + (h_0\partial_t v_0 + v_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))n_y)\, N\, d\gamma \\
&- \frac{\Delta t^2}{12}\oint ((h_1\partial_t u_1 + u_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))n_x \\
&\qquad + (h_1\partial_t v_1 + v_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))n_y)\, N\, d\gamma
\end{aligned}
\tag{2.10}
$$

where the second order spatial derivatives have be eliminated through partial integration.

The boundary term can be written as

$$
\begin{aligned}
&\frac{\Delta t^2}{12}\oint ((h_0\partial_t u_0 + u_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))n_x \\
&\qquad + (h_0\partial_t v_0 + v_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))n_y)\, N\, d\gamma \\
&- \frac{\Delta t^2}{12}\oint ((h_1\partial_t u_1 + u_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))n_x \\
&\qquad + (h_1\partial_t v_1 + v_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))n_y)\, N\, d\gamma \\
&= \frac{\Delta t^2}{12}\oint ((h_0\partial_t u_0 + u_0\partial_t h_0)n_x + (h_0\partial_t v_0 + v_0\partial_t h_0)n_y)\, N\, d\gamma \\
&- \frac{\Delta t^2}{12}\oint ((h_1\partial_t u_1 + u_1\partial_t h_1)n_x + (h_1\partial_t v_1 + v_1\partial_t h_1)n_y)\, N\, d\gamma \\
&= \frac{\Delta t^2}{12}\oint (\partial_t(q_{x0})n_x + \partial_t(q_{0y})n_y)\, N\, d\gamma - \frac{\Delta t^2}{12}\oint (\partial_t(q_{x1})n_x + \partial_t(h_1 v_1)n_y)\, N\, d\gamma \\
&= \frac{\Delta t^2}{12}\oint \partial_t(h_0\boldsymbol{v}_0 - h_1\boldsymbol{v}_1)\cdot \boldsymbol{n} N\, d\gamma
\end{aligned}
$$

showing that it disappears if $\partial_t q = 0$ over the boundary. Experience suggests that this boundary term can be ignored.

## 2.3 Consistent Streamline-Upwind Petrov-Galerkin (SUPG)

The standard SUPG is on the form

$$
< \rho\partial_t h + \nabla \boldsymbol{q} - \rho a \mid N + M >= 0 \tag{2.11}
$$

where $M$ is a perturbation to the test-function space. In the literature various forms for $M$ have been suggested. One such form is

$$M = \tau\, \boldsymbol{v} \cdot \nabla N$$

where $\tau$ is a parameter with the dimension of time. Note that in (2.11) the added term is applied to all terms, including time derivative. This is sometimes referred to as a 'consistent' weighting. The extra terms are interpreted element-wise, as

$$< \rho\partial_t h + \nabla \boldsymbol{q} - \rho a \mid N > +\beta \sum_e < \rho\partial_t h + \nabla q - \rho a \mid \tau\, \boldsymbol{v} \cdot \nabla N >= 0$$

The extra term, which is considered as a correction term, is zero for an exact solution in the classical sense. There is no one single accepted/optimal way of selecting $\tau$, and in the literature various definition has been proposed.

The SUPG was initially introduced for equations on the form $\partial_t h + \boldsymbol{v} \cdot \nabla h - \nabla \cdot (k \nabla h) + g = 0$ and in this case, and for linear elements and regular grids, the optimal value for $\tau$ is

$$\tau = \frac{l}{2\|\boldsymbol{v}\|} \left( \coth \mathrm{Pe} - \frac{1}{\mathrm{Pe}} \right). \tag{2.12}$$

where the Péclet number is

$$\mathrm{Pe} = \frac{\|\boldsymbol{v}\| l}{2k}$$

with $k$ the diffusivity and $l$ is a measure of the (local) element size. In the limiting case where $k \to 0$, the equation becomes hyperbolic, $\mathrm{Pe} \to +\infty$, and $\tau$ approaches

$$\lim_{\mathrm{Pe} \to +\infty} \tau = \tau_s$$

where $\tau_s$ is defined as

$$\tau_s := \frac{l}{2\|\boldsymbol{v}\|}$$

and perturbation term to the test function has the form

$$M = \frac{l}{2} \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \nabla N$$

If on the other hand $\mathrm{Pe} \to 0$ then (2.12) leads to

$$\tau = \frac{l}{2\|\boldsymbol{v}\|} \frac{\mathrm{Pe}}{3} = \frac{l}{2\|\boldsymbol{v}\|} \frac{\|\boldsymbol{v}\| l}{6k} = \frac{l^2}{12k}\,.$$

in which case

$$M = \frac{l^2}{12k} \boldsymbol{v} \cdot \nabla N$$

The value of $\tau$ given by (2.12) does not depend on $\Delta t$. For transient problems with high Péclet number it has been suggested using

$$\tau = \tau_t$$

where

$$\tau_t := \Delta t / 2 \tag{2.13}$$

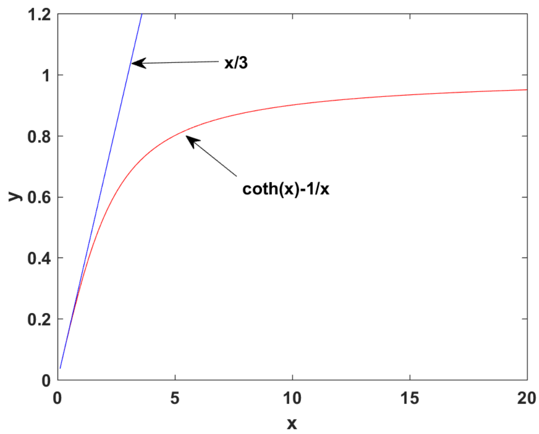
as well as

$$\tau = \tau_s$$

where

$$\tau_s := \frac{l}{2\|\boldsymbol{v}\|} \tag{2.14}$$

The first definition ($\tau_t$) is a temporal criterion while the second ($\tau_s$) is a spatial criterion. The first form is often used in transient situations where diffusion is small and the problem either hyperbolic or close to being hyperbolic. The second form is often used for convection diffusion problems involving temperature such as $\boldsymbol{v} \cdot \nabla T - \nabla(k \nabla T) + f = 0$ (no time dependency). However, as we saw above $\tau_s$ is also the limit of (2.12) for $k \to 0$ and therefore also a potential candidate for $\tau$ in the hyperbolic limit (i.e $\mathrm{Pe} \to +\infty$).



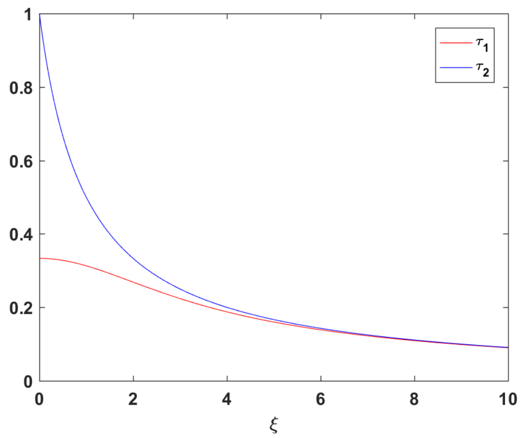




Figure 2.1: SUPG

A guidance as to how to select $\tau$ is looking at some specific limits. The SUPG correction term should vanish if $\Delta t \to 0$, if $l \to 0$, and also if $\|\boldsymbol{v}\| \to 0$. One option for selecting $\tau$, that fulfils these limits, is

$$\tau = \tau_1$$

where $\tau_1$ is defined as

$$\begin{aligned}\tau_1 &:= \tau_s \, \kappa \\ &= \frac{l}{2\,\|\boldsymbol{v}\|}\,\kappa \end{aligned} \tag{2.15}$$

with

$$\kappa = \coth \xi - \frac{1}{\xi} \tag{2.16}$$

and where

$$\begin{aligned}\xi &= \frac{\|\boldsymbol{v}\|}{l}\,\Delta t \\ &= \frac{1}{2\tau_s}\,2\tau_t \\ &= \frac{\tau_t}{\tau_s}\end{aligned} \tag{2.17}$$

is the element Courant number and $l$ is a characteristic local element size. The local element size, $l$, can be defined in a number of similar ways leading to different numerical pre-factors to $\tau$ and $\xi$. The exact functional relationship between $\kappa$ and $\xi$ is also not uniquely defined. We can also write $\tau_1$ as

$$\begin{aligned}\tau_1 :=& \frac{l}{2\,\|\boldsymbol{v}\|}\,\kappa \\ =& \frac{\Delta t}{2}\frac{1}{\xi}\left(\coth(\xi) - \frac{1}{\xi}\right) \\ =& \frac{l}{2\,\|\boldsymbol{v}\|}\left(\coth\left(\frac{\|\boldsymbol{v}\|\,\Delta t}{l}\right) - \frac{l}{\|\boldsymbol{v}\|\,\Delta t}\right)\,.\end{aligned}$$

For $\xi \ll 1$ we have $\kappa \sim \xi/3$, and

$$\tau_1 = \frac{l}{2\,\|\boldsymbol{v}\|}\,\frac{\|\boldsymbol{v}\|\,\Delta t}{3l} = \Delta t/6$$

hence

$$M = \frac{\Delta t}{6}\boldsymbol{v}\cdot\nabla N$$

showing that for $\tau = \tau_1$ the SUPG correction term does indeed go to zero as either $\Delta t \to 0$ and $\|\boldsymbol{v}\| \to 0$. Furthermore, if $l \to 0$ then $\kappa \to 1$ and $\tau_1 \to 0$, so all the above listed limits are obtained with $\tau = \tau_1$ given by (2.15).

Another option of creating a smooth transition between the temporal and spatial criteria 2.13 and 2.14 is to select $\tau$ as

$$\tau = \tau_2$$

where

$$\tau_2 := \left(\frac{1}{\tau_t} + \frac{1}{\tau_s}\right)^{-1} \tag{2.18}$$

which can also be written as

$$\tau_2 := \frac{1}{2}\frac{\Delta t}{1+\xi} \tag{2.19}$$

Expression (2.19) gives the limits

$$\begin{aligned}
M &\to 0 \quad \text{when} \quad \Delta t \to 0\\
M &\to 0 \quad \text{when} \quad \|\boldsymbol{v}\| \to 0\\
M &\to 0 \quad \text{when} \quad l \to 0\\
M &\to \frac{l}{2}\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\cdot\nabla N \quad \text{when} \quad \Delta t \to \infty\\
M &\to \frac{l}{2}\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\cdot\nabla N \quad \text{when} \quad \|\boldsymbol{v}\| \to \infty\\
M &\to \frac{\Delta t}{2}\boldsymbol{v}\cdot\nabla N \quad \text{when} \quad l \to \infty
\end{aligned} \tag{2.20}$$

Apart from a different numerical factor in the last limit, all limits are the same as obtained using definition (2.15). In summary

$$\begin{aligned}
\tau_t &= \frac{\Delta t}{2}\\
\tau_s &= \frac{l}{2\|\boldsymbol{v}\|}\\
\tau_1 &= \frac{l}{2\,\|\boldsymbol{v}\|}\left(\coth\left(\frac{\|\boldsymbol{v}\|\,\Delta t}{l}\right) - \frac{l}{\|\boldsymbol{v}\|\,\Delta t}\right)\\
\tau_2 &= \left(\frac{1}{\tau_t} + \frac{1}{\tau_s}\right)^{-1}.
\end{aligned}$$

The time scales $\tau_1$ and $\tau_2$ can also be written as

$$\tau_1 = \frac{\Delta t}{2}\frac{1}{\xi}\left(\coth\xi - 1/\xi\right) \tag{2.21}$$

$$\tau_2 = \frac{\Delta t}{2}\frac{1}{1+\xi} \tag{2.22}$$

and they are shown in Fig. 2.1b as functions of $\xi$ for $\Delta t/2 = 1$. Only for $\|\boldsymbol{v}\|\Delta t < 2l$ is there any significant difference, and the difference in newer larger than a factor 3 obtained in the limit $\xi \to 0$. It appears unlikely that there will be any significant resulting differences between selecting $\tau = \tau_1$ or $\tau = \tau_2$ (see (2.21) and (2.22)).

For $\tau = \tau_1$ given by (2.15) the SUPG perturbation term is

$$M = \frac{l\kappa}{2\|\boldsymbol{v}\|}\,\boldsymbol{v}\cdot\nabla N$$

Table 2.1: SUPG form function ($M$) limits for different definitions of $\tau$

| | $\Delta t \to 0$ | $\|\boldsymbol{v}\| \to 0$ | $l \to 0$ | $\Delta t \to \infty$ | $\|v\| \to \infty$ | $l \to \infty$ | $\xi \to \infty$ |
|---|---|---|---|---|---|---|---|
| $M = \tau_1 \, \boldsymbol{v} \cdot \nabla N$ | 0 | 0 | 0 | $\frac{l}{2} \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \nabla N$ | $\frac{l}{2} \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \nabla N$ | $\frac{\Delta t}{6} \boldsymbol{v} \cdot \nabla N$ | 0 |
| $M = \tau_2 \, \boldsymbol{v} \cdot \nabla N$ | 0 | 0 | 0 | $\frac{l}{2} \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \nabla N$ | $\frac{l}{2} \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \nabla N$ | $\frac{\Delta t}{2} \boldsymbol{v} \cdot \nabla N$ | 0 |
| $M = \tau_t \, \boldsymbol{v} \cdot \nabla N$ | 0 | 0 | $\frac{\Delta t}{2} \, \boldsymbol{v} \cdot \nabla N$ | $\infty$ | $\infty$ | $\frac{\Delta t}{2} \boldsymbol{v} \cdot \nabla N$ | |
| $M = \tau_s \, \boldsymbol{v} \cdot \nabla N$ | $\frac{l}{2\|\boldsymbol{v}\|} \boldsymbol{v} \cdot \nabla N$ | 0 | 0 | $\frac{l}{2} \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \nabla N$ | $\frac{l}{2} \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \nabla N$ | $\infty$ | |

and the following limits are obtained

$$
\begin{aligned}
M &\to 0 \quad \text{when} \quad \Delta t \to 0 \\
M &\to 0 \quad \text{when} \quad \|\boldsymbol{v}\| \to 0 \\
M &\to 0 \quad \text{when} \quad l \to 0 \\
M &\to \frac{l}{2} \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \nabla N \quad \text{when} \quad \Delta t \to \infty \qquad (2.23) \\
M &\to \frac{l}{2} \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \nabla N \quad \text{when} \quad \|\boldsymbol{v}\| \to \infty \qquad (2.24) \\
M &\to \frac{\Delta t}{6} \boldsymbol{v} \cdot \nabla N \quad \text{when} \quad l \to \infty \qquad (2.25)
\end{aligned}
$$

In the literature it is shown that (2.23) and (2.24) is the optimal choice in the convective limit, i.e. for large element Courant numbers. Limit (2.25) can be justified using Taylor-Galerkin approach (see below).

Implementing SUPG implicitly using the $\Theta$ method leads to

$$
\begin{aligned}
&0 = < \rho(h_1 - h_0)/\Delta t + (1-\Theta)(\nabla_{xy} \cdot \boldsymbol{q}_0 - a_0) + \Theta(\nabla_{xy} \cdot \boldsymbol{q}_1 - a_1) \mid N > \\
&+\beta < \rho(h_1 - h_0)/\Delta t + (1-\Theta)(\nabla_{xy} \cdot q_0 - a_0) + \Theta(\nabla_{xy} \cdot \boldsymbol{q}_1 - a_1) \mid \tau \, ((1-\Theta)\boldsymbol{v}_0 + \Theta \boldsymbol{v}_1) \cdot \nabla_{xy} N >
\end{aligned}
$$

Here the correction/perturbation to the test-function space is a weighted average over the values at the beginning and the end of the time step. This adds another source of non-linearity to the problem. Experience showed this to reduce the radius of convergence considerably and to increase grumpiness on a personal level. Former can be avoided by using the value of perturbation term at the beginning of the time step, i.e.

$$
\begin{aligned}
0 = &< \rho(h_1 - h_0)/\Delta t + (1-\Theta)(\nabla_{xy} \cdot \boldsymbol{q}_0 - a_0) + \Theta(\nabla_{xy} \cdot \boldsymbol{q}_1 - a_1) \mid N > \\
&+ \beta < \rho(h_1 - h_0)/\Delta t + \nabla_{xy} \cdot \boldsymbol{q}_0 - a_0 \mid \tau \, \boldsymbol{v}_0 \cdot \nabla_{xy} N >
\end{aligned}
$$

## 2.4 SIA-motivated diffusion

$$\boldsymbol{q} = \boldsymbol{q}^b + \boldsymbol{q}^d$$

where

$$\boldsymbol{q}^b = \boldsymbol{v} h$$

and

$$\boldsymbol{q}^d = \rho \, D h^{n+2} \, \|\nabla_{xy} s\|^{n-1} \, \nabla s$$

where

$$D = \frac{2A}{n+2} (\rho g)^n$$

$$s = (h + B)\mathcal{H}(h - h_f) + (1 - \mathcal{H}(h - h_f))(S + (1 - \rho/\rho_o) \, h)$$

and therefore (almost)

$$\partial_x s = (\partial_x h + \partial_x B)\mathcal{H}(h - h_f) + (1 - \mathcal{H}(h - h_f))(S + (1 - \rho/\rho_o) \, \partial_x h)$$

This motivates adding a SIA based diffusion term

$$-D < \|\nabla_{xy} s\|^{n-1} \, h^{n+2} \nabla s \mid \nabla_{xy} N >$$

However, this is (currently) not done in $\acute{U}a$.

In 1HD the SIA form of the continuity equation can be written as

$$\rho \partial_t s + \partial_x (k \partial_x s) = \rho a$$

with

$$k := \frac{2\rho A(\rho g)^n}{n+2} \|\partial_x s\|^{n-1} h^{n+2}$$

suggesting a Peclet number

$$\mathrm{Pe} = \frac{uL}{k}$$

## 2.5 Connection between third order Taylor-Galerkin (TG3) and streamline-upwind Petrov-Galerkin (SUPG)

In the context of SUPG, the TG3 system given by (2.8) can be thought of as also introducing extra weighting term beside the standard $N$ term. But those additional weighting terms are only applied to the source term ($a$) and the spatial term, and not to the time-derivative term. Furthermore, as mentioned above in a steady-state these extra weighting terms cancel each other out.

It is instructive to consider as well the case where a second-order forward Taylor expansion

$$h_1 = h_0 + \Delta t \, \partial_t h_0 + \frac{(\Delta t)^2}{2} \partial_{tt}^2 h_0 \tag{2.26}$$

is used instead of the centred expansion given by Eq. (2.4). Inserting (1.18) and (2.7) into (2.26) gives

$$0 = h_1 - h_0 - \Delta t \, (a - \nabla_{xy} \cdot \boldsymbol{q}) - \frac{(\Delta t)^2}{2} (\partial_t a - \nabla_{xy} \cdot (\boldsymbol{v}(a - \nabla_{xy} \cdot \boldsymbol{q}) + h \, \partial_t \, \boldsymbol{v}))$$

The Galerkin system is

$$\begin{aligned} < \rho(h_1 - h_0), N > = \Delta t \ & < \rho a - \nabla_{xy} \cdot \boldsymbol{q}, N > \\ & + \frac{(\Delta t)^2}{2} < \rho \partial_t a, N > - \frac{(\Delta t)^2}{2} < \rho \, h \, \partial_t \boldsymbol{v}, N > \\ & + \frac{(\Delta t)^2}{2} < \rho a - \nabla_{xy} \cdot \boldsymbol{q} \mid \boldsymbol{v} \cdot \nabla_{xy} N > \end{aligned}$$

where a partial integration has been used to get rid of second order derivatives (not writing the boundary terms). The last term is similar to what in some other ad-hoc methods is introduced as a stabilisation term. This term only acts in the direction of flow and is zero transverse to the flow direction. In the above expression all terms are to be evaluated at the beginning of the interval. This approach is usually referred to as the second-order explicit Taylor-Galerkin (TG2e) method. If we evaluate all terms by taking the mean value over the time interval, we get an implicit method, but now the second-order terms do not cancel out in steady-state, and the resulting method is quite similar to the streamline-upwind Petrov-Galerkin.

Dropping the time derivatives of $a$ and $\boldsymbol{v}$, we can rewrite the above system as

$$< \rho(h_1 - h_0)/\Delta t, N > = < \rho a - \nabla_{xy} \cdot \boldsymbol{q}, N + \frac{1}{2} \Delta t \, \boldsymbol{v} \cdot \nabla_{xy} N >$$

showing that the TG2e results in an 'inconsistent' weighting with $\tau = \Delta t$ and $\beta = 1/2$.

Comparing 2.8 with (2.11), TG3 can be interpreted as a some sort of Petrov-Galerkin method where only the spatial terms and the source terms are multiplied by a modified test function. In TG3 the modified test function is

$$N + \frac{\Delta t}{6} \, \boldsymbol{v} \cdot \nabla_{xy} N, \tag{2.27}$$

wheres in SUPG it has the form

$$N + \beta \tau \, \boldsymbol{v} \cdot \nabla_{xy} N. \tag{2.28}$$

The weighting is done inconsistently in TG3, i.e. not over the time-derivative. Apart for the inconsistent weighting used in TG3, the SUPG is equal to TG3 provided the two adjustable parameters $\beta$ and $\tau$ are selected as $\beta = 1/6$ and $\tau = \Delta t$.

The TG3 methods follows automatically from a third-order Taylor expansion and involves no adjustable parameters. The SUPG is in essence a heuristic method.

## 2.6 Implementing fully-implicit forward time integration with respect to velocity and thickness

In a fully implicit approach using the Newton-Raphson iteration the unknowns at time-step 1 are written in incremental form as

$$\begin{aligned} u_1^{i+1} &= \Delta u + u_1^i, \\ v_1^{i+1} &= \Delta v + v_1^i, \\ h_1^{i+1} &= \Delta h + h_1^i, \end{aligned}$$

where $u_1^{i+1}$ is the estimate for $u_1$ at Newton-Raphson iteration step $i$. (Note that $\Delta h \neq h_1 - h_0$ and that $\Delta h \to 0$ with increasing $i$.)

### 2.6.1 First-order fully implicit

Taking only the first-order Taylor terms from (2.10), and only considering the $x$ components for the time being, gives

$$\begin{aligned} 0 = &\frac{\rho}{\Delta t}(\Delta h + h_1^i - h_0) \\ &- \frac{1}{2}(\rho(a_0 + a_1) - \partial_x(\rho u_0 h_0) - \partial_x(\rho(\Delta h + h_1^i)(\Delta u + u_1^i))) \end{aligned}$$

If the specific mass balance is a function of thickness, i.e.

$$a = a(h)$$

then an additional term must be added to the matrix on the left-hand side, and the right-hand side terms must be evaluated within the NR loop every time that the thickness is updated. The first-order equation is then

$$\begin{aligned} 0 = &\frac{\rho}{\Delta t}(\Delta h + h_1^i - h_0) \\ &- \frac{1}{2}\left(\rho\left(a_0(h_0) + a_1(h_1) + \partial_h a_1|_{h_1}\Delta h\right) - \partial_x(q_{x0}) - \partial_x((\Delta h + h_1^i))(\Delta u + u_1^i)\right) \end{aligned}$$

Ignoring second-order terms and taking the terms involving the unknown $\Delta h$ to the left-hand side leads to

$$\begin{aligned} \rho\frac{\Delta h}{\Delta t} &+ \frac{1}{2}\left(\partial_x(\rho u_1^i\,\Delta h + \rho h_1^i\,\Delta u) + \partial_y(\rho v_1^i\,\Delta h + \rho h_1^i\,\Delta v) - \rho\partial_h a|_{h_1}\Delta h\right) \\ &= \frac{\rho}{2}(a_0 + a_1) - \frac{\rho}{\Delta t}(h_1^i - h_0) - \frac{1}{2}(\partial_x(q_{x0}) + \partial_x(\rho h_1^i u_1^i) + \partial_y(q_{0y}) + \partial_y(\rho h_1^i v_1^i)) \end{aligned} \tag{2.29}$$

where the corresponding $y$ terms have been added.

### 2.6.2 Fully implicit SSTREAM time integration with the $\Theta$ method

$$\begin{aligned} \mathcal{F}_x &= 0 \qquad x \text{ Momentum} \\ \mathcal{F}_y &= 0 \qquad y \text{ Momentum} \\ \mathcal{M} &= 0 \qquad \text{Mass} \end{aligned}$$

$$\mathrm{K}^{uu}\,\Delta\boldsymbol{u} := D\boldsymbol{r}_x(\boldsymbol{u}_1^i,\boldsymbol{v}_1^i,\boldsymbol{h}_1^i)[\Delta\boldsymbol{u}]$$

In a fully implicit NR iteration we arrive at a block system with the structure

$$\begin{bmatrix} \mathrm{K}^{\mathcal{F}_x u} & \mathrm{K}^{\mathcal{F}_x v} & \mathrm{K}^{\mathcal{F}_x h} \\ \mathrm{K}^{\mathcal{F}_y u} & \mathrm{K}^{\mathcal{F}_y v} & \mathrm{K}^{\mathcal{F}_y h} \\ \mathrm{K}^{\mathcal{M} u} & \mathrm{K}^{\mathcal{M} v} & \mathrm{K}^{\mathcal{M} h} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{u} \\ \Delta\boldsymbol{v} \\ \Delta\boldsymbol{h} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}_{\mathcal{F}_x} \\ \boldsymbol{r}_{\mathcal{F}_y} \\ \boldsymbol{r}_{\mathcal{M}} \end{bmatrix} \tag{2.30}$$

We go from time step $t = t_0$ to $t = t_1$ and solve for the unknown values for $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{h}$, at $t = t_1$ ($\boldsymbol{u}_1$, $\boldsymbol{v}_1$, and $\boldsymbol{h}_1$) given their respective values at $t = t_0$ ($\boldsymbol{u}_0$, $\boldsymbol{v}_0$, and $\boldsymbol{h}_0$).

$$\begin{aligned} \boldsymbol{u}_1^{i+1} &= \boldsymbol{u}_1^i + \Delta\boldsymbol{u} \\ \boldsymbol{v}_1^{i+1} &= \boldsymbol{v}_1^i + \Delta\boldsymbol{v} \\ \boldsymbol{h}_1^{i+1} &= \boldsymbol{h}_1^i + \Delta\boldsymbol{h} \end{aligned}$$

Note that the increment in velocity and thickness have different physical units. In a transient run it might be better to solve for

$$\Delta\dot{h} := \Delta h/\Delta t$$

and consider the system

$$\begin{bmatrix} \mathrm{K}^{\mathcal{F}_x u} & \mathrm{K}^{\mathcal{F}_x v} & \mathrm{K}^{\mathcal{F}_x h}\Delta t \\ \mathrm{K}^{\mathcal{F}_y u} & \mathrm{K}^{\mathcal{F}_y v} & \mathrm{K}^{\mathcal{F}_y h}\Delta t \\ \mathrm{K}^{\mathcal{M} u} & \mathrm{K}^{\mathcal{M} v} & \mathrm{K}^{\mathcal{M} h}\Delta t \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{u} \\ \Delta\boldsymbol{v} \\ \Delta\dot{\boldsymbol{h}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}_{\mathcal{F}_x} \\ \boldsymbol{r}_{\mathcal{F}_y} \\ \boldsymbol{r}_{\mathcal{M}} \end{bmatrix} \tag{2.31}$$

and

$$\boldsymbol{h}_1^{i+1} = \boldsymbol{h}_1^i + \Delta\dot{\boldsymbol{h}}\,\Delta t$$

For notational simplicity we omit the $i$ superscript and it is to be understood that the values of $\eta$, $\beta^2$, $h$, $u$, and $v$ are the estimated values at iteration $i$.

At element level the matrices are

$$\begin{aligned} [\mathrm{K}^{\mathcal{F}_x u}]_{pq} \quad &= \quad \int_\Omega \big\{4\eta h \partial_x N_p\,\partial_x N_q + h\eta\partial_y N_p\,\partial_y N_q + \mathcal{H}(h-h_f)\beta^2 N_p N_q \\ &\quad +hD_{eu}(4\partial_x u + 2\partial_y v)\,\partial_x N_p + hD_{eu}(\partial_x v + \partial_y u)\,\partial_y N_p \\ &\quad +D_b\,u\,u\,N_p\,N_q\big\}\,dx\,dy \end{aligned}$$

$$\begin{aligned} [\mathrm{K}^{\mathcal{F}_y v}]_{pq} \quad &= \quad \int_\Omega \big\{4\eta h \partial_y N_p\,\partial_y N_q + h\eta\partial_x N_p\,\partial_x N_q + \mathcal{H}(h-h_f)\beta^2 N_p N_q \\ &\quad +hD_{ev}(4\partial_y v + 2\partial_x u)\,\partial_y N_p + hD_{ev}(\partial_x v + \partial_y u)\,\partial_x N_p \\ &\quad +D_b\,v\,v\,N_p\,N_q\big\}\,dx\,dy \end{aligned}$$

$$\begin{aligned} [\mathrm{K}^{\mathcal{F}_x v}]_{pq} \quad &= \quad \int_\Omega \big\{h\eta(2\partial_x N_p\,\partial_y N_q + \partial_y N_p\,\partial_x N_q) \\ &\quad +hD_{ev}(4\partial_x u + 2\partial_y v)\,\partial_x N_p + hD_{ev}(\partial_x v + \partial_y u)\,\partial_y N_p \\ &\quad +D_b\,u\,v\,N_p\,N_q\big\}\,dx\,dy \end{aligned}$$

$$\begin{aligned} [\mathrm{K}^{\mathcal{F}_x v}]_{pq} \quad &= \quad \int_\Omega \big\{h\eta(2\partial_y N_p\,\partial_x N_q + \partial_x N_p\,\partial_y N_q) \\ &\quad +hD_{eu}(4\partial_y v + 2\partial_x u)\,\partial_y N_p + hD_{eu}(\partial_x v + \partial_y u)\,\partial_x N_p \\ &\quad +D_b\,u\,v\,N_p\,N_q\big\}\,dx\,dy \end{aligned}$$

(floating only (original version)

$$
\begin{aligned}
[\mathrm{K}^{xh}]_{pq} \quad = \quad & \int_{\Omega} \big\{ \eta(4\partial_x u + 2\partial_y v)\partial_x N_p \, N_q + \eta(\partial_y u + \partial_x v)\, \partial_y N_p \, N_q \\
& + \delta(h-h_f)\beta^2 u N_p \, N_q \\
& + \rho g(\mathcal{H}(h-h_f) + h\delta(h-h_f))\, ((\rho/\rho_o \partial_x h - \partial_x H)\cos\alpha - \sin\alpha) N_p N_q \\
& + \frac{\rho^2 g}{\rho_o} h \mathcal{H}(h-h_f)\, \cos\alpha \, N_p \, \partial_x N_q \\
& - \varrho g h \cos\alpha \, \partial_x N_p \, N_q \big\} \, dx \, dy
\end{aligned}
$$

)

(floating only (corrected July 2011)

$$
\begin{aligned}
[\mathrm{K}^{\mathcal{F}_x h}]_{pq} \quad = \quad & \int_{\Omega} \big\{ \eta(4\partial_x u + 2\partial_y v)\partial_x N_p \, N_q + \eta(\partial_y u + \partial_x v)\, \partial_y N_p \, N_q \\
& + \delta(h-h_f)\beta^2 u N_p \, N_q \\
& + \rho g \mathcal{H}(h-h_f)\, ((\rho/\rho_o \partial_x h - \partial_x H)\cos\alpha - \sin\alpha) N_p N_q \\
& + \frac{\rho^2 g}{\rho_o} h \mathcal{H}(h-h_f)\, \cos\alpha \, N_p \, \partial_x N_q \\
& + \frac{\rho^2}{\rho_o} \delta(h-h_f)\, \cos\alpha \, h \, \partial_x h \, N_p \, N_q \\
& - \varrho g h \cos\alpha \, \partial_x N_p \, N_q \big\} \, dx \, dy
\end{aligned}
$$

)

(general case

$$
\begin{aligned}
[\mathrm{K}^{\mathcal{F}_x h}]_{pq} = & \int_{\Omega} \big\{ \eta(4\partial_x u + 2\partial_y v)\partial_x N_p \, N_q + \eta(\partial_y u + \partial_x v)\, \partial_y N_p \, N_q \\
& + \delta(h-h_f)\beta^2 u N_p \, N_q \\
& + \rho g \mathcal{H}(h-h_f)\, \partial_x B \, \cos\alpha \, N_p \, N_q - \rho g \sin\alpha N_p \, N_q \\
& - \rho g h \left( 1 - \frac{\rho}{\rho_o} \mathcal{H}(h_f - h) \right) \cos\alpha \, \partial_x N_p \, N_q
\end{aligned}
$$

)

( floating only version:

$$
\begin{aligned}
[\mathrm{K}^{\mathcal{F}_y h}]_{pq} \quad = \quad & \int_{\Omega} \big\{ \eta(4\partial_y v + 2\partial_x u)\partial_y N_p \, N_q + \eta(\partial_x v + \partial_y u)\, \partial_x N_p \, N_q \\
& + \delta(h-h_f)\beta^2 v N_p \, N_q \\
& + \rho g(\mathcal{H}(h-h_f) + h\delta(h-h_f))\, (\rho/\rho_o \partial_y h - \partial_y H)\cos\alpha \, N_p N_q \\
& + \frac{\rho^2 g}{\rho_o} h \mathcal{H}(h-h_f)\, \cos\alpha \, N_p \, \partial_y N_q \\
& - \varrho g h \cos\alpha \, \partial_y N_p \, N_q \big\} \, dx \, dy
\end{aligned}
$$

)

( general case

$$
\begin{aligned}
[\mathrm{K}^{\mathcal{F}_y h}]_{pq} \quad = \quad & \int_{\Omega} \big\{ \eta(4\partial_y v + 2\partial_x u)\partial_y N_p \, N_q + \eta(\partial_x v + \partial_y u)\, \partial_x N_p \, N_q \\
& + \delta(h-h_f)\beta^2 v N_p \, N_q \\
& + \rho g(\mathcal{H}(h-h_f) + h\delta(h-h_f))\, (\rho/\rho_o \partial_y h - \partial_y H)\cos\alpha \, N_p N_q \\
& + \frac{\rho^2 g}{\rho_o} h \mathcal{H}(h-h_f)\, \cos\alpha \, N_p \, \partial_y N_q \\
& - \varrho g h \cos\alpha \, \partial_y N_p \, N_q \big\} \, dx \, dy
\end{aligned}
$$

)

$$[\mathrm{K}^{\mathcal{M}u}]_{pq} = \theta(\partial_x h\, N_q + h\, \partial_x N_q)\, N_p$$

$$[\mathrm{K}^{\mathcal{M}v}]_{pq} = \theta(\partial_y h\, N_q + h\, \partial_y N_q)\, N_p$$

$$[\mathrm{K}^{\mathcal{M}h}]_{pq} = (N_q/\Delta t + \theta(\partial_x u\, N_q + u\partial_x N_q + \partial_y v\, N_q + v\partial_y N_q))\, N_p$$

In the equations the quantities $D_{eu}$, $D_{ev}$, $E$, and $D_b$, which arise because of the linearisation of $\eta$ and $\beta^2$, are given by

$$D_{eu} = E((2\partial_x u + \partial_y v)\partial_x N_q + \frac{1}{2}(\partial_x v + \partial_y u)\partial_y N_q) \tag{2.32}$$

$$D_{ev} = E((2\partial_y v + \partial_x u)\partial_y N_q + \frac{1}{2}(\partial_x v + \partial_y u)\partial_x N_q) \tag{2.33}$$

(2.34)

$$E = \frac{1-n}{4n} A^{-1/n} \dot{\epsilon}^{(1-3n)/n}$$

$$D_b := (1/m - 1)\, C^{-1/m}\, \|\boldsymbol{v}\|^{(1-3m)/m}$$

**Third-order Taylor-Galerkin fully implicit**

The terms in

$$\frac{\Delta t^2}{12} \int (\partial_t a_1 N + (h_1 \partial_t u_1 + u_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))\partial_x N + (h_1 \partial_t v_1 + v_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))\partial_y N)\, dA \tag{2.35}$$

from (2.10), need to be linearised. Starting with

$$h_1 \partial_t u_1 + u_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1}))$$

and and inserting $h^{i+1} = \Delta h + h_1^i$ etc. gives

$$(\Delta h + h_1^i)\partial_t(\Delta u + u_1^i) + (\Delta u + u_1^i)(a_1 - \partial_x((\Delta h + h_1^i)(\Delta u + u_1^i)) - \partial_y((\Delta h + h_1^i)(\Delta v + v_1^i))$$

and first ignoring only some second-order terms

$$\partial_t u_1^i\, \Delta h + h_1^i\, \partial_t \Delta u` + h_1^i \partial_t u_1^i + (a_1\, \Delta u + u_1^i a_1) - (\Delta u + u_1^i)(\partial_x(u_1^i\, \Delta h + h_1^i\, \Delta u + h_1^i u_1^i) + \partial_y(v_1\, \Delta h + h_1^i\, \Delta v + h_1^i v_1^i))$$

and then ignoring the remaining second-order terms

$$\begin{aligned} &((u_1^i - u_0)/\Delta t)(\Delta h + h_1^i) + a_1\, \Delta u + u_1^i a_1 \\ &\qquad - u_1^i(\partial_x(u_1^i\, \Delta h + h_1^i\, \Delta u + h_1^i u_1^i) + \partial_y(v_1\, \Delta h + h_1^i\, \Delta v + h_1^i v_1^i)) \\ &\qquad - (\partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i))\, \Delta u \end{aligned}$$

where $\partial_t u_1^i = (u_1^i - u_0)/\Delta t$ and $\partial_t \Delta u$ has been set to zero. Now shifting the unknowns over to the left-hand side

$$\begin{aligned} &\partial_t u_1^i\, \Delta h + a_1\, \Delta u \\ &- u_1^i(\partial_x(u_1^i\, \Delta h + h_1^i\, \Delta u) + \partial_y(v_1\, \Delta h + h_1^i\, \Delta v)) \\ &- (\partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i))\, \Delta u \\ &= u_1^i(\partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i)) - u_1^i a_1 - \partial_t u_1^i\, h_1^i \end{aligned}$$

and adding the $y$ terms

$$\begin{aligned}
&(\partial_t u_1^i \partial_x N + \partial_t v_1^i \partial_y N)\Delta h + a_1\,(\partial_x N\,\Delta u + \partial_y N\,\Delta v)\\
&\;-(u_1^i \partial_x N + v_1^i \partial_y N)(\partial_x(u_1^i\,\Delta h + h_1^i\,\Delta u) + \partial_y(v_1\,\Delta h + h_1^i\,\Delta v))\\
&\;-(\partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i))\,(\partial_x N\,\Delta u + \partial_y N\,\Delta v)\\
=&(\partial_x N\,u_1^i + \partial_y N\,v_1^i)\,(\partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i)) - a_1(\partial_x N\,u_1^i + \partial_y N\,v_1^i) - (\partial_t u_1^i \partial_x N + \partial_t v_1^i \partial_x N)\,h_1^i
\end{aligned}$$

Then adding the remaining higher-order Taylor terms in (2.10), involving fields from time-step zero, to the right-hand side gives

$$\begin{aligned}
&(\partial_t u_1^i \partial_x N + \partial_t v_1^i \partial_y N)\,\Delta h\\
&\;+(\partial_x N\,\Delta u + \partial_y N\,\Delta v)\,(a_1 - \partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i))\\
&\;-(\partial_x N\,u_1^i + \partial_y N\,v_1^i)\,(\partial_x(u_1^i\,\Delta h + h_1^i\,\Delta u) + \partial_y(v_1\,\Delta h + h_1^i\,\Delta v))\\
=&\partial_t(a_0 - a_1)N\\
&\;-(u_1^i\,\partial_x N + v_1^i\,\partial_y N)(a_1 - \partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i)) - h_1^i\,(\partial_t u_1^i\,\partial_x N + \partial_t v_1^i\,\partial_y N)\\
&\;+(u_0\,\partial_x N + v_0\,\partial_y N)(a_0 - \partial_x(q_{x0}) + \partial_y(q_{0y})) + h_0\,(\partial_t u_0\,\partial_x N + \partial_t v_0\,\partial_y N)
\end{aligned}$$

Now adding first-order Taylor terms from (2.29) to the expression above

$$\begin{aligned}
&\Delta h\,N + \frac{\Delta t}{2}\big(\partial_x(u_1^i\,\Delta h + h_1^i\,\Delta u) + \partial_y(v_1^i\,\Delta h + h_1^i\,\Delta v)\big)\,N\\
&\;+\gamma(\partial_t u_1^i \partial_x N + \partial_t v_1^i \partial_y N)\,\Delta h\\
&\;+\gamma(\partial_x N\,\Delta u + \partial_y N\,\Delta v)\,(a_1 - \partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i))\\
&\;-\gamma(\partial_x N\,u_1^i + \partial_y N\,v_1^i)\,(\partial_x(u_1^i\,\Delta h + h_1^i\,\Delta u) + \partial_y(v_1\,\Delta h + h_1^i\,\Delta v))\\
=&\left(h_0 - h_1^i + \frac{\Delta t}{2}(a_0 + a_1) - \frac{\Delta t}{2}(\partial_x(q_{x0}) + \partial_x(h_1^i u_1^i) + \partial_y(q_{0y}) + \partial_y(h_1^i v_1^i))\right)N\\
&\;+\gamma\partial_t(a_0 - a_1)N\\
&\;-\gamma(u_1^i\,\partial_x N + v_1^i\,\partial_y N)(a_1 - \partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i)) - \gamma h_1^i\,(\partial_t u_1^i\,\partial_x N + \partial_t v_1^i\,\partial_y N)\\
&\;+\gamma(u_0\,\partial_x N + v_0\,\partial_y N)(a_0 - \partial_x(q_{x0}) + \partial_y(q_{0y})) + \gamma h_0\,(\partial_t u_0\,\partial_x N + \partial_t v_0\,\partial_y N)
\end{aligned}$$

which can also be written as

$$\begin{aligned}
&\Delta h\,N + \big(\kappa\,N - \gamma\,(\partial_x N\,u_1^i + \partial_y N\,v_1^i)\big)\,\big(\partial_x(u_1^i\,\Delta h + h_1^i\,\Delta u) + \partial_y(v_1^i\,\Delta h + h_1^i\,\Delta v)\big)\\
&\;+\gamma(\partial_t u_1^i \partial_x N + \partial_t v_1^i \partial_y N)\,\Delta h\\
&\;+\gamma(\partial_x N\,\Delta u + \partial_y N\,\Delta v)\,(a_1 - \partial_x(h_1^i u_1^i) - \partial_y(h_1^i v_1^i))\\
=&(h_0 - h_1^i)\,N\\
&\;+(\kappa\,N + \gamma(u_0\,\partial_x N + v_0\,\partial_y N))\,(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))\\
&\;+\big(\kappa\,N - \gamma(u_1^i\,\partial_x N + v_1^i\,\partial_y N)\big)\,\big(a_1 - \partial_x(h_1^i u_1^i) - \partial_y(h_1^i v_1^i)\big)\\
&\;+\gamma\partial_t(a_0 - a_1)N\\
&\;-\gamma h_1^i\,(\partial_t u_1^i\,\partial_x N + \partial_t v_1^i\,\partial_y N)\\
&\;+\gamma h_0\,(\partial_t u_0\,\partial_x N + \partial_t v_0\,\partial_y N)
\end{aligned}$$

where

$$\gamma = \frac{(\Delta t)^2}{12}$$

and

$$\kappa = \frac{\Delta}{2}$$

### 2.6.3 Semi-implicit: $uv$ explicit, and $h$ implicit

In the semi-implicit approach $h_1$ is treated as the unknown while $h_0$, $u_1$, $v_1$, $h_0$, $v_0$ are assumed to be known.

Taking the unknown $h_1$ in (2.10) to the left-hand side gives

$$
\begin{aligned}
&\int (h_1 + \frac{\Delta t}{2}(\partial_x(q_{x1}) + \partial_y(q_{y1})))\, N\, dA \\
&+ \frac{\Delta t^2}{12}\int ((h_1\partial_t u_1 - u_1(\partial_x(q_{x1}) + \partial_y(q_{y1})))\partial_x N + (h_1\partial_t v_1 - v_1(\partial_x(q_{x1}) + \partial_y(q_{y1})))\partial_y N)\, dA \\
&- \frac{\Delta t^2}{12}\oint ((h_1\partial_t u_1 - u_1(\partial_x(q_{x1}) + \partial_y(q_{y1})))n_x + (h_1\partial_t v_1 - v_1(\partial_x(q_{x1}) + \partial_y(q_{y1})))n_y)\, N\, d\gamma \\
&= \int (h_0 + \frac{\Delta t}{2}(a_0 + a_1 - \partial_x(q_{x0}) - \partial_y(q_{0y})))\, N\, dA \\
&+ \frac{\Delta t^2}{12}\int (\partial_t(a_0 - a_1)N + (h_0\partial_t u_0 - u_1 a_1 + u_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))\partial_x N \\
&\qquad\qquad + (h_0\partial_t v_0 - v_1 a_1 + v_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))\partial_y N)\, dA \\
&- \frac{\Delta t^2}{12}\oint ((h_0\partial_t u_0 - u_1 a_1 + u_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))n_x \\
&\qquad + (h_0\partial_t v_0 - v_1 a_1 + v_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))n_y)\, N\, d\gamma
\end{aligned}
$$

## 2.7 Transient implicit SSHEET/SIA with the Θ method

An implicit method us used where $h_0$ and $h_1$ are the ice thicknesses at the beginning and the end of the time step, respectively. The system is solved using NR, and we write

$$
\begin{aligned}
h^{i+1} &= h^i + \Delta h \\
s^{i+1} &= s^i + \Delta h
\end{aligned}
$$

where $i$ is the number of the non-linear iteration step. Provided the method converges, $\Delta h$ goes to zero with increasing $i$. In the following we simply write $h$ instead of $h^i$.

$$(h + \Delta h - h_0)/\Delta t = -(1-\Theta)\nabla_{xy}\cdot \boldsymbol{q}_0 - \Theta\nabla_{xy}\cdot \boldsymbol{q}_1(h+\Delta h)$$

$$\int_\Omega (h + \Delta h - h_0)N\Omega = -(1-\Theta)\Delta t\int_\Omega N\nabla_{xy}\cdot \boldsymbol{q}_0 - \Theta\Delta t\int_\Omega N\nabla_{xy}\cdot \boldsymbol{q}_1(h+\Delta h) \tag{2.36}$$

To use the above equation we need to know the perturbation in flux due to perturbation in thickness. The simplest way of perturbing $q(h)$ with respect to $h$ is to find

$$\delta q = \lim_{\dot\epsilon\to 0}\frac{d}{d\dot\epsilon}q(h + \dot\epsilon\Delta h).$$

As using SSHEET for floating ice shelves is somewhat questionable we here only consider the case of grounded ice[3]. Where grounded

$$q_x(h) = -\rho\, D\left((\partial_x h + \partial_x B)^2 + (\partial_y h + \partial_y B)^2\right)^{(n-1)/2}\ h^{n+2}\,\partial_x s,$$

and

$$\Delta s = \Delta h,$$

with

$$D = \frac{2A}{n+2}(\rho g)^n$$

[3]In general the surface is related tho the ice thickness and bed through

$$s = (h+B)\mathcal{H}(h-h_f) + (S + (1-\rho/\rho_o)h)\,\mathcal{H}(h_f - h).$$

Therefore

$$
\begin{aligned}
\delta q_x &= \lim_{\dot\epsilon\to 0} \frac{d}{d\dot\epsilon} q_x(h+\dot\epsilon\Delta h) \\
&= -D \lim_{\dot\epsilon\to 0} \frac{d}{d\dot\epsilon} D\left(\partial_x(s+\dot\epsilon\Delta h)^2 + \partial_y(s+\dot\epsilon\Delta h)^2\right)^{(n-1)/2} \; (h+\dot\epsilon\Delta h)^{n+2}\,\partial_x(s+\dot\epsilon\Delta h) \\
&= -D\,\|\nabla_{xy}s\|^{n-1}\, h^{n+2}\,\partial_x\Delta h \\
&\quad - D\,(n+2)\|\nabla_{xy}s\|^{n-1}\; h^{n+1}\,\partial_x s\;\Delta h \\
&\quad - D\,(n-1)\|\nabla_{xy}s\|^{n-3}\,(\partial_x s\,\partial_x\Delta h + \partial_y s\,\partial_y\Delta h)\;\, h^{n+2}\,\partial_x s
\end{aligned}
$$

The first term of the final expression shown above is the perturbation in flux due to increase in slope, the second one the perturbation due to increase in thickness. The third term is non-linear terms that vanishes for $n=1$. This third term represents changes flux caused by a change in effective viscosity due to perturbations in slope. This third terms shows that a change in slope in $y$ direction gives rise to an increase in flux in $x$ direction.

As expected, for a negative unperturbed surface slope ($\partial_x s < 0$), both a positive perturbation in thickness ($\Delta h > 0$), and an increase in (negative) surface slope ($\partial_x\Delta h < 0$), result in a positive perturbation in flux ($\delta q_x > 0$).

### 2.7.1 SSHEET with no-flux natural boundary condition

Using a variant of Gauss theorem given by Eq. (B.3) we write (2.36) on the form

$$
\int_\Omega N\nabla_{xy}\cdot\boldsymbol{q}\,d\Omega = \oint_{\partial\Omega} N\boldsymbol{q}\cdot\hat{\boldsymbol{n}}\,d\Gamma - \int_\Omega \nabla_{xy}N\cdot\boldsymbol{q}\,d\Omega \tag{2.37}
$$

Applying (2.37) on (linearised) (2.36) and assuming that $\boldsymbol{q}\cdot\hat{\boldsymbol{n}} = 0$ on $\partial\Omega$, i.e. ice flux across the boundary is zero (homogeneous Neumann boundary condition), gives

$$
\begin{aligned}
\int_\Omega (h+\Delta h - h_0)\,N\;d\Omega =& (1-\Theta)\Delta t \int \boldsymbol{q}_0\cdot\nabla_{xy}N\;d\Omega \\
&+ \Theta\Delta t\int \boldsymbol{q}_1^i\cdot\nabla_{xy}N\;d\Omega \\
&+ \Theta\Delta t\int \Delta\boldsymbol{q}\cdot\nabla_{xy}N\;d\Omega
\end{aligned}
$$

or

$$
\begin{aligned}
\int_\Omega \Delta h\,N\;d\Omega - \Theta\Delta t\int_\Omega \Delta\boldsymbol{q}\cdot\nabla_{xy}N\;d\Omega& \\
= -\int_\Omega (h-h_0)\,N\;d\Omega& \\
+ (1-\Theta)\Delta t\int_\Omega \boldsymbol{q}_0\cdot\nabla_{xy}N\;d\Omega& \\
+ \Theta\Delta t\int_\Omega \boldsymbol{q}_1^i\cdot\nabla_{xy}N\;d\Omega&
\end{aligned}
$$

where $\boldsymbol{q}^{i+1} = \boldsymbol{q}^i + \Delta\boldsymbol{q}$ is the NR iteration, with

$$
\begin{aligned}
\Delta q_x =& -D(n-1)h^{n+2}\|\nabla_{xy}s\|^{n-3}\partial_x s(\partial_x s\,\partial_x\Delta h + \partial_y s\,\partial_y\Delta h)) \\
& - D(n+2)\|\nabla_{xy}s\|^{n-1}h^{n+1}\,\partial_x s\;\Delta h \\
& - D\|\nabla_{xy}s\|^{n-1}\,h^{n+2}\,\partial_x\Delta h
\end{aligned}
$$

Collecting terms, and writing $s$ and $h$ instead of $s^{i+1}$ and $h^{i+1}$, respectively, and $s_0$ and $h_0$ instead of $s^i$ and $h^i$, gives

$$\begin{aligned}
&\int N_p N_q \, \Delta h_q \, d\Omega \\
+\Theta\Delta t &\int D(n+2)\|\nabla_{xy} s\|^{n-1} h^{n+1} \left(\partial_x N_p \, \partial_x s + \partial_y N_p \, \partial_y s\right) N_q \, \Delta h_q \, d\Omega \\
+\Theta\Delta t &\int D\|\nabla_{xy} s\|^{n-1} \, h^{n+2} \, \left(\partial_x N_p \, \partial_x N_q + \partial_y N_p \, \partial_y N_q\right) \Delta h_q \, d\Omega \\
+\Theta\Delta t &\int D(n-1) h^{n+2} \|\nabla_{xy} s\|^{n-3} (\partial_x N_p \, \partial_x s + \partial_y N_p \, \partial_y s)(\partial_x s \, \partial_x N_q + \partial_y s \, \partial_y N_q) \, \Delta h_q \, d\Omega \\
= &\int (h_0 - h) N_p \, d\Omega \\
+(1-\Theta)\Delta t &\int (q_{x0} \partial_x N_p + q_{y0} \partial_y N_p) \, d\Omega \\
+\Theta\Delta t &\int (q_{x1} \partial_x N_p + q_{y1} \partial_y N_p) \, d\Omega
\end{aligned}$$

where

$$\begin{aligned}
q_{x0} &= D \, \|\nabla_{xy} s_0\|^{(n-1)} \, h_0^{n+2} \, \partial_x s_0 \\
q_{y0} &= D \, \|\nabla_{xy} s_0\|^{(n-1)} \, h_0^{n+2} \, \partial_y s_0 \\
q_{x1} &= D \, \|\nabla_{xy} s_1\|^{(n-1)} \, h_1^{n+2} \, \partial_x s_1 \\
q_{y1} &= D \, \|\nabla_{xy} s_1\|^{(n-1)} \, h_1^{n+2} \, \partial_y s_1
\end{aligned}$$

and

$$D = \frac{2A(\rho g)^n}{n+2}$$

### 2.7.2 Transient SSHEET/SIA with a free-flux natural boundary condition

To arrive at a formulation where free flux is the natural boundary condition we express the flux in terms of the deformational velocity as'

$$\boldsymbol{q} = F \, h \, \boldsymbol{v}_d$$

For a 'free-flux' boundary condition this is the flux at the in and outflow boundaries.

We solve

$$\int_\Omega u_d N \, \Omega = \int_\Omega E \, |\nabla_{xy} s|^{n-1} \, h^{n+1} \, \partial_x s \, d\Omega \tag{2.38}$$

$$\int_\Omega v_d N \, \Omega = \int_\Omega E \, |\nabla_{xy} s|^{n-1} \, h^{n+1} \, \partial_y s \, d\Omega \tag{2.39}$$

$$\int_\Omega (h_1 - h_0) N \, \Omega = -(1-\Theta) \, \Delta t \int_\Omega F N \nabla_{xy} \cdot h_0 \boldsymbol{v}_{d0} \, \Omega - \Theta \, \Delta t \int_\Omega F N \nabla_{xy} \cdot h \boldsymbol{v}_{d1} \, d\Omega \tag{2.40}$$

for $u_d$, $v_d$, and $h$ as unknowns. Writing this as a coupled system with $u_d$, $v_d$, and $h$ all as unknowns gives a system of first order differential equations, rather than one second order equation. This eliminates the need to get rid of a second spatial derivative and the natural boundary conditions now corresponds to a 'free-flux' condition.

NR with

$$\begin{aligned}
u^{i+1} &= u^i + \Delta u \\
v^{i+1} &= v^i + \Delta v \\
h^{i+1} &= h^i + \Delta h \\
s^{i+1} &= s^i + \Delta h
\end{aligned}$$

and linearising gives

$$\begin{aligned} u + \Delta u = E\,\|\nabla_{xy}s\|^{n-1}\,h^{n+1}\,\partial_x s \\ + E(n-1)\|\nabla_{xy}s\|^{n-3}\,h^{n+1}\,\partial_x s\;(\partial_x s\,\partial_x \Delta h + \partial_y s\,\partial_y \Delta h) \\ + E(n+1)\|\nabla_{xy}s\|^{n-1}\,h^n \partial_x s\;\Delta h \\ + E\|\nabla_{xy}s\|^{n-1}\,h^{n+1}\,\partial_x \Delta h \end{aligned}$$

Taking the $\Delta$ terms to one side

$$\begin{aligned} &-\Delta u_q \\ &+ E(n-1)\|\nabla_{xy}s\|^{n-3}\,h^{n+1}\,\partial_x s\;(\partial_x s\,\partial_x \Delta h + \partial_y s\,\partial_y \Delta h) \\ &+ E(n+1)\|\nabla_{xy}s\|^{n-1}\,h^n \partial_x s\;\Delta h \\ &+ E\|\nabla_{xy}s\|^{n-1}\,h^{n+1}\,\partial_x \Delta h \\ &= u - E\,\|\nabla_{xy}s\|^{n-1}\,h^{n+1}\,\partial_x s \end{aligned}$$

Galerkin, $u$ term:

$$\begin{aligned} &- < N_p, N_q > \;\Delta u_q \\ &+ (n-1) < N_p \mid E\|\nabla_{xy}s\|^{n-3}\,h^{n+1}\,\partial_x s(\partial_x s\,\partial_x N_q + \partial_y s\,\partial_y N_q) > \;\Delta h_q \\ &+ (n+1) < N_p \mid E\|\nabla_{xy}s\|^{n-1}\,h^n \partial_x s N_q > \;\Delta h_q \\ &+ < N_p \mid E\|\nabla_{xy}s\|^{n-1}\,h^{n+1} \partial_x N_q > \;\Delta h_q \\ &=< N_p, u - E\,\|\nabla_{xy}s\|^{n-1}\,h^{n+1}\,\partial_x s > \end{aligned}$$

The corresponding $v$ term is obtained by replacing $u$ with $v$ and derivatives with respect to $x$ by derivatives with respect to $y$,

Galerkin $q$ term:

$$\begin{aligned} < N_P \mid N_q > \Delta h_q + \Delta t\;\Theta F < N_p \mid \partial_x u\,N_q + u\,\partial_x N_q + \partial_y v\,N_q + v\,\partial_y N_q > \Delta h_q \\ + \Delta t\;\Theta F < N_p \mid \partial_x h\,N_q + h\partial_x N_q > \Delta u_q \\ + \Delta t\;\Theta F < N_p \mid \partial_y h\,N_q + h\partial_y N_q > \Delta v_q \\ =\Delta t < N_p \mid (1-\Theta)\,a_0 + \Theta\,a_1 > \\ - < N_p \mid (h - h_0) > \\ - \Delta t\,F < N_p, ((1-\Theta)(\partial_x(h_0 u_0) + \partial_y(h_0 u_0)) + \Theta(\partial_x(hu) + \partial_y(hv))) > \end{aligned}$$

## 2.8 Method of characteristics

Think of

$$\partial_t h + \partial_x(uh) = a$$

as

$$D_t h + h\partial_x u = a$$

with

$$D_t h = \partial_t h + u\partial_x h$$

Along the characteristics I have

$$D_t h = \lim_{\Delta t \to 0} \frac{1}{\Delta t}(h(x,t) - h(x - u\,\Delta t \mid t - \Delta t))$$

and if I just write

$$h(x - u\,\Delta t \mid t - \Delta t)) = h(x, t - \Delta t) - d_x h(x, t - \Delta t)\,u\,\Delta t$$

and take the limit I (of course) just get

$$\partial_t h + \partial_x(uh) = a$$

again.

The question seem to me to be about how to approximate the variation along the characteristic. If I write

$$h(x - u\,\Delta t \mid t - \Delta t)) = h_0 - d_x h_0\, u\,\Delta t + \frac{1}{2} d_{xx}^2 h_0 (u\,\Delta t)^2$$

where $h0$ is evaluated at $x$ and at $t - \Delta t$, I get a 'correction' term and the discretized version is

$$\frac{1}{\Delta t}(h_1 - h_0 + d_x h_0\, u\,\Delta t + \frac{1}{2} d_{xx}^2 h_0\,(u\,\Delta t)^2) + h\,d_x u = a$$

It is a bit unclear in the above expression at what time to evaluate $u$. I could go for the average value over the time step, but I will only know it at the previous time step anyhow.

One way of interpreting the above equation is

$$\frac{1}{\Delta t}(h_1 - h_0) + d_x h\, u + \frac{1}{2} u^2\, \Delta t d_{xx}^2 h + h\,d_x u = a$$

and then evaluate all terms at some time within the time step, i.e. the $\Theta$ method. The above equation can be written as

$$\frac{1}{\Delta t}(h_1 - h_0) + \partial_x(hu) + \frac{\Delta t}{2} u^2\,\partial_{xx}^2 h = a$$

In combination with the $\Theta$ method I get

$$\frac{1}{\Delta t}(h_1 - h_0) = \Theta\left(a_1 - \partial_x(q_{x1}) - \frac{1}{2} u_1\,\Delta t\,\partial_{xx}^2(u_1 h_1)\right) + (1 - \Theta)\left(a_0 - \partial_x(q_{x0}) - \frac{1}{2} u_0\,\Delta t\,\partial_{xx}^2(u_0 h_0)\right)$$

(did not write the $a$ correction term)

## 2.9 Taylor-Galerkin

We have

$$\partial_t h + \partial_x(uh) = a$$

we write

$$h_1 = h_0 + \Delta t\,\partial_t h_0 + \frac{(\Delta t)^2}{2}\partial_{tt}^2 h_0$$

which is a second-order accurate Euler method.

Inserting we get

$$\begin{aligned}
h_1 &= h_0 + \Delta t(a - \partial_x(hu)) + \frac{(\Delta t)^2}{2}\partial_t(a - \partial_x(uh)) \\
&= h_0 + \Delta t(a - \partial_x(hu)) + \frac{(\Delta t)^2}{2}(\partial_t a - \partial_{xt}^2(uh)) \\
&= h_0 + \Delta t(a - \partial_x(hu)) + \frac{(\Delta t)^2}{2}(\partial_t a - \partial_x(\partial_h(uh)\partial_t h)) \\
&= h_0 + \Delta t(a - \partial_x(hu)) + \frac{(\Delta t)^2}{2}(\partial_t a - \partial_x(u\partial_t h)) \\
&= h_0 + \Delta t(a - \partial_x(hu)) + \frac{(\Delta t)^2}{2}(\partial_t a - \partial_x(u(a - \partial_x(uh))))
\end{aligned}$$

or

$$\frac{1}{\Delta t}(h_1 - h_0) = a - \partial_x(hu) - \frac{\Delta t}{2}\partial_x(u(a - \partial_x(uh))) + \frac{\Delta t}{2}\partial_t a \tag{2.41}$$

All the therms of the right-hand side of (2.41) refer to time step 0. I get the implicit theta method if I evaluate the right-hand side at both 1 and 0 and weight with $\Theta$.

After a partial integration we get a correction term

$$< \frac{1}{2}\Delta t\,\partial_x u \mid a - \partial_x(uh) >$$

## 2.10 Third order implicit Taylor Galerkin (1HD)

A better justification for evaluation at both time step 1 and 0 comes from writing

$$h_1 = h_0 + \Delta t\, \partial_t h_0 + \frac{(\Delta t)^2}{2} \partial^2_{tt} h_0,$$

$$h_0 = h_1 - \Delta t\, \partial_t h_1 + \frac{(\Delta t)^2}{2} \partial^2_{tt} h_1,$$

adding

$$2(h_1 - h_0) = \Delta t\, (\partial_t h_0 + \partial_t h_1) + \frac{(\Delta t)^2}{2} (\partial^2_{tt} h_0 - \partial^2_{tt} h_1)$$

and simplifying gives

$$\frac{1}{\Delta t}(h_1 - h_0) = \frac{1}{2}(\partial_t h_0 + \partial_t h_1) + \frac{\Delta t}{4} (\partial^2_{tt} h_0 - \partial^2_{tt} h_1) \tag{2.42}$$

Now use

$$\partial_t h = a - \partial_x (hu)$$

and

$$\begin{aligned}
\partial^2_{tt} h &= \partial_t a - \partial^2_{tx}(hu) \\
&= \partial_t a - \partial_t (h\partial_x u + u\partial_x h) \\
&= \partial_t a - (\partial_x h \partial_t u + h\partial^2_{xt} u + \partial_t h \partial_x u + u\partial^2_{xt} h) \\
&= \partial_t a - (\partial_x h \partial_t u + h\partial^2_{xt} u + \partial_x u (a - \partial_x(hu)) + u\partial_x (a - \partial_x(hu))) \\
&= \partial_t a - (\partial_x h \partial_t u + h\partial^2_{xt} u + \partial_x u (a - \partial_x(hu)) + u\partial_x (a - \partial_x(hu))) \\
&= \partial_t a - \partial_x h \partial_t u - h\partial^2_{xt} u - \partial_x (u(a - \partial_x(hu)))
\end{aligned}$$

or

$$\partial^2_{tt} h = \partial_t a - \partial_x (h\partial_t u + ua - u\partial_x(uh)) \tag{2.43}$$

Inserting (2.43) into (2.3) gives

$$\begin{aligned}
0 &= Lh \\
&= \frac{1}{\Delta t}(h_1 - h_0) \\
&- \frac{1}{2}(a_0 - \partial_x(q_{x0}) + a_1 - \partial_x(q_{x1})) \\
&- \frac{\Delta t}{4}(\partial_t a_0 - \partial_x(h_0 \partial_t u_0 + u_0(a_0 - \partial_x(q_{x0}))) - \partial_t a_1 + \partial_x(h_1 \partial_t u_1 + u_1(a_1 - \partial_x(q_{x1}))))
\end{aligned}$$

Galerkin

$$< Lh_1 | N_p >= 0$$

with $u_1 = N_q u_q$, etc. I used partial integration to get rid of second order spatial derivatives

$$\begin{aligned}
0 &= < Lu | N_q > \\
&= \frac{1}{\Delta t} \int (h_1 - h_0) N_q \, dx \\
&- \frac{1}{2} \int (a_0 - \partial_x(q_{x0}) + a_1 - \partial_x(q_{x1})) N_q \, dx \\
&- \frac{\Delta t}{4} \int (\partial_t a_0 - \partial_t a_1) N_q \, dx \\
&- \frac{\Delta t}{4} \int (h_0 \partial_t u_0 + u_0 a_0 - u_0 \partial_x(q_{x0})) - h_1 \partial_t u_1 - u_1 a_1 + u_1 \partial_x(q_{x1}))) \partial_x N_q \, dx \\
&- \frac{\Delta t}{4} (-h_0 \partial_t u_0 - u_0(a_0 - \partial_x(q_{x0})) + h_1 \partial_t u_1 + u_1(a_1 - \partial_x(q_{x1}))) N_q |^{x_r}_{x_l}
\end{aligned}$$

The unknown is $h_1$

$$
\begin{aligned}
&\int (h_1 + \frac{\Delta t}{2}\partial_x(q_{x1}))N_q\,dx \\
+ &\quad \frac{(\Delta t)^2}{4}\int (h_1\partial_t u_1 - u_1\partial_x(q_{x1}))\,\partial_x N_q\,dx + \frac{(\Delta t)^2}{4}(u_1\partial_x(q_{x1}) - h_1\partial_t u_1)N_q|_{x_l}^{x_r} \\
= &\quad \int (h_0 + \frac{\Delta t}{2}(a_1 + a_0 - \partial_x(q_{x0})))N_q\,dx \\
+ &\quad \frac{(\Delta t)^2}{4}\int \partial_t(a_0 + a_1)N_q\,dx \\
+ &\quad \frac{(\Delta t)^2}{4}\int (u_0 a_0 - u_1 a_1 + h_0\partial_t u_0 - u_0\partial_x(q_{x0}))\,\partial_x N_q\,dx \\
+ &\quad \frac{(\Delta t)^2}{4}(u_1 a_1 - u_0 a_0 - h_0\partial_t u_0 + u_0\partial_x(q_{x0}))N_q|_{x_l}^{x_r}
\end{aligned}
$$

Writing out the product terms

$$
\begin{aligned}
&\int (h_1 + \frac{\Delta t}{2}(h_1\partial_x u_1 + u_1\partial_x h_1))N_q\,dx \\
+ &\quad \frac{(\Delta t)^2}{4}\int (\partial_t u_1 h_1 - u_1(h_1\partial_x u_1 + u_1\partial_x h_1))\,\partial_x N_q\,dx \\
+ &\quad \frac{(\Delta t)^2}{4}(u_1(h_1\partial_x u_1 + u_1\partial_x h_1) - h_1\partial_t u_1)N_q|_{x_l}^{x_r} \\
= &\quad \int (h_0 + \frac{\Delta t}{2}(a_1 + a_0 - (h_0\partial_x u_0 + u_0\partial_x h_0)))N_q\,dx \\
+ &\quad \frac{(\Delta t)^2}{4}\int \partial_t(a_0 + a_1)N_q\,dx \\
+ &\quad \frac{(\Delta t)^2}{4}\int (u_0 a_0 - u_1 a_1 + h_0\partial_t u_0 - u_0(h_0\partial_x u_0 + u_0\partial_x h_0))\,\partial_x N_q\,dx \\
+ &\quad \frac{(\Delta t)^2}{4}(u_1 a_1 - u_0 a_0 - h_0\partial_t u_0 + u_0(h_0\partial_x u_0 + u_0\partial_x h_0))N_q|_{x_l}^{x_r}
\end{aligned}
$$

Taking this up to third order

$$\frac{1}{\Delta t}(h_1 - h_0) = \frac{1}{2}(\partial_t h_0 + \partial_t h_1) + \frac{\Delta t}{4}(\partial_{tt}^2 h_0 - \partial_{tt}^2 h_1) + \frac{(\Delta t)^2}{12}(\partial_{ttt}^3 h_0 + \partial_{ttt}^3 h_1)$$

is easy, if we simply approximate the time derivative in the third-order term through finite differences.

$$
\begin{aligned}
\frac{1}{\Delta t}(h_1 - h_0) &= \frac{1}{2}(\partial_t h_0 + \partial_t h_1) + \frac{\Delta t}{4}(\partial_{tt}^2 h_0 - \partial_{tt}^2 h_1) + \frac{(\Delta t)^2}{12\Delta t}(\partial_{tt}^2(h_1 - h_0) + \partial_{tt}^2(h_1 - h_0)) \\
&= \frac{1}{2}(\partial_t h_0 + \partial_t h_1) + \frac{\Delta t}{4}(\partial_{tt}^2 h_0 - \partial_{tt}^2 h_1) + \frac{\Delta t}{6}\,\partial_{tt}^2(h_1 - h_0) \\
&= \frac{1}{2}(\partial_t h_0 + \partial_t h_1) + \frac{\Delta t}{12}\partial_{tt}^2 h_0 - \frac{\Delta t}{12}\partial_{tt}^2 h_1
\end{aligned}
$$

The only thing that changes is the numerical factor of the second-order term. However this is now correct to third order.

# Chapter 3

# Constraints

In *Úa* all essential boundary conditions, and various other constraints on the solution, are enforced using the Lagrange multiplier method. Any multi-linear constraints

$$\boldsymbol{L}\boldsymbol{x} - \boldsymbol{c} = \boldsymbol{0},$$

where $\boldsymbol{x}$ are the unknowns for some $\boldsymbol{L}$ and $\boldsymbol{c}$, can be described. Here $\boldsymbol{L}$ is the *multi-linear constraint* (MLC) matrix and $\boldsymbol{c}$ prescribed nodal values. The matrix $\boldsymbol{L} \in \mathbb{R}^{m\times n}$ has the form

$$\boldsymbol{L} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

where $m$ is the number of constraints and $n$ the number of nodal values. If we, for example, want to enforce the ice thickness at node 1 to be equal to 1, the MLC matrix $\boldsymbol{L} \in \mathbb{R}^{1\times n}$ has the form

$$\boldsymbol{L} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$$

and $\boldsymbol{c} \in \mathbb{R}^{n\times 1}$ is

$$\boldsymbol{c} = (1)$$

## 3.1 Linear system with multi-linear constraints

For a quadratic minimisation problem

$$\min_{\boldsymbol{x}} I(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T \mathrm{A}\boldsymbol{x} - \boldsymbol{b}\boldsymbol{x} \ ,$$

subject to the multi-linear set of constraints

$$\boldsymbol{L}\boldsymbol{x} - \boldsymbol{c} = 0 \ ,$$

the system to be solved is

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{L}^T \\ \boldsymbol{L} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{c} \end{bmatrix}. \tag{3.1}$$

Here $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{L} \in \mathbb{R}^{p\times n}$ where $p$ are the number of constraints. There are various approaches suggested in the literature to solving the system (3.1) such as the full-space, range-space, null-space approaches.

The *range-space* approach is to use the first equation of (3.1) to arrive at

$$\boldsymbol{L}\boldsymbol{x} + \boldsymbol{L}\boldsymbol{A}^{-1}\boldsymbol{L}^T\boldsymbol{\lambda} = \boldsymbol{L}\boldsymbol{A}^{-1}\boldsymbol{b}.$$

Then insert the second equation of (3.1) giving

$$\boldsymbol{L}\boldsymbol{A}^{-1}\boldsymbol{L}^T\boldsymbol{\lambda} = \boldsymbol{L}\boldsymbol{A}^{-1}\boldsymbol{b} - \boldsymbol{c},$$

which can be solved for $\boldsymbol{\lambda}$, and then solve for $\boldsymbol{x}$ using the first equation equation of (3.1) as

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{L}^T\boldsymbol{\lambda}.$$

The *null-space* approach is to find a matrix $\boldsymbol{Z} \in R^{n\times(n-p)}$ which columns for a basis for the null space of $\boldsymbol{L}$ so that

$$\boldsymbol{L}\boldsymbol{Z} = \boldsymbol{0}$$

Let $\hat{\boldsymbol{x}} \in R^n$ be a particular solution for the second equation such that

$$\boldsymbol{L}\hat{\boldsymbol{x}} = \boldsymbol{c}$$

After some manipulations we arrive at the null-space method for solving (3.1) which can be summarised as:

Find $\boldsymbol{Z}$ so that $\boldsymbol{L}\boldsymbol{Z} = \boldsymbol{0}$.
Find $\hat{\boldsymbol{x}}$ so that $\boldsymbol{L}\hat{\boldsymbol{x}} = \boldsymbol{c}$.
Solve $\boldsymbol{Z}^T\boldsymbol{A}\boldsymbol{Z}\boldsymbol{z} = \boldsymbol{Z}^T(\boldsymbol{b} - \boldsymbol{A}\hat{\boldsymbol{x}})$
Set $\boldsymbol{x} = \boldsymbol{Z}\boldsymbol{z} + \hat{\boldsymbol{x}}$
Solve $\boldsymbol{L}^T\boldsymbol{\lambda} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}$

The matrix $\boldsymbol{Z}^T\boldsymbol{A}\boldsymbol{Z}$ is $(n-p)\times n\times n\times n\times n\times(n-p) = (n-p)\times(n-p)$. Therefore if the number of constraints ($p$) is significantly large compared to $n$, $n-p$ is small and this method becomes attractive. However, the matrix $\boldsymbol{Z}^T\boldsymbol{A}\boldsymbol{Z}$ is usually dense.

### 3.1.1 Pre-eliminating point constraints

The system (3.1) has the size $(n+p)\times(n+p)$ where $n$ is the total number of degrees of freedom and $p$ the number of (point) restraints. In some cases it is fairly easy to pre-eliminate the constraints and so arrive at a reduced system of the size $n\times n$.

The system (3.1) is

$$\boldsymbol{A}\boldsymbol{x} + \boldsymbol{L}^T\boldsymbol{\lambda} = \boldsymbol{b} \tag{3.2}$$
$$\boldsymbol{L}\boldsymbol{x} = \boldsymbol{c} \tag{3.3}$$

Here the matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and has rank $n$, while the matrix $\boldsymbol{L} \in \mathbb{R}^{p\times n}$ with $p \le n$ and has rank $p$. The products $\boldsymbol{L}^T\boldsymbol{L} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{L}\boldsymbol{L}^T \in \mathbb{R}^{p\times p}$ both have rank $p$. The matrix $\boldsymbol{L}\boldsymbol{L}^T$ is invertable but $\boldsymbol{L}^T\boldsymbol{L}$ is not.

We consider the particular, but in this context not uncommon, case when

$$\boldsymbol{L}\boldsymbol{L}^T = \boldsymbol{I}_{pp}\,. \tag{3.4}$$

This assumption is not particularly restrictive[1] and allows us to determine $\boldsymbol{\lambda}$ knowing $\boldsymbol{x}$ using Eq. (3.2) as follows

$$\begin{aligned}
&\boldsymbol{L}^T\boldsymbol{\lambda} = (\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})\\
\Longrightarrow\ &\boldsymbol{L}\boldsymbol{L}^T\boldsymbol{\lambda} = \boldsymbol{L}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})\\
\Longrightarrow\ &\boldsymbol{\lambda} = \boldsymbol{L}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})\,.
\end{aligned}$$

And furthermore again using Eq. (3.2)

$$\begin{aligned}
&\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{L}^T\boldsymbol{\lambda}\\
\Longrightarrow\ &\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{L}^T\boldsymbol{L}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})\\
\Longrightarrow\ &\boldsymbol{A}\boldsymbol{x} - \boldsymbol{L}^T\boldsymbol{L}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{L}^T\boldsymbol{L}\,\boldsymbol{b}\\
\Longrightarrow\ &(\boldsymbol{I}_{nn} - \boldsymbol{L}^T\boldsymbol{L})\boldsymbol{A}\boldsymbol{x} = (\boldsymbol{I}_{nn} - \boldsymbol{L}^T\boldsymbol{L})\,\boldsymbol{b}
\end{aligned}$$

[1] Provided a node appears in one constraint only, $\boldsymbol{L}$ can always be scaled so that $\boldsymbol{L}\boldsymbol{L}^T$ equals $\boldsymbol{I}_{pp}$. For example if

$$\boldsymbol{L} = \begin{pmatrix} 1 & 10 & 0 & -2 & 0 & 0 & 0\\ 0 & 0 & 2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 5 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Then scaling each line of $\boldsymbol{L}$ so that the square root of the sum of the squares of the elements of that line is equal to unity results in $\boldsymbol{L}\boldsymbol{L}^T = \boldsymbol{I}_{44}$.

or

$$\boldsymbol{Q}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{Q}\boldsymbol{b} \,, \tag{3.5}$$

where the matrix $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$ defined as

$$\boldsymbol{Q} = \boldsymbol{I}_{nn} - \boldsymbol{L}^T\boldsymbol{L} \,.$$

Multiplying with $\boldsymbol{Q}$ has the effect of 'killing off' the first $p$ equations. Pre-multiply both sides of Eq. (3.3) to obtain

$$\boldsymbol{L}^T\boldsymbol{L} = \boldsymbol{L}^T\boldsymbol{c} \,,$$

and adding to Eq. (3.5) gives

$$(\boldsymbol{Q}\boldsymbol{A} + \boldsymbol{L}^T\boldsymbol{L})\boldsymbol{x} = \boldsymbol{Q}\boldsymbol{b} + \boldsymbol{L}^T\boldsymbol{c} \,,$$

and as

$$\tilde{\boldsymbol{A}} := \boldsymbol{Q}\boldsymbol{A} + \boldsymbol{L}^T\boldsymbol{L} \,,$$

is clearly full rank, we can now solve for $\boldsymbol{x}$ as

$$\boldsymbol{x} = \tilde{\boldsymbol{A}}^{-1}(\boldsymbol{Q}\boldsymbol{b} + \boldsymbol{L}^T\boldsymbol{c}) \,, \tag{3.6}$$

and then calculate $\boldsymbol{\lambda}$ from

$$\boldsymbol{\lambda} = \boldsymbol{L}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) \,. \tag{3.7}$$

A drawback of this approach is that even when the original matrix $\boldsymbol{A}$ is symmetrical, $\tilde{\boldsymbol{A}}$ is generally asymmetrical. While the system can be made symmetrical, in practice doing so in Úa is of limited value as most of the calculations tend to be done using implicit time stepping where $\boldsymbol{A}$ is already non-symmetrical.

Currently (August 2019) this is the approach[2] used in Úa when matrix $\boldsymbol{A}$ is asymmetrical, i.e. the system (3.1) is then solved using Eqs. (3.6) and (3.7) provided Eq. (3.4) holds.

## 3.2 Nodal reactions

I define *effective nodal reactions*, $\boldsymbol{R}$, as

$$\boldsymbol{R} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} = \boldsymbol{L}^T\boldsymbol{\lambda} \,.$$

The effective nodal reactions are defined for all degrees of freedom and for all nodes, i.e. not only for the degrees of freedom and nodes for which BCs are applied to. Clearly the vector $\boldsymbol{R}$ does not represent nodal values in the FE basis. To project $\boldsymbol{R}$ into the FE basis we need to solve

$$< [\boldsymbol{R}^*]_p N_p, N_q >= [\boldsymbol{R}]_q \,,$$

for components of what I refer to as the *physical nodal reactions*, $\boldsymbol{R}^*$. We see that

$$\boldsymbol{R}^* = \boldsymbol{M}^{-1}\boldsymbol{R},$$

or

$$\boldsymbol{R}^* = \boldsymbol{M}^{-1}\boldsymbol{L}^T\boldsymbol{\lambda} \,.$$

Consider re-defining the above system from

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} + \boldsymbol{L}^T\boldsymbol{\lambda} &= \boldsymbol{b} \\ \boldsymbol{L}\boldsymbol{x} &= \boldsymbol{c} \end{aligned}$$

to[3]

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} + (\boldsymbol{L}\boldsymbol{M})^T\boldsymbol{\lambda}^* &= \boldsymbol{b} \\ \boldsymbol{L}\boldsymbol{M}\boldsymbol{x} &= \boldsymbol{L}\boldsymbol{M}\boldsymbol{L}^T\boldsymbol{c} \end{aligned}$$

[2] Presumably this approach will have been derived earlier in the literature, but I have not seen this approach described anywhere.

[3] Alternatively, re-write this as

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} + (\boldsymbol{L}\boldsymbol{M}\boldsymbol{L}^T\boldsymbol{L})^T\boldsymbol{\lambda}' &= \boldsymbol{b} \\ \boldsymbol{L}\boldsymbol{M}\boldsymbol{L}^T\boldsymbol{L}\boldsymbol{x} &= \boldsymbol{L}\boldsymbol{M}\boldsymbol{L}^T\boldsymbol{c} \end{aligned}$$

where

$$\boldsymbol{L}^T\boldsymbol{\lambda} = (\boldsymbol{L}\boldsymbol{M}\boldsymbol{L}^T\boldsymbol{L})^T\boldsymbol{\lambda}' \,,$$

in which case

$$(\boldsymbol{LM})^T\boldsymbol{\lambda}^* = \boldsymbol{L}^T\boldsymbol{\lambda}$$

or

$$\boldsymbol{L}^T\boldsymbol{\lambda}^* = \boldsymbol{M}^{-1}\boldsymbol{L}^T\boldsymbol{\lambda}$$

hence

$$\boldsymbol{R}^* = \boldsymbol{L}^T\boldsymbol{\lambda}^*$$

The above reformulation requires all links to be homogeneous. Dirichlet BCs can be non-homogeneous.

## 3.3 Non-linear system with non-linear constraints

If we have a non-linear minimisation problem

$$\min_{\boldsymbol{x}} I(x)$$

and non-linear set of constraints $l(x) = 0$, we minimise the extended cost function

$$I(x) + < \lambda(x) \mid l(x) >,$$

or once we have discretized the problem,

$$I(x) + \boldsymbol{\lambda}^T\boldsymbol{l}\ .$$

A stable equilibrium point is a minimum wither respect to $x$ and a maximum with respect to $\lambda$. Setting the derivatives with respect to $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ to zero leads to

$$\begin{aligned}\partial_{\boldsymbol{x}} I(x) + \boldsymbol{\lambda}^T\partial_{\boldsymbol{x}}\boldsymbol{l}(\boldsymbol{x}) &= \boldsymbol{0}\\ \boldsymbol{l}(\boldsymbol{x}) &= \boldsymbol{0}\end{aligned}$$

If this non-linear system is again solved using Newton-Raphson method then we use first-order Taylor expansion and write

$$\begin{aligned}\partial_{\boldsymbol{x}} I(\boldsymbol{x}) &= \partial_{\boldsymbol{x}} I_0 + \boldsymbol{\lambda}_0^T\,\partial_{\boldsymbol{x}}\boldsymbol{l}_0 + \partial^2_{\boldsymbol{xx}} I_0\,\Delta\boldsymbol{x} + \boldsymbol{\lambda}_0\,\partial_{\boldsymbol{xx}}\boldsymbol{l}_0\,\Delta\boldsymbol{x} + \partial_x\boldsymbol{l}_0^T\,\Delta\boldsymbol{\lambda}\\ \boldsymbol{l}(\boldsymbol{x}) &= \boldsymbol{l}(\boldsymbol{x}_0) + \partial_{\boldsymbol{x}}\boldsymbol{l}\,\Delta\boldsymbol{x}\end{aligned}$$

and repeatedly solve

$$\begin{bmatrix} \boldsymbol{H} + \partial^2_{xx}\boldsymbol{l}\,\boldsymbol{\lambda}_0 & \partial_{\boldsymbol{x}}\boldsymbol{l}^T \\ \partial_{\boldsymbol{x}}\boldsymbol{l} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{x} \\ \Delta\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -\partial_{\boldsymbol{x}} I_0 - \partial_{\boldsymbol{x}}\boldsymbol{l}^T\,\boldsymbol{\lambda}_0 \\ -\boldsymbol{l}_0 \end{bmatrix} \tag{3.8}$$

where again $\boldsymbol{H} = \partial_{\boldsymbol{xx}} I(x)$ is the Hessian matrix.

If the constraints are linear, we can write

$$\boldsymbol{l} = \boldsymbol{Lx} - \boldsymbol{c} = \boldsymbol{0}$$

in which case

$$\partial_{\boldsymbol{x}}\boldsymbol{l} = \boldsymbol{L} \quad \text{and} \quad \partial^2_{\boldsymbol{xx}}\boldsymbol{l} = \boldsymbol{0}$$

and the system to be solved is

$$\begin{bmatrix} \boldsymbol{H} & \boldsymbol{L}^T \\ \boldsymbol{L} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -\partial_{\boldsymbol{x}} I_0 - \partial_{\boldsymbol{x}}\boldsymbol{l}^T\,\boldsymbol{\lambda}_0 \\ -\boldsymbol{Lx}_0 + \boldsymbol{c} \end{bmatrix}$$

---

hence

$$\boldsymbol{\lambda}' = (\boldsymbol{LL}^T)^{-1}\boldsymbol{M}^{-1}(\boldsymbol{LL}^T)^{-1}\boldsymbol{L}^T\boldsymbol{\lambda}\ .$$

We find that

$$\boldsymbol{\lambda}' = (\boldsymbol{LL}^T)^{-1}\boldsymbol{LR}^*$$

Note that for $m$ linear independent constraints, the matrix $\boldsymbol{L} \in \mathbb{R}^{m\times n}$ has the rank $m$. The product $\boldsymbol{LL}^T$ is in $\mathbb{R}^{m\times m}$ and

$$\text{rank}(\boldsymbol{LL}^T) = \text{rank}(\boldsymbol{L}^T\boldsymbol{L}) = \text{rank}(\boldsymbol{L}) = m$$

and therefore the symmetric matrix $(\boldsymbol{LL}^T) \in \mathbb{R}^{m\times m}$ is full rank and its inverse exists. However, $(\boldsymbol{L}^T\boldsymbol{L}) \in \mathbb{R}^{n\times n}$ is not full rank and therefore not invertable.

## 3.4 FE formulation of the Newton-Raphson method with multi-linear constraints

We solve

$$< f(u), N_p >= 0 \quad \text{subject to} \quad g_q(u) = 0 \quad \text{where} \quad q = 1 \dots N$$

Assume there is a scalar $I(u)$ such that $< f(u), N_p >$ is derivative of $I$ with respect to $u_p$, then minimising

$$I(u)+ < \lambda, g(u) >$$

gives, with $u = N_p u_p$ and $\lambda = N_p \lambda_p$

$$< f(u), N_q > + < \lambda, \partial g/\partial u \, N_q > = 0 \tag{3.9}$$
$$< N_q, g(u) > = 0 \tag{3.10}$$

In general, the $< f(u), N_q >$ system can be expected to be non-linear. Assuming for the moment the constraints $g(u) = 0$ are linear ($g_q = A_{qr} u_r - b_q = 0$), and the resulting system is solved using the Newton-Raphson method, gives

$$< f(u_0), N_q > + < \partial f/\partial u \, \Delta u, N_q > + < \lambda, \partial g/\partial u \, N_q > = 0 \tag{3.11}$$
$$< N_q, \partial g/\partial u \; (u_0 + \Delta u) - b > = 0 \tag{3.12}$$

or

$$< f(u_0), N_q > + < \partial f/\partial u \, N_p \mid N_q > \Delta u_p + < N_p, \partial g/\partial u N_q > \lambda_p = 0 \tag{3.13}$$
$$< N_q, \partial g/\partial u N_p > u_{0p} + < N_q \mid \partial g/\partial u N_p > \Delta u_p - < N_q, N_p > b_p = 0 \tag{3.14}$$

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{L}^T \\ \boldsymbol{L} & 0 \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{f}(\boldsymbol{u}_0) \\ \boldsymbol{S}\boldsymbol{b} - \boldsymbol{L}\boldsymbol{u}_0 \end{bmatrix}$$

where

$$f_q :=< f(u_0), N_q >$$

and

$$S_{qp} :=< N_q, N_p >$$

and

$$L_{pq} =< N_p, \partial g/\partial u N_q >$$

or if $\boldsymbol{\lambda} = \Delta\boldsymbol{\lambda} + \boldsymbol{\lambda}_0$,

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{L}^T \\ \boldsymbol{L} & 0 \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{u} \\ \Delta\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{f}(\boldsymbol{u}_0) - \boldsymbol{L}^T\boldsymbol{\lambda}_0 \\ \boldsymbol{S}\boldsymbol{b} - \boldsymbol{L}\boldsymbol{u}_0 \end{bmatrix}$$

If the constraints can be written as $Au - b = 0$ then

$$L_{pq} =< N_p, A_{pq} N_q >$$

(no summation implied). If the form functions are delta functions then, $\boldsymbol{S} = \mathbf{1}$ and $\boldsymbol{L} = \boldsymbol{A}$.

## 3.5 Automated thickness-positivity constraints (active set method)

Assuming the existence of a potential $P$ the problem can be written as a constraint minimisation problem

$$\min_{\boldsymbol{h},\boldsymbol{v}} P(\boldsymbol{h}, \boldsymbol{v})$$
$$\text{s.t} \quad h(x, y) \geq= 0 \quad \text{on} \quad \Omega$$

The Lagrangian

$$\mathcal{L}P + \int_\Omega \lambda(x, y) h(x, y) \, d\Omega$$

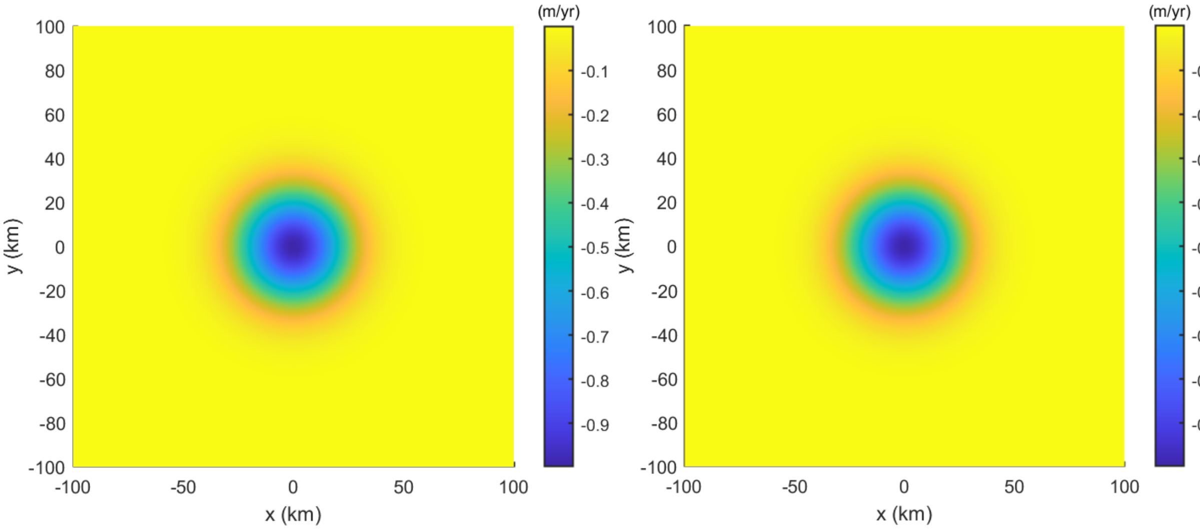



Figure 3.1: Active set method, as described in section 3.5, used to ensure positive ice thicknesses. To the left are the thickness reactions as calculated by *Úa* for a situation where ice thicknesses are zero everywhere, and using the surface-mass balance distribution shown in the right-hand side panel. The ice geometry is a simple horizontal slab. Hence, surface velocities are everywhere equal to zero. The active set method is here used to ensure that despite non-zero melt being applied over an area where the thickness is everywhere zero, the ice thickness does not become negative. Note that the (nodal) thickness reactions need to be pre-multiplied by the mass matrix and divided by the ice density for them to become 'mesh independent' and comparable to the surface mass balance. As is evident from comparing the two panels, the active set method creates a fictitious mass balance distribution that exactly counteracts the applied mass balance. Consequently, the ice thickness remains zero everywhere despite non-zero melt being applied along the upper surface.

Karush-Kuhn-Tucker conditions are

$$D_{\delta \boldsymbol{h}}[\mathcal{L}(\boldsymbol{h}, \boldsymbol{\lambda})] = 0 \tag{3.15}$$

$$\lambda_q \leq 0 \tag{3.16}$$

$$h_q \geq 0 \tag{3.17}$$

$$\boldsymbol{\lambda}^T \cdot \boldsymbol{h} = 0 \tag{3.18}$$

$$\tag{3.19}$$

The condition $\boldsymbol{\lambda}^T \cdot \boldsymbol{h} = 0$ is the complementary condition, if thickness is positive then the corresponding $\boldsymbol{\lambda}$ nodal values must be zero, and where $\lambda$ is negative the thickness must be zero.

If the system of field equations can not be derived from a minimum energy principle, as is the case here, the KKT conditions are introduced into the weak form.

Selecting the correct active set is a non-trivial problem and identifying it can be time consuming.

The thickness constraint is

$$h(x, y) = h_p \phi_p(x, y) \geq 0,$$

everywhere, of in the FE context

$$< h \mid \xi_q >= 0 \qquad \text{for} \qquad i = 1 \dots n \,,$$

where $n$ is the number of test functions (here equal to the number of nodes). We can also write this as

$$\boldsymbol{M}\boldsymbol{h} \geq \boldsymbol{0}$$

where $[\boldsymbol{M}]_{ij} = \int_\Omega \phi_i(x, y)\xi_j(x, y)\, d\Omega$ and $\boldsymbol{h}$ the vector of nodal thickness values. In the Galerkin method we use $\phi_q = \xi_q$. We solve the resulting inequality problem using a simple iterative procedure usually referred to as the active-set method. The active set, $\mathbb{A}$, is the set of nodes for which the equality constraint $h_p = 0$ is applied at current iteration. These nodes are referred to as active, while the remaining nodes where $h_p > 0$ are considered inactive. At a given iteration we solve the resulting (equality) constrained problem using the method of Lagrange multipliers and then update the active set as described below. Over the active set we have the equality constraint

$$
\begin{aligned}
0 &= \int_\Omega \sum_{q\in\mathbb{A}} h_q \phi_q(x,y)\xi_p(x,y)\,d\Omega \\
&= \int_\Omega \text{Ł}_{qr} h_r \phi_q\,\xi_p\,d\Omega \qquad \text{for} \qquad p = 1\ldots n\,,
\end{aligned}
$$

where $r$ ranges from 1 to $n$ and $L_{qr} = 1$ if node $r$ is active in the constraint $q \in \mathbb{A}$, and zero otherwise. We can also write the above equation as

$$\boldsymbol{LMh} = \mathbf{0}\,.$$

In addition to the active-set equality equations we have the field equations, which in in this particular case are the conservation equations for momentum and mass, written in discretized form as

$$\boldsymbol{F}(\boldsymbol{x}) = \mathbf{0}\,.$$

These are linearised as

$$\boldsymbol{K}\Delta\boldsymbol{x} = -\boldsymbol{F}(\boldsymbol{x}_0)\,,$$

where $\boldsymbol{K}$ is the directional derivative of $\boldsymbol{F}$, and solved using the Newton-Raphson method. We thus arrive at the KKL system

$$
\begin{bmatrix} \boldsymbol{K} & \boldsymbol{M}^T\boldsymbol{L}^T \\ \boldsymbol{LM} & \mathbf{0} \end{bmatrix}
\begin{bmatrix} \Delta\boldsymbol{h} \\ \Delta\boldsymbol{\lambda}^* \end{bmatrix}
=
\begin{bmatrix} -\boldsymbol{F}(\boldsymbol{x}_0) - \boldsymbol{LM}\boldsymbol{\lambda}_0^* \\ -\boldsymbol{LM}\boldsymbol{h}_0 \end{bmatrix}\,. \tag{3.20}
$$

Physically, $\boldsymbol{\lambda}^*$ can the though of as the thickness, increment, or additional mass influx per time unit, required to ensure positive thickness. At any given active-set iteration the active is therefore updated based on the sign of $\lambda_q^*$ at node $q$. Those nodes for which $\lambda_q^*$ is negative become inactive and are taken out of the active set, nodes where $\lambda_q^*$ is positive are kept in the active set, and previously inactive nodes where $h_q$ becomes negative are added to the active set. The iteration is continued until no further changes to the active set are required. Occasionally, the active set becomes cyclic, and the iteration the then terminated with all cyclic nodes included in the active set.

As commonly done in numerical models, linear multi-points constrains are implemented in Úa as $\boldsymbol{Lx} = \boldsymbol{c}$ where $\boldsymbol{L}$ is the where $\boldsymbol{L} \in \mathbb{R}^{p\times n}$ is an arbitrary MPCs matrix, where $p$ is the number of linear nodal constraints and $n$ the total degrees of freedom. When combined with Newton-Raphson method, this results in a KKL-type matrix system

$$
\begin{bmatrix} \boldsymbol{K} & \boldsymbol{L}^T \\ \boldsymbol{L} & \mathbf{0} \end{bmatrix}
\begin{bmatrix} \Delta\boldsymbol{x} \\ \Delta\boldsymbol{\lambda} \end{bmatrix}
=
\begin{bmatrix} -\boldsymbol{F}(\boldsymbol{x}_0) - \boldsymbol{L}\boldsymbol{\lambda}_0 \\ -\boldsymbol{L}\boldsymbol{c}_0 \end{bmatrix}\,. \tag{3.21}
$$

Note that when solving implicitly for velocities and thickness, as done in Úa , $\Delta\boldsymbol{x}$ stands for both velocity components and thickness, i.e.

$$\Delta\boldsymbol{x} = \begin{pmatrix} \Delta\boldsymbol{u} \\ \Delta\boldsymbol{v} \\ \Delta\boldsymbol{h} \end{pmatrix}\,.$$

The key difference between systems (3.20) and (3.21) is the inclusion of the mass matrix $\boldsymbol{M}$ in the former. Rather than modifying the commonly used framework (3.21) for implementing MPCs in finite elements to include the mass matrix $\boldsymbol{M}$, we now solve (3.21) and then calculate $\lambda^*$ afterwards from the solution of that system as

$$\boldsymbol{\lambda}^* = (\boldsymbol{LL}^T)^{-1}\boldsymbol{LM}^{-T}\boldsymbol{L}^T\boldsymbol{\lambda}\,. \tag{3.22}$$

Both $\boldsymbol{\lambda}^*$ and $\boldsymbol{\lambda}$ are in $\mathbb{R}^p$ while $\boldsymbol{M}^{-T}\boldsymbol{L}^T\boldsymbol{\lambda}$ is in $\mathbb{R}^n$. Note that if each of the columns of $\boldsymbol{L} \in \mathbb{R}^{p\times n}$ have exactly one non-zero entry and that entry is equal to unity, as will be the case if $\boldsymbol{L}$ is a thickness constraint, then $\boldsymbol{LL}^T$ is a $p \times p$ unity matrix irrespectively of how the thickness constraints have been labelled when constructing $\boldsymbol{L}$. It is easy to see[4] that $\boldsymbol{\lambda}^*$ is indeed identical to solution for $\boldsymbol{\lambda}$ of the KKL-system including the mass matrix, i.e. Eq. (3.20). However, only $\boldsymbol{\lambda}^*$ obtained by solving Eq. (3.20) has

[4]Consider

$$
\begin{aligned}
\boldsymbol{Fx} + \boldsymbol{L}^T\boldsymbol{\lambda} &= 0 \\
\boldsymbol{Fx} + \boldsymbol{ML}^T\boldsymbol{\lambda}^* &= 0
\end{aligned}
$$

from which (3.22) follows.

the physical interpretation mentioned above, i.e. it is the thickness increment required at a given node in the active set $\mathbb{A}$ to enforce thickness positivity. Hence, updating the active set must be based on the sign of $\boldsymbol{\lambda}^*$ and not on the sign of $\boldsymbol{\lambda}$. For linear elements all the entries of the mass matrix $\boldsymbol{M}$ are positive by construction and therefore, in that particular case, the active set update can be based on either $\lambda_q$ or $\lambda_q^*$. However, for higher order elements, e.g. quadratic or cubic, using $\boldsymbol{\lambda}$ instead of $\boldsymbol{\lambda}^*$ will, in general, results in an incorrect active-set update.

The active-set is updated after each NR iteration, and the the active-set update continued until convergence. In transient runs, only one or two updates are typically required. Updating the active set within the Newton loop was tried but did not result in any computational gains.

### 3.5.1 Thickness barrier

$$I = \gamma_h \lambda_h e^{-(h-h_{\min})/\lambda_h)}$$

$$\partial_h I = -\gamma_h e^{-(h-h_{\min})/\lambda_h}$$

$$\partial^2_{hh} I = \frac{\gamma_h}{\lambda_h} e^{-(h-h_{\min})/\lambda_h}$$

If the problem were self-adjoint then this amounts to adding a term to the prognostic equations, i.e.

$$\partial_t h + \nabla_{xy} \cdot \boldsymbol{q}_h + \partial_h I = a$$

or

$$\partial_t h + \nabla_{xy} \cdot \boldsymbol{q}_{xy} = a - \partial_h I$$

which shows that the method is equivalent to adding a fictitious mass-balance term. For $\gamma_h = 1$, $\lambda = 1$, $h_{\min} = 0$ and $h = 100$ the numerical value is about $10^{-44}$ and about $10^{-5}$ for $h = 1$.

If $a = 0$ and $\boldsymbol{v}_h = 0$

$$\partial_t h = -\partial_h I = \gamma_h e^{-(h-h_{\min})/\lambda_h}$$

and $\gamma_h$ has the units length per time and can be thought of as the fictitious mass balance at $h = h_{\text{hmin}}$.

Solving

$$\partial_t h + \nabla_{xy} \cdot \boldsymbol{q}_h - \gamma_h e^{-(h-h_{\min})/\lambda_h} = a$$

implicitly using NR with respect to $h$ where

$$h_{n+1} = h^i_{n+1} + \Delta h$$

is $h$ at time step $n+1$ and $i$ is the NR iteration number, gives

$$\frac{1}{\Delta t}(\Delta h + h^i_{n+1} - h_n) - \frac{1}{2}\left(\gamma_h e^{-(h^i_{n+1}-h_{\min})/\lambda_h} - \frac{\gamma_h}{\lambda_h} e^{-(h^i_{n+1}-h_{\min})/\lambda_h}\,\Delta h + \gamma_h e^{-(h_n-h_{\min})/\lambda_h}\right) = 0$$

where I have omitted writing the flux and the accumulation terms, i.e.

$$\left(\frac{1}{\Delta t} + \frac{1}{2}\frac{\gamma_h}{\lambda_h} e^{-(h^i_{n+1}-h_{\min})/\lambda_h}\right)\Delta h = -\frac{1}{\Delta t}(h^i_{n+1} - h_n) + \frac{\gamma_h}{2}\left(e^{-(h^i_{n+1}-h_{\min})/\lambda_h} + \gamma_h e^{-(h_n-h_{\min})/\lambda_h}\right)$$

# Chapter 4

# The non-linear FE system its solution

## 4.1 Variational formulation

For an elliptic PDE

$$\mathcal{L}u = f \qquad \text{(Ignoring BCs in this example)}$$

the unique solution $u$ minimises the functional

$$J(w) = \frac{1}{2}\int_\Omega w\mathcal{L}w \, d\Omega - \int_\Omega fw \, d\Omega \ ,$$

provided (1) $\mathcal{L}$ is linear, (2) self-adjoint

$$\int_\Omega v\mathcal{L}u \, d\Omega = \int_\Omega u\mathcal{L}v \, d\Omega \ ,$$

and (3), positive definite

$$\int_\Omega u\mathcal{L}u \geq 0 \ .$$

The resulting system for $u$ is

$$\frac{1}{2}\int_\Omega \mathcal{L}u \, \delta u \, d\Omega - \int_\Omega f \, \delta u \, d\Omega = 0 \ ,$$

and this is the system solved in the finite element method. The discrete from is

$$Au = F$$

where $[A]_{ij} = a_{ij}$ and $F_j = l_j$ and $a_{ij} = a(\phi_i, \xi_k)$ , $l_j = l(\xi_j)$ with

$$a_{ij} = \int_\Omega \phi_i \mathcal{L}\xi_j \, d\Omega$$

Note that in the FE method tend to construct the system $Au = F$ directly as a weighted residuals problem, i.e.

$$\int_\Omega (\mathcal{L}u - f)\phi_i \, d\Omega = 0 \ .$$

Only if the PDE is self-adjoint and positive-definite, can we interpret the discrete system $\boldsymbol{Au} = \boldsymbol{F}$ as resulting from a minimisation problem. If $\boldsymbol{A}$ is positive-definite we can work backwards from the discretized problem and interpret $\boldsymbol{Au} = \boldsymbol{F}$ as the solution to

$$\min_{\boldsymbol{x}} \left\{ \frac{1}{2}\boldsymbol{u}^T\boldsymbol{Au} - \boldsymbol{Fu} \right\} \ .$$

## 4.2 Non-linear self-adjoint problem

If the operator is self-adjoint and positive definite the resulting non-linear problem is a convex optimisation problem. We solve

$$\min_x J(\boldsymbol{x}) \;, \tag{4.1}$$

where $J : \boldsymbol{R}^n \to \boldsymbol{R}$. If, as we here assume, the problem is convex and differentiable we can in principle solve

$$\nabla J(x) = 0 \;. \tag{4.2}$$

For a convex function $J$, the two problems (4.1) and (4.2) are equivalent. A useful fact is that the function $J$ is convex on a convex set if and only if the Hessian is positive semi-definite.

For either solving a non-linear problem, or a non-linear convex optimisation problem, the method of choice is the Newton-Raphson method (Boyd and Vandenberghe, 2004). We expand

$$J(\boldsymbol{x}) = J(\boldsymbol{x}_0) + \nabla J(\boldsymbol{x}_0)\, \Delta \boldsymbol{x} + \frac{1}{2} \Delta \boldsymbol{x}^T\, \nabla^2 J(\boldsymbol{x}_0)\, \Delta \boldsymbol{x} + O(\|\delta \boldsymbol{x}^3\|)$$

where $\Delta \boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_0$, and solve for

$$\min_x \left\{ \nabla J(\boldsymbol{x}_0)\, \Delta \boldsymbol{x} + \frac{1}{2} \Delta \boldsymbol{x}^T \nabla^2 J(\boldsymbol{x}_0)\, \Delta \boldsymbol{x} \right\}$$

resulting in the Newton system

$$\nabla^2 J(\boldsymbol{x}_0)\, \Delta \boldsymbol{x} = -\nabla J(\boldsymbol{x}_0) \;,$$

and the Newton update

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{d}_k \;,$$

where

$$\boldsymbol{d}_k = -\left(\boldsymbol{H}_k\right)^{-1} \nabla J(\boldsymbol{x}_k) \;,$$

is the Newton step, and

$$\hat{\boldsymbol{d}}_k = \boldsymbol{d}_k / \|\boldsymbol{d}_k\| \;,$$

is the Newton direction, and

$$\boldsymbol{H}_k := \nabla^2 J(\boldsymbol{x}_k) \;,$$

is the Hessian matrix

However, unless the function $J(\boldsymbol{x})$ can be approximated reasonably well locally around $\boldsymbol{x}_0$ as a quadratic function there is no guarantee that $J(\boldsymbol{x}_{k+1}) \le J(\boldsymbol{x}_k)$ where $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{d}_k$. Should this happen, we solve the line-search sub-problem

$$\min_\gamma \left\{ J(\boldsymbol{x}_k + \gamma \boldsymbol{d} \right\} \;,$$

with $0 < \gamma < 1$. This line-search sub-problem has a local minima for some $\gamma$ within the bounds $0 \le \gamma < 1$ provided $\boldsymbol{d}_k$ is a direction of descent. For $\boldsymbol{d}_k$ to be a direction of descent requires

$$\nabla J(\boldsymbol{x}_k)^T \cdot \boldsymbol{d}_k < 0 \;. \tag{4.3}$$

From

$$\nabla J(\boldsymbol{x}_k)^T \cdot \boldsymbol{d}_k = -\nabla J(\boldsymbol{x}_k)^T \boldsymbol{H}^{-1}\, \nabla J(\boldsymbol{x}_k) \;,$$

we see that

$$\nabla J(\boldsymbol{x}_k)^T \cdot \boldsymbol{d}_k < 0 \;, \tag{4.4}$$

is indeed fulfilled, provided $\boldsymbol{H}^{-1}$ is positive definite.

At the minimum of its quadratic approximation $J(x)$ has the value

$$\begin{aligned} J(x) &\approx J(\boldsymbol{x}_k) + \nabla J(\boldsymbol{x}_k)\, \Delta \boldsymbol{x} + \frac{1}{2} \Delta \boldsymbol{x}^T\, \boldsymbol{H}_k\, \Delta \boldsymbol{x} \\ &= J(\boldsymbol{x}_k) - \nabla J(\boldsymbol{x}_k)\, (\boldsymbol{H}_k)^{-1}\, \nabla J(\boldsymbol{x}_k) + \frac{1}{2} ((\boldsymbol{H}_k)^{-1}\, \nabla J(\boldsymbol{x}_k))^T\, \boldsymbol{H}_k\, (\boldsymbol{H}_k)^{-1}\, \nabla J(\boldsymbol{x}_k) \\ &= J(\boldsymbol{x}_k) - \frac{1}{2} \nabla J(\boldsymbol{x}_k)\, (\boldsymbol{H}_k)^{-1}\, \nabla J(\boldsymbol{x}_k) \;. \end{aligned}$$

This motivates the definition of the *Newton decrement* $D_k$ as

$$D_k := \sqrt{\nabla J(\boldsymbol{x}_k)\,(\boldsymbol{H}_k)^{-1}\,\nabla J(\boldsymbol{x}_k)}\,,$$

and $D_k^2/2$ is an approximate bound on the suboptimality gap at iteration $k$. Having solved the Newton system, i.e. $\boldsymbol{d}_k = -(\boldsymbol{H}_k)^{-1}\,\nabla J(\boldsymbol{x}_k)$, we can calculate the Newton increment as

$$D_k^2 = -\nabla J(\boldsymbol{x}_k)\cdot\boldsymbol{d}_k.$$

Positive definiteness of $\nabla^2 J$ implies $D > 0$. We can also write the Newton decrement as

$$\begin{aligned}
D &= \sqrt{\nabla J(\boldsymbol{x}_k)^T\,(\boldsymbol{H}_k)^{-1}\,\nabla J(\boldsymbol{x}_k)} \\
&= (\nabla J^T \boldsymbol{H}^{-T}\boldsymbol{H}^T\boldsymbol{H}^{-1}\nabla J) \\
&= ((\nabla H^{-1}\nabla J)^T\boldsymbol{H}^T\boldsymbol{H}^{-1}\nabla J) \\
&= (\boldsymbol{d}^T\boldsymbol{H}\boldsymbol{d})^{1/2} \\
&= \|\boldsymbol{d}\|_{\boldsymbol{H}}
\end{aligned}$$

showing that the Newton decrement is the norm of the Newton step in the Hessian norm. The line search problem is finding $\gamma$ so that

$$\begin{aligned}
\frac{d}{d\gamma}J(\boldsymbol{x}+\gamma\boldsymbol{d}) &= \nabla J(\boldsymbol{x})\cdot\boldsymbol{d} \\
&= -\nabla J(\boldsymbol{x})\,(\boldsymbol{H})^{-1}\,\nabla J(\boldsymbol{x}) \\
&= -D^2\,.
\end{aligned}$$

Again we see that if $\boldsymbol{H}$ is positive definite, the slope at $\gamma = 0$ is negative.

For a self-adjoint problem the Newton's increment is a natural candidate for a stopping criterion, i.e. quit iteration if $D^2/2 \le \epsilon$, where $\epsilon$ is some prescribed sub-optimality tolerance.

## 4.3 Non-linear non-self-adjoint problem

For a non-self-adjoint differential operator no (scalar) cost function can be defined in a similar way as for the self-adjoint case.

Now the Newton-Raphson system is

$$\boldsymbol{K}\,\boldsymbol{d} = -\boldsymbol{R}\,,$$

where

$$\boldsymbol{K} = \frac{\partial\boldsymbol{R}}{\partial\boldsymbol{d}}\,,$$

is the Jakobian. We can also write this as

$$R_{p,q}\,d_q = -R_p\,.$$

The vector $\boldsymbol{R}$ are the nodal residuals and $\boldsymbol{d}$ the increments during each NR iteration. Provided the system converges, both approach zeros as the iteration progresses. In FE, common convergence criteria are based on

| | |
|---|---|
| $\lVert\boldsymbol{R}\rVert$ | Force Residuals |
| $\lVert\boldsymbol{d}\rVert$ | Displacements |
| $-\boldsymbol{R}\cdot\boldsymbol{d}$ | Work Residuals |

For self-adjoint problem where $\boldsymbol{R} = \nabla J$ and $\boldsymbol{K} = \boldsymbol{H}$, the work and the Newton decrement criteria

$$\begin{aligned}
D^2 &= -\boldsymbol{R}\cdot\boldsymbol{d} \\
&= \boldsymbol{R}\cdot\boldsymbol{K}^{-1}\boldsymbol{R} \\
&= \nabla J\cdot\boldsymbol{H}^{-1}\,\nabla J \\
&\ge 0
\end{aligned}$$

are identical. However, for a non-self adjoint problem $D^2$ can be negative as well as positive. Instead of minimising $D^2$ we find the solution where the increment, $\boldsymbol{d}$ is normal to the residual force $\boldsymbol{R}$, i.e. we minimise

$$D^2 = |\boldsymbol{R} \cdot \boldsymbol{d}| \ .$$

This corresponds to a minimum in the work function, for example because either force and/or displacement increments are small, if the force increments are orthogonal to displacement increments.

## 4.4 Constraint self-adjoint problem

Again if the problem is self-adjoint there exist a scalar function $J$ such that $\nabla J = \boldsymbol{R}$ and $\boldsymbol{H} = \nabla^2 J$. Now consider the constraint optimisation problem

$$\begin{aligned} \min_x \quad & J(x) \\ \text{s.t.} \quad & \boldsymbol{L}\boldsymbol{x} = \boldsymbol{c} \end{aligned}$$

where $\boldsymbol{L} \in \mathbb{R}^{p\times n}$. A solution $\boldsymbol{x}^*$ is optimal if and only if there exist a $\boldsymbol{\lambda}$ such that

$$\begin{aligned} \nabla J(\boldsymbol{x}^*) + \boldsymbol{L}^T\boldsymbol{\lambda} &= \boldsymbol{0} \\ \boldsymbol{L}\boldsymbol{x}^* &= \boldsymbol{c} \end{aligned}$$

where $\boldsymbol{R} = \nabla J$ (Boyd and Vandenberghe, 2004). Second-order expansion

$$\begin{aligned} \nabla J(\boldsymbol{x} + \Delta\boldsymbol{x}) + \boldsymbol{L}^T\boldsymbol{\lambda} &\approx \nabla J(\boldsymbol{x}) + \nabla^2 J\,\Delta\boldsymbol{x} + \boldsymbol{L}^T(\boldsymbol{\lambda} + \Delta\boldsymbol{\lambda}) = 0 \\ \boldsymbol{L}(\boldsymbol{x} + \Delta\boldsymbol{x}) &= \boldsymbol{c} \end{aligned}$$

and solving for the minimum of the quadratic term, gives the KKT system

$$\begin{bmatrix} \boldsymbol{H} & \boldsymbol{L}^T \\ \boldsymbol{L} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{x} \\ \Delta\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{R}(\boldsymbol{x}_0) - \boldsymbol{L}^T\boldsymbol{\lambda}_0 \\ \boldsymbol{c} - \boldsymbol{L}\boldsymbol{x}_0 \end{bmatrix} \tag{4.5}$$

where $\boldsymbol{H} = \nabla^2 J$ is the Hessian matrix. The $\Delta\boldsymbol{x}$ and $\Delta\boldsymbol{\lambda}$ are referred to as the primal and the dual steps, respectively.

Note that since the constraints are linear, they are fulfilled exactly at each iteration.[1] The second equation, i.e. the condition $\boldsymbol{L}\Delta\boldsymbol{x} = \boldsymbol{c} - \boldsymbol{L}\boldsymbol{x}_0$ ensures that the Newton step $\Delta\boldsymbol{x}$ is always a feasible direction.[2] It follows that if $\boldsymbol{x}_0$ is a feasible point, i.e. if $\boldsymbol{L}(\boldsymbol{x}_0) = \boldsymbol{c}$, and if $\Delta\boldsymbol{x}$ is in the null space of $\boldsymbol{L}$, i.e. $\boldsymbol{L}\Delta\boldsymbol{x} = \boldsymbol{0}$, then every point $\boldsymbol{x}_0 + \gamma\Delta\boldsymbol{x}$ for any $\gamma$ is also a feasible point because

$$\begin{aligned} \boldsymbol{L}(\boldsymbol{x}_0 + \gamma\Delta\boldsymbol{x}) &= \boldsymbol{L}(\boldsymbol{x}_0) + \boldsymbol{L}\gamma\Delta\boldsymbol{x} \\ &= \boldsymbol{c} + \gamma\boldsymbol{L}\Delta\boldsymbol{x} \\ &= \boldsymbol{c} \ . \end{aligned}$$

Hence, when conducting a line search the constraints are automatically fulfilled and there is no need to re-solve the KKL system during the line search. However, if the constraints are non-linear this is no longer case. In particular, if the constraints are not equality constraints but inequality constraints, it is possible that some of the constraints may become active/inactive during the line search.

The Newton step, $\boldsymbol{d}$,

$$\boldsymbol{d} = \Delta\boldsymbol{x}$$

is defined as a solution $\Delta\boldsymbol{x}$ to the two KKL equations

$$\begin{aligned} \boldsymbol{H}\Delta\boldsymbol{x} + \boldsymbol{L}^T\boldsymbol{\lambda} &= -\boldsymbol{R}(\boldsymbol{x}_0) \\ \boldsymbol{L}\Delta\boldsymbol{x} &= \boldsymbol{0} \end{aligned}$$

[1] We could, in fact, start the iteration with $\boldsymbol{x} = \boldsymbol{x}_0$ where $\boldsymbol{x}_0$ is a particular solution of $\boldsymbol{L}\boldsymbol{x}_0 = \boldsymbol{c}$ and then insist that any addition to $x$ is in the null-space of $\boldsymbol{L}$, i.e. $\boldsymbol{L}\Delta\boldsymbol{x} = \boldsymbol{0}$.

[2] Since the constraints are linear, we can demand that they are fulfilled exactly at each iteration, in particular we then have $\boldsymbol{L}\boldsymbol{x}_0 = \boldsymbol{c}$ and therefore $\boldsymbol{L}\Delta\boldsymbol{x} = \boldsymbol{0}$. Hence, for linear constraints the KKL system can also be written as

$$\begin{bmatrix} \boldsymbol{H} & \boldsymbol{L}^T \\ \boldsymbol{L} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{x} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{R}(\boldsymbol{x}_0) \\ \boldsymbol{0} \end{bmatrix} \ . \tag{4.6}$$

or

$$\begin{aligned}\Delta\boldsymbol{x} &= -\boldsymbol{H}^{-1}\left(\boldsymbol{R} + \boldsymbol{L}^T(\boldsymbol{\lambda}_0 + \Delta\boldsymbol{\lambda})\right) \\ &= -\boldsymbol{H}^{-1}\left(\nabla J + \boldsymbol{L}^T\boldsymbol{\lambda}\right) \ .\end{aligned}$$

Taking the full Newton step, the reduction in $J$ at each iteration is therefore

$$\begin{aligned}J(x) - J(\boldsymbol{x}_k) =& \nabla J(\boldsymbol{x}_k)\,\Delta\boldsymbol{x} + \frac{1}{2}\Delta\boldsymbol{x}^T\,\boldsymbol{H}_k\,\Delta\boldsymbol{x} \\ =& \nabla J^T\,\boldsymbol{H}^{-1}\left(-\nabla J - \boldsymbol{L}^T\boldsymbol{\lambda}\right) + \frac{1}{2}(\boldsymbol{H}^{-1}\left(-\nabla J - \boldsymbol{L}^T\boldsymbol{\lambda}\right))^T\,\boldsymbol{H}_k\,\boldsymbol{H}^{-1}\left(-\nabla J - \boldsymbol{L}^T\boldsymbol{\lambda}\right) \\ =& \nabla J^T\,\boldsymbol{H}^{-1}\left(-\nabla J - \boldsymbol{L}^T\boldsymbol{\lambda}\right) + \frac{1}{2}(-\nabla J - \boldsymbol{L}^T\boldsymbol{\lambda})^T\,\boldsymbol{H}^{-1}(-\nabla J - \boldsymbol{L}^T\boldsymbol{\lambda}) \\ =& -\nabla J^T\,\boldsymbol{H}^{-1}\nabla J - \nabla J\boldsymbol{H}^{-1}\boldsymbol{L}^T\boldsymbol{\lambda} + \frac{1}{2}(-\nabla J - \boldsymbol{L}^T\boldsymbol{\lambda})^T\,\boldsymbol{H}^{-1}(-\nabla J - \boldsymbol{L}^T\boldsymbol{\lambda}) \\ =& -\nabla J^T\,\boldsymbol{H}^{-1}\nabla J - \nabla J\boldsymbol{H}^{-1}\boldsymbol{L}^T\boldsymbol{\lambda} \\ &+ \frac{1}{2}\left(\nabla J^T\boldsymbol{H}^{-1}\nabla J + (\boldsymbol{L}^T\boldsymbol{\lambda})^T\boldsymbol{H}^{-1}\boldsymbol{L}^T\boldsymbol{\lambda} + \nabla J\boldsymbol{H}^{-1}\boldsymbol{L}^T\boldsymbol{\lambda} + (\boldsymbol{L}^T\boldsymbol{\lambda})^T\boldsymbol{H}^{-1}\nabla J\right) \\ =& -\frac{1}{2}(\nabla J^T + (\boldsymbol{L}^T\boldsymbol{\lambda})^T)\,\boldsymbol{H}^{-1}(\nabla J^T + \boldsymbol{L}^T\boldsymbol{\lambda}) \\ &- \nabla J\boldsymbol{H}^{-1}\boldsymbol{L}^T\boldsymbol{\lambda} + \frac{1}{2}\left(\nabla J^T\boldsymbol{H}^{-1}\boldsymbol{L}^T\boldsymbol{\lambda} + (\boldsymbol{L}^T\boldsymbol{\lambda})^T\boldsymbol{H}^{-1}\nabla J\right)\end{aligned}$$

The last line equals zero if $\boldsymbol{H}$ symmetrical, hence as in the unconstrained case the Newton decrement is a measure of sub-optimality, i.e.

$$\begin{aligned}J(x) - J(\boldsymbol{x}_k) =& -\frac{1}{2}(\nabla J^T + (\boldsymbol{L}^T\boldsymbol{\lambda})^T)\,\boldsymbol{H}^{-1}(\nabla J + \boldsymbol{L}^T\boldsymbol{\lambda}) \\ =& -D^2/2\end{aligned}$$

where Newton decrement $D$ is

$$\begin{aligned}D^2 &= (\nabla J + (\boldsymbol{L}^T\boldsymbol{\lambda})^T\cdot\Delta\boldsymbol{x} \\ &= (\nabla J + \boldsymbol{L}^T\boldsymbol{\lambda})^T\boldsymbol{H}^{-1}(\nabla J + \boldsymbol{L}^T\boldsymbol{\lambda}) \\ &= (\nabla J + \boldsymbol{L}^T\boldsymbol{\lambda})^T\boldsymbol{H}^{-1}\boldsymbol{H}\boldsymbol{H}^{-1}(\nabla J + \boldsymbol{L}^T\boldsymbol{\lambda}) \\ &= (\nabla J + \boldsymbol{L}^T\boldsymbol{\lambda})^T\boldsymbol{H}^{-T}\boldsymbol{H}\boldsymbol{H}^{-1}(\nabla J + \boldsymbol{L}^T\boldsymbol{\lambda}) \\ &= (\boldsymbol{H}^{-1}(\nabla J + \boldsymbol{L}^T\boldsymbol{\lambda}))^T\boldsymbol{H}\boldsymbol{H}^{-1}(\nabla J + \boldsymbol{L}^T\boldsymbol{\lambda}) \\ &= \boldsymbol{d}^T\boldsymbol{H}\boldsymbol{d}\end{aligned}$$

When conducting a line search we find, similarly to the unconstrained case, that

$$\begin{aligned}\frac{d}{d\gamma}J(\boldsymbol{x} + \gamma\boldsymbol{d}) &= \nabla J(\boldsymbol{x})\cdot\boldsymbol{d} \\ &= \boldsymbol{R}^T\cdot\boldsymbol{d} \\ &= -(\boldsymbol{H}\boldsymbol{d} + \boldsymbol{L}^T\boldsymbol{\lambda})^T\cdot\boldsymbol{d} \\ &= -\boldsymbol{d}^T\cdot\boldsymbol{H}^T\boldsymbol{d} - \boldsymbol{\lambda}^T\boldsymbol{L}\cdot\boldsymbol{d} \\ &= -\boldsymbol{d}^T\cdot\boldsymbol{H}^T\boldsymbol{d} - \boldsymbol{\lambda}^T(\boldsymbol{c} - \boldsymbol{L}\boldsymbol{x}_0) \\ &= -D^2 + (\boldsymbol{L}\boldsymbol{x}_0 - \boldsymbol{c})^T\cdot\boldsymbol{\lambda}\ .\end{aligned}$$

Thus if $\boldsymbol{H}$ is symmetrical and positive definite, and furthermore $\boldsymbol{x}_0$ a feasible point, then the slope at $\gamma = 0$ is negative. However, the Newton direction at an infeasible point is not necessarily a decent direction for $J$. This point, and other similar are also discussed in Boyd and Vandenberghe (2004).

## 4.5 Convergence criteria

If the Newton-Raphson method converges then $\Delta\boldsymbol{x} \to 0$ and $\Delta\boldsymbol{\lambda} \to 0$. This suggests defining a convergence criteria in terms of the norm of $\Delta\boldsymbol{x}$ and $\Delta\boldsymbol{\lambda}$, i.e. in terms of the size of the update to the solution.

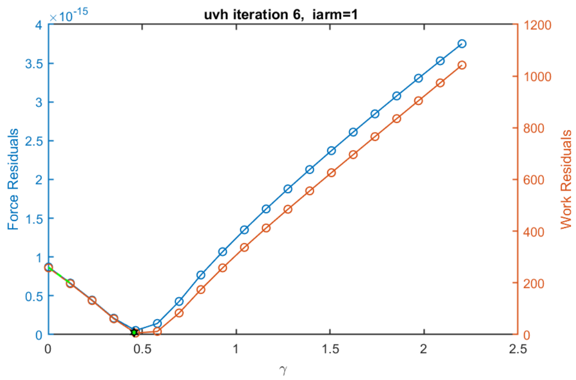



Figure 4.1: An example of the variation of $r^2_{\text{Force}}$ and $r^2_{\text{Work}}$ as a function of $\gamma$ during a line-search. Here the backtracking algorithm selects $\gamma \approx 1/2$. For clarity, the circles show some additional function values not calculated during the line search.

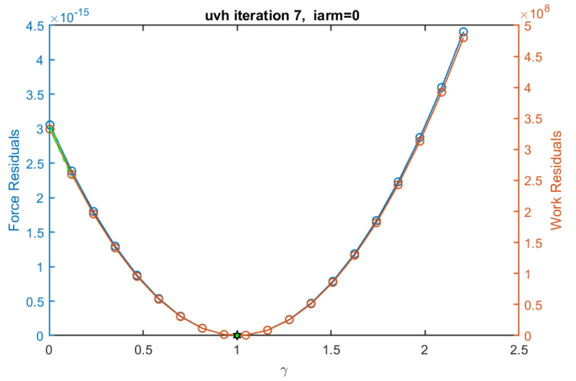



Figure 4.2: Another example of the variation of $r^2_{\text{Force}}$ and $r^2_{\text{Work}}$ as a function of $\gamma$. Here the full Newton step ($\gamma = 1$) was accepted. Also shown are the slopes at $\gamma = 0$ calculated as $2r$ at $\gamma = 0$.

Another convergence criteria is the norm of the force residuals $\boldsymbol{R}$ or the size of the Newton increment $\boldsymbol{R} \cdot \boldsymbol{d}$ (work residuals). The primary convergence criteria in Úa is the size of the force residuals. It is possibly to define a convergence criteria based on force residuals, work residuals and increments, and some combinations thereof. Defining a convergence criteria in terms of increments can be quite problematic as smallness of increments does not imply convergence.

### 4.5.1 Force residuals

Writing the residual vector $\boldsymbol{R}$ as

$$\boldsymbol{R} = \boldsymbol{T}(\boldsymbol{x}) - \boldsymbol{F} \,,$$

where $\boldsymbol{T}$ and $\boldsymbol{F}$ are the internal and the external nodal forces, respectively facilitates the definition of a normalised *force residuals* as

$$r^2 := \frac{(\boldsymbol{R} + \boldsymbol{L}^T \boldsymbol{\lambda})^T (\boldsymbol{R} + \boldsymbol{L}^T \boldsymbol{\lambda}) + (\boldsymbol{c} - \boldsymbol{L}\boldsymbol{x})^T (\boldsymbol{c} - \boldsymbol{L}\boldsymbol{x})}{\boldsymbol{F}^T \boldsymbol{F}} \tag{4.7}$$

$$= \frac{(\boldsymbol{R} + \boldsymbol{L}^T \boldsymbol{\lambda})^T (\boldsymbol{R} + \boldsymbol{L}^T \boldsymbol{\lambda})}{\boldsymbol{F}^T \boldsymbol{F}} \,, \tag{4.8}$$

where in (4.8) is has been assumed that $\boldsymbol{c} - \boldsymbol{L}\boldsymbol{x} = \boldsymbol{0}$ at each iteration. If the Newton step is started from a feasible point, then the Newton direction is a decent direction of (4.8). However if the point is infeasible, (4.7) must be used.

If at some iteration a full Newton step can be taken, the iterate becomes feasible and as all subsequent steps are in the null space of $\boldsymbol{L}$, all following iterates will be feasible as well.

At start of an iteration using as start values $\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{\lambda} = 0$, the internal forces are all zero, $\boldsymbol{T} = \boldsymbol{0}$, and hence $r = 1$. The iteration is continued until $r$ drops below a prescribed tolerance. By default this tolerance is

$$r^2 < 10^{-15} \,.$$

### 4.5.2 Work residuals

The *work residuals* are defined in terms of the Newton decrement as

$$r := -(\boldsymbol{R} + \boldsymbol{L}^T \boldsymbol{\lambda})^T \cdot \boldsymbol{d}$$

where $\boldsymbol{d}$ is the (full) Newton step.

$$\boldsymbol{d} = -\boldsymbol{H}^{-1} (\boldsymbol{R} + \boldsymbol{L}^T \boldsymbol{\lambda}) \,.$$

The actual minimisation is based on the value $r^2$ which is always positive.

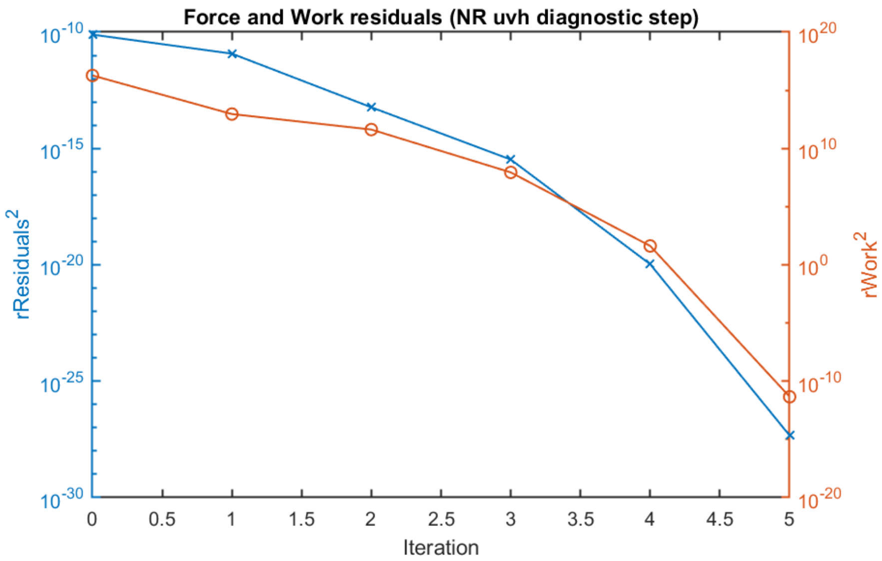



Figure 4.3: Need to correct the title, An example of $r^2_{\text{Force}}$ and $r^2_{\text{Work}}$ as a function of Newton-Raphson iteration number.

### 4.5.3 Increments

The increments are quantified through

$$\Delta = \sqrt{\frac{1}{2\mathcal{A}} \int \left( \frac{\Delta x}{|x| + x_0} \right)^2 \, d\mathcal{A}} \,,$$

where

$$\mathcal{A} = \int d\mathcal{A} \,,$$

and $x_0$ is a regularisation parameter. So for example when solving for velocities the size of the velocity increments is defined as

$$\Delta_{uv} = \sqrt{\frac{1}{2\mathcal{A}} \int \left( \left( \frac{\Delta u}{|u| + u_0} \right)^2 + \left( \frac{\Delta v}{|v| + u_0} \right)^2 \right) d\mathcal{A}} \,.$$

where $u_0$ is some small constant with the dimension velocity.

## 4.6 Summary of convergence criteria

Ignoring constraints and possible additional normalisation constants, the three convergence criteria are:

$$\begin{aligned} J^2_{\text{Force}} :=& \boldsymbol{R}^T \cdot \boldsymbol{R} \\ J^2_{\text{Work}} :=& (\boldsymbol{R}^T \cdot \boldsymbol{d})^2 \\ =& (\boldsymbol{R}^T \cdot \boldsymbol{H}^{-1} \boldsymbol{R})^2 \\ J^2_{\text{Displacements}} :=& \boldsymbol{d} \cdot \boldsymbol{d} \end{aligned}$$

Note that since $\boldsymbol{H}$ is not guaranteed to be positive definite, $(\boldsymbol{R}^T \cdot \boldsymbol{H}^{-1} \boldsymbol{R})$ is not necessarily positive and we can not define a corresponding inner product as we could in the self-adjoint positive-definite case considered above. The work criteria is the only one independent of coordinates (affine invariance).

The Newton step is a direction of decent if Eq. (4.3) is fulfilled for those cost functions. We have already seen that the Newton step is a direction of decent if the problem is self-adjoint and positive definite (see Eq. 4.4). For $J_{\text{Force}}$ we find

$$\begin{aligned}\nabla J^2_{\text{Force}} \cdot \boldsymbol{d} =& (R_p R_p)_{,q}\, d_q \\ =& (R_p R_{p,q} + R_{p,q} R_p)\, d_q \\ =& 2 R_p R_{p,q}\, d_q \\ =& -2 R_p R_p \\ =& -J_{\text{Force}} \\ \leq& 0 \;,\end{aligned}$$

and similarly for $J_{\text{Work}}$

$$\begin{aligned}\nabla J^2_{\text{Work}} \cdot \boldsymbol{d} =& (R_p d_p R_s d_s)_{,q}\, d_q \\ =& (R_{p,q} d_p R_s d_s + R_p d_p R_{s,q} d_s)\, d_q \\ =& (R_{p,q} d_p R_s d_s + R_s d_s R_{p,q} d_p)\, d_q \\ =& 2 R_{p,q} d_p R_s d_s\, d_q \\ =& 2 R_{p,q} d_q d_p R_s d_s \\ =& -2 R_p d_p R_s d_s \\ =& -2 J_{\text{Work}} \\ \leq& 0 \;,\end{aligned}$$

so the Newton step is a direction of descent for both $J^2_{\text{Force}}$ and $J^2_{\text{Work}}$. Same calculations show that when conducting a line search, the slopes at $\gamma = 0$ are

$$\begin{aligned}\frac{d}{d\gamma} J^2_{\text{Force}}(\boldsymbol{x} + \gamma \boldsymbol{d})\bigg|_{\gamma=0} =& \nabla J^2_{\text{Force}} \cdot \boldsymbol{d} \\ =& -2 J^2_{\text{Force}}(\boldsymbol{x}) \;,\end{aligned}$$

and

$$\begin{aligned}\frac{d}{d\gamma} J^2_{\text{Work}}(\boldsymbol{x} + \gamma \boldsymbol{d})\bigg|_{\gamma=0} =& \nabla J^2_{\text{Work}} \cdot \boldsymbol{d} \\ =& -2 J^2_{\text{Work}}(\boldsymbol{x}) \;.\end{aligned}$$

In *Úa* the letter $r$ is used for these costs functions to avoid confusion with the inverse cost function denoted with $J$. The cost functions can also be written as inner products as

$$\begin{aligned}J^2_{\text{Force}} = r^2_{\text{Force}} &= \boldsymbol{R}^T \boldsymbol{R} = < \boldsymbol{R} \mid \boldsymbol{R} >_{l^2} \\ J^2_{\text{Work}} = r^2_{\text{Work}} &= \boldsymbol{R}^T \boldsymbol{H}^{-1} \boldsymbol{R} = < \boldsymbol{R} \mid \boldsymbol{R} >_{\boldsymbol{H}^{-1}} \\ &= \boldsymbol{R}^T \boldsymbol{H}^{-T} \boldsymbol{H} \boldsymbol{H}^{-1} \boldsymbol{R} = < \boldsymbol{d} \mid \boldsymbol{d} >_{\boldsymbol{H}} \quad \text{(if symmetric and positive definite)} \\ J^2_{\text{Displacements}} = r^2_{\text{Displacements}} &= < \boldsymbol{d} \mid \boldsymbol{d} >_{L^2}\end{aligned}$$

# Chapter 5

# Inverse modelling

We denote the control and the state variables by $p$ and $q$, respectively. The 'state variable' ($q$) is any variable calculated by the model, such as velocity, rates of elevation change, etc. The 'control' variable ($p$) is any model input variable required by the model to calculate the state variables (e.g. $A$ and $C$, $B$, etc.).

We write the forward model as

$$F(q(p), p) = 0,$$

where $p$ are model parameters and $q$ the state variable.

A 'forward calculation' consists in finding a solution $q$ to the above equation for some parameters $p$. Roughly speaking, an 'inverse problem' is the opposite problem of finding $p$ given $q$. Inverse problems encountered in geophysics typically have at least vigintillion[1] and one possible solutions. However, some of those might be more likely than others. Selecting the more likely one out of the vigintillion and one possible ones is commonly done by introducing some constraints on $p$.

We consider the problem of minimising an objective function $J$ with respect to $p$. Typically the objective function $J$ can be thought of as a sum of two terms

$$J(q(p), p) = I(q(p)) + R(p),$$

where $I$ is a misfit term and $R$ a regularisation term.

When inverting for the (distributed) model parameter $p$ we refer to it as as the control variable to distinguish it from any other model parameters.

## 5.1 General inverse methodology

Bayesian framework provides a general methodology for inverse problems. The inverse problem we like to solve if that of determining some model parameters, $p$, given some measurement data, $d$. Solving an inverse problem is to determine the conditional distribution

$$P(p|d)$$

that is, we want to determine the probability distribution for our model parameters, $p$, given our measurement data, $d$, where we have a forward model, $f$, providing a functional relationship between the model parameters and the data

$$d = f(p) \; .$$

More generally we can write the forward model as

$$d(x, t) = f(x, t_0; p)$$

to stress the fact that the model output is a function of space, $x$, and variable with time, $t$. When running a transient model we start the run at $t = t_0$.

If our model is perfect, the measurements error free and we are using the perfectly correct values for all the model parameters $p$, our model output will match the measurements perfectly. More realistically our data will have some errors and we write

$$\tilde{d} = d + \epsilon$$

[1] 1 Vigintillion = $10^{63}$.

Table 5.1: Notation used in inverse modelling (I)

| | |
|---|---|
| $F(d(p), p) = 0$ | state equation |
| $d = f(p)$ | forward model |
| $p$ | model parameters, control variable (design variable), e.g. $A$, $C$, $B$ |
| $d$ | state variable, e.g. $u$, $v$, $\partial_t h$ |
| $p_{\text{prior}}$ and $\tilde{p}$ | *a priori* estimates of model parameters |
| $d_{\text{meas}}$ and $\tilde{d}$ | estimates of the state variable (measurements) |
| $J = I + R$ | objective function |
| $R$ | regularisation term |
| $I$ | misfit term |
| $\mathcal{L} = I + R + < \lambda \mid F(d(p), p) >$ | Lagrangian or the 'extended' objective function. |
| $K$ | covariance matrix |
| $\mathcal{A} = \int dA$ | domain area |

where $\tilde{d}$ are our measurements and $d$ are the (unknown to us) error free data. Similarly, we may have some prior estimates for the model parameters. We can think of these as measurements of $p$ with some associated errors and write

$$\tilde{p} = p + \epsilon$$

where $\tilde{p}$ are our measurements (prior estimates) of the model parameters, and $p$ are the corresponding 'true' values. Finally, the forward model might in inexact and we write

$$\tilde{f} = f + \epsilon$$

where $\tilde{f}$ is the forward model we are using, and $f$ the perfectly correct (and unknown to us) forward model.

Specifying the error structure of $\tilde{d}$ and $\tilde{p}$ can be either easy or difficult depending on the situation. Typically $\epsilon$ might be assumed to be a Gaussian process with some covariance, i.e. $P(x) = \mathcal{N}(x|\tilde{x}, \Sigma)$, and this now introduces new covariance parameters.

We can think of the inverse model as a *retrieval function* $\mathcal{R}$ which gives us an estimate, $\hat{p}$, for the model parameters give our measurement data, prior estimates of the model parameters and our description of all associated errors,

$$\hat{p} = \mathcal{R}(f, \tilde{d}, \tilde{p}, \mu)$$

Here $\mu$ are parameters of our error models. For example if we assume Gaussian normal distribution of errors $\mu$ would be a vector with two elements, the elements being the mean and the variance.

The Bayesian theorem

$$P(p|d) = \frac{P(d|p)\,P(p)}{P(d)}$$

is a basic statement about conditional probabilities. Here it allows us to 'invert' the conditional probability $P(p|d)$ and write it in terms of $P(d|p)$ which, given the relation $y = f(p)$, can be done by solving the forward model for a given $p$.

The term

$$P(d|p)$$

is referred to as the *likelihood*, and

$$P(p)$$

is the *prior*,

$$P(d)$$

is the *marginal likelihood*, and

$$P(p|d)$$

is the *posterior* distribution.

Ideally we would like to determine the posterior distribution, $P(x|y)$, but computational limitations might force us to only calculate some of its aspect such as its maximum with respect to $x$, i.e. the *maximum a posteriori* (MAP) estimate of $x$.

$$p_{\text{MAP}} = \max_p P(p|d) \; .$$

If we are only interested in the general shape of $P(p|d)$ as a function of $p$ we do not need to estimate the marginal likelihood, $P(d)$, as it is not dependent of $p$. Our primary focus is then to determine the likelihood, $P(d|p)$, and the prior, $P(p)$.

### 5.1.1 Example of Bayesian estimation

To clarify some of the concepts that follow we give a quick summary of Bayesian analysis of a linear model $f$ where

$$f(x) = \sum_{1}^{n} f_q \phi_q(x) \tag{5.1}$$

or for any location $x = x^*$

$$f(x_*) = \boldsymbol{\phi}(x_*)^T \boldsymbol{f}$$

We would like to determine the $n$ coefficients $f_q$ from $m$ available measurements $y_q$ taken at the locations $x_q$, $q = 1 \dots m$ where

$$y = f(x) + \epsilon$$

and

$$\epsilon \sim \mathcal{N}(0, \sigma^2)$$

hence,

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_n(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_m) & \phi_2(x_m) & \cdots & \phi_n(x_m) \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} + \begin{pmatrix} \epsilon \\ \epsilon \\ \vdots \\ \epsilon \end{pmatrix}$$

or

$$\boldsymbol{y} = \boldsymbol{\Phi} \boldsymbol{f} + \boldsymbol{\epsilon} \tag{5.2}$$

where $\boldsymbol{y}$ is the *measurement vector*. Our objective is to determine the vector $\boldsymbol{f}$ from our measurements $\boldsymbol{y}$ given our forward model expressed by Eq. (5.2) and $\boldsymbol{\Phi} \in \mathbb{R}^{m \times n}$.

The Bayesian approach to this is to determine

$$P(\boldsymbol{f}|\boldsymbol{y}) = \frac{P(\boldsymbol{y}|\boldsymbol{f})P(\boldsymbol{f})}{P(\boldsymbol{y})}$$

The data errors are assumed to be Gaussian white noise

$$\mathbb{E}[\epsilon(x)\epsilon(x')] = \sigma^2 \delta(x - x')$$

where $\mathbb{E}$ is the expectation operator i.e. $\epsilon \sim \mathcal{N}(0, \sigma^2)$. The *noise model* or the *likelihood* is then

$$P(\boldsymbol{y}|\boldsymbol{f}) = \mathcal{N}(\boldsymbol{\Phi}\boldsymbol{f}, \sigma^2 \boldsymbol{I})$$

which we can see as follows

$$\begin{aligned} P(\boldsymbol{y}|\boldsymbol{f}) &= \prod_{i=1}^{n} P(y_i|\boldsymbol{f}) \\ &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y_i - [\boldsymbol{\Phi}]_{ij} f_j)^2}{2\sigma^2}} \\ &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{f}\|^2}{2\sigma^2}} \\ &= \mathcal{N}(\boldsymbol{\Phi}\boldsymbol{f}|\sigma^2 \boldsymbol{I}) \end{aligned}$$

We now need to specify the prior. There are many options available to us. Sometimes a prior is set on the coefficients $\boldsymbol{f}$ and these are then thought of as 'weights'. This approach is often used in linear regression and then referred to as *ridge regression*. A more common approach in inverse methodology is to set a prior on the function $f(x)$ itself.

**Prior on function coefficients (ridge regression)**

We put a zero mean Gaussian prior with the covariance matrix $K_p$ on the expansion coefficients $\boldsymbol{f}$

$$\boldsymbol{f} \sim \mathcal{N}(0, K_a)$$

and obtain for

$$\begin{aligned} P(\boldsymbol{f}|\boldsymbol{y}) &\propto e^{-\frac{\|\boldsymbol{y}-\boldsymbol{\Phi f}\|^2}{2\sigma^2}} e^{-\frac{1}{2}\boldsymbol{f}^T \boldsymbol{K}_a^{-1} \boldsymbol{f}} \\ &\propto e^{-\frac{1}{2}(\boldsymbol{f}-\bar{\boldsymbol{f}})^T \hat{\boldsymbol{K}} (\boldsymbol{f}-\bar{\boldsymbol{f}})} \end{aligned}$$

where

$$\begin{aligned} \hat{\boldsymbol{f}} &= \sigma^{-2} \left(\sigma^{-2}\boldsymbol{\Phi}^T\boldsymbol{\Phi} + \boldsymbol{K}_a^{-1}\right)^{-1} \boldsymbol{\Phi}^T \boldsymbol{y} \\ \hat{\boldsymbol{K}} &= \left(\sigma^{-2}\boldsymbol{\Phi}^T\boldsymbol{\Phi} + \boldsymbol{K}_a^{-1}\right)^{-1} \end{aligned}$$

for the posterior distribution where $\hat{\boldsymbol{K}}$ is the posterior covariance matrix.

We can generalise these expressions to include the case of a general covariance $\boldsymbol{K}_{\dot{\epsilon}}$ rather than $\boldsymbol{K}_{\dot{\epsilon}} = \sigma^2 \mathbf{1}$ as done above) for the data errors, and non zero mean value for $f$, i.e.

$$\begin{aligned} \dot{\boldsymbol{\epsilon}} &\sim \mathcal{N}(\mathbf{0}, \boldsymbol{K}_{\dot{\epsilon}}) \\ \boldsymbol{f} &\sim \mathcal{N}(\boldsymbol{f}_a, \boldsymbol{K}_a) \end{aligned}$$

and obtain[2]

$$\begin{aligned} \hat{\boldsymbol{f}} &= \boldsymbol{f}_a + \left(\boldsymbol{\Phi}^T \boldsymbol{K}_{\dot{\epsilon}}^{-1} \boldsymbol{\Phi} + \boldsymbol{K}_a^{-1}\right)^{-1} \boldsymbol{\Phi}^T \boldsymbol{K}_{\dot{\epsilon}}^{-1} (\boldsymbol{y} - \boldsymbol{\Phi f}_a) \\ &= \boldsymbol{f}_a + \boldsymbol{K}_a \boldsymbol{\Phi}^T \left(\boldsymbol{\Phi K}_a \boldsymbol{\Phi} + \boldsymbol{K}_{\dot{\epsilon}}\right)^{-1} (\boldsymbol{y} - \boldsymbol{\Phi f}_a) \\ \hat{\boldsymbol{K}} &= \left(\boldsymbol{\Phi}^T \boldsymbol{K}_{\dot{\epsilon}}^{-1} \boldsymbol{\Phi} + \boldsymbol{K}_a^{-1}\right)^{-1} \\ &= \boldsymbol{K}_a - \boldsymbol{K}_a \boldsymbol{\Phi}^T \left(\boldsymbol{\Phi K}_a \boldsymbol{\Phi}^T + \boldsymbol{K}_{\epsilon}\right)^{-1} \boldsymbol{\Phi K}_a \ . \end{aligned}$$

Note that $\boldsymbol{f}$, $\hat{\boldsymbol{f}}$ and $\boldsymbol{f}_a$ are coefficients or weights in the expansion for $f(x)$ given by Eq. (5.1) and that $f(x)$ at a given location $x = x_*$ is given by

$$f(x_*) = \boldsymbol{\phi}_*^T \boldsymbol{f}_a + \boldsymbol{\phi}_*^T \boldsymbol{K}_a \boldsymbol{\Phi}^T \left(\boldsymbol{\Phi K}_a \boldsymbol{\Phi} + \boldsymbol{K}_{\dot{\epsilon}}\right)^{-1} (\boldsymbol{y} - \boldsymbol{\Phi f}_a) \ ,$$

where

$$\boldsymbol{\phi}_* := \boldsymbol{\phi}(x_*) \ .$$

## 5.2 Sparse precision matrices

In Úa the likelihood is assumed to be on the form

$$P(d|p) = \mathcal{N}(p|\tilde{p}, \Sigma) := (2\pi)^{-N/2} |\boldsymbol{Q}_I|^{1/2} e^{-\frac{1}{2}(\tilde{\boldsymbol{d}} - \boldsymbol{f}(\boldsymbol{p}))^T \boldsymbol{Q}_I (\tilde{\boldsymbol{d}} - \boldsymbol{f}(\boldsymbol{p}))} \ , \tag{5.3}$$

and similarly for the prior

$$P(p) = \mathcal{N}(p|\tilde{p}, \Sigma) := (2\pi)^{-N/2} |\boldsymbol{Q}_R|^{1/2} e^{-\frac{1}{2}(\boldsymbol{p} - \tilde{\boldsymbol{p}})^T \boldsymbol{Q}_R (\boldsymbol{p} - \tilde{\boldsymbol{p}})} \ , \tag{5.4}$$

where the *precision matrices* $\boldsymbol{Q}_I$ and $\boldsymbol{Q}_R$ is defined as the inverse of the corresponding covariance matrices, i.e.

$$\boldsymbol{Q} = \boldsymbol{\Sigma}^{-1} \ .$$

[2]Also note that if $\boldsymbol{f}_a = \mathbf{0}$ and $\boldsymbol{K}_a \to \boldsymbol{\infty}$ we get the usual least-squares estimate

$$\begin{aligned} \hat{\boldsymbol{f}} &= \left(\boldsymbol{\Phi}^T \boldsymbol{\Phi}\right)^{-1} \boldsymbol{\Phi}^T \boldsymbol{y} \\ \hat{\boldsymbol{K}}^{-1} &= \boldsymbol{\Phi}^T \boldsymbol{K}_{\epsilon}^{-1} \boldsymbol{\Phi} \ . \end{aligned}$$

These are both Gaussian processes but with different precision matrices.[3]

Typically the covariance is written as a sum of two terms, uncorrelated noise component and a spatially correlated component. We might for example assume that the only contribution to the covariance of the likelihood $P(y|x)$ are the measurements errors, that is

$$\mathrm{cov}(\tilde{d} - f(p)) = \mathrm{cov}(d - \epsilon - d) = \mathrm{cov}(\epsilon)$$

This assumes that the model is perfect, i.e. $\mathrm{cov}(d - f(p)) = 0$. If, on the other hand, the data is perfect but the model is not, the resulting *structural errors* might be expected to be spatially correlated. We now need to select a covariance function which is both flexible enough to describe this correlation realistically, and at the same time results in a sparse precision matrix.

Currently in *Úa*, and as described in more detail in a following section, the precision matrices are specified as a sum of uncorrelated and correlated errors. For uncorrelated errors the precision matrix is

$$\boldsymbol{Q} = \boldsymbol{\epsilon}^{-1} \boldsymbol{M} \boldsymbol{\epsilon}^{-1}$$

where $\boldsymbol{M}$ is the mass matrix. The correlated error component is modelled using a Matérn covariance, where the precision matrix is

$$\boldsymbol{Q} = \kappa^2 \boldsymbol{M} + \boldsymbol{D} \,. \tag{5.5}$$

Here $\boldsymbol{D}$ is the stiffness matrix and $\kappa^2$ a parameter that can be related to the correlation length. These precision matrices are automatically sparse in the FE basis.

Furthermore, currently in *Úa*, and as is for in large-scale geophysical inverse problems, one does not calculate the distribution

$$P(p|d) \propto P(d|p)P(p)$$

Instead, by taking the negative log we arrive at

$$J = \gamma_1(\tilde{d} - f(p))Q_l(\tilde{d} - f(p)) + \gamma_2(p - \tilde{p})Q_p(p - \tilde{p})$$

and this cost function $J$ is then minimised with respect to $p$ using a gradient-based optimisation method where the gradient is calculated in a computationally efficient manner using the adjoint method. Here $Q_I$ is the precision matrix of the likely hood, and $Q_R$ the precision matrix for the prior. The two terms above are often referred to as the misfit and the regularisation term, respectively. The MAP estimate is then

$$P_{\mathrm{MAP}} = \max_p J(p)$$

More precisely

$$-\log P(\boldsymbol{d}|\boldsymbol{p}) = \frac{N}{2}\log 2\pi + \frac{1}{2}\log|\boldsymbol{Q}_I| + \frac{1}{2}(\tilde{\boldsymbol{d}} - \boldsymbol{f}(\boldsymbol{p}))^T \boldsymbol{Q}_I \, (\tilde{\boldsymbol{d}} - \boldsymbol{f}(\boldsymbol{p})) \,,$$

and

$$-\log P(\boldsymbol{p}) = \frac{N}{2}\log 2\pi + \frac{1}{2}\log|\boldsymbol{Q}_R| + \frac{1}{2}(\tilde{\boldsymbol{p}} - \boldsymbol{p})^T \boldsymbol{Q}_R \, (\tilde{\boldsymbol{p}} - \boldsymbol{p}) \,,$$

or

$$-\log P(\boldsymbol{p}|\boldsymbol{d}) = \underbrace{\frac{1}{2}(\tilde{\boldsymbol{d}} - \boldsymbol{f}(\boldsymbol{p}))^T \boldsymbol{Q}_I \, (\tilde{\boldsymbol{d}} - \boldsymbol{f}(\boldsymbol{p}))}_{I} + \underbrace{\frac{1}{2}(\tilde{\boldsymbol{p}} - \boldsymbol{p})^T \boldsymbol{Q}_R \, (\tilde{\boldsymbol{p}} - \boldsymbol{p})}_{R} + \frac{1}{2}\log|\boldsymbol{Q}_I| + \frac{1}{2}\log|\boldsymbol{Q}_R| + N\log 2\pi$$

and the cost-function is defined as

$$J = I + R$$

[3]If we have two estimates of the same quantity each with their own PDF, then the joint PDF is simply the product

$$\begin{aligned} P(\boldsymbol{f}) &\propto e^{-\frac{1}{2}\boldsymbol{f}^T \boldsymbol{K}_1^{-1} \boldsymbol{f}} \, e^{-\frac{1}{2}\boldsymbol{f}^T \boldsymbol{K}_2^{-1} \boldsymbol{f}} \\ &= e^{-\frac{1}{2}\boldsymbol{f}^T (\boldsymbol{K}_1^{-1} + \boldsymbol{K}_2^{-1}) \boldsymbol{f}} \end{aligned}$$

and therefore the joint covariance is

$$\boldsymbol{K}_{1+2} = \left(\boldsymbol{K}_1^{-1} + \boldsymbol{K}_2^{-1}\right)^{-1}$$

or in terms of precision matrices

$$\boldsymbol{Q}_{1+2} = \boldsymbol{Q}_1 + \boldsymbol{Q}_2$$

so in this context we can simply add precision matrices.

The precision matrices will depend on a number of parameters and we refer to these as *hyper parameters*. We lump all these hyper-parameters into a vector $\boldsymbol{\mu}$

Including the hyper-parameters, the posterior over the model parameters $p$ is

$$P(\boldsymbol{p}|\boldsymbol{d},\boldsymbol{\mu}) = \frac{P(\boldsymbol{d}|\boldsymbol{p})P(\boldsymbol{p}|\boldsymbol{\mu})}{P(\boldsymbol{d}|\boldsymbol{\mu})}$$

where

$$P(\boldsymbol{d}|\boldsymbol{\mu}) = \int P(\boldsymbol{d}|\boldsymbol{p})P(\boldsymbol{p}|\boldsymbol{\mu})\, d\boldsymbol{p}$$

is the marginal likelihood or the *evidence*. We can now use Bayes theorem again and write the posterior over the hyper-parameters as

$$P(\boldsymbol{\mu}|\boldsymbol{d}) = \frac{P(\boldsymbol{d}|\boldsymbol{\mu})P(\boldsymbol{\mu})}{P(\boldsymbol{d})}$$

Question: Can this be used to estimate the hyper-parameters? Is this feasible or is calculating the evidence too complicated?

### 5.2.1 Prediction error

Currently in *Úa* one only determines a point estimate for $p$. Preferably we would like to calculate the distribution

$$P(p|d)$$

How can this be done?

Furthermore we would very much like to be able to estimate the uncertainty in any predictions, i.e. when conducting a transient run

$$d(t) = f(d(t=0), p)$$

we would like to estimate the *predictive distribution*

$$P(d(t)) = \int P(d(t)|p)\; P(p|d(t_0))\; dx$$

which we can think of as the uncertainty in our prediction $y(t)$ due to the *posterior* distribution. In general we might have many other factors contributing to the uncertainty in $y(t)$ as well.

## 5.3 Inversion in *Úa*

Using the inverse capabilities of *Úa* it is possible to invert for $A$ and/or $C$, and bed geometry ($B$) using measurements of horizontal surface velocities ($u_s, v_s$) and/or rates of thickness changes ($\partial_t h$). One can invert jointly for $A$ and $C$ or any combination thereof such as $\log A$ and $\log C$. But currently joint inversion with $B$ are not implemented. When inverting for $A$ and $C$, regularisation can be applied on $A$, $C$ or $\log A$ and $\log C$. Again here all combinations are possible, so one can, for example, invert for $\log A$ and $C$ with regularisation applied on $A$ and $\log C$, if one so prefers. Typical use involves inverting for $\log A$ and $\log C$ simultaneously with regularisation applied on $\log A$ and $\log C$. When inverting for $B$, regularisation is applied on $B$. Note that regularisation is always applied on the difference between $p$ and its prior $\tilde{p}$, i.e. on $p - \tilde{p}$. In addition, strict limits can put on the range of the control parameter $p$. One can, for example, but strict upper and lower limits on $B$ at each location.

One can invert for $A$ and $C$ either over nodes or over elements. Typically inversion is done over nodes. Inversion for $B$ is done over nodes only.

The forward model is the SSTREAM momentum equation. Note that if one includes $\dot{h}$ as measurements, in effect, the mass balance equation is used as well. It is a question of viewpoint if the forward model then includes the mass conservation in addition to the conservation of momentum, or just the momentum equation with the mass conservation used to calculate $\dot{h}$ as a part of the evaluation of the cost function. Both viewpoints are mathematically equivalent.

Table 5.2: Notation used in inverse modelling (II)

| Control variables | State variables | Priors | Measurements | Forward model |
|---|---|---|---|---|
| $p$ | $q$ | $\tilde{q}$ | $\tilde{p}$ | $F(q(p),p)=0$ |
| $A$, $C$, $\log A$, $\log C$ | $u_s$, $v_s$ , $\dot{h}$ | $\tilde{A}$, $\tilde{C}$ | $\tilde{u}_s$, $\tilde{v}_s$, $\tilde{\dot{h}}$ | SSTREAM (SSA) |
| $b$, $B$ | $u_s$, $v_s$ , $\dot{h}$ | $\tilde{b}$ , $\tilde{B}$ | $\tilde{u}_s$, $\tilde{v}_s$, $\tilde{\dot{h}}$ | SSTREAM (SSA) |

## 5.4 Objective functions

The objective function, $J$, is a sum of (squared) distances between a) measurements and model outputs (data misfit, $I$), and b) the parameter values and the priors (regularisation, $R$), i.e.

$$J = \underbrace{\|q-\tilde{q}\|^2}_{I} + \underbrace{\|p-\tilde{p}\|^2}_{R} \; .$$

In Úa the objective function $J$ is dimensionless.

Conceptually, and depending on the situation, we might think quite differently about $I$ and $R$. But these two terms are in some ways also quite similar: Both can be thought of as measures of the distances from model outputs and model parameters to measurements ($\tilde{q}$) of the state variables ($q$) and measurements ($\tilde{p}$) of the model parameters ($p$), respectively.

The data misfit is the distance between model output and measurements and it could, for example, be measured as

$$I(f) = \|f\|^2 = \frac{1}{2\mathcal{A}} \iint f(x,y)\,\gamma(x,y,x',y')\,f(x',y')\;dx\,dy\,dx'\,dy', \tag{5.6}$$

where $\gamma(x,y,x',y')$ is the covariance kernel and $\mathcal{A}$ is the domain area. Defined in this manner, $I$ is dimensionless. In the particular case of uncorrelated fields $\gamma(x,y,x',y') = c\,\delta(x-x')\,\delta(y-y')$

$$I = \frac{1}{2\mathcal{A}} \int (f(x,y)/e(x,y))^2\;dxdy, \tag{5.7}$$

where $e(x,y) = 1/\sqrt{c}$ are the data errors.

If $\tilde{q}$ denotes estimates of $q$ then a typical misfit term might be on the form

$$I = \|q-\tilde{q}\|^2 = \frac{1}{2\mathcal{A}} \int ((q-\tilde{q})/e_u)^2\,dA,$$

where $e_u$ are measurement errors.

In Bayesian context the regularisation term has the same form as $I(f)$ and is a measure of the distance between the system state and the a prior. In a discrete form the misfit term could, for example, be written as

$$R = (\boldsymbol{p}-\tilde{\boldsymbol{p}})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{p}-\tilde{\boldsymbol{p}})$$

where $\boldsymbol{\Sigma}$ is a covariance matrix, $\boldsymbol{p}$ the model parameters, and $\tilde{\boldsymbol{p}}$ the a prior estimates of those model parameters. Apart from often having only a very limited knowledge of the covariance matrix $\boldsymbol{\Sigma}$, problems with this formulation can, for example, arise if the inverse of $\boldsymbol{\Sigma}$ is not sparse. Practical reasons often influence the form of regularisation term. A pragmatic approach is to select a differential operator having a sparse representation. Preferable such a sparse operator is also the inverse of a 'reasonable' covariance matrix.

## 5.5 Misfit functions in *Úa*

Currently the misfit function $I$ has the form

$$I = I_u + I_v + I_{\dot{h}}$$

or

$$
\begin{aligned}
I =& \frac{1}{2\mathcal{A}} \int ((u - u_{\text{meas}})/u_{\text{error}})^2 \, dA \\
&+ \frac{1}{2\mathcal{A}} \int ((v - v_{\text{meas}})/v_{\text{error}})^2 \, dA \\
&+ \frac{1}{2\mathcal{A}} \int ((\dot{h} - \dot{h}_{\text{meas}})/\dot{h}_{\text{error}})^2 \, dA
\end{aligned}
$$

where $u$ and $v$ are the horizontal velocity components, and $\dot{h}$ is the rate of thickness change, while

$$\mathcal{A} = \int dA,$$

is the total area.

The rate of thickness change is calculated as

$$\dot{h} = -(a - \partial_x(uh) - \partial_y(vh)),$$

and in the adjoint method the gradient of the cost function with respect to the state variable $\boldsymbol{v}$ acquires an additional term:

$$2I = \|u - \tilde{u}\|^2 + \|\dot{h}_d - (a - \partial_x(uh))\|,$$

or

$$I_{\dot{h}} = \frac{1}{2\mathcal{A}} \int \left( (a - \partial_x(uh) - \partial_y(vh) - \dot{h}_{\text{meas}})/\dot{h}_{\text{error}} \right)^2 \, dx \, dy,$$

with the corresponding directional derivative with respect to $u$ and $v$ given by

$$\delta_{uv} I_{\dot{h}} = \frac{1}{\mathcal{A}} \int \left( (\dot{h} - \dot{h}_{\text{meas}})/\dot{h}^2_{\text{error}} \right) (\partial_x(h\delta u) + \partial_y(h\delta v)) \, dx \, dy$$

which, like the cost function itself, is dimensionless.

## 5.6 Regularisation in *Úa*

In *Úa* the regularisation can be done either using a Bayesian or Tikhonov regularisation.

### 5.6.1 Bayesian approach

Bayesian estimation of, for example, the (distributed) parameter $x$ in the forward model $y = f(x)$ given the prior $\tilde{x}$ and measurements $\tilde{y}$ involves calculating

$$P(x|y) = \frac{P(y|x) \, P(x)}{P(y)}$$

where one might assume

$$y = \tilde{y} + \epsilon$$

where $\epsilon$ is uncorrelated measurement noise, and

$$x = \tilde{x} + \epsilon$$

where $\epsilon$ is a Gaussian field with some covariance, i.e. $P(x) = \mathcal{N}(x|\tilde{x}, \Sigma)$. A typical Gaussian assumption in $d$ dimensions is on the form

$$P(x) = \mathcal{N}(x|\tilde{x}, \Sigma) := (2\pi)^{-d/2} |\boldsymbol{Q}|^{1/2} e^{-\frac{1}{2}(\boldsymbol{x}-\tilde{\boldsymbol{x}})^T \boldsymbol{Q} \, (\boldsymbol{x}-\tilde{\boldsymbol{x}})} \,, \tag{5.8}$$

where the *precision matrix* $\boldsymbol{Q}$ is defined as the inverse of the covariance matrix, i.e.

$$\boldsymbol{Q} = \boldsymbol{\Sigma}^{-1} \,.$$

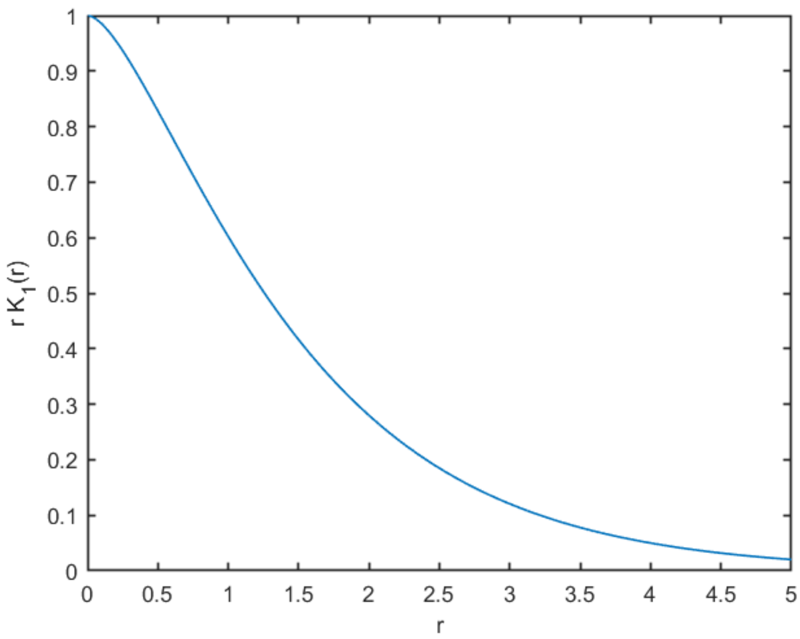



Figure 5.1: The function $rK_1(r)$ appearing in Eq. (5.14)

Using this Bayesian formulation, the regularisation term for the (distributed) model parameter $p$ has, has the discretized form

$$R = (\boldsymbol{p} - \tilde{\boldsymbol{p}})^T \boldsymbol{\Sigma}_{pp}^{-1} (\boldsymbol{p} - \tilde{\boldsymbol{p}}), \tag{5.9}$$

where

$$\boldsymbol{\Sigma}_{pp} = \operatorname{cov}(\boldsymbol{p}, \boldsymbol{p})$$

where

$$\operatorname{cov}(f, g) := \mathbb{E}\left((f - \mathbb{E}(f))(g - \mathbb{E}(g))\right)$$

where $\mathbb{E}$ is the expectation operator. Generally our prior is here the expected value, and for the model parameter $\boldsymbol{p}$, the elements of the covariance matrix are then

$$[\boldsymbol{\Sigma}_{pp}]_{ij} = \mathbb{E}(\boldsymbol{p}(\boldsymbol{r}_i) - \tilde{\boldsymbol{p}}(\boldsymbol{r}_i), \boldsymbol{p}(\boldsymbol{r}_j) - \tilde{\boldsymbol{p}}(\boldsymbol{r}_j)) \ .$$

The Bayesian approach has a clear statistical interpretation, but, in general, requires an inversion of the covariance matrix $\boldsymbol{\Sigma}$ which can be impractical for large problems. In *Úa* one can specify any covariance matrix $\boldsymbol{\Sigma}$ but in practice this is not a computationally feasible approach for large problems unless $\boldsymbol{\Sigma}$ is selected so that it has some computationally advantageous structure. One of the practical difficulties with directly prescribing a covariance matrix in the objective function is that the inverse of the resulting matrix is, in general, not sparse even if the matrix itself is sparse. For that reason the general Bayesian approach in *Úa* , where one specifies $\Sigma$ directly, can only be used for fairly small problems.

A possible alternative approach is to limit the form of the covariance function to a form that allows for the construction of a sparse inverse directly within the FE context. For example consider the discretisation of the (self adjoint) Helmholtz equation

$$\mathcal{L} f(\boldsymbol{r}) = 0 \, , \quad \boldsymbol{r} \in \mathbb{R}$$

where

$$\mathcal{L} := \nabla^2 - \kappa^2 \, ,$$

$f$ is some scalar function, and $k$ a wave number. This gives the inner-product induced norm

$$\|f\|^2 =< \mathcal{L}^{1/2} f \mid \mathcal{L}^{1/2} f > \tag{5.10}$$
$$=< f \mid \mathcal{L} \mid f > \tag{5.11}$$
$$= \boldsymbol{f}^T (\kappa^2 \boldsymbol{M} + \boldsymbol{D}) \boldsymbol{f} \tag{5.12}$$

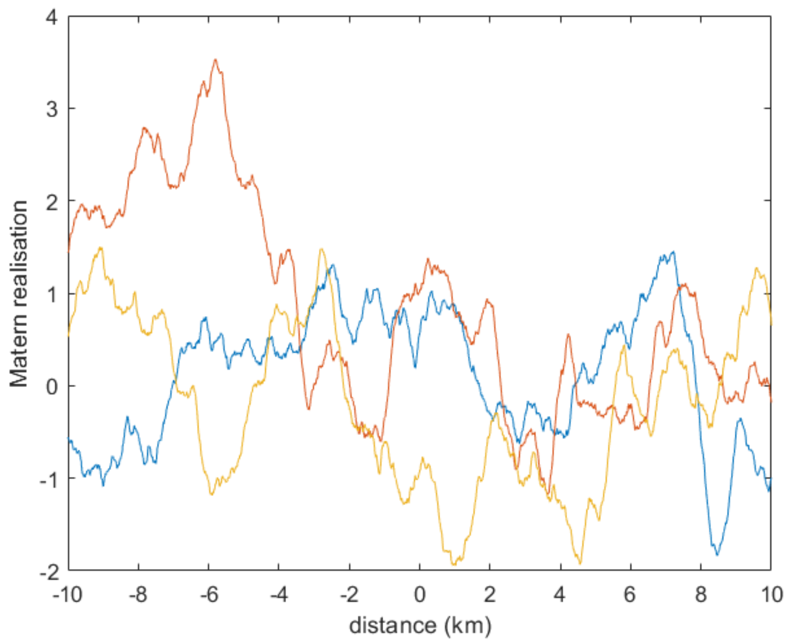



Figure 5.2: Three examples of Matérn realisations for $\rho = 3$ km, $\nu = 1$, d=2 and $\sigma = 1$ .

and hence a precision matrix

$$\boldsymbol{Q} = \kappa^2 \boldsymbol{M} + \boldsymbol{D} \,, \tag{5.13}$$

which is by construction sparse. Note that minimising $\|f\|$ with respect to $f$ results in the system

$$(\kappa^2 \boldsymbol{M} + \boldsymbol{D})\boldsymbol{f} = \boldsymbol{0} \,,$$

so $\|f\|$ is minimised when $f$ is the solution to the Helmholtz equation.

The question is now what the resulting covariance looks like. Consider now a solution to the stochastic partial differential equation (here Helmholtz equation)

$$(\nabla^2 - \kappa^2) f(\boldsymbol{r}) = \epsilon(\boldsymbol{r}) \,, \quad \boldsymbol{r} \in \mathbb{R}$$

where the innovation process $\epsilon(\boldsymbol{r})$ is a spatial Gaussian white noise. As shown by Whittle (1954), the auto-covariance is then

$$\mathrm{cov}(f(\boldsymbol{r}_1), f(\boldsymbol{r}_2)) = \sigma^2 \kappa \|\boldsymbol{r}_2 - \boldsymbol{r}_1\| \; K_1(\kappa \|\boldsymbol{r}_2 - \boldsymbol{r}_1\|) \,,$$

where the marginal variance is

$$\sigma^2 = \frac{1}{4\pi\kappa^2}$$

or

$$\mathrm{cov}(f(\boldsymbol{r}_1), f(\boldsymbol{r}_2)) = \frac{\|\boldsymbol{r}_2 - \boldsymbol{r}_1\|}{4\pi\kappa} \; K_1(\kappa \|\boldsymbol{r}_2 - \boldsymbol{r}_1\|) \,, \tag{5.14}$$

where $K_1$ is the modified Bessel function of the second kind and order one. Note that

$$\lim_{r \to 0} r \, K_1(r) = 1 \,.$$

The shape of the covariance function (5.14) is shown in Fig. 5.1. As pointed out by Whittle (1954) the covariance function is similar to $e^{-\kappa r}$ but differs in that it is flat at the origin and its rate of decay slower.

The covariance shown above is a special case of the Matérn covariance function

$$r(x) = \frac{\sigma^2}{2^{\nu-1}\Gamma(\nu)} (\kappa \, x)^\nu K_\nu(\kappa x)$$

for $\alpha = 2$ and $\nu = \alpha - d/2 = 1$ for $d = 2$ where $d$ is the number of spatial dimensions. As shown by Whittle (1954), and discussed and explained in great detail by Lindgren et al. (2011), the Matérn covariance function of a Gaussian process is a solution to the linear fractional stochastic partial differential equation

$$(\kappa^2 - \nabla^2)^{\alpha/2} f(x) = g(x), \quad x \in \mathbb{R}^d$$

where the $g(x)$ is a spatial Gaussian process with unit variance. Hence, for the Gaussian probability in Eq. (5.8) with a Matérn covariance function, the precision matrix in the FE basis is given by Eq. (5.13).

We can easily make this framework more flexible by allowing the coefficients in the Helmholtz equation to vary spatially, i.e. writing

$$\mathcal{L} f = \kappa^2 f - \nabla(a \nabla \cdot f) \ ,$$

where $a$ and $\kappa^2$ are both functions of space. The FE formulation using

$$\begin{aligned} a(x,y) &= a_q \phi(x,y) \\ \kappa(x,y) &= k_q \phi(x,y) \end{aligned}$$

then gives the Galerkin system

$$\begin{aligned} 0 &= - < (\nabla((a_r \nabla \phi_r) f_p \phi_p) + (k_s \phi_s)^2 f_p \phi_p, \phi_q > \\ &=< ((a_r \phi_r) f_p \nabla \phi_p, \nabla \phi_q > + < (k_s \phi_s)^2 f_p \phi_p, \phi_q > \\ &=< ((a_r \phi_r) \nabla \phi_p, \nabla \phi_q > f_p + < (k_s \phi_s)^2 \phi_p, \phi_q > f_p \\ &= [\tilde{D}]_{pq} f_q - [\tilde{M}]_{pq} f_p \end{aligned}$$

where the last line defines the elements of the matrices $\tilde{\boldsymbol{D}}$ and $\tilde{\boldsymbol{M}}$. Hence

$$< f \mid \mathcal{L} \mid f >= \boldsymbol{f}^T (\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{M}}) \boldsymbol{f}$$

In the particular case when $a_q = 1$ and $k_q = \kappa$ for all $q$, we have $a_q \phi_q = 1$ and $k_q^2 \phi_1 = \kappa$ and $\tilde{\boldsymbol{M}} = \boldsymbol{M}$ and $\tilde{\boldsymbol{D}} = \boldsymbol{D}$.

We might also imagine a situation where we have 'direct' estimates of the parameter $p$. For example if we are inverting for the bedrock $B$ (i.e. $p = B$) we might have some 'direct' estimates based on radar soundings, seismic or gravity measurements. We of course quickly get rid of all estimates based on gravity measurements, but we should at least consider the possibility that seismic and radar data may provide some useful information on bedrock topography. We can express this by writing

$$\text{cov}(B_i, B_j) = [\boldsymbol{\Sigma}_{BB}]_{ij} + \sigma_i^2 \delta_{ij} \quad \text{(no summation implied)}$$

where $\sigma_i$ is the data error variance at location $\boldsymbol{r}_i$. We can think of $\boldsymbol{\Sigma}_{BB}$ as the covariance matrix for the noise-free latent, i.e. unobserved, $B$.

$$\int f(x) \kappa(x - x') f(x') \, dx; dx'$$

### 5.6.2 Tikhonov regularisation

Instead of using the statistical Bayesian framework, one can use the arguably more pragmatic Tikhonov regularisation approach where one, for example, penalises amplitude and/or slope writing

$$R = \gamma_a^2 \|p - \tilde{p}\| + \gamma_s^2 \|\nabla(p - \tilde{p})\|$$

or

$$R = \frac{1}{2\mathcal{A}} \int \left( \gamma_s^2 \left(\nabla(p - \tilde{p})\right)^2 + \gamma_a^2 \left(p - \tilde{p}\right)^2 \right) \, dA \tag{5.15}$$

with the $s$ and $a$ subscripts being mnemonics for slope and amplitude, respectively. As is presumably already clear from the above expression, this type of Tikhonov regularisation and the Bayesian approach using the Matérn covariance are almost identical and the difference in many ways just a question of semantics.

The inversion can be done directly with respect to the variable $p$, or with respect to the logarithm of the variable, i.e. $\log_{10} p$. If done with respect to the logarithm of $p$, the Tikhonov regularisation term has the form

$$
\begin{aligned}
R &= \frac{1}{2\mathcal{A}} \int \left( \gamma_s^2 \left( \nabla \left( \log_{10}(p) - \log_{10}(\tilde{p}) \right) \right)^2 + \gamma_a^2 \left( \log_{10}(p) - \log_{10}(\tilde{p}) \right)^2 \right) dA \\
&= \frac{1}{2\mathcal{A}} \int \left( \gamma_s^2 \left( \nabla \log_{10}(p/\tilde{p}) \right)^2 + \gamma_a^2 \log_{10}^2(p/\tilde{p}) \right) dA \qquad (5.16)
\end{aligned}
$$

Now $\gamma_a$ is dimensionless and the dimension of $\gamma_s$ is length, i.e.

$$
\begin{aligned}
[\gamma_a] &= [] \\
[\gamma_s] &= [l]
\end{aligned}
$$

As shown in section 5.8 the Tikhonov approach leads to

$$
\begin{aligned}
R &= \frac{1}{2} (\boldsymbol{p} - \tilde{\boldsymbol{p}})^T \left( \gamma_a^2 \boldsymbol{M} + \gamma_s^2 (\boldsymbol{D}_x + \boldsymbol{D}_y) \right) (\boldsymbol{p} - \tilde{\boldsymbol{p}}) \qquad (5.17) \\
&= \frac{1}{2} (\boldsymbol{p} - \tilde{\boldsymbol{p}})^T \boldsymbol{\Sigma}_{pp}^{-1} (\boldsymbol{p} - \tilde{\boldsymbol{p}}) \qquad (5.18)
\end{aligned}
$$

where

$$
\boldsymbol{\Sigma}_{pp}^{-1} = \frac{1}{2} \left( \gamma_a^2 \boldsymbol{M} + \gamma_s^2 (\boldsymbol{D}_x + \boldsymbol{D}_y) \right) \qquad (5.19)
$$

and $\boldsymbol{M}$ is the mass matrix and $\boldsymbol{D}_x$ and $\boldsymbol{D}_y$ the stiffness matrices. Hence, with the Tikhonov approach we are calculating directly the inverse of a matrix $\boldsymbol{\Sigma}_{pp}$ as (5.19). By construction this inverse will be sparse. However, it is in general not clear if this is an inverse of a covariance matrix. One can, for example, show that for $\gamma_a = 0$ and $\gamma_s \neq 0$, $\boldsymbol{\Sigma}$ as defined by Eq. (5.19) is not a covariance matrix.

## 5.7 The adjoint method

The adjoint method is a simple trick to speed up the calculation of gradients of the objective function with respect to the control variables.

Assume we want to solve the minimisation problem

$$
\min_{p \in P, q \in U} J(q(p))
$$

subject to forward model

$$
F(q(p), p) = 0.
$$

The objective function is

$$
J : U \times P \to \mathbb{R}
$$

and the forward model, i.e. the state equation

$$
F : U \times P \to \mathbb{W}
$$

the model control parameter space $P$ (also referred to as control space), the state space $U$ and the image space $W$ are Banach spaces.

We want to determine the sensitivity of cost function $J$ with respect to the (distributed) control parameter $p$.

We define the Lagrange function, or the 'extended' objective function, $\mathcal{L}$ as

$$
\mathcal{L} = U \times V \times W^* \to \mathbb{R}
$$

as

$$
\mathcal{L}(q(p), p, \lambda) = J(q(p), p) + < \lambda \mid F(q(p), p) >_{W^*, W} \qquad (5.20)
$$

The adjoint variable $\lambda$ is in the dual of the image space $\mathbb{W}$.

Equation (5.20) provides no constraints on the adjoint variable $\lambda$ because the second term is always equal to zero for any value of $p$. Therefore

$$
\mathcal{L}(q(p), p, \lambda) = J(q(p), p)
$$

and

$$D\mathcal{L}(p)[\phi] = DJ(p)[\phi] =< \nabla_p \mathcal{L} \mid \phi >$$

Also note that

$$d_p F = \partial_p F + \partial_q F \, d_p q = 0.$$

Introducing

$$j(p) = J(q(p), p) = \mathcal{L}(q(p), p, \lambda),$$

we have

$$j'(p) = (\partial q/\partial p)^* \partial \mathcal{L}(q(p), p, \lambda)/\partial q + \partial \mathcal{L}(q(p), p, \lambda)/\partial p, \tag{5.21}$$

Now we chose $\lambda$ such that

$$\partial \mathcal{L}(q(p), p, \lambda)/\partial q = 0.$$

Hence $\lambda$ must be a solution to

$$\partial J(q(p), p)/\partial q + (\partial F/\partial q)^* \lambda = 0, \tag{5.22}$$

and then the direction derivative $j'(p)$ is

$$j'(p) = \partial J(q(p), p)/\partial p + (\partial F/\partial p)^* \lambda \tag{5.23}$$

Eq. (5.21) can also be written as

$$< j'(p), \phi >_{P^*,P} =< \partial_u \mathcal{L} \mid \partial_p q \, \phi >_{U^*,U} + < \partial_p \mathcal{L} \mid \phi >_{P^*,P},$$

and the directional derivative is then

$$< \partial_u \mathcal{L}, \partial_p q \, \phi >_{U^*,U} = 0,$$

for all $\phi$.

Another approach: Differentiating Eq. (5.20) with respect to the control variable $p$ we obtain

$$\begin{aligned} DJ(p)[\phi] = D\mathcal{L}(p)[\phi] &=< \partial_q J \mid d_p q \, \phi > + < \partial_p J, \phi > + < \lambda \mid \partial_q F \, d_p q \, \phi > + < \lambda \mid \partial_p F \, \phi > + < d_p \lambda \mid F > \\ &=< \partial_q J \mid d_p q \, \phi > + < (\partial_q F)^* \lambda \mid d_p q \, \phi > + < \lambda \mid \partial_p F \, \phi > + < \partial_p J, \phi > \\ &=< \partial_q J + (\partial_q F)^* \lambda \mid d_p q \, \phi > + < \lambda \mid \partial_p F \, \phi > + < \partial_p J, \phi > \end{aligned}$$

We now use the freedom that $\lambda$ has not been specified and now determine $\lambda$ by setting

$$< \partial_q J + (\partial_q F)^* \lambda \mid \phi >= 0,$$

and therefore

$$DJ(p)[\phi] =< (\partial_p F)^* \lambda + \partial_p J \mid \phi >, \tag{5.24}$$

which is identical to (5.23) which we derived earlier using a somewhat different approach.[4]

This now gives us a three-step method for calculating the directional gradient of object function $J$ with respect to $p$:

1. Solve the state equation, i.e. the forward problem

$$< F(q(p), p) | \phi >_{W^*,W} = 0,$$

for the state variable $q$.

This, in general, is a non-linear problem that can be solved iteratively using the Newton-Raphson system, i.e.

$$< \partial_q F \, \Delta q \mid \phi >= - < F(q(p), p) \mid \phi >,$$

and can be written in discrete form as

$$\boldsymbol{K} \boldsymbol{\Delta q} = \boldsymbol{b},$$

where

$$[\boldsymbol{K}]_{pq} =< \partial_{q_p} F \mid \phi_q >,$$

and

$$[\boldsymbol{b}]_q = - < F(q(p), p) \mid \phi_q > .$$

[4] Apparently the 'adjoint' method gets its name from the fact that the resulting expression of the derivative of the cost function involves an adjoint of an operator. The adjoint involved is the adjoint of the directional derivative of the forward model $F$ with respect to the control variable $p$. However, in general there is no need to introduce this adjoint and we can just as well write

$$DJ(p)[\phi] =< \lambda \mid \partial_p F \, \phi > + < \partial_p J, \phi > .$$

However, when solving for $\lambda$ we do need the adjoint of the directional derivative of the forward model with respect to $q$.

2. Solve the adjoint problem for $< \partial_q J + (\partial_q F)^* \lambda \mid \phi >= 0$ for $\lambda \in W^*$, i.e.

$$< (\partial_q F)^* \lambda \mid \phi >_{U^*,U} = - < \partial_q J \mid \phi >_{U^*,U}$$

   for the adjoint variable $\lambda$. If the forward tangential model $(\partial_q F)$ is self adjoint, this involves solving

$$\boldsymbol{K\lambda} = \boldsymbol{b}$$

3. Calculate the directional derivative of $j$ as

$$DJ(p)[\phi] =< j'(p) \mid \phi >_{P^*,P} =< (\partial_p F)^* \lambda + \partial_p J \mid \phi >_{P^*,P}$$

   In discrete form the derivative can be evaluated as

$$j'(p) = \boldsymbol{P\lambda} + \boldsymbol{Q}$$

   where

$$[\boldsymbol{P}]_{ij} =< \partial_{p_i} F \mid \phi_j >$$

   and

$$[\boldsymbol{Q}]_i =< \partial_p J \mid \phi_i >$$

   However, $< (\partial_p F)^* \lambda \mid \phi >$ can usually be evaluated directly within the assembly look without the need of ever forming the matrix $\boldsymbol{P}$. Furthermore, the forward model is solved in a weak form and this often involves some manipulations of the $< \lambda \mid F >$ term in Eq. (5.20).

Usually only the regularisation term is an explicit function of the control variable, i.e.

$$J(F(p, q(p), p) = I(F(p, q(p))) + R(p)$$

and therefore

$$\partial_p J = \partial_p R$$

The adjoint variable $\lambda$ is the gradient of the objective function with respect to the state variable $q$

$$\lambda = \nabla_q J$$

In the adjoint method we need to calculate a number of derivatives. These are:

1. The directional derivative of the forward model with respect to the state and control variables, i.e $\partial_q F$ and $\partial_p F$.

2. The directional derivative of the objective function with respect to the state and control variables, i.e. $\partial_q J$ and $\partial_p J$.

### 5.7.1 Summarising the adjoint approach

Summarising, and using a somewhat inexact notation, we can calculate the (total) directional derivative $d_p J$ of the objective function $J$ with respect to the control parameters $p$, as follows:

$$\begin{aligned} F(q(p), p) &= 0 & (5.25)\\ \partial_q F^* \lambda &= \partial_q J & (5.26)\\ d_p J &= \partial_p F^* \lambda + \partial_p J & (5.27) \end{aligned}$$

A bit more precise notation for the adjoint problem and the calculation of the derivative is

$$\begin{aligned} < \delta_q F^* \mid \lambda > &= \delta_q J & (5.28)\\ d_p J &=< \delta_p F^* \mid \lambda > + \delta_p J & (5.29) \end{aligned}$$

## 5.8 Evaluating objective functions and their directional derivatives

If $\phi_i$ are the basis functions then

$$[\boldsymbol{M}]_{ij} =< \phi_i \mid \phi_j >$$

is the mass matrix (also known as the Gramian matrix), and

$$[\boldsymbol{D}_x]_{ij} =< \nabla_x \phi_i \mid \nabla_x \phi_j >$$
$$[\boldsymbol{D}_y]_{ij} =< \nabla_y \phi_i \mid \nabla_y \phi_j >$$

the stiffness matrices.

For

$$I = \frac{1}{2}\|f\|^2 = \frac{1}{2}\int f(x,y)\, f(x,y)\; dxdy \tag{5.30}$$

and

$$f(x,y) = f_i \phi_i(x,y)$$

we find

$$\begin{aligned} I &= \frac{1}{2}\|f\|^2 \\ &= \frac{1}{2} < f \mid f > \\ &= \frac{1}{2} < f_p\phi_p \mid f_q\phi_q > \\ &= \frac{1}{2} f_p < \phi_p \mid \phi_q > f_q \\ &= \frac{1}{2} f_p M_{pq} f_q \end{aligned}$$

or

$$I = \frac{1}{2}\boldsymbol{f}\boldsymbol{M}\boldsymbol{f}$$

where

$$[\boldsymbol{f}]_i = f_i$$

The directional derivative is

$$\begin{aligned} DI(f,\phi_q) &= \lim_{\epsilon\to 0}\frac{d}{d\epsilon} I(f+\epsilon\phi_q) \\ &= \frac{1}{2}\frac{d}{d\epsilon} < f+\epsilon\phi_q \mid f+\epsilon\phi_q > \\ &=< f \mid \phi_q > \\ &=< f_p\phi_p \mid \phi_q > \\ &=< \phi_q \mid \phi_p > f_p \\ &= M_{qp} f_p \\ &= \boldsymbol{M}\boldsymbol{f} \end{aligned}$$

or

$$DI(f,\phi_q) = [\boldsymbol{M}\boldsymbol{f}]_q$$

The $p$ component of the directional derivative represents the (linear) rate-of-change in $I$ as the value of $f$ is perturbed by $\epsilon\phi_p$.

We can also write

$$DI(f,\phi_q) =< f_p\phi_p \mid \phi_q >$$

and therefore by the definition of a gradient as

$$\frac{d}{d\epsilon} J(f+\epsilon\delta f)|_{\epsilon=0} =< \mathrm{grad} J(f), \delta f >$$

the gradient of $I$ in Eq. (5.30) is $f$ (as it of course should be).

Similarly if

$$I = \frac{1}{2} < \nabla f \mid \nabla f >= \frac{1}{2}(< \partial_x f, \partial_x f > + < \partial_y f, \partial_y f >)$$

then

$$\begin{aligned} I_x &= \frac{1}{2} < \partial_x f, \partial_x f > \\ &= \frac{1}{2} < f_p \partial_x \phi_p, f_q \partial_x f_q > \\ &= \frac{1}{2} f_p < \partial_x \phi_p, \partial_x \phi_q > f_q \\ &= \frac{1}{2} \boldsymbol{f} \boldsymbol{D}_x \boldsymbol{f} \end{aligned}$$

and

$$\begin{aligned} DI_x(f, \phi_p) &= \frac{1}{2} \lim_{\epsilon \to 0} \frac{d}{d\epsilon} < \partial_x (f + \epsilon \phi_p) \mid \partial_x (f + \epsilon \phi_p) > \\ &=< \partial_x f \mid \partial_x \phi_p > \\ &=< f_q \partial_x \phi_q \mid \partial_x \phi_p > \\ &=< \partial_x \phi_p \mid \partial_x \phi_q > f_q \\ &= \boldsymbol{D}_x \boldsymbol{f} \end{aligned}$$

or

$$DI_x(f, \phi_p) = \boldsymbol{D}_x \boldsymbol{f}$$

Summarising, if we have a regularisation term on the form

$$R = \frac{1}{2} \|f\|^2_{L^2(\Omega)} + \frac{1}{2} \|\nabla f\|^2_{L^2(\Omega)}$$

it can be evaluated knowing the mass and the stiffness matrices as

$$\begin{aligned} R &= \frac{1}{2} \|f\|^2_{L^2(\Omega)} + \frac{1}{2} \|\nabla f\|^2_{L^2(\Omega)} \\ &= \frac{1}{2} \boldsymbol{f} \boldsymbol{M} \boldsymbol{f} + \frac{1}{2} \boldsymbol{f} (\boldsymbol{D}_x + \boldsymbol{D}_y) \boldsymbol{f} \end{aligned}$$

and direction derivative is

$$\frac{d}{d\epsilon} I(f + \epsilon \phi_p) = [\boldsymbol{M} \boldsymbol{f} + (\boldsymbol{D}_x + \boldsymbol{D}_y) \boldsymbol{f}]_p$$

And the regularisation term (**??**) can be evaluated similarly as

$$\begin{aligned} R &= \frac{1}{2\mathcal{A}} \int \left( \left( \gamma_s^2 \left( \nabla (p - \tilde{p}) \right)^2 \right) + \gamma_a^2 \left( p - \tilde{p} \right)^2 \right) \, dA \\ &= \frac{1}{2\mathcal{A}} \left( (\boldsymbol{p} - \tilde{\boldsymbol{p}})^T \gamma_s^2 \boldsymbol{D} \, (\boldsymbol{p} - \tilde{\boldsymbol{p}}) + \gamma_a^2 (\boldsymbol{p} - \tilde{\boldsymbol{p}})^T \boldsymbol{M} (\boldsymbol{p} - \tilde{\boldsymbol{p}}) \right) \\ &= \frac{1}{2\mathcal{A}} \left( (\boldsymbol{p} - \tilde{\boldsymbol{p}})^T (\gamma_s^2 \boldsymbol{D} + \gamma_a^2 \boldsymbol{M}) (\boldsymbol{p} - \tilde{\boldsymbol{p}}) \right) \end{aligned}$$

and the directional derivative is

$$d_{\boldsymbol{p}} R = \frac{1}{\mathcal{A}} (\gamma_s^2 \boldsymbol{D} + \gamma_a^2 \boldsymbol{M}) (\boldsymbol{p} - \tilde{\boldsymbol{p}})$$

## 5.9 Simple example of a gradient calculation of an objective function

Consider, as an illustration, the problem of inverting for ice thickness ($h$) using rates of thickness change ($\dot{h}$) as observations. Here the control variable is $h$ and the field variable $\dot{h}$. We have measurements $\tilde{\dot{h}}$ and an *a priori* estimate $\tilde{h}$.

Hence, we have

$$p = h, \tag{5.31}$$
$$q = \dot{h}. \tag{5.32}$$
$$\tag{5.33}$$

The object function $J$ is

$$J = \frac{1}{2\mathcal{A}} \int \left(\dot{h} - \tilde{\dot{h}}\right)^2 \, dA + \frac{1}{2\mathcal{A}} \int \left(h - \tilde{h}\right)^2 \, dA,$$

where the first term is the misfit term and the second one the regularisation term. The forward model $F = 0$ is

$$F = \dot{h} - (a - \partial_x(uh))$$

where $\dot{h}$ is the unknown, although using FE we would solve this as

$$\begin{aligned} 0 =< F|\phi_j > &= \int \left(\dot{h} - (a - \partial_x(uh))\right) \, \phi_j \, dA \\ &= \int \left(\dot{h}_i \phi_i - (a - \partial_x(uh))\right) \, \phi_j \, dA, \end{aligned}$$

for each $j$, or

$$\int \dot{h}_i \phi_i \phi_j \, dA = \int (a - \partial_x(uh)) \, \phi_j \, dA,$$

that is

$$\dot{\boldsymbol{h}} = \boldsymbol{M}^{-1} \boldsymbol{b},$$

where

$$[\boldsymbol{M}]_{ij} = \int \phi_i \phi_j \, dA \tag{5.34}$$
$$[\boldsymbol{b}]_j = \int (a - \partial_x(uh)) \, \phi_j \, dA. \tag{5.35}$$
$$\tag{5.36}$$

and therefore in discrete form the forward model

$$\boldsymbol{F} = 0.$$

with

$$\boldsymbol{F} = \boldsymbol{M}\dot{\boldsymbol{h}} - \boldsymbol{b}.$$

We also find that using FE approach that

$$J = \frac{1}{2\mathcal{A}} (\dot{\boldsymbol{h}} - \tilde{\dot{\boldsymbol{h}}})^T \boldsymbol{M} (\dot{\boldsymbol{h}} - \tilde{\dot{\boldsymbol{h}}}) + \frac{1}{2\mathcal{A}} (\boldsymbol{h} - \tilde{\boldsymbol{h}})^T \boldsymbol{M} (\boldsymbol{h} - \tilde{\boldsymbol{h}})$$

The task at hand is to calculate the directional derivative (see Appendix C) of the object function $J$ with respect to the control variable $h$, i.e.

$$\delta_h J = DJ(h)[\delta h] = \lim_{\epsilon \to 0} \frac{d}{d\epsilon} J(h + \epsilon \, \delta h)$$

### 5.9.1 Direct approach

Since the forward model is so simple we can just take the (total) derivative directly by inserting the forward model $F$ into the objective function $J$, leading to

$$d_h J = \frac{1}{\mathcal{A}} \int \left((a - \partial_x(uh)) - \tilde{\dot{h}}\right) \, \partial_x(u \, \delta h) \, dA + \frac{1}{\mathcal{A}} \int \left(h - \tilde{h}\right) \delta h \, dA. \tag{5.37}$$

This is the desired directional derivative and one can now do an inversion for $h$ using this derivative.

Note that if we set $\delta h = \phi$, which is a typical approach and the one used in Úa, this is the derivative with respect to the perturbation in the nodal values, i.e. $\delta h = \phi$. This derivative is clearly intimately related to the FE basis. There is then the separate, but related and quite an important question as to what the *coefficients of the directional derivative* are in a given FE setting. Assuming that all variables are expanded in the same FE basis, i.e. $f(x, y) = f_i \phi_i(x, y)$ where $f$ is any of $u$, $h$, $\dot{h}$ and also

$$d_h J = (d_h J)_i \, \delta h_i = (d_h J)_i \, \phi_i$$

we must solve for these coefficients, i.e. $(d_h J)_i$ in the expansion $d_h J = (d_h J)_i \phi_i$, in the usual FE manner by requiring

$$< (d_h J)_i \phi_i - d_h J | \phi_j >= 0,$$

for all $j$ values, and therefore[5]

$$\int (d_h J)_i \, \phi_i \, \phi_j \, dA = \frac{1}{\mathcal{A}} \int \left(\dot{h} - \tilde{\dot{h}}\right) (\partial_x u \, \phi_j + u \, \partial_x \phi_j) \, dA + \frac{1}{\mathcal{A}} \int \left(h - \tilde{h}\right) \phi_j \, dA. \tag{5.38}$$

Hence, the directional derivative vector, formed by ordering the coefficients $(\delta_h J)_i$ according to nodal numbers, is

$$\boldsymbol{d_h J} = \boldsymbol{M}^{-1} \boldsymbol{k},$$

where

$$[\boldsymbol{k}]_j = \frac{1}{\mathcal{A}} \int \left(\dot{h} - \tilde{\dot{h}}\right) (\partial_x u \, \phi_j + u \, \partial_x \phi_j) \, dA + \frac{1}{\mathcal{A}} \int \left(h - \tilde{h}\right) \phi_j \, dA.$$

### 5.9.2 Adjoint approach

We follow the recipe provided in section 5.7.1.

**First, solve the forward model:**

We calculate

$$\partial_q F \, \Delta q = -F(q) \qquad \text{with } q \to q + \Delta q \text{ and } \Delta q \to 0,$$

which in this particular case where $q = \dot{h}$ is

$$1 \, \Delta \dot{h} = -\dot{h}_0 + a - \partial_x (u h_0),$$

since $\partial_{\dot{h}} F = 1$, or

$$h_1 = \Delta \dot{h} + \dot{h}_0 = a - \partial_x (u h_o),$$

since this is a linear forward model the NR iteration converges in first iteration.

Of course, as explained above, the FE formulation of the problem leads to

$$\boldsymbol{M} \, \boldsymbol{\Delta \dot{h}} = -(\boldsymbol{M} \, \boldsymbol{\dot{h}} - \boldsymbol{b})$$

so here $1 = |\phi >< \phi| = \boldsymbol{M}$.

[5] There appears to be some confusion in the literature regarding this point and sometimes (often, practically always) the expression given for the directional derivative is something like

$$(d_h J)_j = \frac{1}{\mathcal{A}} \int \left(\dot{h} - \tilde{\dot{h}}\right) (\partial_x u \, \phi_j + u \, \partial_x \phi_j) \, dA + \frac{1}{\mathcal{A}} \int \left(h - \tilde{h}\right) \phi_j \, dA, \quad \text{(wrong!)}$$

and in fact inversion can be successfully done using this derivative. However, this 'derivative' will be intimately related to the structure of the FE approximation used, both the basis and the discretisation. This expression for the 'derivative' is simply an example of a FE assembly, and it does not represent the values of the coefficients of the derivative in the FE basis. The correct expression is Eq. (5.38).

**Secondly, solve the adjoint problem:**

The adjoint problem is

$$\partial_{\dot{h}} F^* \lambda = \partial_{\dot{h}} J,$$

or

$$1^* \lambda = \frac{1}{\mathcal{A}} \int \left(\dot{h} - \tilde{\dot{h}}\right) \delta \dot{h} \, dA.$$

We can ask what the 1 actually stands for in this context, and the answer is that this is the unity operator in the relevant basis, i.e. the inner product between the unit vectors spanning the space, or the Gramian matrix. In the FE context, where the basis functions are $\phi_i$, this matrix is the mass matrix $[\boldsymbol{M}]_{ij} =< \phi_i \mid \phi_j >$.

To see this more clearly let us calculate the directional derivative from the FE formulation of the forward problem while realising that $\partial_{\dot{h}}^* \lambda$ represents an inner product, i.e. we need to evaluate

$$\begin{aligned} < \delta_{\dot{h}} F \mid \lambda > &= \int \delta_{\dot{h}} \left(\dot{h} - (a - \partial_x(uh))\right) \lambda \, dA \\ &= \int \phi_i \, \lambda \, dA \\ &= \int \phi_i \, \phi_j \lambda_j \, dA \\ &= [\boldsymbol{M}]_{ij} \lambda_j, \end{aligned}$$

showing that the expression $1\lambda$ is, in the FE context, given by $\boldsymbol{M}\boldsymbol{\lambda}$. Also note that the term $\partial_{\dot{h}} J$ is in fact a directional derivative and, again in FE context, reads

$$\delta_{\dot{h}} = \frac{1}{\mathcal{A}} \int \left(\dot{h} - \tilde{\dot{h}}\right) \phi_j \, dA.$$

Hence, the adjoint problem is

$$\boldsymbol{M}\boldsymbol{\lambda} = \boldsymbol{l},$$

where

$$[\boldsymbol{l}]_j = \frac{1}{\mathcal{A}} \int \left(\dot{h} - \tilde{\dot{h}}\right) \phi_j \, dA.$$

The vector $\boldsymbol{l}$ is

$$\begin{aligned} \boldsymbol{l} &= \frac{1}{\mathcal{A}} \int \left(\dot{h}_i - \tilde{\dot{h}}_i\right) \phi_i \phi_j \, dA \\ &= \left(\dot{h}_i - \tilde{\dot{h}}_i\right) \frac{1}{\mathcal{A}} \int \phi_i \phi_j \, dA \\ &= \frac{1}{\mathcal{A}} \boldsymbol{M}(\dot{\boldsymbol{h}} - \tilde{\dot{\boldsymbol{h}}}), \end{aligned}$$

and the adjoint equations therefore simply $\boldsymbol{\lambda} = \frac{1}{\mathcal{A}}(\dot{\boldsymbol{h}} - \tilde{\dot{\boldsymbol{h}}})$.

We can also arrive at the adjoint equation directly from the discrete form of the equations, as derived above, where the forward model and the object function are given by

$$\begin{aligned} &\boldsymbol{F} = \boldsymbol{M}\dot{\boldsymbol{h}} - \boldsymbol{b}, \\ &J = \frac{1}{2\mathcal{A}}(\dot{\boldsymbol{h}} - \tilde{\dot{\boldsymbol{h}}})^T \boldsymbol{M}(\dot{\boldsymbol{h}} - \tilde{\dot{\boldsymbol{h}}}) + \frac{1}{2\mathcal{A}}(\boldsymbol{h} - \tilde{\boldsymbol{h}})^T \boldsymbol{M}(\boldsymbol{h} - \tilde{\boldsymbol{h}}). \end{aligned}$$

It follows that $\partial_{\dot{h}} \boldsymbol{F} = \boldsymbol{M}$ and $\partial_{\dot{h}} J = \frac{1}{\mathcal{A}} \boldsymbol{M}(\dot{\boldsymbol{h}} - \tilde{\dot{\boldsymbol{h}}})$ and the adjoint equation is

$$\boldsymbol{M}\boldsymbol{\lambda} = \frac{1}{\mathcal{A}} \boldsymbol{M}(\dot{\boldsymbol{h}} - \tilde{\dot{\boldsymbol{h}}}),$$

and we again arrive at

$$\boldsymbol{\lambda} = \frac{1}{\mathcal{A}}(\dot{\boldsymbol{h}} - \tilde{\dot{\boldsymbol{h}}}).$$

**Thirdly, evaluate the directional derivative:**

The directional derivative is the given by

$$d_h J = \partial_h F^* \lambda + \partial_h J.$$

We are now suffering from the notation in section 5.7.1 being rather relaxed. Since $F$ is an operator we need to remember that the first term on the right-hand side is an inner product and we need to evaluate

$$d_h J =< \delta_h F^* \mid \lambda > + \delta_h J.$$

where the $\delta$ symbols indicates that these are directional derivative. This leads to

$$\begin{aligned} d_h J &= \int \lambda \, \partial_x (u \, \delta h) \, dA + \frac{1}{\mathcal{A}} \int \left( h - \tilde{h} \right) \, \delta h \, dA, \\ &= \frac{1}{\mathcal{A}} \int \left( \dot{h} - \tilde{\dot{h}} \right) \partial_x (u \, \delta h) \, dA + \frac{1}{\mathcal{A}} \int \left( h - \tilde{h} \right) \, \delta h \, dA, \end{aligned}$$

which is same as (5.37). The second term on the right-hand side can be evaluated directly from the discrete from as

$$\delta_h J = \frac{1}{\mathcal{A}} \boldsymbol{M} (\boldsymbol{h} - \tilde{\boldsymbol{h}}).$$

## 5.10 $B$ inversion

Cost function:

$$J = J_{uv} + J_{\dot{h}} + J_A + J_B + J_C \tag{5.39}$$

Forward models, momentum and mass conservation:

$$\begin{aligned} F_{uv} &= F_{uv}(u, b, h(b), d(b)) && \text{(momentum)} \\ F_{\dot{h}} &= F_{\dot{h}}(u, h(b)) && \text{(mass conservation)} \\ F_b &= F_b(s, S, B) && \text{(flotation)} \end{aligned}$$

where $F_{uv}$ are here the SSA momentum equations, and

$$F_{\dot{h}} = \dot{h} + \partial_x (u \, (s - b)) - a = 0$$

and $b$ is calculated using the floating condition from $s$, $S$ and $B$ given $\rho$ and $\rho_o$ as

$$F_b = b - \mathcal{G} \, B - (1 - \mathcal{G}) \frac{\rho s - \rho_o S}{\rho - \rho_o} \tag{5.40}$$

where

$$\mathcal{G} = \mathcal{H}((s - b) - \rho_o (S - B)/\rho).$$

as explained further in section 5.12 this is a non-linear system and needs to be solved iteratively.

The cost functions are

$$J_{uv} = \frac{1}{2\mathcal{A}} \int \left( ((u_s - \tilde{u}_s)/e_u)^2 + ((v_s - \tilde{v}_s)/e_v)^2 \right) \, dA \tag{5.41}$$

$$J_{\dot{h}} = \frac{1}{2\mathcal{A}} \int \left( (\dot{h} - \tilde{\dot{h}})/e_{\dot{h}} \right)^2 \, dA \tag{5.42}$$

and $J_B$, $J_C$ and $J_A$ are regularisation terms (e.g Eq. 5.15).

The extended cost function becomes

$$\mathcal{L} = J_{uv} + J_{\dot{h}} + J_A + J_B + J_C + (F_{uv}, \lambda_{uv}) + (F_{\dot{h}}, \lambda_{\dot{h}}) + (F_b, \lambda_b)$$

and we wish to determine

$$d_B J = \delta_B J_B + (\delta_B F^*_{uv}, \lambda_{uv}) + (\delta_B F^*_{\dot{h}}, \lambda_{\dot{h}}) + (\delta_B F^*_b, \lambda_b) \tag{5.43}$$

We require

$$\delta_{uv}\mathcal{L} = 0$$
$$\delta_{\dot{h}}\mathcal{L} = 0$$
$$\delta_b\mathcal{L} = 0$$

and we get the adjoint equations are (see also Eq. 5.22)

$$(\delta_{uv}F^*_{uv}, \lambda_{uv}) + (\delta_{uv}F^*_{\dot{h}}, \lambda_{\dot{h}}) + (\delta_{uv}F^*_b, \lambda_b) = -\delta_{uv}J_{uv} - \delta_{uv}J_{\dot{h}} \tag{5.44}$$
$$(\delta_{\dot{h}}F^*_{uv}, \lambda_{uv}) + (\delta_{\dot{h}}F^*_{\dot{h}}, \lambda_{\dot{h}}) + (\delta_{\dot{h}}F^*_b, \lambda_b) = -\delta_{\dot{h}}J_{uv} - \delta_{\dot{h}}J_{\dot{h}} \tag{5.45}$$
$$(\delta_b F^*_{uv}, \lambda_{uv}) + (\delta_b F^*_{\dot{h}}, \lambda_{\dot{h}}) + (\delta_b F^*_b, \lambda_b) = -\delta_b J_{uv} - \delta_b J_{\dot{h}} \tag{5.46}$$

For the cost function Eq. (5.39) and its terms as defined by Eqs. (5.41) and (5.42), $J_{\dot{h}}$ is not an explicit function of either $b$ or $u$ and $v$, and therefore

$$\delta_b J_{uv} = 0$$
$$\delta_{\dot{h}} J_{uv} = 0$$
$$\delta_b J_{\dot{h}} = 0$$
$$\delta_{uv} J_{\dot{h}} = 0$$

Also from the definition of $F_b$ by Eq. (5.40) and $F_{uv}$

$$(\delta_{uv}F^*_b, \lambda_b) = 0$$
$$(\delta_{\dot{h}}F^*_b, \lambda_b) = 0$$
$$(\delta_{\dot{h}}F^*_{uv}, \lambda_{\dot{h}}) = 0$$

The adjoint system Eqs. (5.44) to (5.46) therefore becomes

$$(\delta_{uv}F^*_{uv}, \lambda_{uv}) + (\delta_{uv}F^*_{\dot{h}}, \lambda_{\dot{h}}) = -\delta_{uv}J_{uv} \tag{5.47}$$
$$(\delta_{\dot{h}}F^*_{\dot{h}}, \lambda_{\dot{h}}) = -\delta_{\dot{h}}J_{\dot{h}} \tag{5.48}$$
$$(\delta_b F^*_{uv}, \lambda_{uv}) + (\delta_b F^*_{\dot{h}}, \lambda_{\dot{h}}) + (\delta_b F^*_b, \lambda_b) = 0 \tag{5.49}$$

Note that if, alternatively, $F_{\dot{h}}$ is included directly in $J_{\dot{h}}$, i.e.

$$J_{\dot{h}} = \frac{1}{2\mathcal{A}} \int \left( (a - (\partial_x(uh) + \partial_y(vh))) - \tilde{\dot{h}} \right)^2 \, dA$$

then $F_{\dot{h}}$ is no longer required to be a part of the extended cost function (the Lagrangian), and the total derivative $d_B J$ can be calculated as

$$d_B J = \delta_B J_B + (\delta_B F^*_{uv}, \lambda_{uv}) + (\delta_B F^*_b, \lambda_b)$$

However, now $J_{\dot{h}}$ is an explicit function of $b$, so $\delta_b J_{\dot{h}} \neq 0$ and the adjoint system becomes

$$(\delta_{uv}F^*_{uv}, \lambda_{uv}) = -\delta_{uv}J_{uv} - \delta_{uv}J_{\dot{h}}$$
$$(\delta_b F^*_{uv}, \lambda_{uv}) + (\delta_b F^*_{\dot{h}}, \lambda_{\dot{h}}) + (\delta_b F^*_b, \lambda_b) = 0$$

and $\lambda_{uv}$ is a solution to

$$(\delta_{uv}F^*_{uv}, \lambda_{uv}) = -\delta_{uv}J_{uv} - \delta_{uv}J_{\dot{h}}$$

## 5.11 Inverting for $b$ using a fixed floating mask

Here we consider the option of inverting only for $b$ for a given flotation mask $\tilde{\mathcal{G}}$. We assume the upper surface ($s$), is known from measurements $\tilde{s}$.

Cost function:

$$J = J_{uv} + J_{\dot{h}} + J_A + J_b + J_C \tag{5.50}$$

Forward models, momentum and mass conservation:

$$
\begin{aligned}
F_{uv} &= F_{uv}(u, b, d(b)) && \text{(momentum)} \\
F_{\dot{h}} &= F_{\dot{h}}(u, b) && \text{(mass conservation)} \\
\mathcal{G} &= \tilde{\mathcal{G}} && \text{(flotation)}
\end{aligned}
$$

where $F_{uv}$ are here the SSA momentum equations, and

$$F_{\dot{h}} = \dot{h} + \partial_x(u\,(s-b)) - a = 0$$

The cost functions are

$$J_{uv} = \frac{1}{2\mathcal{A}} \int \left( ((u_s - \tilde{u}_s)/e_u)^2 + ((v_s - \tilde{v}_s)/e_v)^2 \right) dA \tag{5.51}$$

$$J_{\dot{h}} = \frac{1}{2\mathcal{A}} \int \left( \left( (a - (\partial_x(uh) + \partial_y(vh))) - \tilde{\dot{h}} \right) / e_{\dot{h}} \right)^2 dA \tag{5.52}$$

and $J_b$, $J_C$ and $J_A$ are regularisation terms (e.g Eq. 5.15) on the form

$$J_b = \frac{1}{2\mathcal{A}} \int \left( (b - \tilde{b})/e_b \right)^2 dA$$

Since we here include $F_{\dot{h}}$ in the cost function, it does not need to be included in the extended cost function, therefore

$$\mathcal{L} = J_{uv} + J_{\dot{h}} + J_A + J_b + J_C + (F_{uv}, \lambda_{uv}).$$

We wish to determine $d_b J$,

$$d_b J = \delta_b J_b + \delta_b J_{\dot{h}} + (\delta_b F^*_{uv}, \lambda_{uv}) \tag{5.53}$$

The term $(\delta_b F^*_{uv}, \lambda_{uv})$ accounts for the effect of $b$ on on $d_b J$ (see Eq. 5.50) due to the implicit dependency of $u$ and $v$ on $b$.

We require

$$\delta_{uv} \mathcal{L} = 0$$

and we get the adjoint equation (see also Eq. 5.22)

$$(\delta_{uv} F^*_{uv}, \lambda_{uv}) = -\delta_{uv} J_{uv} - \delta_{uv} J_{\dot{h}} \tag{5.54}$$

We assume the floating mask $\mathcal{G}$ is given as an input and we denote the prescribed/measured floating mask as $\tilde{\mathcal{G}}$. We furthermore require that any changes in $b$ do not lead to any changes in the floating mask. Therefore, for the flotation mask not to change during the inversion for $b$, the ice thickness $h$ must be above the flotation limit where $\tilde{\mathcal{G}} < 1/2$ and below it where $\tilde{\mathcal{G}} < 1/2$, i.e.

$$
\begin{aligned}
h > h_f \quad &\text{where} \quad \tilde{\mathcal{G}} \geq 1/2 \\
h < h_f \quad &\text{where} \quad \tilde{\mathcal{G}} < 1/2
\end{aligned}
$$

or

$$
\begin{aligned}
\rho(s-b) > \rho_o(S-B) \quad &\text{where} \quad \tilde{\mathcal{G}} \geq 1/2 \\
\rho(s-b) < \rho_o(S-B) \quad &\text{where} \quad \tilde{\mathcal{G}} < 1/2
\end{aligned}
$$

We calculate the bedrock elevation ($B$) and the draft ($d$), given $\rho$ and $\rho_o$, as

$$B = \begin{cases} b & \text{for} \quad \tilde{\mathcal{G}} \geq 1/2 \\ \min(b, \tilde{B}) & \text{otherwise} \end{cases}$$

Since $b = B$ where $\tilde{\mathcal{G}} \geq 1/2$ we have

$$
\begin{aligned}
b < \frac{\rho s - \rho_o S}{\rho - \rho_o} \quad &\text{where} \quad \tilde{\mathcal{G}} \geq 1/2 \\
b > s - \rho_o(S-B)/\rho \quad &\text{where} \quad \tilde{\mathcal{G}} < 1/2
\end{aligned}
$$

Provided these constraints on $b$ are enforced, the floating mask ($\mathcal{G}$) remains unchanged as $b$ is updated during the inversion. All directional derivatives of $\mathcal{G}$ with respect to $b$ are therefore automatically equal to zero. However, as $b$ can change downstream of the grounding line within these constraints, the surface elevation $s$ can change as well according to

$$s = (1 - \rho_o/\rho) b + \frac{\rho_o}{\rho} S, \quad \text{where} \quad \tilde{\mathcal{G}} < 1/2$$

Such changes can be suppressed using an appropriate regularisation on $b$ using as prior

$$\tilde{b} = \frac{\rho \tilde{s} - \rho_o S}{\rho - \rho_o} = \frac{\rho}{\rho_o - \rho} S - \frac{\rho_o}{\rho_o - \rho} \tilde{s} \quad \text{where} \quad \tilde{\mathcal{G}} < 1/2$$

where $\tilde{s}$ are measurements of $s$, and prescribing errors $e_b$ in $b$ derived from the observational errors $e_s$ of $s$ over floating areas as

$$e_b = e_s / |1 - \rho_o/\rho|$$

Alternatively an additional term $J_s$ to the cost function $J$ can be added as

$$J_s = \frac{1}{2\mathcal{A}} \int \left\{ (1 - \mathcal{G}) \left[ (1 - \rho_o/\rho) b + \rho_o S/\rho - \tilde{s} \right] / e_s \right\}^2 \, dA$$

The directional derivative of the draft $d$ defined as

$$d = \mathcal{H}(S - B)(S - b)$$

with respect to $b$ is required. Note that where $S = B$, the ice is grounded for any $h > 0$, so therefore $b = B$ and in a weak sense $\int (\delta_b \mathcal{H}(S - b))(s - b) \, dA = 0$, hence

$$\delta_b d = -\mathcal{H}(S - B)$$

## 5.12 Inverting for bedrock elevation $B$ with varying flotation mask

When inverting for bedrock $B$ the elevations of the upper and lower ice surfaces ($s$ and $b$ respectively) need to be recalculated as $B$ is updated. We assume we have reasonably accurate measurements, $\tilde{s}$, of the surface elevation. When updating $b$ we therefore consider $s$, $S$, and $B$ given, and calculate $b$ and $h$ from $s$, $S$, and $B$, i.e.

$$b = b(s, S, B, \rho, \rho_o)$$
$$h = h(s, S, B, \rho, \rho_o)$$

as

$$b = \mathcal{G}\, B + (1 - \mathcal{G}) \frac{\rho s - \rho_o S}{\rho - \rho_o}, \tag{5.55}$$

$$h = \mathcal{G}\, (s - B) + (1 - \mathcal{G}) \frac{s - S}{1 - \rho/\rho_o}, \tag{5.56}$$

where

$$\mathcal{G} = \mathcal{H}(h - \rho_o(S - B)/\rho)\ . \tag{5.57}$$

This is a non-linear system because $\mathcal{G}$ depends on $h$ and solving this system is discussed in Sec. 1.9.3

We need to know the directional derivatives of various geometrical variables and quantities with respect to $B$. These include

$$\begin{aligned} \delta_B \mathcal{G} &= \lim_{\epsilon \to 0} \frac{d}{d\epsilon} \mathcal{H}(s - (B + \epsilon\, \delta B) - \rho_o(S - (B + \epsilon\, \delta B))/\rho) \\ &= \lim_{\epsilon \to 0} \left( \delta(s - (B + \epsilon\, \delta B) - \rho_o(S - B - \epsilon\, \delta B)/\rho)(-\delta B + \frac{\rho_o}{\rho} \delta B \right) \\ &= \delta(h - h_f)\ (\rho_o/\rho - 1)\ \delta B \end{aligned}$$

and

$$\begin{aligned}\delta_B b &= \mathcal{G}\,\delta B + \delta_B G\, B - \frac{\rho s - \rho_o S}{\rho - \rho_o}\delta_B \mathcal{G} \\ &= \mathcal{G}\,\delta B + \left(B - \frac{\rho s - \rho_o S}{\rho - \rho_o}\right)\delta_B \mathcal{G} \\ &= \mathcal{G}\,\delta B + (\rho_o/\rho - 1)\left(B - \frac{\rho s - \rho_o S}{\rho - \rho_o}\right)\delta(h - h_f)\,\delta B\end{aligned}$$

and

$$\delta_B h = -\delta_B b$$

Also from

$$d = \mathcal{H}(S - B)(S - b)$$

we have

$$\delta_B d = -\,\delta(S - B)\,(S - b)\,\delta B - \mathcal{H}(S - B)\,\delta_B b$$

When calculating gradients with the adjoint method we get a term of the form

$$(\delta_B(\partial_x b), \lambda)$$

It's presumably best to use an integral theorem here and write this as

$$(\delta_B(\partial_x b), \lambda) = (\partial_x(\delta_B b), \lambda) = -(\delta_B b, \partial_x \lambda)$$

where $\lambda = 0$ along the boundary, for example something like

$$\begin{aligned}((\rho h - \rho_o d)\delta_B(\partial_x b), \lambda) &= -(\delta_B b, \partial_x\left((\rho h - \rho_o d)\lambda\right)) \\ &= -(\delta_B b, (\rho \partial_x h - \rho_o \partial_x d)\lambda + (\rho h - \rho_o d)\partial_x \lambda)\end{aligned}$$

## 5.13 Gradients of objective functions with respect to control variables

In the following we assume that all variables involved, such as $A$, $C$, $b$ and $\lambda$ are represented in the same basis, i.e.

$$\begin{aligned}C &= A_p\,\phi_p(x, y) \\ A &= A_p\,\phi_p(x, y) \\ b &= b_p\,\phi_p(x, y) \\ \lambda &= \lambda_q\,\phi_q(x, y)\end{aligned}$$

### 5.13.1 Gradient calculation in 1HD with respect to $C$

As an example we consider the calculation of the gradient of the objective function $J$ with respect to slipperiness. The only term of the momentum equations containing $C$ is the basal drag term

$$\boldsymbol{t}_b = \mathcal{H}(h - h_f)\; C^{-1/m} \|\boldsymbol{v}_b\|^{1/m-1}\,\boldsymbol{v}_b$$

We need to evaluate

$$\begin{aligned}DJ(C)[\phi] &=< (\partial_C F)^* \lambda + \partial_C J \mid \phi > \\ &=< (\partial_C \boldsymbol{t}_b)^* \lambda + \partial_C J \mid \phi >\end{aligned}$$

giving

$$DJ(C)[\phi] =< \frac{1}{m}\mathcal{H}(h - h_f) C^{-1/m-1} \|\boldsymbol{v}_b\|^{1/m-1}\,\boldsymbol{v}_b \lambda \mid \phi > + < \partial_C J \mid \phi >$$

In the above listed expression one needs to form a sum between $\boldsymbol{v}_b$ and $\lambda$ for each value of $C$. The adjoint variable $\lambda$ is a solution of the adjoint equation and can be considered as a vector variables with $x$ and $y$ components similarly to $\boldsymbol{v}$.

### 5.13.2 Gradient calculation in 1HD with respect to $A$

The directional derivative can be calculated (see Eq. 5.23) as

$$j'(A) = \partial J(q(p),p)/\partial A + (\partial F/\partial A)^* \lambda \tag{5.58}$$

Focusing on the second term

$$\begin{aligned}
(\partial_A F)^* \lambda &=< \partial_A \left( 2\partial_x (A^{-1/n}\, h\, |\partial_x u|^{(1-n)/n} \partial_x u) - t_{bx} - \frac{1}{2} g \partial_x (\rho h^2 - \rho_o d^2) + g\, \mathcal{H}(h-h_f)(\rho h - \rho_o H^+)\partial_x B \right) \mid \lambda > \\
&= - < 2\partial_A \left( A^{-1/n}\, h\, |\partial_x u|^{(1-n)/n} \partial_x u \right) \mid \partial_x \lambda > \\
&= - < \frac{2}{n} A^{-1/n-1}\, h\, |\partial_x u|^{(1-n)/n} \partial_x u\, \delta A, \partial_x \lambda >
\end{aligned}$$

where we have omitted writing the boundary term assuming that $\lambda$ is set to zero along the boundary (or periodic boundary conditions for periodic domains.) Hence

$$\begin{aligned}
(\partial_A F)^* \lambda &=< \frac{2}{n} A^{-1/n-1}\, h\, |\partial_x u|^{(1-n)/n} \partial_x u\, \delta A, \partial_x \lambda > \\
&=< \frac{2}{n} A^{-1/n-1}\, h\, |\partial_x u|^{(1-n)/n} \partial_x u\, \phi_p, \lambda_q \partial_x \phi_q > \\
&=< \frac{2}{n} A^{-1/n-1}\, h\, |\partial_x u|^{(1-n)/n} \partial_x u\, \phi_p, \partial_x \phi_q > \lambda_q
\end{aligned}$$

or

$$(\partial_A F)^* \lambda = \boldsymbol{K\lambda}$$

where

$$K = (\partial F/\partial A)^*$$

is

$$K_{pq} =< \frac{2}{n} A^{-1/n-1}\, h\, |\partial_x u|^{(1-n)/n} \partial_x u\, \phi_p, \partial_x \phi_q > \lambda_q$$

however it is more efficient to calculate this matrix-vector product directly without ever forming the matrix as

$$\boldsymbol{K\lambda} =< \frac{2}{n} A^{-1/n-1}\, h\, |\partial_x u|^{(1-n)/n} \partial_x u\, \phi_p, \partial_x \lambda >$$

### 5.13.3 Gradient calculation in 1HD with respect to $b$

The directional derivative can be calculated (see Eq. 5.23) as

$$j'(b) = \partial J(q(p),p)/\partial b + (\partial F/\partial b)^* \lambda \tag{5.59}$$

Simplifying the notation a bit and just considering the grounded ice situation the $x$ term of the SSA equation is

$$F = 2\partial_x \left( A^{-1/n}\, h\, |\partial_x u|^{(1-n)/n} \partial_x u \right) - \mathcal{H}(h-h_f) C^{-1/m} \|u\|^{1/m-1} u - \rho g h \partial_x s - \frac{1}{2} g h^2 \partial_x \rho = 0 \tag{5.60}$$

where $h = s - b$.

Considering initially the first term of Eq. (5.60)

$$\begin{aligned}
(\partial_b F)^* \lambda &=< \partial_b \left( 2\partial_x (A^{-1/n}\, h\, |\partial_x u|^{(1-n)/n} \partial_x u) \right) \mid \lambda > \\
&= - < 2\partial_b \left( A^{-1/n}\, (s-b)\, |\partial_x u|^{(1-n)/n} \partial_x u \right) \mid \partial_x \lambda > \\
&=< 2A^{-1/n}\, |\partial_x u|^{(1-n)/n} \partial_x u\, \delta b, \partial_x \lambda >
\end{aligned}$$

or

$$[(\partial_b F)^* \lambda]_p =< 2A^{-1/n}\, |\partial_x u|^{(1-n)/n} \partial_x u\, \phi_p, \partial_x \phi_q > \lambda_q$$

where we again have omitted writing the boundary term assuming that $\lambda$ is set to zero along the boundary (or periodic boundary conditions applied for periodic domains.). Using Eq. (1.49)

$$\eta = \frac{1}{2} A^{-1/n} \, |\partial_x u|^{(1-n)/n}$$

leads to

$$[(\partial_b F)^* \lambda]_p = < 4\eta \partial_x u \, \phi_p, \partial_x \phi_q > \lambda_q$$

Second term of Eq. (5.60) leads to

$$[(\partial_b F)^* \lambda]_p = < \delta(h - h_f) \, C^{-1/m} \|u\|^{1/m-1} u \, \phi_p, \phi_q > \lambda_q$$

(Note: I've ignored here the fact that where the ice is grounded and $b = B$, $h_f = \rho_o H/\rho = \rho_o(S-B)/\rho = \rho_o(S-b)/\rho$ is a function of $b$)

More generally one can write this as

$$[(\partial_b F)^* \lambda]_p = < (\partial_b t_{bx}) \phi_p, \phi_q > \lambda_q$$

And the last two terms of Eq. (5.60) give

$$\begin{aligned}
(\partial_b F)^* \lambda &= - < \partial_b \left( \rho g h \partial_x s + \frac{1}{2} g h^2 \partial_x \rho \right), \lambda > \\
&= < (\rho g \partial_x s + g h \partial_x \rho) \, \phi, \lambda >
\end{aligned}$$

or

$$[(\partial_b F)^* \lambda]_p = < (\rho g \partial_x s + g h \partial_x \rho) \, \phi_p \mid \phi_q > \lambda_q$$

Usually only the regularisation term is an explicit function of the control variable, i.e.

$$J(F(p, q(p), p) = I(F(p, q(p))) + R(p)$$

and therefore

$$\partial_p J = \partial_p R$$

But if we solve for $b$ and use $p_t h$ as a constraint, calculated as $a - \partial_x(uh)$ we now have $b$ explicitly both in the misfit term $I$ and in the regularisation term $R$, so $\partial_b J$ has now this additional $I$ term.

## 5.14 Inverting for $\log p$

To lessen the chances of a strictly positive parameter $p$ becoming negative in the course of the inversion we can make a change of variables writing

$$p = 10^\gamma = e^{\gamma \ln(10)}$$

or

$$\log_{10} p = \gamma$$

We have

$$\begin{aligned}
\frac{\partial J}{\partial \gamma} &= \frac{\partial J}{\partial p} \frac{\partial p}{\partial \gamma} \\
&= \frac{\partial J}{\partial p} \ln(10) \, 10^\gamma \\
&= \ln(10) \, p \frac{\partial J}{\partial p}
\end{aligned}$$

showing that the change of variables causes a rescaling of the gradient, making the gradient go to zero as $p \to 0$. However, for a finite step size this rescaling of the gradient alone does not guarantee that $p$ will not become negative (and $\log p$ complex) during the optimisation. In $\acute{U}a$ one therefore also enforces the positivity (non complex) constraint when inverting for $\log p$ of a strictly positive parameter.

## 5.15 Simplified gradients using a *fixed-point* approach.

Experience has shown that it is sometimes useful to start an inverse iteration sequence with some ad-hoc estimation of the inverted field, or some crude initial gradients. Such an approach can, for example, be used to create a rough estimate of the inverted field directly from measurements.

In *Úa* the resulting gradients are referred to as *fixed-point gradients*. The argument for using these ad-hoc approaches, instead of the correctly calculated gradients by the adjoint method, is that they have shown to provide useful starting points for an inversion and to speed up the initial steps of an inverse iteration.

Consider an inversion of $B$ and let's estimate what $B = s - h$ might look like if we ignore horizontal stress transmission, i.e.

$$u_s \approx C\tau^m,$$

with

$$\tau \approx \rho g h \partial_x s.$$

A rough initial estimate $h_0$ for $h$ might be

$$h_0 = \frac{u}{C(\rho g \partial_x s)^m}.$$

This clearly has plenty of potential to go hopelessly wrong, but might be OK as a rough initial estimate. A more robust approach would be to approximate the gradient of the cost function

$$J = \frac{1}{2}\|u - \tilde{u}\| + \frac{1}{2}\|h - \tilde{h}\|,$$

giving

$$\delta_h J = (u - \tilde{u})\frac{Cm}{h}\tau^m + (h - \tilde{h}),$$

where for notational simplicity we have assumed $\|f\| = f^2$. This is a much more robust approach as here $J$ is recalculated using the forward model and the (incorrect) gradient is used in a robust line-search step. This approach guarantees that the formally specified misfit function is actually being minimised. At worst using this approach the iteration will stagnate, in which case one simply switches to the (correct) adjoint gradient.

A similar approach can be used to estimate $\delta_C J$, and using the $C$ fixed-point gradient in *Úa* for the first 1 or 2 iterations often results quite dramatic reductions in the cost function.

## 5.16 The form of the adjoint equations for Bayesian approach using Gaussian statistics

We anticipate using a Bayesian approach assuming Gaussian statistics and therefore that the cost function might be on the form

$$\begin{aligned}\min_p I(q(p)) &=< q - \tilde{q} \mid K_q^{-1} \mid q - \tilde{q} > + < p - \tilde{p} \mid K_P^{-1} \mid p - \tilde{p} > \\ &=< K_q^{-T/2}(q - \tilde{q}) \mid K_q^{-1/2}(q - \tilde{q}) > + < K_p^{-T/2}(p - \tilde{p}) \mid K_p^{-1/2}(p - \tilde{p}) >\end{aligned}$$

where $K$ is a covariance matrix (and therefore positive definite).

Repeating the calculations required in the adjoint approach for this particular case,

$$\begin{aligned}d_p I = d_p \mathcal{L} &=< K_q^{-1/2}(q - \tilde{q}) \mid d_p q > + < \lambda \mid \partial_q F d_p q + \partial_p F > + < \partial_p \lambda \mid F > \\ &=< K_q^{-1/2}(q - \tilde{q}) \mid d_p q > + < \lambda \mid \partial_q F d_p q + \partial_p F > \\ &=< K_q^{-1/2}(q - \tilde{q}) + \lambda(\partial_q r)^* \mid d_p q > + < \lambda \mid \partial_p F >\end{aligned}$$

where we have omitted the $< p - \tilde{p} \mid K_P^{-1} \mid p - \tilde{p} >$ term for the time being. We now use the flexibility of $\lambda$ not having been specified and require that

$$< K_q^{-1/2}(q - \tilde{q}) + \lambda(\partial_q F)^* \mid \delta p >= 0$$

and therefore

$$d_p I = < \lambda \mid \partial_p F >$$

For a cost function on the form

$$\min_p I(q(p)) = < q - \tilde{q} \mid K_q^{-1} \mid q - \tilde{q} > + < p - \tilde{p} \mid K_P^{-1} \mid p - \tilde{p} >$$

we would arrive at

$$d_p I = < \lambda \mid \partial_p F > + < K_p^{-1/2}(p - \tilde{p}) \mid \delta p >$$

One might ask why we don't just calculate the cost gradient as

$$d_p = < K_q^{-1/2}(q - \tilde{q}) \mid d_p u >$$

But this would require calculating $\partial_p q$ which is a pain in the neck and requires $N$ solutions for the forward problem, where $N+1$ is the number of discrete control parameters. Using the adjoint method we only need to solve the forward problem twice.

## 5.17 Adjoint equations (Bayesian case with constraints on vertical velocity)

We want to minimise a cost function $\tilde{J}$ on the form

$$\tilde{J}(u, v, w, p) = I(u, v, w) + F(p)$$

where $I$ is a data discrepancy functional, and $R$ a regularisation term

As a misfit function we use

$$I = I_u + I_v + I_o,$$

where each term has the form

$$I_u = < C_{uu}^{-1/2} u - \tilde{(u)} \mid C_{uu}^{-1/2} u - \tilde{(u)} >$$

with $C_{uu}^{-1/2}$ being an error covariance matrix.

The regularisation term has the form

$$R = < C_{pp}^{-1/2} p \mid C_{pp}^{-1/2} p >$$

We minimise $\tilde{J}$ subject to the conditions

$$F(u(p), v(p), p) = 0$$

and

$$w_s = f(u, v, h, b)$$

where $r$ are the diagnostic equations, $f$ is a function giving the vertical surface velocity $w_o$ as a function of the variables of the diagnostic equations, and where $p$ stands for some control variable (distributed model parameter) such as the basal slipperiness $C$ or the rate factor $A$. We therefore consider the extended cost function

$$J(u, v, w, \lambda, \mu, p) = I(u, v, w) + F(p) + < \lambda \mid F(q(p), v(p), p) > + < \mu \mid w - f(u, v, h, b) >$$

where $\lambda$ and $\mu$ are Lagrange multipliers.

The directional derivative of $J$ with respect to $\lambda$ in the direction of $\delta\lambda$ is defined as

$$\frac{d}{d\epsilon} J(\lambda + \epsilon\, \delta\lambda) \mid_{\epsilon=0}$$

and is denoted by $\delta J(\lambda, \delta\lambda)$

The directional derivatives of $J$ are

$$\delta J(\lambda, \delta\lambda) = \delta_\lambda J = < \delta\lambda, F(u, v, p) >$$

$$J(\mu, \delta\mu) = < \delta\mu, w - f(u, v, h, b) >$$

$$J(u, \delta u) = < C_{uu}^{-1/2}(u - \tilde{u}, C^{-1/2} \delta u > + < \lambda, \nabla_u r\, \delta u > - < \mu \mid \nabla_u f\, \delta u >$$

## 5.18 Prognostic equations are formally self-adjoint

The SSTREAM equations are formally[6] self-adjoint as we will now show.

Define the inner product

$$r =< f_x \mid \lambda > + < f_y \mid \mu >$$

where

$$\begin{aligned} f_x =& \partial_x(h\eta(4\partial_x u + 2\partial_y v)) + \partial_y(h\eta(\partial_y u + \partial_x v)) - \mathcal{H}(h-h_f)t_{bx} \\ &- \frac{1}{2}g\partial_x(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h-h_f)(\rho h - \rho_o H^+)\partial_x B \end{aligned} \tag{5.61}$$

$$\begin{aligned} f_y =& \partial_y(h\eta(4\partial_y v + 2\partial_x u)) + \partial_x(h\eta(\partial_x v + \partial_y u) - \mathcal{H}(h-h_f)t_{by} \\ &- \frac{1}{2}g\partial_y(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h-h_f)(\rho h - \rho_o H^+)\partial_y B \end{aligned} \tag{5.62}$$

or

$$\begin{aligned} r =& \iint \big\{(\partial_x(h\eta(4\partial_x u + 2\partial_y v)) + \partial_y(h\eta(\partial_y u + \partial_x v))) - \mathcal{H}(h-h_f)t_{bx} \\ &- \frac{1}{2}g\partial_x(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h-h_f)(\rho h - \rho_o H^+)\partial_x B\big\}\,\lambda\,dx\,dy \\ &+ \iint \big\{(\partial_y(h\eta(4\partial_y v + 2\partial_x u)) + \partial_x(h\eta(\partial_x v + \partial_y u)) - \mathcal{H}(h-h_f)t_{by} \\ &- \frac{1}{2}g\partial_y(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h-h_f)(\rho h - \rho_o H^+)\partial_y B\big\}\mu\,dx\,dy \end{aligned}$$

The use of Green's theorem gives

$$\begin{aligned} r =& -\iint_\Omega \big\{h\eta(4\partial_x u + 2\partial_y v)\partial_x\lambda + h\eta(\partial_y u + \partial_x v)\partial_y\lambda + \mathcal{H}(h-h_f)\beta^2 u\,\lambda \\ &- \frac{1}{2}g(\rho h^2 - \rho_o d^2)\partial_x\lambda + \lambda\,g\,\mathcal{H}(h-h_f)(\rho h - \rho_o H^+)\partial_x B\big\}\,dx\,dy \\ &+ \oint_\Gamma (h\eta(4\partial_x u + 2\partial_y v)\,\lambda\,n_x + h\eta(\partial_y u + \partial_x v)\,\lambda\,n_y - \frac{1}{2}g(\rho h^2 - \rho_o d^2)\,\lambda n_x)\;d\Gamma \\ &- \iint_\Omega \big\{h\eta(4\partial_y v + 2\partial_x u)\partial_y\mu + h\eta(\partial_x v + \partial_y u)\partial_x\mu + \mathcal{H}(h-h_f)\beta^2 v \\ &- \frac{1}{2}g(\rho h^2 - \rho_o d^2)\partial_y\mu + \mu\,g\,\mathcal{H}(h-h_f)(\rho h - \rho_o H^+)\partial_x B\big\}\,dx\,dy \\ &+ \oint_\Gamma (h\eta(4\partial_y v + 2\partial_x u)\,\mu\,n_y + h\eta(\partial_x v + \partial_y u)\,\mu\,n_x - \frac{1}{2}g(\rho h^2 - \rho_o d^2)\,\mu n_y)\;d\Gamma \end{aligned}$$

and a second use of Green's theorem gives after some rearrangements

[6] Here 'formally' stands for 'if ignoring boundary conditions'.

$$
\begin{aligned}
r = &\iint_\Omega \big\{\partial_x(h\eta(4\partial_x\lambda + 2\partial_y\mu))\,u + \partial_y(h\eta(\partial_y\lambda+\partial_x\mu))\,u - \mathcal{H}(h-h_f)\beta^2 u\,\lambda \\
&+ \frac{1}{2}g(\rho h^2 - \rho_o d^2)\partial_x\lambda - \lambda\, g\,\mathcal{H}(h-h_f)(\rho h - \rho_o H^+)\partial_x B\big\}\,dx\,dy \\
&+ \oint_\Gamma (h\eta(4\partial_x u + 2\partial_y v)\,\lambda\, n_x + h\eta(\partial_y u + \partial_x v)\,\lambda\, n_y - \frac{1}{2}g(\rho h^2 - \rho_o d^2)\,\lambda n_x)\;d\Gamma \\
&+ \iint_\Omega \big\{\partial_y(h\eta(4\partial_x\mu + 2\partial_y\lambda))\,v + \partial_x(h\eta(\partial_x\mu+\partial_y\lambda))\,v - \mathcal{H}(h-h_f)\beta^2 u\,\lambda \\
&+ \frac{1}{2}cdg(\rho h^2 - \rho_o d^2)\partial_y\mu - \mu\, g\,\mathcal{H}(h-h_f)(\rho h - \rho_o H^+)\partial_x B\big\}\,dx\,dy \\
&+ \oint_\Gamma (h\eta(4\partial_y v + 2\partial_x u)\,\mu\, n_y + h\eta(\partial_x v + \partial_y u)\,\mu\, n_x - \frac{1}{2}g(\rho h^2 - \rho_o d^2)\,\mu n_y)\;d\Gamma \\
&- \oint_\Omega (uh\eta(4\partial_x\lambda + 2\partial_y\mu)n_x + uh\eta(\partial_y\lambda + \partial_x\mu)n_y)\,d\Gamma \\
&- \oint_\Omega (vh\eta(4\partial_y\mu + 2\partial_y\lambda)n_y + vh\eta(\partial_x\mu + \partial_y\lambda)n_x)\,d\Gamma
\end{aligned}
$$

If we ignore the BCs terms, the equations are clearly self-adjoint.

In *Úa* the BCs conditions for the adjoint problem (the boundary terms shown above) are generated automatically from the BCs of the forward problem using some sensible assumptions such as homogenisation of the adjoint BCs if Dirichlet and natural BCs are applied to the forward problem, and periodic BCs for the adjoint problem if periodic BCs are applied to the forward problem. The user can overwrite these assumptions if needed.

The adjoint approach is based on the use of the adjoint of the linearised/tangent forward model around the converged solution of non-linear forward model. The non-linear forward model is

$$F(u) = \mathbf{0}$$

and the tangent model is the directional derivative of the forward model in the direction $\delta\boldsymbol{u}$.

$$K = D\boldsymbol{F}(\boldsymbol{u})[\delta\boldsymbol{u}]$$

Often this is simply be written as

$$K = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{u}}$$

Define

$$<\boldsymbol{L}\,\delta U \mid \Lambda> = < U \mid \boldsymbol{L}\,\Lambda>$$

where $U = (\delta u, \delta v)^T$ and $\boldsymbol{L}$ is the operator acting on $U$ as given by the system (1.84) and (1.85).

$$\begin{aligned}
< \boldsymbol{L}\,\delta u \mid \Lambda >= & \iint_\Omega \big\{(\partial_x(h\eta(4\partial_x\delta u + 2\partial_y\delta v)) + \partial_y(h\eta(\partial_y\delta u + \partial_x\delta v)))\,\lambda \\
& +(\partial_y(h\eta(4\partial_y\delta v + 2\partial_x\delta u)) + \partial_x(h\eta(\partial_x\delta v + \partial_y\delta u)))\,\mu\big\}\,dx\,dy \\
= & -\iint_\Omega \big\{h\eta(4\partial_x\delta u + 2\partial_y\delta v)\partial_x\lambda + h\eta(\partial_y\delta u + \partial_x\delta v)\partial_y\lambda \\
& h\eta(4\partial_y\delta v + 2\partial_x\delta u)\partial_y\mu + h\eta(\partial_x\delta v + \partial_y\delta u)\partial_x\mu\big\}\,dx\,dy \\
& + \oint_\Gamma \big\{h\eta(4\partial_x\delta u + 2\partial_y\delta v)\,\lambda\,n_x + h\eta(\partial_y\delta u + \partial_x\delta v)\,\lambda\,n_y) \\
& +h\eta(4\partial_y\delta v + 2\partial_x\delta u)\,\mu\,n_y + h\eta(\partial_x\delta v + \partial_y\delta u)\,\mu\,n_x\big\}\,d\Gamma \\
= & \iint_\Omega \big\{(\partial_x(h\eta(4\partial_x\lambda + 2\partial_y\mu)) + \partial_y(h\eta(\partial_y\lambda + \partial_x\mu)))\,\delta u \\
& (\partial_y(h\eta(4\partial_x\mu + 2\partial_y\lambda)) + \partial_x(h\eta(\partial_x\mu + \partial_y\lambda)))\,\delta v\big\}\,dx\,dy \\
& + \oint_\Gamma \big\{(h\eta(4\partial_x\delta u + 2\partial_y\delta v)\,n_x + h\eta(\partial_y\delta u + \partial_x\delta v)\,n_y)\lambda \\
& +(h\eta(4\partial_y\delta v + 2\partial_x\delta u)\,n_y + h\eta(\partial_x\delta v + \partial_y\delta u)\,n_x)\mu\big\}\,d\Gamma \\
& - \oint_\Gamma (h\eta(4\partial_x\lambda + 2\partial_y\mu)n_x + h\eta(\partial_y\lambda + \partial_x\mu)n_y)\delta u\,d\Gamma \\
& - \oint_\Gamma (h\eta(4\partial_y\mu + 2\partial_y\lambda)n_y + h\eta(\partial_x\mu + \partial_y\lambda)n_x)\delta v\,d\Gamma
\end{aligned}$$

Once the forward model has been solved the velocity field fulfils given the BCs to a high degree of accuracy. The boundary conditions on the $\delta$ fields follow from above

## 5.19 Covariance kernels

$$F(f) = \int\int f(x)\,\kappa(x,x')\,f(x')\;dx\;dx'$$

where $\kappa$ is the covariance kernel.

Assuming isotropic, stationary, and translation invariance, i.e.

$$\kappa(x,x') = \kappa(|x-x'|)$$

a multipole expansion of an exponentially decaying covariance is on the form

$$e^{-|x-x_i|^2/4T} = \sum_{n_1,n_2=0}^{\infty} \Theta_{n1n_2}(x-c)...$$

# Chapter 6

# Level-set method

> Level-set methods (LSM) are a conceptual framework for using level sets as a tool for numerical analysis of surfaces and shapes. The advantage of the level-set model is that one can perform numerical computations involving curves and surfaces on a fixed Cartesian grid without having to parameterise these objects.
>
> *Wikipedia*
> *Level-set method*

We use an implicit formulation to describe the position of the calving front front $\mathcal{C}$ is defined as the solution to

$$\varphi(\mathcal{C}, t) = 0 \tag{6.1}$$

for all times $t$, where $\varphi$ is a function $\varphi : \mathbb{R}^2 \times \mathbb{R}$. The function $\varphi$ is referred to as the *level-set function*. By definition the curve moves with the (prescribed) speed $u$ the direction $\nabla\varphi$, i.e.

$$\boldsymbol{u} = u \frac{\nabla\varphi}{\|\nabla\varphi\|}$$

As $\boldsymbol{u}$ is the velocity of the curve, the value of $\varphi$ must not change for any point along the curve travelling with the velocity $\boldsymbol{u}$, and therefore the material derivative of $\varphi$ is zero, i.e.

$$\partial_t\varphi + \boldsymbol{u} \cdot \nabla\varphi = 0\ . \tag{6.2}$$

Let $\boldsymbol{v}$ be a material velocity of the ice and $\boldsymbol{c}$ the calving velocity

$$\boldsymbol{c} = c\,\hat{\boldsymbol{n}}$$

where the normal $\hat{\boldsymbol{n}}$ points outwards from the calving front. We define the sign of $\varphi$ such that $\varphi$ is positive within the ice. Hence,

$$\hat{\boldsymbol{n}} = -\frac{\nabla\varphi}{\|\nabla\varphi\|}$$

and

$$\boldsymbol{c} = -c\,\frac{\nabla\varphi}{\|\nabla\varphi\|}$$

where $c$ is the calving rate. (The calving rate is a scalar quantity, defined as the difference between the advance/retreat rate of the calving front and the material velocity, $\boldsymbol{v}$, of ice at the calving front in normal direction.) The velocity $\boldsymbol{u}$ of the calving front is the difference between the material velocity, $\boldsymbol{v}$, of ice at the calving front and the calving velocity, i.e.

$$\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{c}\ ,$$

and therefore

$$\partial_t\varphi + (\boldsymbol{v} - \boldsymbol{c}) \cdot \nabla\varphi = 0\ ,$$

which can be modified somewhat as follows

$$\begin{aligned}\partial_t \varphi + \boldsymbol{v} \cdot \nabla \varphi &= \boldsymbol{c} \cdot \nabla \varphi \\ &= -c \frac{\nabla \varphi}{\|\nabla \varphi\|} \cdot \nabla \varphi \, , \\ &= -c \frac{\|\nabla \varphi\|^2}{\|\nabla \varphi\|} \, ,\end{aligned}$$

to give

$$\partial_t \varphi + \boldsymbol{v} \cdot \nabla \varphi = -c \|\nabla \varphi\| \, . \tag{6.3}$$

and

$$\partial_t \varphi + (\boldsymbol{v} - \boldsymbol{c}) \cdot \nabla \varphi = 0 \, . \tag{6.4}$$

Eqs. (6.3) and 6.4 are equivalent formulations. We refer to those as the *kinematic calving front condition*, providing an evolutionary equation for the level-set function $\varphi$ for a given material velocity $\boldsymbol{v}$ of the material particles at the calving front, and a calving rate $c$.[1] The level set can be initialised based on the (signed) distance to the calving front $\mathcal{C}$ at $t = 0$. Initialising with signed-distance function implies $\|\nabla \varphi\| = 1$ as the curve is crossed.

Here, and in contrast to many other applications of the level set function, the (horizontal) velocity field, $\boldsymbol{v} = (u, v)$ is not a conservative flow field and the total area on each side of $\mathcal{C}$ therefore not conserved.

The level-set equations (6.3) and (6.4) are non-linear hyperbolic equations. As an example of type of behaviour that can be expected consider the one-dimentional case where

$$\partial_t \varphi + (v - c) \, \partial_x \varphi = 0 \, . \tag{6.6}$$

If $v - c$ is spatially constant, Eq. (6.6) is the simple linear advection equation and the solution is a travelling shape-preserving wave

$$\varphi(x, t) = \varphi_0(x - (v - c)t)$$

However, if $v - c$ is spatially variable, for example if $v - c = \epsilon \, (x - x_0)$, where $\epsilon$ is some (possibly small) constant, then the solution to (6.6) becomes

$$\varphi(x, t) = -(x - x_c) \, e^{\epsilon(t - t_0)}$$

This example is particularly pertinent because, as mentioned above, the level set is often initialized as a signed distance function, which in this one-dimentional example implies

$$\varphi(x, t = t_0) = (x - x_c)$$

where here $x_c$ stands for the position of the calving front at $t = t_0$. This example suggests that if $u - c$ is not spacially constant, one can expect $\phi$ to grow without bounds with time. In partcular, $\varphi$ will with time cease being a (signed) distance function with $\|\nabla \varphi\| = 1$. At the same time there are good reasons for wanting $\varphi$ to be at least approximately a signed distance function at all times. Changes in physical properties as the curve $\mathcal{C}$ is crossed can, for example, then be easily parameterised as a function of normal distance to $\mathcal{C}$ using $\varphi$.

As shown by Barles et al. (1993) signed-distance functions are not solution to Eq. (6.2) and with time the solution to Eq. (6.2) will no longer fulfil $\|\nabla \varphi\| = 1$. Gomes and Faugeras (2000) gives simple intuitive examples showing that evolving $\varphi$ using Eq. (6.2) can generally be expected to dangerously increase $\|\varphi\|$ with time, eventually leading to shock formation. Any numerical formulation that can not deal with formation of such shocks, such as the finite-element method using continuous form functions, may therefore be expected to eventually fail or to produce highly inaccurate solutions. Furthermore, determining the position of the curve $\mathcal{C}$ implicitly by finding the roots of Eq. (6.1) can be expected

[1] In the literature the *Level-set equation* is usually written as

$$\partial_t \varphi + F \, \|\nabla \varphi\| = 0, \tag{6.5}$$

where the scalar $F$ is the speed in outward normal direction. We can bring Eq. (6.3) to this standard form by defining $F$ as

$$F = \boldsymbol{v} \cdot \hat{\boldsymbol{n}} - c \, .$$

Generally, one finds that the speed function $F$ needs to be both physically and numerically motivated. Hence, we prefer the standard form (6.5) where $F$ is some specified function.

to become increasingly difficult as the magnitude of the gradient of $\varphi$ grows with time and eventually becomes unbounded.

Several methods have been proposed in the literature to deal with this problem. These include initialisation whereby $\varphi$ is either periodically reset to be equal to the signed distance function, or where a correction procedure is applied to 'push' the level set towards the signed distance function (e.g. Sussman et al., 1994). A level set method based on a variational principle can be derived by adding a pertubation

$$\mathcal{P} = \frac{1}{2}\int_A (\|\nabla\varphi\| - 1)^2 \, dA$$

to the energy potential where $p$ is some postive even integer (e.g. Luo et al., 2019). Minimizing this additional potential term involved adding the corresponing directional derivative with respect to $\varphi$ to the level set equation. The directional derivative of $\mathcal{P}$ in the direction $\delta\varphi$ is

$$\begin{aligned} D_{\delta\varphi}\mathcal{P} &= \int_A (\|\nabla\varphi\| - 1)\frac{\nabla\varphi \cdot \nabla\delta\varphi}{\|\nabla\varphi\|} \, dA \\ &= \int_A \left(1 - \frac{1}{\|\nabla\varphi\|}\right) \nabla\varphi \cdot \nabla\delta\varphi \, dA \end{aligned}$$

where we have used that

$$\begin{aligned} D_{\delta\varphi}\|\nabla(\varphi)\| &= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\|\nabla(\varphi + \epsilon\delta\varphi)\| \\ &= \frac{\nabla\varphi \cdot \nabla\delta\varphi}{\|\nabla\varphi\|} \end{aligned}$$

This is a weak form of a diffusion term, and in strong from the augmented equation is

$$\partial_t\varphi + \boldsymbol{v}\cdot\nabla\varphi - \nabla\cdot(\kappa\nabla\varphi) = -c\|\nabla\varphi\| \,.$$

where $\kappa$ is a diffusion coefficient which we write as

$$\kappa = \mu k$$

where $k$ is nondimentional and has the form

$$\begin{aligned} k &= \frac{(\|\nabla\varphi\| - 1)}{\|\nabla\varphi\|} \\ &= \left(1 - \frac{1}{\|\nabla\varphi\|}\right) \end{aligned} \tag{6.7}$$

while $\mu$ is some suitable selected constant with the dimensions distance$^2$ $\times$ time$^{-1}$.

As discussed by Luo et al. (2019) this diffusion term has the undesirable effect that the diffusion rate becomes unbounded for $\|\nabla\varphi\| = 0$. This issue can readily be addressed by instead selecting a pertubation term on the form

$$\mathcal{P} = \frac{1}{pq}\int_A (\|\nabla\varphi\|^q - 1)^p \, dA \tag{6.8}$$

for which

$$\begin{aligned} D_{\delta\varphi}\mathcal{P} &= \int_A (\|\nabla\varphi\|^q - 1)^{p-1}\,\|\nabla\varphi\|^{q-1}\,\frac{\nabla\varphi \cdot \nabla\delta\varphi}{\|\nabla\varphi\|} \, dA \\ &= \int_A (\|\nabla\varphi\|^q - 1)^{p-1}\;\|\nabla\varphi\|^{q-2}\,\nabla\varphi \cdot \nabla\delta\varphi \, dA \end{aligned}$$

giving

$$k = (\|\nabla\varphi\|^q - 1)^{p-1}\;\|\nabla\varphi\|^{q-2} \tag{6.9}$$

which is bounded for $\|\nabla\varphi\| = 0$ provided $q > 1$. The diffusion term defined by Eq. (6.9) can be both negative and positive and is an example of a *forward-and-backward* (FAB) diffusion.

The term defined by Eq. (6.8) correspondes to adding a regularisation term on the form

$$\mathcal{P} =< f - f_o \mid f - f_o >_{L^{p/2}}$$

to the energy functional, where

$$f = \|\nabla\varphi\|$$

and

$$f_0 = \|\nabla\varphi\|_0$$

is the prior, and where we have set $f_0 = 1$. Regularisation terms of this type are commonly used in inverse theory. In contrast to expression (6.7) the resulting diffusivity (Eq. 6.9) is bounded as $\|\nabla\varphi\| \to 0$. Ensuring boundness in this limits, seems to have been one of the motivations of (e.g. Li et al., 2010; Touré and Soulaïmani, 2016) for introducing their modified expressions for $\kappa$.

Another approach is to introduce a suitable non-linear diffusion term that depends on the deviation of $\varphi$ from the distance function. The diffusion will be either positive and negative, depending on whether $\|\varphi\|$ is greater or less than unity. The resulting non-linear diffusion term is referred to as forward-and-backwards (FAB) diffusion. Adding a diffusion term to the hyperbolic Eq. (6.3), gives

$$\partial_t\varphi + \boldsymbol{v}\cdot\nabla\varphi - \nabla\cdot(\kappa\nabla\varphi) = -c\|\nabla\varphi\| \,. \tag{6.10}$$

and

$$\partial_t\varphi + (\boldsymbol{v}-\boldsymbol{c})\cdot\nabla\varphi - \nabla\cdot(\kappa\nabla\varphi) = 0 \,. \tag{6.11}$$

Eqs. (6.10) and (6.11) are mathematically equivivalent.

Various suggestions for a suitable FAB diffusion have been proposed (e.g. Li et al., 2010; Touré and Soulaïmani, 2016). Li et al. (2010), for example suggest

$$\kappa = \mu\, k(\|\nabla\varphi\|) \,,$$

with

$$k(x) = \begin{cases} 1 - 1/x & \text{for} \quad x \geq 1 \\ \frac{1}{2\pi}\frac{\sin(2\pi x)}{x} & \text{for} \quad x < 1 \end{cases} \tag{6.12}$$

If $\varphi$ has the units distnace, the diffusion coefficient $\mu$ has the units distance$^2$/time. Touré and Soulaïmani (2016) suggest setting

$$\mu = \beta\frac{\|\boldsymbol{u}\|\, l^2}{2}$$

where $\beta$ is close to unity, $l$ is the local element length but this expression appears to have wrong units!

Also, why

$$\begin{aligned}\boldsymbol{u} &= \boldsymbol{v} - \boldsymbol{c} \\ &= \boldsymbol{v} + c\frac{\nabla\varphi}{\|\nabla\varphi\|}\end{aligned}$$

Like the level-set function, the calving rate, $c$, is a function $c : \mathbb{R}^2 \times \mathbb{R}$, and is, hence, defined over the whole computational domain. Although there is in glaciology currently no general consensus regarding the form of the calving rate function, $(c)$, the calving rate is often taken to be a function of either ice thickness or stress. Extending, $c$ over the whole computational domain therefore poses no particular problem.

## 6.1 Numerical implementation

In *Úa* the Level Set is evolved by solving Eq. (6.10) implicitly with respect to $\varphi$ using the Newton-Raphson method with SUPG weighting. See section 2.3 on various options for selecting $\tau$.

The NR approach, where we solve for $\Delta\varphi$ where $\varphi_1^i$ at time step 1 is updated as

$$\varphi_1^i = \varphi_1^{i-1} + \Delta\varphi \,,$$

requires the linearisation of both the non-linear $\|\nabla\varphi\|$ term and the non-linear FAB diffusion term.

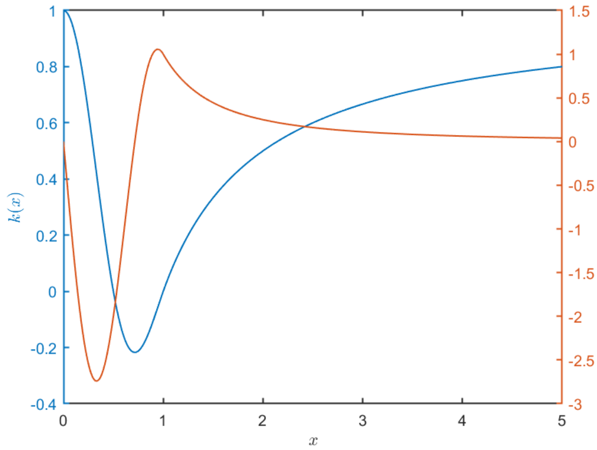



Figure 6.1: The $k(x)$ function as given by Eq. 6.12. This term introduces forward-and-backward diffusion to Eq. (6.10).

We find

$$
\begin{aligned}
D_{\delta\varphi}\|\nabla(\varphi+\epsilon\delta\varphi)\| &= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\|\nabla(\varphi+\epsilon\delta\varphi)\| \\
&= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\ \left((\partial_x\varphi+\epsilon\partial_x\delta\varphi)^2+(\partial_y\varphi+\epsilon\partial_y\delta\varphi)^2\right)^{1/2} \\
&= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\ \left((\partial_x\varphi)^2+(\partial_y\varphi)^2+2\epsilon\partial_x\varphi\,\partial_x\delta\varphi+\epsilon^2(\partial_x\delta\varphi)^2+2\epsilon\partial_y\varphi\,\partial_y\delta\varphi+\epsilon^2(\partial_y\delta\varphi)^2\right)^{1/2} \\
&= \frac{1}{2}\left((\partial_x\varphi)^2+(\partial_y\varphi)^2\right)^{-1/2}2(\partial_x\varphi\,\partial_x\delta\varphi+\partial_y\varphi\,\partial_y\delta\varphi) \\
&= \frac{\nabla\varphi\cdot\nabla\delta\varphi}{\|\nabla\varphi\|} \\
&= -\hat{\boldsymbol{n}}\cdot\nabla\delta\varphi
\end{aligned}
$$

or

$$\|\nabla(\varphi+\Delta\varphi)\| = \|\nabla\varphi\| + \hat{\boldsymbol{n}}\cdot\nabla\Delta\varphi\ .$$

The time derivative is discretized using the $\theta$ method. Discretized with time and linearised the $\|\varphi\|$ term becomes

$$\theta\|\nabla(\varphi_1^i+\Delta\varphi)\| + (1-\theta)\|\nabla\varphi_0\| = \theta\left(\|\nabla\varphi_1^i\| + \hat{\boldsymbol{n}}\cdot\nabla\Delta\varphi\right) + (1-\theta)\|\nabla\varphi_0\|\ .$$

and Eq. (6.3) therefore

$$\frac{\varphi_1^i+\Delta\varphi-\varphi_0}{\Delta t} + \theta\boldsymbol{v}_1\cdot(\nabla\varphi_1^i+\nabla\Delta\varphi) + (1-\theta)\boldsymbol{v}_0\cdot\nabla\varphi_0 = \theta\left(c_1\,\|\nabla\varphi_1^i\| + c_1\hat{\boldsymbol{n}}\cdot\nabla\Delta\varphi\right) + (1-\theta)c_0\,\|\nabla\varphi_0\|\ ,$$

or

$$\left(\frac{1}{\Delta t}+\theta\,(\boldsymbol{v}_1-\boldsymbol{c}_1)\cdot\nabla\right)\Delta\varphi = -\frac{\varphi_1^i-\varphi_0}{\Delta t} - \theta\,(\boldsymbol{v}_1-\boldsymbol{c}_1)\cdot\nabla\varphi_1^i - (1-\theta)(\boldsymbol{v}_0-\boldsymbol{c}_0)\cdot\nabla\varphi_0) \tag{6.13}$$

where the only non-linear contribution stems from the dependence of $\boldsymbol{c}_1$ on $\varphi$.

Furthermore, linearizing the FAB term

$$\kappa = \mu\,k(\|\nabla\varphi\|)\ ,$$

gives

$$k(\|\nabla(\varphi+\delta\varphi)\|) = k(\|\nabla\varphi\|) + k'(\|\nabla\varphi\|)\,\hat{\boldsymbol{n}}\cdot\nabla\delta\varphi\ .$$

and the corresponding linearised FAB diffusion terms are then added to Eq. (6.13) to produce a linearised and time-discretized version of Eq. (6.10).

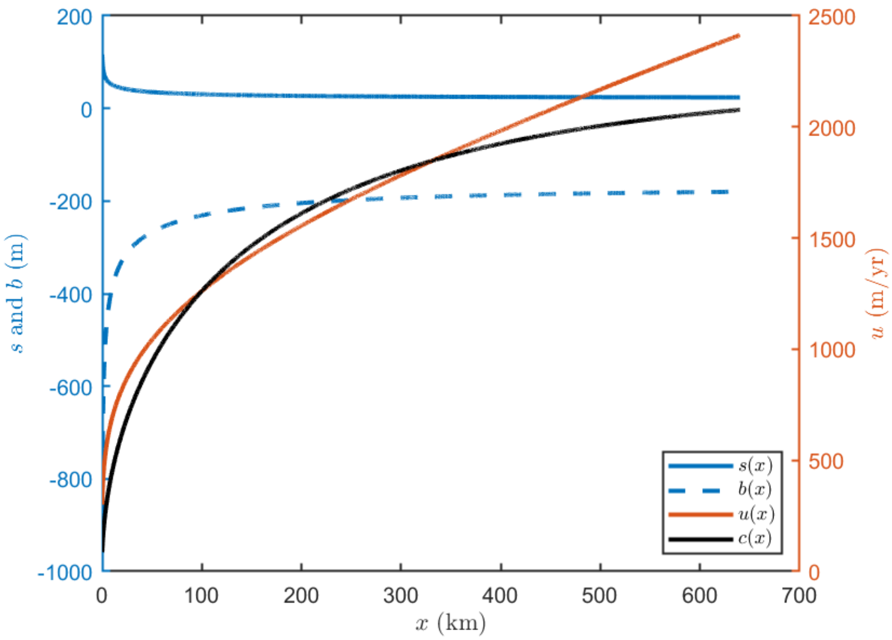



Figure 6.2: Geometry and velocities for a one-dimensional unconfined ice shelf. The grounding line is at $x = 0$ and $h_{\rm gl} = 300\,\rm m$ and $u_{\rm gl} = 1000\,\rm m\,a^{-1}$. A calving relationship on the form $c = kh^{-2}$ is shown as a black line with $k = 0.086320\,\rm km^3\,a^{-1}$. For this calving relationship and this particular value of $k$, the ice velocity $(u(x))$ and calving rate $(c(x))$ are equal at $x = x_1 = 100\,\rm km$ and at $x = x_2 = 331\,\rm km$. A calving front is table at $x = x_1$, and unstable at $x = x_2$. The ice rate factor $(A)$ is set to $A = 1.1461 \times 10^{-8}\,\rm a^{-1}\,kPa^{-3}$, which corresponds to an ice temperature of -10 degrees Celsius (Smith & Morland, 1982). Ice density $\rho = 910\,\rm kg\,m^3$, ocean density $\rho = 1030\,\rm kg\,m^3$ and gravitational acceleration $g = 9.81\,\rm m\,s^{-2}$ and constant surface mass balance of $a_s = 0.3\,\rm m\,a^{-1}$.

As commonly done in finite-elements, a weak form is obtained by forming the $L^2$ inner product of the equation to be solved with suitable form functions. The diffusion term is weighted using functions from the same space as used to expand $\varphi$ (Bubnov-Galerkin method), and the remaining terms are weighted using the streamline-upwind Petrov-Galerkin approach. The diffusion term is integrated by parts and as a result the natural boundary condition for $\varphi$ is the free outflow/inflow boundary condition. Generally, a convection-diffusion equation such as (6.10) requires a Dirichlet type boundary condition along the inflow boundary with the (natural) free-outflow boundary condition applied over the remaining parts of the boundary.

## 6.2 Testing the calving implementation

In one dimension, the position of the calving front $x_c$ is governed by the first-order system

$$\dot{x}_c = f(x_c) \; , \tag{6.14}$$

with

$$f(x) = u(x) - c(x) \; .$$

A simple test case for the calving implementation can be construed by prescribing the calving rate using the analytical solution for an unconstrained ice shelf.

Solutions for the flow of a one-dimensional unbuttressed ice shelf are derived in section 9.5 where it is showed that the velocity is given by

$$u(x) = \left( \frac{K + \gamma(q_{gl} + ax)^{n+1}}{a} \right)^{1/(n+1)} \tag{9.46}$$

and the constants $K$ and $\gamma$ are given by Eqs. (9.42) and (9.45). By prescribing the calving rate as a function of thickness both $u$ and $c$ become known quantities of distance and the position of the calving

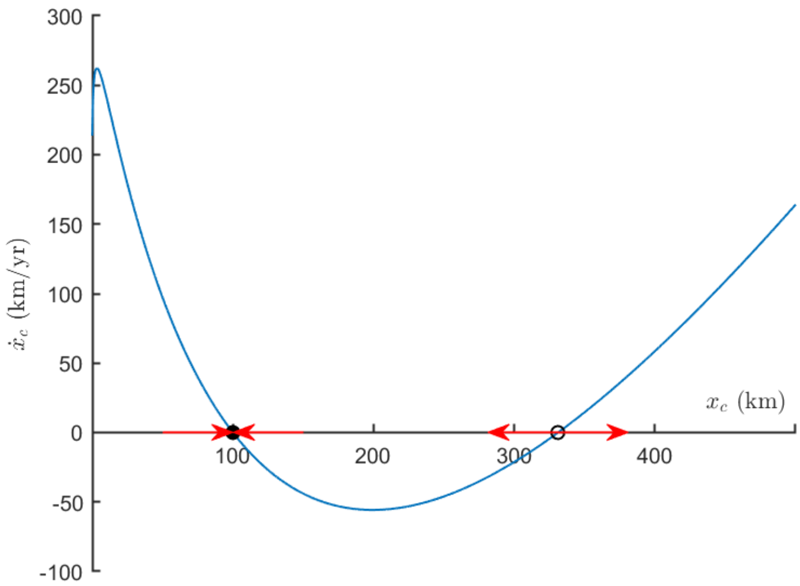



Figure 6.3: Phase portrait for calving law (6.15) with parameter values same as in Fig. 6.2. The steady-state at $x_c = x_1 = 100\,\mathrm{km}$ is stable, and the one at $x_c = x_2 = 331\,\mathrm{km}$ unstable.

front, where $u = c$, can easily be determined. For example prescribing

$$c(x) = k\, h^p \tag{6.15}$$

where $k$ and $p$ are parameters with some suitably selected values. An example is given in Fig. 6.2 where $k$ and $p$ have been selected to allow two possible steady state calving front positions at $x_1$ and $x_2$.

A fixed-point of the first-order system (6.14) is stable for $f'(x) < 0$, and unstable for $f'(x) > 0$. In this particular case, we see that a calving front is only stable if, locally around the steady-state position, the calving rate increases faster in along-flow direction than the ice velocity. Any positive perturbation in the calving-front position will then lead to a greater increase in the calving rate than in ice velocity. The calving front therefore retreats until the original steady-state position is found. The calving front position ast $x_1 = 100\,\mathrm{km}$ is therefore stable, while the one to the right at $x_2 = 331\,\mathrm{k}$ is unstable. This is also evident from the phase portrait shown in Fig. 6.3. If an initial calving front is set somewhere between $x = 0$ and $x = x_1$, the calving front will migrate with time and approach $x = x_1$. If the initial calving front is set at $x > x_2$ the calving front will grow to infinity at an accelerating rate.

Note that for there to be a stable (steady-state) calving front position, the calving rate must both equal the ice velocity at that location, and also locally increase slower than the ice velocity in the direction normal to the calving front. For a one-dimensional unbuttressed ice shelf, the ice velocity increases and the ice thickness decreases monotonically with down-stream distance. Hence, for a calving law on the form (6.15), where calving rate is a function of ice thickness, all calving front positions are unstable for any $p > 0$. No steady-state solutions for unconfined ice-shelves are therefore possible where the calving-rate increases with ice thickness, or for any calving law based on a quantity that in turns increases with ice thickness, such as (horizontal) strain rates or stresses. In Figs. 6.2 and 6.3, the exponent is set to $p = -2$ to allow for the existence of a least one stable steady state.

The calving-rate expression used in this example is not motivated by any physical considerations and is only selected to allow for a convenient testing of the numerical calving implementation against an analytical solution. It is interesting to note that many experimentally motivated calving laws suggest calving rate to be an increasing function of ice thickness, i.e. with $p > 0$.

$$x_c(t) = \int_{t=t_0}^{t} \left(u(x) - c(x)\right)\, dx + x_c(t_0)$$

We can express this as

$$\hat{\boldsymbol{n}} \cdot \nabla \left( (\boldsymbol{v} - \boldsymbol{c}) \cdot \hat{\boldsymbol{n}} \right) < 0 \qquad \text{(stable calving front)}$$

where again

$$\hat{n} = \frac{\nabla\varphi}{\|\nabla\varphi\|} \ .$$

In the particular case of a one-dimensional ice-shelf, and here assumed aligned with the $x$ axis, this condition is

$$\frac{\partial(u-c)}{\partial x} < 0 \ .$$

# Chapter 7

# Virtual crack closure technique

The *finite crack extension method*, the *crack closure technique* (CCT), and the closely related *virtual crack closure technique* (VCCT), are some of the more commonly used methods for calculating stress intensity factors (SIF) in finite-elements. Of these, the VCCT is easiest to implement and quite accurate provided the domain at the crack tip is sufficiently well resolved. The methods are all based on the original energy balance proposed by Irwin whereby the energy release is assumed equal to the work ($W$) required to close the crack. The Irwin expression is

$$W = \frac{1}{2} \int_0^l u(x)\ \sigma(l-x)\ dx\ .$$

Here $l$ is a small crack increment, $u$ the displacement normal to the crack plane (for mode I), $\sigma$ the stress, and it has been assumed that the crack is aligned with the $x$ direction. The energy release rate, $G$, is therefore

$$G = \lim_{l \to 0} \frac{1}{2l} \int_0^l u(x) \sigma(l-x)\ dx$$

This expression for the energy release rate is sometimes referred to as the *crack-closure integral method* of Irwin. (Note that the 'rate' in the term 'energy release rate', is with respect to changes in crack length. So 'rate' is here not with respect to time but with respect to distance, i.e. the crack increment).

In fracture mechanics $G$ is referred to as the crack driving force, despite not having the units of force. $G$ has the SI units of the product of stress and displacement, or $\mathrm{Pa\,m} = \mathrm{N/m} = \mathrm{J/m^2}$. The work done by external forces, $W$ is equal to the change in stored deformational energy and therefore $G$ is also equal to the derivative of the potential energy of the body with respect to the crack length.

*For purely viscous media we take a very similar viewpoint, but replace displacements and strains with velocities and strain rates, and work and potential energy with rate-of-work and change in potential energy with time.* This perfect analogy between elastic and viscous fracture mechanics, whereby

$$\begin{aligned} \epsilon &\to \dot{\epsilon} \\ d &\to u \\ J &\to C^* \end{aligned}$$

where $d$ is displacement and $J$ and $C^*$ are integrals (see below) is discussed in detail by Gross and Seelig (2006).

Calculating the work of external forces in FE is a simple summation exercise

$$\dot{W} = \frac{1}{2} \sum_i (F_x u + F_u v)_i$$

where the sum is over nodes and $F_x$ and $F_y$ the equivalent nodal forces, and now $u$ and $v$ are the $x$ and $y$ velocity components. Most FE programs determine the nodal forces as a part of the matrix assembly (In Úa these are calculated as the SSA equation is assembled).

In the finite crack extension method, the work $W$ is calculated for several different crack extensions $l$. For example using central differences the crack driving force is evaluated as

$$\dot{G} \approx \frac{\dot{W}_{+l} - \dot{W}_{-l}}{2l}$$

where here the crack has been extended and closed by the distance $l$. In principle, one can calculate $W$ for a range of $l$ values and plot $W$ as a function of $l$ as done in Antunes et al. (1999) to determine how well the limit $l \to 0$ is approximated. The SI units of $\dot{G}$ are the same as those of the produce of stress and velocity, or $\mathrm{Pa\,m\,s^{-1}} = \mathrm{N\,m^{-1}\,s^{-1}}$ and $\dot{G}$ can be thought of as the crack driving force per time unit.

The VCCT method is a clever trick where one uses the (internal) equivalent nodal forces between elements in contact ahead of the crack tip and the relative nodal displacements behind the crack tip to evaluate $W$. This eliminates the the need to open the crack by the distance $l$ (but requires a somewhat different internal estimate of the nodal forces as these will always be close to zero for a structure in mechanical equilibrium). For viscous material we use the jump in velocity across the crack instead of the nodal displacements.

So the general idea is to use the discontinuity in velocity normal to the crack together and the equivalent nodal loads ahead of the crack to estimate the strain-rate energy release rate, $\dot{G}$, as the crack is extended.

Typically the criteria for crack growth in a elastic medium is

$$G > G^*$$

where $G^*$ is some material parameter. Here I'm not primarily interested in this 'on/off' question and instead write

$$\frac{dl}{dt} = B\left(\frac{\dot{G}}{\dot{G}^*}\right)^q$$

where $B$, $G^*$, and $q$ are some parameters.

## 7.1 $J$ and $C^*$ integrals for elastic and viscous fracture

Elastic $J$ integral

$$J = \int_\Gamma (W dy - T_i d_{i,1} dx)$$

where $d$ are displacements, and $W$ the strain energy density

$$W = \int_0^\epsilon \sigma_{pq}\, d\epsilon_{qp}$$

$T_i$ are the components of the traction vector, $T_i = \sigma_{ij} n_j$ The integration path $\Gamma$ a continuous and differentiable curve surrounding the crack tip, not including the edges of the crack. In the absence of body forces, or if the body forces are acting out of the plane, $J = 0$ for any closed curve. (for prof see: https://en.wikipedia.org/wiki/J-integral). Some further details related to the calculation of $J$ integral in FE are found in IPCC (2019).

For creep fracture it is common to use instead the creep $C^*$ integral

$$C^* = \int_\Gamma \left(\dot{W} dy - T_i u_{i,1}\right)$$

where $\dot{W}$ is the stress power

$$\dot{W} = \int_0^{\dot{\epsilon}} \sigma_{pq}\, d\dot{\epsilon}_{pq}$$

and $u_i$ the velocity components. The $C^*$ is path independent and completely analogous to the $J$ integral, but with displacements replaced by velocities and energy density with energy dissipation rate. $C^*$ is the rate equivalent of the $J$ contour integral.

$C$ represents the power difference as the crack length, $l$, is increased incrementally

$$C^* = -\frac{1}{w}\frac{dU}{dl}$$

$U$ is the stress-power dissipation rate.

The solution close to the crack tip, for a stationary crack, is given by the HRR-field

$$\sigma \sim \left(\frac{C^*}{Ar}\right)^{1/(n+1)}$$
$$\dot{\epsilon} \sim A\left(\frac{C^*}{Ar}\right)^{n/(n+1)}$$
$$u \sim A^{1/(n+1)}\,(C^*)^{n/(n+1)}\; r^{1/(n+1)}$$

as $r \to 0$, where $r$ is the radial distance from the crack tip (Saxena, 1993, 2015; Gross and Seelig, 2006). and $C^*$ can be therefore related to the strength of the stress singularity

$$C^* = \lim_{r\to 0} \frac{u^{(n+1)/n}}{(Ar)^{1/n}} \tag{7.1}$$

In principle this provides a way of calculating $C^*$ from the SSA by evaluating this limit numerically, or from plotting $\dot{\epsilon}$ and/or $u$ as a function of radial distance, although in practice this may require very finely resolved mesh. Note that Eq. (7.1) has the right physical dimensions, i.e. the dimensions of $C^*$ are those of stress times velocity or $\mathrm{Pa\,m\,s^{-1}}$.

Creep crack growth rate is then often assumed to be given by

$$\frac{dl}{dt} = B\,(C^*)^q$$

where $B$ and $q$ as some material constants.

# Chapter 8

# Further technical FE implementation details

## 8.1 Only the (fully) floating condition as a natural boundary condition

Here I am ignoring possible gradients in density and the treatment of the boundary term only includes the fully floated case as a natural condition.

Note that

$$\begin{aligned}\rho g h\,\partial_x s &= \rho g h\,\partial_x s + \frac{1}{2}\varrho g\,\partial_x h^2 - \rho g h(1-\rho/\rho_o)\,\partial_x h\\ &= \frac{1}{2}\varrho g\,\partial_x h^2 + \rho g h\,\partial_x(s - S - (1-\rho/\rho_o)\,h)\end{aligned}$$

hence

$$\rho g h\,\partial_x s = \frac{1}{2}\varrho g\,\partial_x h^2 + \rho g h\,\partial_x s^{'} \tag{8.1}$$

with

$$s^{'} := s - S - (1-\rho/\rho_o)h$$

and

$$\varrho = \rho(1-\rho/\rho_o),$$

The field equations can therefore also be written as

$$\begin{aligned}\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) + \partial_y(h\eta(\partial_x v + \partial_y u)) - \beta^2 u &= \rho g h(\partial_x s^{'}\cos\alpha - \sin\alpha) + \frac{1}{2}\varrho g\cos\alpha\,\partial_x h^2,\\ \partial_y(4h\eta\partial_y v + 2h\eta\partial_x u) + \partial_x(h\eta(\partial_y u + \partial_x v)) - \beta^2 v &= \rho g h\partial_y s^{'}\cos\alpha + \frac{1}{2}\varrho g\cos\alpha\,\partial_y h^2 \mid\end{aligned}$$

### 8.1.1 Remark

To see that the right-hands sides of (1.81) and (8.1) i.e.

$$\begin{aligned}\rho g h\,\partial_x s &= \frac{1}{2}\varrho g\,\partial_x h^2 + \rho g h\,\partial_x s^{'}\\ &= \frac{1}{2}g\partial_x(\rho h^2 - \rho_o d^2) + g(\rho h - \rho_o d)\,\partial_x b\end{aligned}$$

are equal (ignoring spatial gradients in density) we consider the three cases:

1. Fully floating: In that case $s^{'} = 0$ and $\rho h = \rho_o d$ and both sides are equal.
2. Fully grounded: We have $d = 0$

$$\begin{aligned}\frac{1}{2}g\partial_x(\rho h^2 - \rho_o d^2) + g(\rho h - \rho_o d)\,\partial_x b &= \frac{1}{2}g\rho\partial_x h^2 + g\rho h\,\partial_x b\\ &= \rho g h\partial_x s\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\varrho g\,\partial_x h^2 + \rho g h\,\partial_x s' &= \frac{1}{2}\varrho g\,\partial_x h^2 + \rho g h\,\partial_x (s - S - (1-\rho/\rho_o)h)\\ &= \rho g h \partial_x s\end{aligned}$$

3. Partly floating:

$$\begin{aligned}\frac{1}{2} g\partial_x(\rho h^2 - \rho_o d^2) + g(\rho h - \rho_o d)\,\partial_x b &= \rho g h\,\partial_x h - \rho_o g d\,\partial_x d + g(\rho h - \rho_o d)\partial_x b\\ &= \rho g h\,\partial_x s - \rho_o g d\,\partial_x d - \rho_o g d\,\partial_x b\\ &= \rho g h\,\partial_x s - \rho_o g d(\partial_x d + \partial_x b)\\ &= \rho g h\,\partial_x s - \rho_o g d(\partial_x S - \partial_x b + \partial_x b)\\ &= \rho g h\,\partial_x s\end{aligned}$$

### 8.1.2 FE formulation

**$x$ direction**

$$\int_\Omega (\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) - \frac{1}{2}\varrho g\cos\alpha\,\partial_x h^2 + \partial_y(h\eta(\partial_x v + \partial_y u)) - t_{bx} - \rho g h(\partial_x s'\cos\alpha - \sin\alpha))N\,dx\,dy = 0$$

with Weertman type Neumann BC on $\Gamma_2$

$$(4h\eta\partial_x u + 2h\eta\partial_y v)n_x + \eta h(\partial_x v + \partial_y u)n_y = \frac{1}{2}\rho(1-\rho/\rho_o)gh^2 n_x$$

Green's theorem used to get rid of second derivatives gives

$$\begin{aligned}-&\int_\Omega((4h\eta\partial_x u + 2h\eta\partial_y v)\partial_x N - \frac{1}{2}\varrho g\cos\alpha\,h^2\partial_x N + h\eta(\partial_x v + \partial_y u)\partial_y N)\,dx\,dy\\ &-\int_\Omega (t_{bx} + \rho g h(\partial_x s'\cos\alpha - \sin\alpha)N\,dx\,dy\\ &+\int_\Gamma((4h\eta\partial_x u + 2h\eta\partial_y v - \frac{1}{2}\varrho g\cos\alpha h^2)n_x + h\eta(\partial_x v + \partial_y u)n_y)N\,d\Gamma = 0\end{aligned}$$

If the von Neumann boundary condition is of Weertman type, the boundary integral along $\Gamma_2$ is equal to zero (for $\alpha = 0$), and zero on the remaining part of the boundary if we set the weight functions to zero and determine the values of the unknowns using Dirichlet boundary conditions.

We are left with

$$\begin{aligned}-\int_\Omega (h\eta(4\partial_x u + 2\partial_y v)\partial_x N + h\eta(\partial_x v + \partial_y u)\partial_y N)\,dx\,dy - \int_\Omega t_{bx}N\,dx\,dy\\ = \rho g\int_\Omega h((\partial_x s - (1-\rho/\rho_o)\partial_x h)\cos\alpha - \sin\alpha)N\,dx\,dy - \frac{1}{2}\varrho g\cos\alpha\int_\Omega h^2\,\partial_x N\,dx\,dy\end{aligned}$$

**$y$ direction**

$$\int_\Omega \partial_y((4h\eta\partial_y v + 2h\eta\partial_x u) - \frac{1}{2}\varrho g\cos\alpha\,\partial_y h^2 + \partial_x(h\eta(\partial_y u + \partial_x v)) - t_{by} - \rho g h\partial_y s'\cos\alpha)N\,dx\,dy$$

with Weertman type boundary condition

$$\eta h(\partial_x v + \partial_y u)n_x + (4\eta h\partial_y v + 2\eta h\partial_x u)n_y = \frac{1}{2}\rho(1-\rho/\rho_o)gh^2 n_y$$

We have

$$\begin{aligned}-&\int_\Omega((4h\eta\partial_y v + 2h\eta\partial_x u)\partial_y N - \frac{1}{2}\varrho g\cos\alpha\,h^2\partial_y N + h\eta(\partial_y u + \partial_x v)\partial_x N)\,dx\,dy\\ &-\int_\Omega(t_{by} + \rho g h\partial_y s'\cos\alpha)N\,dx\,dy\\ &+\int_\Gamma((4h\eta\partial_y v + 2h\eta\partial_x u - \frac{1}{2}\varrho g\cos\alpha\,h^2)n_y + \eta h(\partial_y u + \partial_x v)n_x)N\,d\Gamma = 0 \qquad (8.2)\end{aligned}$$

Again we can ignore the boundary integral as it is identically equal to zero.

$$\begin{aligned}
&-\int_\Omega ((4h\eta\partial_y v + 2h\eta\partial_x u)\partial_y N + h\eta(\partial_y u + \partial_x v)\partial_x N)\,dx\,dy - \int_\Omega t_{by} N\,dx\,dy \\
&= \quad \rho g\cos\alpha\int_\Omega h(\partial_y s - (1-\rho/\rho_o)\partial_y h)N\,dx\,dy - \frac{1}{2}\varrho g\cos\alpha\int_\Omega h^2\partial_y N\,dx\,dy \qquad (8.3)
\end{aligned}$$

### 8.1.3 2HD FE diagnostic equation written in terms of $h$ (suitable for fully coupled approach)

Where the ice is afloat, $s - S = (1-\rho/\rho_o)\,h$ and $s^{'} = 0$, hence

$$s^{'} = \begin{cases} s - S - (1-\rho/\rho_o)\,h, & \text{if}\, h > h_f \\ 0, & \text{if}\, h \le h_f \end{cases}$$

i.e.

$$s^{'} := \mathcal{H}(h-h_f)(s - S - (1-\rho/\rho_o)h)$$

where

$$h_f := (S-B)\rho_o/\rho$$

We can also write $s^{'}$ as

$$\begin{aligned}
s^{'} &= \mathcal{H}(h-h_f)(s - S - (1-\rho/\rho_o)h) \\
&= \mathcal{H}(h-h_f)(h\rho/\rho_o + s - S - h) \\
&= \mathcal{H}(h-h_f)(h\rho/\rho_o + h + b - S - h) \\
&= \mathcal{H}(h-h_f)(h\rho/\rho_o + h + B - S - h) \\
&= \mathcal{H}(h-h_f)(h\rho/\rho_o + B - S) \\
&= \mathcal{H}(h-h_f)(h\rho/\rho_o - H)
\end{aligned}$$

i.e.

$$s^{'}(x) = \mathcal{H}(h-h_f)(\rho/\rho_o h - H) \qquad (8.4)$$

where we used the fact that $b = B$ whenever $\mathcal{H}(h-h_f) = 1$. This expression for $s^{'}$ is needed for linearisation around $h$.

The field equations can therefore be written as before, i.e. as

$$\begin{aligned}
\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) + \partial_y(h\eta(\partial_x v + \partial_y u)) - \beta^2 u &= \rho g h(\partial_x s^{'}\cos\alpha - \sin\alpha) + \frac{1}{2}\varrho g\cos\alpha\,\partial_x h^2, \\
\partial_y(4h\eta\partial_y v + 2h\eta\partial_x u) + \partial_x(h\eta(\partial_y u + \partial_x v)) - \beta^2 v &= \rho g h\partial_y s^{'}\cos\alpha + \frac{1}{2}\varrho g\cos\alpha\,\partial_y h^2,
\end{aligned}$$

but the linearisation with respect to $h$ needed in a fully coupled approach requires (8.4).

The FE formulation for the prognostic equation is the $\theta$ method, i.e.

$$R_p^h = \int_\Omega \{\frac{1}{\Delta t}(h_1-h_0)+\theta\partial_x(u_1h_1)+(1-\theta)\partial_x(u_0h_0)+\theta\partial_y(v_1h_1)+(1-\theta)\partial_y(u_0h_0)-a\}N_p\,dx\,dy = 0 \quad (8.5)$$

where $0 \le \theta \le 1$.

$$a := a_s + a_b$$

## 8.2 Element integrals

$$x = x_p N_P(\xi,\eta)$$

For 3-node element:

Nodal $u$ displacement vector of a particular element

$$u_e := \begin{pmatrix} u1 \\ u2 \\ u3 \end{pmatrix}$$

$$\text{fun} := \begin{pmatrix} N_1(\xi,\eta) \\ N_2(\xi,\eta) \\ N_3(\xi,\eta) \end{pmatrix}$$

$$u(x,y) = u_e^T \text{ fun}$$

$$\text{der} = \begin{pmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} \end{pmatrix}$$

$$\text{coo} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_4 \end{pmatrix}$$

$$\boldsymbol{J} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

$$\boldsymbol{J}^{-1} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial y} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} = \boldsymbol{J} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \boldsymbol{J}^{-1} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix}$$

$$\boldsymbol{J} = \begin{pmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_4 \end{pmatrix}$$

$$\boldsymbol{J} = \text{der coo}$$

$$\boldsymbol{D} = \begin{pmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \end{pmatrix} = \boldsymbol{J}^{-1} \begin{pmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} \end{pmatrix}$$

$$\boldsymbol{D} = \boldsymbol{J}^{-1} \text{ der}$$

Change of integral, example:

$$\int_{A_e} u_p N_p(x,y) N_q(x,y)\, dx\, dy = \int_{\Delta} u_p N_p(\xi,\eta) N_q(\xi,\eta) \det \boldsymbol{J}\, d\xi\, d\eta$$

Example:

$$\int \partial_x u\, \partial_x N\, dx\, dy = (\int D_{1p} D_{1q}\, |\boldsymbol{J}|\, d\eta\, d\xi)\, u_p$$

## 8.3 Edge integrals

We have integrals on the form

$$\int_\Gamma u(x,y)N(x,y)\, n_x \, d\Gamma$$

with

$$\boldsymbol{n} = \begin{pmatrix} n_x \\ n_y \end{pmatrix}$$

If the boundary is parameterised such that $(x,y) = (x(\gamma), y(\gamma))$ as $\gamma$ goes form 0 to 1 then

$$\boldsymbol{n} = \frac{1}{\sqrt{(\partial_\gamma x)^2 + (\partial_\gamma y)^2}} \begin{pmatrix} -\partial_\gamma y \\ \partial_\gamma x \end{pmatrix}$$

and

$$d\Gamma = \sqrt{(\partial_\gamma x)^2 + (\partial_\gamma y)^2}\; d\gamma$$

and therefore

$$\boldsymbol{n}\, d\Gamma = \begin{pmatrix} -\partial_\gamma y \\ \partial_\gamma x \end{pmatrix} d\gamma$$

Hence

$$\int_\Gamma u(x,y)N(x,y)\, n_x \, d\Gamma = -\int_0^1 u(x(\gamma), y(\gamma))\, N(x(\gamma), y(\gamma))\, \partial_\gamma y \, d\gamma$$

### 8.3.1 Edge 12

For edge 12, $\eta = 0$. I parameterise it as $(\xi, \eta) = (1-\gamma, 0)$ as this takes me from node 1 to node 2 in clockwise direction (that is how I order the nodes, most FE do it the other way around)

$$x = x_p N_p(1-\gamma, 0) \quad \text{and} \quad y = y_P N_P(1-\gamma, 0)$$

The normal is

$$\boldsymbol{n}\, d\Gamma = \begin{pmatrix} -\partial_\gamma y \\ \partial_\gamma x \end{pmatrix} d\gamma$$

and

$$\begin{aligned} \frac{1}{\partial\gamma} &= \frac{1}{\partial\xi}\frac{\partial\xi}{\partial\gamma} + \frac{1}{\partial\eta}\frac{\partial\eta}{\partial\gamma} \\ &= -\frac{1}{\partial\xi} \end{aligned}$$

and therefore

$$\partial_\gamma y = -\partial_\xi y = -J_{12}$$

and

$$\boldsymbol{n}\, d\Gamma = \begin{pmatrix} \partial_\xi y \\ -\partial_\xi x \end{pmatrix} d\gamma = \begin{pmatrix} J_{12} \\ -J_{11} \end{pmatrix} d\gamma$$

or simply

$$\boldsymbol{n}\, d\Gamma = \begin{pmatrix} y_q \, \partial_\xi N_q(\xi, 0) \\ -x_q \, \partial_\xi N_q(\xi, 0) \end{pmatrix} d\gamma$$

For the linear triangle, for example, I get

$$\begin{aligned} x &= x_1\xi + x_2(1-\xi) + x_3\,\eta \\ &= x_1(1-\gamma) + x_2(1-(1-\gamma)) + x_3\, 0 \\ &= x_1(1-\gamma) + x_2\,\gamma \end{aligned}$$

$$y = y_1(1-\gamma) + y_2\,\gamma$$

and

$$\begin{aligned} \partial_\gamma x &= -x_1 + x_2 \\ \partial_\gamma y &= -y_1 + y_2 \end{aligned}$$

and a normal

$$\boldsymbol{n}\, d\Gamma = \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix} d\gamma$$

### 8.3.2 Edge 23

For edge 23, $\xi = 0$. I parameterise the edge as $(\xi, \eta) = (0, \gamma)$, this takes me from node 2 to 3 as $\gamma$ varies from 0 to 1.

$$\boldsymbol{n}\, d\Gamma = \left( \begin{array}{c} -\partial_\gamma y \\ \partial_\gamma x \end{array} \right) d\gamma$$

$$\partial_\gamma = \partial_\eta$$

and therefore

$$\boldsymbol{n}\, d\Gamma = \begin{pmatrix} -\partial_\eta y \\ \partial_\eta x \end{pmatrix} = \begin{pmatrix} -J_{22} \\ J_{21} \end{pmatrix} d\gamma$$

For the linear triangle, for example, I get

$$x = x_1 0 + x_2(1-\gamma) + x_3\gamma$$

$$y = y_1 0 + y_2(1-\gamma) + y_3\gamma$$

and

$$\partial_\gamma x = -x_2 + x_3$$

$$\partial_\gamma y = -y_2 + y_3$$

and a normal

$$\boldsymbol{n}\, d\Gamma = \left( \begin{array}{c} y_2 - y_3 \\ x_3 - x_2 \end{array} \right) d\gamma$$

### 8.3.3 Edge 32

For edge 32 is parameterised as $(\xi, \eta) = (\gamma, 1-\gamma)$, and

$$\boldsymbol{n}\, d\Gamma = \left( \begin{array}{c} -\partial_\gamma y \\ \partial_\gamma x \end{array} \right) d\gamma$$

and

$$\begin{aligned} \frac{1}{\partial\gamma} &= \frac{1}{\partial\xi}\frac{\partial\xi}{\partial\gamma} + \frac{1}{\partial\eta}\frac{\partial\eta}{\partial\gamma} \\ &= \frac{1}{\partial\xi} - \frac{1}{\partial\eta} \end{aligned}$$

and therefore

$$\boldsymbol{n}\, d\Gamma = \left( \begin{array}{c} -y_q(\partial_\xi N_q - \partial_\eta N_q) \\ x_q(\partial_\xi N_q - \partial_\eta N_q) \end{array} \right) d\gamma = \begin{pmatrix} J_{22} - J_{12} \\ J_{11} - J_{21} \end{pmatrix} d\gamma$$

For the linear triangle, for example, I get

$$x = x_1\gamma + x_2(1 - \gamma - (1-\gamma)) + x_3(1-\gamma)$$

or

$$x = x_1\gamma + x_3(1-\gamma)$$

$$y = y_1\gamma + y_3(1-\gamma)$$

and

$$\partial_\gamma x = x_1 - x_3$$

$$\partial_\gamma y = y_1 - y_3$$

and a normal

$$\boldsymbol{n}\, d\Gamma = \left( \begin{array}{c} y_3 - y_1 \\ x_1 - x_3 \end{array} \right) d\gamma$$

## 8.4 Various directional derivatives

### 8.4.1 Directional derivative of draft with respect to ice thickness

For implicit forward time integration with respect to $h$ using the NR method, various directional derivatives with respect to $h$ must be calculated.

Using Eq. (1.61) we find that the directional derivative of the draft $d$ with respect to $h$ is

$$\begin{aligned} Dd(h)[\Delta h] &= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}d(h+\epsilon\Delta h) \\ &= \mathcal{H}(h_f-h)\rho\,\Delta h\,/\rho_o - \rho h\,\delta(h_f-h)\,\Delta h\,/\rho_o + \mathcal{H}(H)H\,\delta(h-h_f)\,\Delta h \\ &= \frac{\rho}{\rho_o}\mathcal{H}(h_f-h)\,\Delta h + \delta(h-h_f)(H-\frac{\rho}{\rho_o}h)\,\Delta h \end{aligned}$$

When integrated the second term in the above expression integrates to zero, because where $h = h_f$ we have $H = \rho h/\rho_o$, hence[1]

$$Dd(h)[\Delta h] = \frac{\rho}{\rho_o}\mathcal{H}(h_f-h)\,\Delta h. \tag{8.6}$$

Using Eq. (8.6) the directional derivative of

$$D(\frac{1}{2}g(\rho h^2-\rho_o d^2))[\Delta h]$$

with respect to perturbation in $h$ is found to be

$$\begin{aligned} D\left(\frac{1}{2}g(\rho h^2-\rho_o d^2)\right)[\Delta h] &= g(\rho(h\,\Delta h - \rho_o d\frac{\rho}{\rho_o}\mathcal{H}(h_f-h)\,\Delta h) \\ &= \rho g\,(h-\mathcal{H}(h_f-h)\,d)\,\Delta h \end{aligned}$$

The directional derivative of $g(\rho h-\rho_o d)\partial_x b$ with respect to $h$ is found to be

$$D\left(g(\rho h-\rho_o d)\partial_x b\right)[\Delta h] = \rho g\,\mathcal{H}(h-h_f)\,\partial_x B\,\Delta h. \tag{8.7}$$

To see this first notice that using Eq. (1.61)

$$\begin{aligned} g(\rho h-\rho_o d)\partial_x b &= g(\rho h - \mathcal{H}(h_f-h)\rho h - \rho_o\mathcal{H}(H)\mathcal{H}(h-h_f)H)\partial_x b \\ &= g(\rho h-(1-\mathcal{H}(h-h_f))\rho h - \rho_o\mathcal{H}(H)\mathcal{H}(h-h_f)H)\partial_x b \\ &= g(\rho h + \mathcal{H}(h-h_f)(\rho h-\rho_o H^+) - \rho h)\partial_x b \\ &= g\mathcal{H}(h-h_f)(\rho h-\rho_o H^+)\partial_x b \\ &= g\mathcal{H}(h-h_f)(\rho h-\rho_o H^+)\partial_x B. \end{aligned}$$

Using the above expression we now calculate the directional derivative of $g(\rho h-\rho_o d)\partial_x b$ with respect to $h$ and find

$$\begin{aligned} D\left(g(\rho h-\rho_o d)\partial_x b\right)[\Delta h] &= \mathcal{H}(h_f-h)\rho g\,\partial_x B\,\Delta h + g\,\delta(h-h_f)(\rho h-\rho_o H^+)\partial_x B\,\Delta h \\ &= \left(\rho\mathcal{H}(h-h_f)+\delta(h-h_f)(\rho h-\rho_o H^+)\right)\,g\,\partial_x B\,\Delta h \\ &= \rho g\,\mathcal{H}(h-h_f)\,\partial_x B\,\Delta h \end{aligned}$$

where the last step is correct when the expression is evaluated under an integral, thus demonstrating the correctness of Eq. (8.7).

(Not sure where to put this, but keep it here for the time being.) The lower ice surface is related to thickness through

$$b = \mathcal{H}(h-h_f)B + \mathcal{H}(h_f-h)(S-\rho h/\rho_o)$$

and therefore

$$\partial_h b = \delta(h-h_f)B - \delta(h_f-h)(S-\rho h/\rho_o) - \mathcal{H}(h_f-h)\rho/\rho_o$$

[1] This argument does not hold if the Heaviside function and the Dirac delta functions are approximated. In that case the full expression must be used. Important for getting quadratic convergence in NR.

and

$$\partial_x b = \partial_h b \, \partial_x h = \delta(h - h_f) B \, \partial_x h - \delta(h_f - h)(S - \rho h/\rho_o) \, \partial_x h - \mathcal{H}(h_f - h)\rho \, \partial_x h/\rho_o$$

and assuming

$$\frac{\partial^2 b}{\partial h \partial x} = \frac{\partial^2 b}{\partial x \partial h}$$

If $f(x)$ is a test function

$$\partial_x \int \delta(x) \, \partial_x f(x) \, dx = -\partial_x \int \mathcal{H}(x) f(x) \, dx = -f(0) - \int \mathcal{H}(x) \, \partial_x f(x) \, dx$$

### 8.4.2 Linearisation of the 2HD forward problem needed for the adjoint method

For the adjoint method we need

$$\mathrm{K}^{xC} \, \Delta C_q := -D\boldsymbol{F}_x(\boldsymbol{u}_1^i, \boldsymbol{v}_1^i, \boldsymbol{h}_1^i)[\Delta C_q]$$

Here $\Delta C_q$ is the nodal value itself, the perturbation in $C$ is $\Delta C_q N_q$ (no summation).

$$[\mathrm{K}^{xC}]_{pq} = \frac{1}{m} \int_\Omega \mathcal{H}(h - h_f) C^{-1/m-1} \, \|\boldsymbol{v}_{xy}\|^{1/m-1} \, u N_p \, N_q \, dxdy \tag{8.8}$$

and

$$[\mathrm{K}^{yC}]_{pq} = \frac{1}{m} \int_\Omega \mathcal{H}(h - h_f) C^{-1/m-1} \, \|\boldsymbol{v}_{xy}\|^{1/m-1} \, v N_p \, N_q \, dxdy \tag{8.9}$$

where

$$\boldsymbol{v}_{xy} = \begin{pmatrix} u \\ v \end{pmatrix}$$

and

$$\beta^2 = C^{-1/m} \|\boldsymbol{v}_{xy}\|^{1/m-1}$$

$$\mathrm{K}^C = \begin{bmatrix} \mathrm{K}^{xC} \\ \mathrm{K}^{yC} \end{bmatrix}$$

is $2N \times n$ where $N$ are degrees of freedom.

If the cost function $I$ is calculated using FE type inner product , then the gradient of the cost function is then given by

$$\begin{aligned} \frac{\partial I}{\partial C_q} &= [\mathrm{K}^C]_{pq} \lambda_p \\ &= \frac{1}{m} \int_\Omega \mathcal{H}(h - h_f) C^{-1/m-1} \, \|\boldsymbol{v}_{xy}\|^{1/m-1} \, (u\lambda + v\mu) \, N_q \, dxdy \end{aligned}$$

If $I$ is calculated as a discrete sum over values then

$$\frac{\partial I}{\partial C_q} = \frac{1}{m} \mathcal{H}(h - h_f) C^{-1/m-1} \, \|\boldsymbol{v}_{xy}\|^{1/m-1} \, (u\lambda + v\mu)$$

Note: Perturbing on particular nodal value in $C = C_r N_r$ can be written as

$$(C_r + \Delta C \delta_{rp}) N_r$$

$C = C_r N_r$, $\Delta C = \Delta \hat{C} N_q$ with $\Delta \hat{C} = \Delta C_q$

$$\begin{aligned} (C + \Delta C)^m &= (C_r N_r + \Delta \hat{C} N_q)^m \\ &= (C_r N_r + \Delta \hat{C} N_q (C_j N_j)/(C_i N_i))^m \\ &= (C_r N_r)^m (1 + \Delta \hat{C} N_q/(C_i N_i))^m \\ &\approx (C_r N_r)^m (1 + m \Delta \hat{C} N_q/(C_i N_i)) \\ &= (C_s N_s)^m + m (C_s N_s)^{m-1} \Delta \hat{C} N_q \\ &= (C_s N_s)^m + m (C_s N_s)^{m-1} \Delta C \end{aligned}$$

I can write the perturbation as

$$K\Delta\hat{C} = m(C_sN_s)^{m-1}N_q\Delta\hat{C}$$

where

$$K = m(C_sN_s)^{m-1}N_q$$

## 8.5 FE formulation and linearisation for the 1HD Problem

### 8.5.1 Field equations and boundary conditions (1HD)

$$4\partial_x(h\eta\partial_x u) - \beta^2 u = \rho g h(\partial_x s \cos\alpha - \sin\alpha)$$

$$\beta^2 = C^{-1/m}\|u\|^{1/m-1}$$

with the sliding law written on the form

$$u = C|t_{bx}|^{m-1}\, t_{bx}$$

We have

$$t_{bx} = C^{-1/m}|u|^{1/m-1}\, u.$$

Glen's flow law is

$$\dot{\epsilon}_{ij} = A\tau^{n-1}\tau_{ij},$$

where

$$\tau = \sqrt{\tau_{ij}\tau_{ij}/2}$$

The flow law can also be written as

$$\tau_{ij} = A^{-1/n}\,\dot{\epsilon}^{(1-n)/n}\,\dot{\epsilon}_{ij},$$

where

$$\dot{\epsilon} = \sqrt{(\partial_x u)^2} = |\partial_x u|$$

If we write

$$\tau_{ij} = 2\eta\dot{\epsilon}_{ij}$$

then $\eta$ is the effective viscosity given by

$$\eta = \frac{1}{2}A^{-1/n}\,\dot{\epsilon}^{(1-n)/n} = \frac{1}{2}A^{-1/n}\,\|\partial_x u\|^{(1-n)/n}$$

## 8.6 Linearisation of field equations (1HD)

In the non-linear case using Newton's method I need to linearise the equation.

Linearised with respect to $u$ by writing $u = \bar{u} + \Delta u$, $\eta = \bar{\eta} + \partial_u\eta\Delta u$, and $\beta^2 = \bar{\beta}^2 + \partial_u(\beta^2)\Delta u$

$$4\partial_x(h(\bar{\eta} + \partial_u\eta\Delta u)\partial_x(\bar{u} + \Delta u)) - (\bar{\beta}^2 + D\beta^2(u)[\Delta u])(\bar{u} + \Delta u) = \rho g h(\partial_x s\cos\alpha - \sin\alpha)$$

or

$$4\partial_x(h(\bar{\eta}\partial_x\bar{u} + \partial_x\bar{u}\,D\eta(u)[\Delta u] + \bar{\eta}\partial_x\Delta u)) - \bar{\beta}^2\bar{u} - \bar{\beta}^2\Delta u - \bar{u}\,D\beta^2(u)[\Delta u] = \rho g h(\partial_x s\cos\alpha - \sin\alpha) \quad (8.10)$$

where

$$\eta = \frac{1}{2}A^{-1/n}\,\dot{\epsilon}^{(1-n)/n} = \frac{1}{2}A^{-1/n}\,|\partial_x u|^{(1-n)/n}$$

$$\begin{aligned}
D\eta(u)[\Delta u] &= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\eta(u+\epsilon\Delta u) && (8.11)\\
&= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\frac{1}{2}A^{-1/n}\,\|\partial_x(u+\epsilon\Delta u)\|^{(1-n)/n}\\
&\underset{\partial_x u>0}{=} \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\frac{1}{2}A^{-1/n}\,(\partial_x u+\epsilon\partial_x\Delta u))^{(1-n)/n}\\
&= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\frac{1}{2}A^{-1/n}\,((\partial_x u)^{(1-n)/n}+(\partial_x u)^{(1-2n)/n}\frac{1-n}{n}\epsilon\partial_x\Delta u+\ldots))\\
&= \frac{1}{2}A^{-1/n}\,((\partial_x u)^{(1-2n)/n}\frac{1-n}{n}\partial_x\Delta u)\\
&= \frac{1-n}{2n}A^{-1/n}\,(\partial_x u)^{(1-2n)/n}\partial_x\Delta u && (8.12)
\end{aligned}$$

Doing the same for $\partial_x u<0$ shows that

$$D\eta(u)[\Delta u]=\frac{1-n}{2n}A^{-1/n}\,\|\partial_x u\|^{(1-2n)/n}\text{sign}(\partial_x u)\,\partial_x\Delta u$$

(old result)

$$\partial_u\eta=\frac{1-n}{2n}\,A^{-1/n}\|\partial_x u\|^{1/n-2}\,\text{sign}(\partial_x u)$$

The directional derivative of $\beta^2$ is

$$\begin{aligned}
D\beta(u)[\Delta u] &= (1/m-1)\,C^{-1/m}\|u\|^{(1-3m)/m}\,u\,\Delta u\\
&= (1/m-1)\,C^{-1/m}\|u\|^{(1-2m)/m}\,\text{sign}(u)\,\Delta u
\end{aligned}$$

old result

$$\partial_u\beta^2=(1/m-1)\,C^{-1/m}\|u\|^{1/m-2}\,\text{sign}(u)$$

Inserting into (8.10) gives

$$4\partial_x(h(\bar\eta\partial_x\bar u+\bar u E\partial_x\Delta u+\bar\eta\partial_x\Delta u))-\bar\beta^2\bar u-\bar\beta^2\Delta u-\bar u B\Delta u=\rho g h(\partial_x s\cos\alpha-\sin\alpha)$$

where

$$B=(1/m-1)\,C^{-1/m}\|u\|^{(1-2m)/m}\,\text{sign}(u)$$

and

$$E=\frac{1-n}{2n}A^{-1/n}\,\|\partial_x u\|^{(1-2n)/n}\text{sign}(\partial_x u)$$

which can be written on the form

$$4\partial_x(h(\bar\eta+\bar u E)\partial_x\Delta u)-(\bar\beta^2+\bar u B)\Delta u=\rho g h(\partial_x s\cos\alpha-\sin\alpha)-4\partial_x(h\bar\eta\partial_x\bar u)+\bar\beta^2\bar u$$

### 8.6.1 Newton Raphson

$$4\partial_x(\eta h\partial_x u)-\mathcal{H}(h-h_f)\beta^2 u=\frac{1}{2}g\cos\alpha\rho(1-\rho/\rho_o)\partial_x h^2+\rho g h\cos\alpha\,\mathcal{H}(h-h_f)\partial_x s^{'}-\rho g h\sin\alpha, \quad (8.13)$$

$$s^{'}(x)=\mathcal{H}(h-h_f)(\rho/\rho_o h-H)$$

where

$$H=S-B,$$

or as

$$4\partial_x(\eta h\partial_x u)-\mathcal{H}(h-h_f)\beta^2 u=\frac{\varrho g}{2}\partial_x h^2\cos\alpha+\rho g h\mathcal{H}(h-h_f)(\rho/\rho_o\partial_x h-\partial_x H)\cos\alpha-\rho g h\sin\alpha \quad (8.14)$$

The FE formulation is

$$\begin{aligned}
R_p^u &= \int \{4\eta h \partial_x u \partial_x N_p + \mathcal{H}(h-h_f)\beta^2 u N_p \\
&\quad -\frac{\varrho g}{2} h^2 \partial_x N \cos\alpha + \rho g h \mathcal{H}(h-h_f)(\rho/\rho_o \partial_x h - \partial_x H) N_p \cos\alpha - \rho g h N_p \sin\alpha\}\, dx \\
&\quad -h(4\eta\partial_x u - \varrho g h/2) u N_p|_{x_0}^{x_l} = 0 \qquad (8.15)
\end{aligned}$$

where

$$u = N_p u_p$$

If all von Neumann BC are of Weertman type, the boundary term is zero because at the calving front

$$8\eta\partial_x u = \varrho g h.$$

I also have

$$\partial_t h + \partial_x(uh) = a \qquad (8.16)$$

where

$$a = a_s + a_b$$

The FE formulation used is the $\theta$ method, i.e.

$$R_p^h = \int \{\frac{1}{\Delta t}(h_1 - h_0) + \theta\partial_x(u_1 h_1) + (1-\theta)\partial_x(u_0 h_0) - a_s - a_b\} N_p\, dx = 0 \qquad (8.17)$$

where $0 \leq \theta \leq 1$.

I go from time $t = t_0$ to time $t = t_1$ where $t_1 > t_0$, and I assume that the values for $u$ at $h$ at $t = t_0$ (i.e. $u_0$ and $h_0$) are known. I iteratively solve for corrections to the values at time step $t = t_1$

$$u_1 = \bar{u}_1 + \Delta u$$
$$h_1 = \bar{h}_1 + \Delta h$$

using the Newton-Raphson method.

For Newton-Raphson I need to take the directional derivative of this equation with respect to $u$ and $h$,

$$K(\Delta u, \Delta v)^T = \lim_{\epsilon\to 0} \frac{d}{d\epsilon} \boldsymbol{R}(\boldsymbol{v} + \epsilon\Delta\boldsymbol{v}, \boldsymbol{h} + \epsilon\Delta\boldsymbol{h})$$

where I have now ordered the discrete values of $u$ and $h$ into a vector

$$D(\mathcal{H}(h-h_f)\beta^2(u))[\Delta h] = \delta(\bar{h} - h_f)\beta^2(\bar{u})\,\Delta h$$

Directional derivative of right-hand term with respect to $h$.

$$\begin{aligned}
D\{\rho g h \partial_x s'\}[\Delta h] =& D\{\rho g h \partial_x(\mathcal{H}(h-h_f)(\rho h/\rho_o - H))\}[\Delta h] \\
=& \lim_{\epsilon\to 0} \partial_\epsilon \left(\rho g (h + \epsilon\,\Delta h)\, \partial_x(\mathcal{H}(h + \epsilon\,\Delta h - h_f)(\rho(h+\epsilon\,\Delta h)/\rho_o - H))\right) \\
=& \rho g \partial_x(\mathcal{H}(h-h_f)(\rho h/\rho_o - H))\,\Delta h \\
&+ \rho g h \partial_x(\mathcal{H}(h-h_f)\rho\Delta h/\rho_o) \\
&+ \rho g h \partial_x(\delta(h-h_f)\Delta h\,(\rho h/\rho_o - H)) \quad (=0) \\
=& \rho g \mathcal{H}(h-h_f)(\rho\partial_x h/\rho_o - \partial_x H)\,\Delta h \\
&+ \frac{\rho^2}{\rho_o} g h \mathcal{H}(h-h_f)\,\partial_x\Delta h \\
&+ \frac{\rho^2}{\rho_o}\delta(h-h_f)\,h\,\partial_x h\,\Delta h
\end{aligned}$$

$$\begin{aligned}
[\mathrm{K}uh]_{pq}\Delta h_q = DR_p^u(u,h)[\Delta h_q] \quad = & \int\{4\bar{\eta}\partial_x\bar{u}\partial_x N_p \\
&+\delta(\bar{h} - h_f)\bar{\beta}^2\bar{u}N_p \\
&-\varrho g\bar{h}\partial_x N_p\cos\alpha \\
&+\rho g\mathcal{H}(\bar{h} - h_f)(\rho/\rho_o\partial_x\bar{h} - \partial_x H)N_p\cos\alpha \\
&+\rho g\bar{h}\delta(\bar{h} - h_f)(\rho/\rho_o\partial_x\bar{h} - \partial_x H)N_p\cos\alpha \\
&+\rho g\bar{h}\mathcal{H}(\bar{h} - h_f)\rho/\rho_o N_p\cos\alpha\,\partial_x \\
&-\rho g N_p\sin\alpha\}\,\Delta h_q\,dx
\end{aligned}$$

$$
\begin{aligned}
[\mathrm{K}uu]_{pq}\Delta u_q = DR^u_p(u,h)[\Delta u_q] &= \int \{4\bar{h}\partial_u\eta(\bar{u})\,\partial_x\bar{u}\,\partial_x N_p\partial_x \\
&\quad +4\eta(\bar{u})\bar{h}\partial_x N_p\partial_x \\
&\quad +\mathcal{H}(h-h_f)\partial_u\beta^2(\bar{u})N_p \\
&\quad +\mathcal{H}(h-h_f)\beta^2(\bar{u})N_p\}\,\Delta u_q\,dx
\end{aligned}
$$

Linearising (8.17) gives

$$
\int\{(\Delta h+\bar{h}-h_0)/\Delta t+\theta\partial_x((\bar{u}+\Delta u)(\bar{h}+\Delta h))+(1-\theta)\partial_x(u_0h_0)-a_s-a_b\}N_p=0
$$

or

$$
\begin{aligned}
0 = &\int\{\Delta h/\Delta t+\theta\partial_x((\bar{u}\Delta h+\bar{h}\Delta u))\}N_p\,dx+ \\
&\int\{(\bar{h}-h_0)/\Delta t+\theta\partial_x(\bar{u}\bar{h})+(1-\theta)\partial_x(u_0h_0)-a_s-a_b\}N_p\,dx
\end{aligned}
\tag{8.18}
$$

or

$$
[\mathrm{K}hu]_{pq}\Delta u_q = \theta(\partial_x\bar{h}+\bar{h}\partial_x)\Delta u_q\,N_p
$$

$$
[\mathrm{K}hh]_{pq} = (1/\Delta t+\theta(\partial_x\bar{u}\,N_q+\bar{u}\partial_x N_q))\,N_p
$$

$$
\begin{bmatrix}\mathrm{K}uu & \mathrm{K}uh\\ \mathrm{K}hu & \mathrm{K}hh\end{bmatrix}\begin{bmatrix}\Delta\boldsymbol{v}\\ \Delta\boldsymbol{h}\end{bmatrix}=\begin{bmatrix}\boldsymbol{R}^u\\ \boldsymbol{R}^h\end{bmatrix}
\tag{8.19}
$$

where

$$
\boldsymbol{r}^h=\boldsymbol{T}^h-\boldsymbol{F}^h
$$

where $\boldsymbol{T}$ and $\boldsymbol{F}$ are the internal and external nodal forces, respectively, and $\boldsymbol{R}$ is the residual or out-of-balance nodal forces.

$$
F^h=-\int\{a_s+a_b-(\bar{h}-h_0)/\Delta t\}N_p\,dx
$$

and

$$
T^h=\int\{\theta\partial_x(\bar{u}\bar{h})+(1-\theta)\partial_x(u_0h_0)\}N_p\,dx
$$

$$
T^u_p=\int\{4\eta h\partial_x u\partial_x N_p+\mathcal{H}(h-h_f)\beta^2 uN_p)N_p\}\,dx
$$

$$
F^u_p=\int\{\frac{\varrho g}{2}h^2\partial_x N+\rho gh\mathcal{H}(h-h_f)(\rho/\rho_o\partial_x h-\partial_x H)N_p\}\,dx
$$

### 8.6.2 Connection to Piccard iteration

If instead of writing

$$
4\partial_x(h(\bar{\eta}\partial_x\bar{u}+\partial_u\eta\partial_x\bar{u}\Delta u+\bar{\eta}\partial_x\Delta u))-(\bar{\beta}^2\bar{u}+(\bar{\beta}^2+\partial_u\beta^2\bar{u})\Delta u)=\rho gh(\partial_x s\cos\alpha-\sin\alpha)
$$

we ignore the dependency of $\eta$ and $\beta^2$ on $u$ we get

$$
4\partial_x(h(\bar{\eta}\partial_x\bar{u}+\bar{\eta}\partial_x\Delta u))-(\bar{\beta}^2\bar{u}+\bar{\beta}^2\Delta u)=\rho gh(\partial_x s\cos\alpha-\sin\alpha)
$$

or simply

$$
4\partial_x(h\bar{\eta}\partial_x(\bar{u}+\Delta u))-\bar{\beta}^2(\bar{u}+\Delta u)=\rho gh(\partial_x s\cos\alpha-\sin\alpha)
$$

which can be solved directly for $\bar{u}+\Delta u$. This is the Piccard iteration, i.e. an incomplete Newton iteration.

## 8.7 Linearisation in 2HD

### 8.7.1 Drag-term linearisation (2HD)

Basal drag is generally a function of basal sliding velocity and flotation conditions, i.e.

$$\boldsymbol{t}_b = f(h, h_f, \boldsymbol{v})$$

Using Weertman sliding law, basal drag is

$$\boldsymbol{t}_b = \mathcal{H}(h - h_f)\beta^2\, \boldsymbol{v}_b$$

or

$$\begin{pmatrix} t_{xb} \\ t_{xy} \end{pmatrix} = \mathcal{H}(h - h_f)\, \beta^2 \begin{pmatrix} u \\ v \end{pmatrix} \tag{8.20}$$

were

$$\beta^2 = C^{-1/m} \|\boldsymbol{v}_b\|^{1/m-1}$$

$$\boldsymbol{t}_b = (g(\rho h - \rho_o H))^{q/m}\, C^{-1/m}\, \|\boldsymbol{v}_b\|^{1/m-1}\, \boldsymbol{v}_b, \tag{8.21}$$

therefore

$$\tau_b = \tau_b(u, v, h)$$

We therefore need to linearise $\boldsymbol{t}_b$ with respect to $u$, $v$, and $h$.

We start by linearising $\beta^2$ with respect to $u$ and $v$ obtaining[2]

$$\begin{aligned}
\beta(u + \epsilon\Delta u, v + \epsilon\Delta v) &= C^{-1/m}((u + \epsilon\Delta u)^2 + (v + \epsilon\Delta v)^2)^{(1/m-1)/2} \\
&= C^{-1/m}((u^2 + v^2 + 2\epsilon(u\,\Delta u + v\,\Delta v))^{(1/m-1)/2} \\
&= C^{-1/m}((u^2 + v^2)^{(1/m-1)/2} + (1/m - 1)(u^2 + v^2)^{(1/m-1)/2-1}\, \epsilon(u\,\Delta u + v\,\Delta v)) \\
&= \beta^2(u, v) + C^{-1/m}(1/m - 1)(u^2 + v^2)^{(1/m-3)/2}\, \epsilon(u\,\Delta u + v\,\Delta v))
\end{aligned}$$

The directional derivative is defined as

$$D\beta(\boldsymbol{v})[\Delta \boldsymbol{v}_{xy}] = \lim_{\epsilon \to 0} \frac{d}{d\epsilon} \beta^2(u + \epsilon\Delta u, v + \epsilon\Delta v)$$

and we arrive at[3]

$$D\beta(\boldsymbol{v})[\Delta u, \Delta v] = (1/m - 1)\, C^{-1/m} \|\boldsymbol{v}\|^{(1-3m)/m}\, (u\,\Delta u, v\,\Delta v)$$

and the directional derivatives of $\boldsymbol{t}_b$ with respect to $u$ and $v$ are therefore

$$\begin{aligned}
D\boldsymbol{t}_{xb}[\Delta u, \Delta v] &= \beta^2 \Delta u + D\beta^2[\Delta u]\, u + D\beta^2[\Delta v]\, u \\
&= \beta^2 \Delta u + (1/m - 1)\, C^{-1/m} \|\boldsymbol{v}\|^{(1-3m)/m}\, (u^2 \Delta u + uv\,\Delta v) \\
D\boldsymbol{t}_{yb}[\Delta u, \Delta v] &= \beta^2 \Delta v + (1/m - 1)\, C^{-1/m} \|\boldsymbol{v}\|^{(1-3m)/m}\, (v^2 \Delta v + uv\,\Delta u)
\end{aligned}$$

which can also be written on the form

$$\begin{pmatrix} \Delta t_{xb} \\ \Delta t_{xy} \end{pmatrix} = \mathcal{H}(h - h_f) \begin{pmatrix} \beta^2 + \mathcal{D}\, u^2 & \mathcal{D}\, uv \\ \mathcal{D}\, uv & \beta^2 + \mathcal{D}\, v^2 \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} \tag{8.22}$$

---

[2] If $y \ll x$ then $(x + y)^m \approx x^n + m\, x^{m-1} y$.

[3] In 1d we get

$$\begin{aligned}
D\beta(\boldsymbol{v})[\Delta \boldsymbol{v}] &= (1/m - 1)\, C^{-1/m} \|u\|^{(1-3m)/m}\, u\, \Delta u \\
&= (1/m - 1)\, C^{-1/m} \|u\|^{(1-2m)/m}\, \mathrm{sign}(u)\, \Delta u
\end{aligned}$$

where we have now added back the $\mathcal{H}(h-h_f)$ factor, and where

$$\mathcal{D} := (1/m-1)\, C^{-1/m} \|\boldsymbol{v}\|^{(1-3m)/m}$$

Note that because of the non-linearity of the sliding law, the basal drag term in $x$ direction depends on both $u$ and $v$ and this is reflected in the directional derivatives above.

In a fully implicit treatment we also must include the effect of grounding/un-grounding the basal drag term, or

$$\begin{aligned} D\boldsymbol{t}_{xb}[\Delta h] &= \lim_{\epsilon\to 0} \frac{d}{d\epsilon} \boldsymbol{t}_{xb}(h+\epsilon\Delta h) \\ &= \lim_{\epsilon\to 0} \frac{d}{d\epsilon} \left(\mathcal{H}(h+\epsilon\Delta h - h_f)\, \beta^2\, u\right) \\ &= \lim_{\epsilon\to 0} \delta(h+\epsilon\Delta h - h_f)\Delta h\, \beta^2\, u \\ &= \delta(h-h_f)\, \beta^2\, u\, \Delta h \end{aligned}$$

and therefore for the Weertman sliding law we have:

$$\begin{pmatrix} \Delta t_{bx} \\ \Delta t_{by} \end{pmatrix} = \begin{pmatrix} \mathcal{H}(h-h_f)(\beta^2 + \mathcal{D}\, u^2) & \mathcal{H}(h-h_f)\mathcal{D}\, uv & \delta(h-h_f)\, \beta^2\, u \\ \mathcal{H}(h-h_f)\mathcal{D}\, uv & \mathcal{H}(h-h_f)(\beta^2+\mathcal{D}\, v^2) & \delta(h-h_f)\, \beta^2\, v \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta h \end{pmatrix} \tag{8.23}$$

where again

$$\mathcal{D} := (1/m-1)\, C^{-1/m} \|\boldsymbol{v}\|^{(1-3m)/m}$$

For the Budd sliding law (see sec 1.5.2), assuming perfect hydrological connection between the subglacial hydrological system and the ocean,

$$N = \rho g\, \mathcal{H}(h-h_f)\, (h-h_f)$$

we have

$$\boldsymbol{t}_b = \quad N^{q/m}\, \beta^2\, \boldsymbol{v}_b, \tag{8.24}$$

and linearising with respect to $h$ now gives

$$\begin{aligned} D\boldsymbol{t}_{xb}[\Delta h] &= \lim_{\epsilon\to 0} \frac{d}{d\epsilon} \boldsymbol{t}_{xb}(h+\epsilon\Delta h) \\ &= \lim_{\epsilon\to 0} \frac{d}{d\epsilon} \left((\rho g(h+\epsilon\Delta h - h_f))^{q/m}\ \beta^2 u\right) \\ &= \lim_{\epsilon\to 0} \frac{q}{m}\, (\rho g(h+\epsilon\Delta h - h_f))^{q/m-1}\ \ \rho g\, \Delta h\, \beta^2 u \\ &= \rho g \frac{q}{m}\, (\rho g(h-h_f))^{q/m-1}\ \ \beta^2 u\, \Delta h \end{aligned}$$

and therefore for the Budd sliding law we have:

$$\begin{pmatrix} \Delta t_{bx} \\ \Delta t_{by} \end{pmatrix} = N^{q/m} \begin{pmatrix} N^{q/m}\, (\beta^2 + \mathcal{D}\, u^2) & N^{q/m}\, \mathcal{D}\, uv & \mathcal{E}\, \beta^2\, u \\ N^{q/m}\, \mathcal{D}\, uv & N^{q/m}\, (\beta^2 + \mathcal{D}\, v^2) & \mathcal{E}\, \beta^2\, v \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta h \end{pmatrix} \tag{8.25}$$

$$\tag{8.26}$$

where

$$\mathcal{E} = \frac{q}{m} N^{q/m-1} \rho g\, (\delta(h-h_f)\, (h-h_f) + \mathcal{H}(h-h_f))$$

We see that for example

$$\begin{aligned} \frac{\partial t_{bx}}{\partial h} &= \frac{q}{m} N^{q/m-1} \rho g \beta^2 u \\ &= \frac{q}{m} (\rho g(h-h_f))^{q/m-1} \rho g\, \beta^2 u \end{aligned}$$

where $h > h_f$, and in the linear case when $q/m = m = 1$, we have

$$\frac{\partial t_{bx}}{\partial h} = \frac{\rho g}{C} u$$

This is the partial derivative with respect to $h$ and as such does not include the effect of changes in $h$ on $u$. This shows that with Budd sliding law, the basal drag increases with ice thickness everywhere upstream of the grounding line. For Weertman law no similar statement can be made and the basal drag only changes with thickness through the implicit dependency of the velocity on thickness.

Note that for $q/m < 1$, $\Delta t_b \to \infty$ as $\Delta h \to h_f$.

**Ocean drag term**

We add an ocean drag term over the floating section of the form

$$\boldsymbol{t}_b^o = \mathcal{H}(h_f - h)\beta_o^2 \, (\boldsymbol{v}_b - \boldsymbol{v}_o)$$

with

$$\beta_o^2(u,v) = C_o^{-1/m_o} |\boldsymbol{v}_b - \boldsymbol{v}_o|^{1/m_o - 1}$$

The total drag is a sum of that due to basal sliding and ocean currents.

$$\boldsymbol{t}_b = \mathcal{H}(h - h_f)\, \beta^2 \, \boldsymbol{v} + \mathcal{H}(h_f - h)\, \beta_o^2 \, (\boldsymbol{v} - \boldsymbol{v}_o)$$

So

$$t_{bx}^o = \mathcal{H}(h_f - h) C_o^{-1/m_o} \left( (u - u_o)^2 + (v - v_0)^2 \right)^{(1-m)/2m} (u - u_o)$$
$$t_{yx}^o = \mathcal{H}(h_f - h) C_o^{-1/m_o} \left( (u - u_o)^2 + (v - v_0)^2 \right)^{(1-m)/2m} (v - v_o)$$

and hence

$$D\boldsymbol{t}_{bx}^o[\Delta u] = \mathcal{H}(h_f - h) C_o^{-1/m_o} \left( \|\boldsymbol{v}_b - \boldsymbol{v}_o\|^{(1-m_o)/m_o} + (1/m - 1) \|\boldsymbol{v}_b - \boldsymbol{v}_o\|^{(1-3m_o)/m_o} (u - u_o)^2 \right) \Delta u$$
$$D\boldsymbol{t}_{bx}^o[\Delta v] = \mathcal{H}(h_f - h) C_o^{-1/m_o} (1/m - 1) \|\boldsymbol{v}_b - \boldsymbol{v}_o\|^{(1-3m_o)/m_o} (v - v_o)(u - u_o)\, \Delta v$$
$$D\boldsymbol{t}_{by}^o[\Delta u] = \mathcal{H}(h_f - h) C_o^{-1/m_o} (1/m - 1) \|\boldsymbol{v}_b - \boldsymbol{v}_o\|^{(1-3m_o)/m_o} (v - v_o)(u - u_o)\, \Delta u$$
$$D\boldsymbol{t}_{by}^o[\Delta v] = \mathcal{H}(h_f - h) C_o^{-1/m_o} \left( \|\boldsymbol{v}_b - \boldsymbol{v}_o\|^{(1-m_o)/m_o} + (1/m - 1) \|\boldsymbol{v}_b - \boldsymbol{v}_o\|^{(1-3m_o)/m_o} (v - v_o)^2 \right) \Delta v$$
$$D\boldsymbol{t}_{bx}^o[\Delta h] = -\delta(h_f - h) C_o^{-1/m_o} \|\boldsymbol{v}_b - \boldsymbol{v}_o\|^{(1-m_o)/m_o} (u - u_o) \Delta h$$
$$D\boldsymbol{t}_{by}^o[\Delta h] = -\delta(h_f - h) C_o^{-1/m_o} \|\boldsymbol{v}_b - \boldsymbol{v}_o\|^{(1-m_o)/m_o} (v - v_o) \Delta h$$

### 8.7.2 Flow law linearisation (2HD)

Using Glen's flow law, deviatoric stresses are related to strain rates through Eq. (1.46), i.e.

$$\tau_{ij} = A^{-1/n} \, \dot{\epsilon}^{(1-n)/n} \, \dot{\epsilon}_{ij},$$

where

$$\dot{\epsilon} = \sqrt{\dot{\epsilon}_{ij}\dot{\epsilon}_{ij}/2}$$

which in the Shallow Ice Stream Approximation takes the form

$$\dot{\epsilon} = \sqrt{(\dot{\epsilon}_{xx})^2 + (\dot{\epsilon}_{yy})^2 + \dot{\epsilon}_{xx}\, \dot{\epsilon}_{yy} + (\dot{\epsilon}_{xy})^2} \tag{8.27}$$
$$= ((\partial_x u)^2 + (\partial_y v)^2 + \partial_x u \, \partial_y v + (\partial_x v + \partial_y u)^2/4)^{1/2}. \tag{8.28}$$

If we write

$$\tau_{ij} = 2\eta \dot{\epsilon}_{ij}$$

where $\eta$ is the effective viscosity given by

$$\eta = \frac{1}{2} A^{-1/n} \, \dot{\epsilon}^{(1-n)/n}$$

then, in the Shallow Ice Stream Approximation, $\eta$ is

$$\eta = \frac{1}{2}A^{-1/n}\left((\partial_x u)^2 + (\partial_y v)^2 + \partial_x u\,\partial_y v + (\partial_x v + \partial_y u)^2/4\right)^{(1-n)/2n}.$$

More generally we can express the stresses as a function of velocities as

$$\tau_{ij} = f(u_q).$$

We now need to linearise each of the stress components with respect to the unknown velocity components $u$ and $v$ velocities. We start with $\tau_{xx}$ where we have

$$\begin{aligned}\tau_{xx} &= A^{-1/n}\,\dot\epsilon^{(1-n)/n}\,\dot\epsilon_{xx} && (8.29)\\ &= A^{-1/n}\,\dot\epsilon^{(1-n)/n}\,\partial_x u && (8.30)\end{aligned}$$

and we linearise

$$\begin{aligned}D\tau_{xx}[u;\Delta u] &= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\tau_{xx}(u;\Delta u) && (8.31)\\ &= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\left(2\eta\,\partial_x u\right) && (8.32)\\ &= 2\eta\,\partial_x\Delta u + (2D\eta[\Delta u])\,\partial_x u && (8.33)\end{aligned}$$

$$\eta = \frac{1}{2}A^{-1/n}\left((\partial_x u)^2 + (\partial_y v)^2 + \partial_x u\,\partial_y v + (\partial_x v + \partial_y u)^2/4\right)^{(1-n)/2n}$$

$$\begin{aligned}&(\partial_x(u+\Delta u))^2 + (\partial_y(v+\Delta v))^2 + \partial_x(u+\Delta u)\,\partial_y(v+\Delta v) + (\partial_x(v+\Delta v) + \partial_y(u+\Delta u))^2/4\\
&\quad = (\partial_x u)^2 + 2\partial_x u\partial_x\Delta u\\
&\quad +(\partial_y v)^2 + 2\partial_y v\partial_y\Delta v\\
&\quad +(\partial_x u + \partial_x\Delta u)(\partial_y v + \partial_y\Delta v)\\
&\quad +(\partial_x v + \partial_x\Delta v + \partial_y u + \partial_y\Delta u)^2/4\\
&\quad = (\partial_x u)^2 + 2\partial_x u\partial_x\Delta u\\
&\quad +(\partial_y v)^2 + 2\partial_y v\partial_y\Delta v\\
&\quad +\partial_x u\partial_y v + \partial_x u\,\partial_y\Delta v + \partial_y v\,\partial_x\Delta u\\
&\quad +(\partial_x v + \partial_y u)/4 + (\partial_x v + \partial_y u)(\partial_x\Delta v + \partial_y\Delta u)/2\\
&\quad = e^2 + \delta de^2\end{aligned}$$

where

$$\begin{aligned}e^2 &= (\partial_x u)^2 + (\partial_y v)^2 + \partial_x u\partial_y v + (\partial_x v + \partial_y u)/4\\ &= \dot\epsilon_{xx}^2 + \dot\epsilon_{yy}^2 + \dot\epsilon_{xx}\dot\epsilon_{yy} + \dot\epsilon_{xy}^2\end{aligned}$$

$$\delta e^2 = 2\partial_x u\partial_x\Delta u + 2\partial_y v\partial_y\Delta v + \partial_x u\,\partial_y\Delta v + \partial_y v\,\partial_x\Delta u + (\partial_x v + \partial_y u)(\partial_x\Delta v + \partial_y\Delta u)/2$$

or

$$\begin{aligned}\delta e^2 &= (2\partial_x u + \partial_y v)\,\partial_x\Delta u + (2\partial_y v + \partial_x u)\,\partial_y\Delta v + \frac{1}{2}(\partial_x v + \partial_y u)\,(\partial_x\Delta v + \partial_y\Delta u)\\ &= (2\dot\epsilon_{xx} + \dot\epsilon_{yy})\,\partial_x\Delta u + (2\dot\epsilon_{yy} + \dot\epsilon_{xx})\,\partial_y\Delta v + \dot\epsilon_{xy}\,(\partial_x\Delta v + \partial_y\Delta u)\end{aligned}$$

The directional derivative of $\eta$ is

$$\begin{aligned}D\eta(u,v)[\Delta u,\Delta v] &= E\left((2\dot\epsilon_{xx} + \dot\epsilon_{yy})\,\partial_x\Delta u + (2\dot\epsilon_{yy} + \dot\epsilon_{xx})\,\partial_y\Delta v + \dot\epsilon_{xy}\,(\partial_x\Delta v + \partial_y\Delta u)\right)\\ &= E\left((2\partial_x u + \partial_y v)\,\partial_x\Delta u + (2\partial_y v + \partial_x u)\,\partial_y\Delta v + \frac{1}{2}(\partial_x v + \partial_y u)\,(\partial_x\Delta v + \partial_y\Delta u)\right)\end{aligned}$$

where

$$E := \frac{1-n}{4n} A^{-1/n} e^{(1-3n)/n}$$

which I can also write as

$$D\eta(u,v)[\Delta u, \Delta v] = d_u \eta \, \Delta u + d_v \eta \, \Delta v \tag{8.34}$$

where

$$d_u \eta = E\left((2\dot\epsilon_{xx} + \dot\epsilon_{yy})\,\partial_x + \dot\epsilon_{xy}\,\partial_y\right)$$

and

$$d_v \eta = E\left((2\dot\epsilon_{yy} + \dot\epsilon_{xx})\,\partial_y + \dot\epsilon_{xy}\,\partial_x\right)$$

where

$$E := \frac{1-n}{4n} A^{-1/n} e^{(1-3n)/n}$$

or as

$$D\eta(u,v)[\Delta u, \Delta v] = d_{xu}\partial_x \Delta u + d_{yu}\partial_y \Delta u + d_{yv}\partial_y \Delta v + d_{xv}\partial_x \Delta v \tag{8.35}$$

where

$$d_{xu} = E\,(2\dot\epsilon_{xx} + \dot\epsilon_{yy})$$

and

$$d_{yu} = E\,\dot\epsilon_{xy}$$

and

$$d_{yv} = E\,(2\dot\epsilon_{yy} + \dot\epsilon_{xx})$$

and

$$d_{xv} = E\,\dot\epsilon_{xy}$$

$$\begin{pmatrix} D\eta_x \\ D\eta_y \end{pmatrix} = \begin{pmatrix} E(2\dot\epsilon_{xx} + \dot\epsilon_{yy})\partial_x & E\dot\epsilon_{xy}\partial_y & 0 \\ E\dot\epsilon_{xy}\partial_x & E(\dot\epsilon_{xx} + 2\dot\epsilon_{yy})\partial_y & 0 \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta h \end{pmatrix} \tag{8.36}$$

In the 1d case we get

$$\frac{1-n}{4n} A^{-1/n} |\partial_x u|^{(1-3n)/n}\, 2\partial_x u\, \partial_x \Delta u = \frac{1-n}{2n} A^{-1/n} |\partial_x u|^{(1-2n)/n} \mathrm{sign}(\partial_x u)\, \partial_x \Delta u$$

### 8.7.3 Field-equation linearisation

linearising

$$\begin{aligned} &\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) + \partial_y(h\eta(\partial_x v + \partial_y u)) - \beta^2 u \\ &= \frac{1}{2}\varrho g \cos\alpha\,\partial_x h^2 + \rho g h(\partial_x s^{'} \cos\alpha - \sin\alpha) \end{aligned}$$

gives

$$\begin{aligned} &\partial_x(4h(\eta\partial_x u + D\eta\,\partial_x u + \eta\,\partial_x \Delta u) + 2h(\eta\partial_y v + D\eta\,\partial_y v + \eta\,\partial_y \Delta v)) \\ &\quad + \partial_y(h(\eta\partial_x v + D\eta\,\partial_x v + \eta\,\partial_x \Delta v) + h(\eta\partial_y u + D\eta\,\partial_y u + \eta\,\partial_y \Delta u)) \\ &\quad - (\beta^2 u + D\beta^2\, u + \beta^2\,\Delta u) \\ &\quad = \frac{1}{2}\varrho g \cos\alpha\,\partial_x h^2 + \rho g h(\partial_x s^{'} \cos\alpha - \sin\alpha) \end{aligned}$$

or

$$\begin{aligned} &\partial_x(4h(D\eta\,\partial_x u + \eta\,\partial_x \Delta u) + 2h(D\eta\,\partial_y v + \eta\,\partial_y \Delta v)) \\ &\quad + \partial_y(h(D\eta\,\partial_x v + \eta\,\partial_x \Delta v) + h(D\eta\,\partial_y u + \eta\,\partial_y \Delta u)) \\ &\quad - (D\beta^2\, u + \beta^2\,\Delta u) \\ &\quad = \frac{1}{2}\varrho g \cos\alpha\,\partial_x h^2 + \rho g h(\partial_x s^{'} \cos\alpha - \sin\alpha) - \partial_x(4h\eta(\partial_x u + \partial_y v)) - \partial_y(h\eta(\partial_x v + \partial_y u)) + \beta^2 u \end{aligned}$$

after having done the partial integration I get within the integral

$$
\begin{aligned}
&(4h(D\eta\,\partial_x u + \eta\,\partial_x \Delta u) + 2h(D\eta\,\partial_y v + \eta\,\partial_y \Delta v))\,\partial_x N_i \qquad (8.37)\\
&\quad +h((D\eta\,\partial_x v + \eta\,\partial_x \Delta v) + h(D\eta\,\partial_y u + \eta\,\partial_y \Delta u)\,\partial_y N_i\\
&\quad +(D\beta^2\, u + \beta^2\,\Delta u)N_i\\
&\quad = +\frac{1}{2}\varrho g\cos\alpha\,\partial_x h^2 \partial_x N_i - \rho g h(\partial_x s^{'}\cos\alpha - \sin\alpha)N_i - 4h\eta(\partial_x u + 2\partial_y v)\partial_x N_i - h\eta(\partial_x v + \partial_y u)\partial_y N_i - \beta^2 u N_i \qquad (8.38)
\end{aligned}
$$

Inserting (8.34)

$$
\begin{aligned}
&(4h(\eta\,\partial_x \Delta u + \partial_x u(d_u\eta\,\Delta u + d_v\eta\,\Delta v)) + 2h(\eta\,\partial_y \Delta v + \partial_y v(d_u\eta\,\Delta u + d_v\eta\,\Delta v)))\,\partial_x N_i\\
&\quad +(h(\eta\,\partial_x \Delta v + \partial_x v(d_u\eta\,\Delta u + d_v\eta\,\Delta v)) + h(\eta\,\partial_y \Delta u + \partial_y u(d_u\eta\,\Delta u + d_v\eta\,\Delta v)))\,\partial_y N_i\\
&\quad +(\beta^2\,\Delta u + u\,(d_u\beta^2\;\Delta u + d_v\beta^2\,\Delta v))N_i\\
&\quad = +\frac{1}{2}\varrho g\cos\alpha\,\partial_x h^2 \partial_x N_i - \rho g h(\partial_x s^{'}\cos\alpha - \sin\alpha)N_i - 4h\eta(\partial_x u + 2\partial_y v)\partial_x N_i - h\eta(\partial_x v + \partial_y u)\partial_y N_i - \beta^2 u N_i
\end{aligned}
$$

I take all coefficients in front of $\Delta u$ and put them in the 11 part of the matrix, and everything in front of $\Delta v$ and put that in the 12 part of the matrix.

To make this clear insert (8.35) into (8.38)

$$
\begin{aligned}
&\quad 4h(\eta\,\partial_x \Delta u + \partial_x u(d_{xu}\partial_x \Delta u + d_{yu}\partial_y \Delta u + d_{yv}\partial_y \Delta v + d_{xv}\partial_x \Delta v))\partial_x N_i\\
&+ \quad 2h(\eta\,\partial_y \Delta v + \partial_y v(d_{xu}\partial_x \Delta u + d_{yu}\partial_y \Delta u + d_{yv}\partial_y \Delta v + d_{xv}\partial_x \Delta v))\,\partial_x N_i\\
&+ \quad h(\eta\,\partial_x \Delta v + \partial_x v(d_{xu}\partial_x \Delta u + d_{yu}\partial_y \Delta u + d_{yv}\partial_y \Delta v + d_{xv}\partial_x \Delta v))\partial_y N_i\\
&+ \quad h(\eta\,\partial_y \Delta u + \partial_y u(d_{xu}\partial_x \Delta u + d_{yu}\partial_y \Delta u + d_{yv}\partial_y \Delta v + d_{xv}\partial_x \Delta v))\partial_y N_i\\
&+ \quad (\beta^2\,\Delta u + u\,(d_u\beta^2\;\Delta u + d_v\beta^2\,\Delta v))N_i\\
&\quad = +\frac{1}{2}\varrho g\cos\alpha\,\partial_x h^2 \partial_x N_i - \rho g h(\partial_x s^{'}\cos\alpha - \sin\alpha)N_i - 4h\eta(\partial_x u + 2\partial_y v)\partial_x N_i - h\eta(\partial_x v + \partial_y u)\partial_y N_i - \beta^2 u N_i
\end{aligned}
$$

We have $\Delta u = N_j \Delta u_j$ and $\Delta v = N_j \delta v_j$

$$
\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} \Delta u_j \\ \Delta v_j \end{pmatrix}
$$

$$
\begin{aligned}
[K_{12}]_{ij} \;&=\; h\eta\,\partial_x N_j \partial_y N_i\\
&\quad +4h\partial_x u\, d_{yv}\,\partial_y N_j \partial_x N_i + 4h\partial_x u\, d_{xv}\,\partial_x N_j \partial_x N_i\\
&\quad +2h\partial_y v\, d_{yv}\,\partial_y N_j \partial_x N_i + 2h\partial_y v\, d_{xv}\,\partial_x N_j \partial_x N_i\\
&\quad +h\partial_x v\, d_{yv}\,\partial_y N_j \partial_y N_j + h\partial_x v\, d_{xv}\,\partial_x N_j \partial_y N_i\\
&\quad +h\partial_y u\, d_{yv}\,\partial_y N_j \partial_y N_i + h\partial_y u\, d_{xv}\,\partial_x N_j \partial_y N_i\\
&=\; h\eta\,\partial_x N_j \partial_y N_i\\
&\quad +(4h\partial_x u\, d_{xv} + 2h\partial_y v\, d_{xv})\,\partial_x N_j \partial_x N_i\\
&\quad +(h\partial_x v\, d_{yv} + h\partial_y u\, d_{yv})\,\partial_y N_j \partial_y N_i\\
&\quad +(h\partial_x v\, d_{xv} + h\partial_y u\, d_{xv})\,\partial_x N_j \partial_y N_i\\
&\quad +(4h\partial_x u\, d_{yv} + 2h\partial_y v\, d_{yv})\,\partial_y N_j \partial_x N_i\\
&=\; h\eta\,\partial_x N_j \partial_y N_i\\
&\quad +2Eh(2\dot\epsilon_{xx} + \dot\epsilon_{yy})\dot\epsilon_{xy}\,\partial_x N_j \partial_x N_i\\
&\quad +Eh2\dot\epsilon_{xy}(2\dot\epsilon_{yy} + \dot\epsilon_{xx})\,\partial_y N_j \partial_y N_i\\
&\quad +Eh2\dot\epsilon_{xy}\dot\epsilon_{xy}\,\partial_x N_j \partial_y N_i\\
&\quad +2Eh(2\dot\epsilon_{xx} + \dot\epsilon_{yy})(2\dot\epsilon_{yy} + \dot\epsilon_{xx})\,\partial_y N_j \partial_x N_i
\end{aligned}
$$

If we swap $u$ and $v$ and $x$ and $y$ and then $i$ and $j$ (transpose) we get the same matrix, hence

$$
K_{12} = K_{21}'
$$

$$\begin{aligned}
[K_{11}]_{ij} &= 4h(\eta\,\partial_x N_j + \partial_x u(d_{xu}\partial_x N_j + d_{yu}\partial_y N_j))\,\partial_x N_i \\
&\quad +2h\partial_y v(d_{xu}\partial_x N_j + d_{yu}\partial_y N_j)\,\partial_x N_i \\
&\quad +h(\partial_x v(d_{xu}\partial_x N_j + d_{yu}\partial_y N_j)\,\partial_y N_i \\
&\quad +h(\eta\,\partial_y N_j + \partial_y u(d_{xu}\partial_x N_j + d_{yu}\partial_y N_j))\,\partial_y N_i \\
&= h(4\eta + (4\partial_x u + 2\partial_y v)d_{xu})\partial_x N_j\,\partial_x N_i \\
&\quad +h(\eta + (\partial_y u + \partial_x v)d_{yu})\partial_y N_j \partial_y N_i \\
&\quad +h(4\partial_x u + 2\partial_y v)d_{yu}\partial_y N_j \partial_x N_i \\
&\quad +h(\partial_x v + \partial_y u)d_{xu}\partial_x N_j \partial_y N_i \\
&= h(4\eta + E2(2\dot\epsilon_{xx} + \dot\epsilon_{yy})(2\dot\epsilon_{xx} + \dot\epsilon_{yy}))\partial_x N_j\,\partial_x N_i \\
&\quad +h(\eta + 2E\dot\epsilon_{xy}\dot\epsilon_{xy})\partial_y N_j \partial_y N_i \\
&\quad +Eh2(2\dot\epsilon_{xx} + \dot\epsilon_{yy})\dot\epsilon_{xy}\partial_y N_j \partial_x N_i \\
&\quad +Eh2\dot\epsilon_{xy}(2\dot\epsilon_{xx} + \dot\epsilon_{yy})\partial_x N_j \partial_y N_i \\
&= 4h\eta\partial_x N_j\,\partial_x N_i + h\eta\partial_y N_j \partial_y N_i \\
&\quad +2hE(2\dot\epsilon_{xx} + \dot\epsilon_{yy})^2\partial_x N_j\,\partial_x N_i \\
&\quad +2Eh\dot\epsilon_{xy}^2\partial_y N_j \partial_y N_i \\
&\quad +2Eh(2\dot\epsilon_{xx} + \dot\epsilon_{yy})\dot\epsilon_{xy}(\partial_y N_j \partial_x N_i + \partial_x N_j \partial_y N_i)
\end{aligned}$$

so $K_{11}$ and $K_{22}$ are symmetrical.

One might expect that the $u d_u \beta^2\,\Delta v$ makes the matrix unsymmetrical, but in fact

$$u\,d_v\beta^2 = u(1/m-1)C^{-1/m}\|\boldsymbol{v}\|^{(1-3m)/m}v = v(1/m-1)C^{-1/m}\|\boldsymbol{v}\|^{(1-3m)/m}u = v\,d_u\beta^2$$

so the contributions to 12 and 21 are equal, and hence this term does not give rise to an unsymmetrical matrix.

The tangent matrix $K$ is symmetrical for non-linear flow including both the non-linear effects of $\beta^2$ and $\eta$.

Note: In 1D I get

$$\begin{aligned}
&4h(D\eta\,\partial_x u + \eta\,\partial_x \Delta u)\partial_x N_i + (D\beta^2\,u + \beta^2\,\Delta u)N_i \\
&\qquad = +\frac{1}{2}\varrho g\cos\alpha\,\partial_x h^2\,\partial_x N_i - \rho g h(\partial_x s'\cos\alpha - \sin\alpha)\,N_i - 4h\eta\partial_x u\partial_x N - \beta^2 u\,N
\end{aligned}$$

## 8.8 Weak form

$x$ **direction**

$$\int_\Omega (\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) + \partial_y(h\eta(\partial_x v + \partial_y u)) - t_{bx} - \rho g h(\partial_x s\cos\alpha - \sin\alpha))N\,dx\,dy = 0$$

with von Neumann BC on $\Gamma_2$

$$(4h\eta\partial_x u + 2h\eta\partial_y v)n_x + \eta h(\partial_x v + \partial_y u)n_y = \frac{1}{2}\rho g h(h-H)n_x$$

and

$$\eta h(\partial_x v + \partial_y u)n_x + (4\eta h\partial_y v + 2\eta h\partial_x u)n_y = \frac{1}{2}\rho g h(h-H)n_y$$

Green's theorem used to get rid of second derivatives gives

$$\begin{aligned}
&-\int_\Omega ((4h\eta\partial_x u + 2h\eta\partial_y v)\partial_x N + h\eta(\partial_x v + \partial_y u)\partial_y w)\,dx\,dy \\
&\quad -\int_\Omega (t_{bx} + \rho g h(\partial_x s\cos\alpha - \sin\alpha)N\,dx\,dy + \int_\Gamma ((4h\eta\partial_x u + 2h\eta\partial_y v)n_x + h\eta(\partial_x v + \partial_y u)n_y)N\,d\Gamma = 0
\end{aligned}$$

Using the BC we have

$$
\begin{aligned}
-\int_\Omega & ((4h\eta\partial_x u + 2h\eta\partial_y v)\partial_x N + h\eta(\partial_x v + \partial_y u)\partial_y w)\,dx\,dy \\
& - \int_\Omega (t_{bx} + \rho g h(\partial_x s\cos\alpha - \sin\alpha))N\,dx\,dy + \int_{\Gamma_2} \frac{1}{2} g\rho(1-\rho/\rho_o)h^2 n_x N\,d\Gamma = 0
\end{aligned}
$$

### $y$ direction

$$
\int_\Omega \partial_y((4h\eta\partial_y v + 2h\eta\partial_x u) + \partial_x(h\eta(\partial_y u + \partial_x v)) - t_{by} - \rho g h\partial_y s\cos\alpha)N\,dx\,dy
$$

Green's

$$
\begin{aligned}
-\int_\Omega & ((4h\eta\partial_y v + 2h\eta\partial_x u)\partial_y N + h\eta(\partial_y u + \partial_x v)\partial_x w)\,dx\,dy \\
& - \int_\Omega (t_{by} + \rho g h\partial_y s\cos\alpha)N\,dx\,dy \\
& + \int_\Gamma ((4h\eta\partial_y v + 2h\eta\partial_x u)n_y + \eta h(\partial_y u + \partial_x v)n_x)N\,d\Gamma
\end{aligned} \tag{8.39}
$$

the von Neumann BC is

$$
\eta h(\partial_x v + \partial_y u)n_x + (4h\eta\partial_y v + 2h\eta\partial_x u)n_y = \frac{1}{2}\rho g h(h-H)n_y
$$

hence

$$
\begin{aligned}
-\int_\Omega & ((4h\eta\partial_y v + 2h\eta\partial_x u)\partial_y N + h\eta(\partial_y u + \partial_x v)\partial_x N)\,dx\,dy \\
& - \int_\Omega (t_{by} + \rho g h\partial_y s\cos\alpha)N\,dx\,dy + \int_{\Gamma_2} \frac{1}{2} g\rho(1-\rho/\rho_o)h^2 n_y w\,d\Gamma = 0
\end{aligned}
$$

### Ice shelf

$$
\int_\Omega (\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) + \partial_y(h\eta(\partial_x v + \partial_y u)) - t_{bx} - \frac{1}{2}\rho(1-\rho/\rho_o)g\partial_x h^2 N\,dx\,dy = 0
$$

On $\Gamma_2$ we write the von Neumann BC as

$$
(4\eta h\partial_x u + 2\eta h\partial_y v)n_x + \eta h(\partial_x v + \partial_y u)n_y = \frac{1}{2}\rho(1-\rho/\rho_o)gh^2 n_x
$$

and

$$
\eta(\partial_x v + \partial_y u)n_x + (4\eta\partial_y v + 2\eta\partial_x u)n_y = \frac{1}{2}\rho(1-\rho/\rho_o)gh^2 n_y
$$

we consider the term

$$
\int_\Omega (\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) + \partial_y(h\eta(\partial_x v + \partial_y u)) - \frac{1}{2}\rho(1-\rho/\rho_o)g\partial_x h^2)N\,dx\,dy = \tag{8.40}
$$

$$
-\int_\Omega (4h\eta\partial_x u + 2h\eta\partial_y v + h\eta(\partial_x v + \partial_y u) - \frac{1}{2}\rho(1-\rho/\rho_o)gh^2)\partial_x N\,dx\,dy \tag{8.41}
$$

$$
+\int_\Gamma ((4h\eta\partial_x u + 2h\eta\partial_y v)n_x + h\eta(\partial_x v + \partial_y u)n_y - \frac{1}{2}\rho(1-\rho/\rho_o)gh^2 n_x)N\,d\Gamma \tag{8.42}
$$

Along $\Gamma_2$, the path integral disappears and along $\Gamma_1$ we set $w^* = 0$, hence

$$
-\int_\Omega (4h\eta\partial_x u + 2h\eta\partial_y v + h\eta(\partial_x v + \partial_y u) - \frac{1}{2}\rho(1-\rho/\rho_o)gh^2)\partial_x w - t_{bx}N)\,dx\,dy = 0 \tag{8.43}
$$

## 8.9 Thoughts about ice shelf von Neumann BC

### 8.9.1 1d case

Field equation:

$$4\partial_x(\eta h \partial_x u) - t_x - \rho g h \partial_x s \cos\alpha + \rho g h \sin\alpha = 0$$

Boundary condition

$$4\eta\partial_x u = \frac{1}{2}\rho g(1-\rho/\rho_o)h \tag{8.44}$$

which for $\eta = \frac{1}{2}A^{-1/n}|\partial_x u|^{(1-n)/n}$ can also be written as

$$\partial_x u = A(\rho g f/4)^n$$

where $f = (1-\rho/\rho_o)h(x_c)$, and $x_c$ is the location of the calving front. We write the field equation as

$$4\partial_x(\eta h \partial_x u) - \beta^2 u - \rho g h \partial_x(s^{'} + (1-\rho/\rho_o)h)\cos\alpha + \rho g h \sin\alpha = 0$$

with

$$s^{'} := f - (1-\rho/\rho_o)h = s - S - (1-\rho/\rho_o)h$$

or as

$$4\partial_x(\eta h \partial_x u) - \frac{1}{2}g\cos\alpha\rho(1-\rho/\rho_o)\partial_x h^2 - \rho g h\cos\alpha\partial_x s^{'} - \beta^2 u + \rho g h \sin\alpha = 0,$$

using $\partial_x S = 0$. Here $S$ is the elevation of sea level (usually the coordinate system would be defined so that $S = 0$), and $s$ the surface elevation of the upper ice surface.

When deriving the weak form we do integration by terms on the first two terms

$$\begin{aligned}
&\int (4\partial_x(\eta h\partial_x u) - \frac{1}{2}g\cos\alpha\rho(1-\rho/\rho_o)\partial_x h^2 - \rho g h \cos\alpha\partial_x s^{'} - t_x + \rho g h \sin\alpha)N\,dx \\
&= \quad (4\eta h \partial_x u - \frac{1}{2}g\cos\alpha\rho(1-\rho/\rho_o)h^2)N|_{x_1}^{x_2} \\
&- \quad \int (4\eta h\partial_x u - \frac{1}{2}g\cos\alpha\rho(1-\rho/\rho_o)h^2)\partial_x N\,dx \\
&- \quad \int (\rho g\cos\alpha h\partial_x s^{'} + t_x - \rho g h\sin\alpha)N\,dx
\end{aligned}$$

The neat thing about this formulation is that for the usual BC at the ice-shelf edge, the 'boundary integral term' is zero.

The quantity $s^{'}$ is the difference between the actual surface altitude above sea level, and the surface altitude above sea level if floating. On a floating ice shelf $s^{'}$ is equal to zero everywhere.

If all von Neumann boundary conditions are of the type (8.44) we only have to solve

$$\int ((-4\eta h\partial_x u + \frac{1}{2}g\cos\alpha\rho(1-\rho/\rho_o)h^2)\partial_x w - (\rho g\cos\alpha h\partial_x s^{'} + t_x - \rho g h\sin\alpha)N)\,dx = 0$$

with

$$s^{'} = s - S - (1-\rho/\rho_o)h$$

or

$$-\int 4\eta h\partial_x u\,\partial_x N\,dx - \int t_x\,N\,dx = \rho g\cos\alpha\int h\,\partial_x s^{'} N\,dx - \frac{1}{2}\rho g\cos\alpha(1-\rho/\rho_o)\int h^2\partial_x N\,dx - \rho g\sin\alpha\int hN\,dx$$

# Part II

# Some aspects of glacier mechanics, possibly of interest to *Úa* uses, but not specifically related to *Úa*

Here I've put some rather random bits related to glacier mechanics. The selection is based both on what many *Úa* users might find useful, but also reflects somewhat the topics I've covered in previous lectures that I've given on glacier mechanics, especially lectures given at Caltech in 2014.

# Chapter 9

# An ice shelf in one horizontal dimension (1HD).

We consider an ice shelf in one horizontal dimension (1HD) under plain-strain conditions

$$\dot{\epsilon}_{yy} = 0,$$

and in addition we assume $v = 0$, and that all other transverse gradients to be zero as well. This ice shelf is laterally confined, , i.e. it can not spread out in $y$ direction, but there is no friction along the sides. It is free to deform in $x$ direction only.

The vertical stress is, as always in the SSA approximation, given by

$$\sigma_{zz} = -\rho g(s - z), \tag{9.1}$$

where $s$ is the upper surface. The traction at the lower surface, where $z = b$, must equal the ocean pressure giving

$$\rho(s - b) = \rho_o(S - b),$$

from which various other floating relationships follow:

$$h = \rho_o d/\rho = \frac{s - S}{1 - \rho/\rho_o} = \frac{\rho_o}{\rho}(S - b), \tag{9.2}$$

$$b = \frac{\rho s - \rho_o S}{\rho - \rho_o} = S - \frac{\rho}{\rho_o} h, \tag{9.3}$$

$$s = S + (1 - \rho/\rho_o)h = (1 - \rho_o/\rho)b + \frac{\rho_o}{\rho} S, \tag{9.4}$$

$$f = (1 - \rho/\rho_o)h. \tag{9.5}$$

Note that we write $d = S - b$ for the ice shelf draft, $f = S - s$ for the freeboard, and $h = s - b$ and $H = S - B$.

The SSTREAM/SSA equation to be solved is in this case simply

$$4\partial_x(h\eta\partial_x u) = \rho g(1 - \rho/\rho_o)h\,\partial_x h, \tag{9.6}$$

where the effective viscosity $\eta$ is given by

$$\eta = \frac{1}{2} A^{-1/n} |\partial_x u|^{(1-n)/n}.$$

Eq. (9.6) is the vertical integrated form of the momentum equation in $x$ direction, and reflects the equilibrium of vertically integrated forces in the horizontal direction . The equation can also written in terms of the horizontal deviatoric stress $\tau_{xx}$ as

$$2\partial_x(h\tau_{xx}) = \rho g(1 - \rho/\rho_o)h\,\partial_x h, \tag{9.7}$$

or as

$$\partial_x(h\tau_{xx}) = \frac{1}{4} g\varrho\,\partial_x h^2, \tag{9.8}$$

where

$$\varrho = \rho\,(1 - \rho/\rho_o).$$

If $\varrho$ is independent of $x$ (an assumption that has already been made in the derivation of Eq. 9.6) then we can integrate (9.8) on both sides, and we find that

$$\tau_{xx} = \frac{1}{4} g\varrho\, h + K, \tag{9.9}$$

where $K$ is independent of $x$.

Note that the freeboard $f = s - S$ is equal to

$$f = (1 - \rho/\rho_o)h,$$

so we can also write (9.9) as,

$$\tau_{xx} = \frac{1}{4} g\rho f + K. \tag{9.10}$$

The integration constant $K$ follows from the boundary conditions, and we will show below that $K = 0$.

Eq. (9.10) shows that the horizontal deviatoric stresses are directly proportional to the freeboard. In some ways $g\rho f/4$ is the 'driving stress' that drives the deformation of the ice shelf, and it plays a similar role to the driving stress $\rho g h \partial_x s$ in the SIA.

Knowing $\tau_{xx}$ and $\sigma_{zz}$, we can now determine the pressure $p$ from

$$p = \tau_{zz} - \sigma_{zz} = -\tau_{xx} - \sigma_{zz}, \tag{9.11}$$

using the incompressibility condition $\dot\epsilon_{xx} + \dot\epsilon_{zz} = 0$ and the flow law. The horizontal stress $\sigma_{xx}$ is then

$$\sigma_{xx} = \tau_{xx} - p = \tau_{xx} - (-\tau_{xx} - \sigma_{zz}) = 2\tau_{xx} + \sigma_{zz}, \tag{9.12}$$

Once we have determined all stresses, we can determine the deformation using the Glen-Steinemann flow law

$$\dot\epsilon_{ij} = A\tau^{n-1}\tau_{ij}, \tag{9.13}$$

where $A$ and $n$ are some rheological parameters, and $\tau$ is the second invariant of the deviatoric stress tensor given by

$$\tau = (\tau_{pq}\tau_{pq}/2)^{1/2}\,. \tag{9.14}$$

## 9.1 Boundary condition at the calving front

Across the (vertical) interface between ice and ocean the traction must be continuous, i.e.

$$(\sigma_{\text{ice shelf}} - \sigma_{\text{ocean}})\hat{\boldsymbol{n}} = 0,$$

where $\hat{\boldsymbol{n}}$ is a unit normal vector pointing horizontally outwards from the calving front. The position of the calving front is given by $x = x_c$.

We ignore bending forces at the calving front and only require continuity of integrated values. Hence we require

$$\underbrace{\int_b^s \sigma_{xx}\,dz}_{\text{ice shelf}} = \underbrace{\int_b^S \sigma_{xx}\,dx}_{\text{ocean}} \tag{9.15}$$

The vertically integrated force of the ocean on the ice at $x = x_c$ is given by the integral

$$\underbrace{\int_b^s \sigma_{xx}\,dx}_{\text{ocean}} = -\int_b^S \rho_o g(S - z)\,dz = -\frac{1}{2}\rho_o g(S - b)^2, \tag{9.16}$$

where $S$ is the ocean surface.

This integrated force must equal the vertical integrated horizontal stress within the ice shelf along the calving front given by the integral

$$\int_b^s \sigma_{xx}\,dz,$$

at $x = x_c$. We know $\sigma_{zz}$ within the ice shelf from Eq. (9.1). Using Eq. (9.12) to express $\sigma_{xx}$ in terms of $\sigma_{zz}$ and $\sigma_{xx}$, we find

$$
\begin{aligned}
\underbrace{\int_b^s \sigma_{xx}\, dz}_{\text{ice shelf}} &= \int_b^s (2\tau_{xx} + \sigma_{zz})\, dz \\
&= 2h\tau_{xx} + \rho g \int_b^s (z - s)\, dz \\
&= 2h\tau_{xx} + \rho g \left((s^2 - b^2)/2 - s(s - b)\right) \\
&= 2h\tau_{xx} - \frac{1}{2}\rho g h^2, \qquad (9.17)
\end{aligned}
$$

where we have made use of the fact that $\tau_{xx}$ is independent of $z$. Hence, equality of vertical integrated horizontal stresses at the calving front requires

$$2h\tau_{xx} - \frac{1}{2}\rho g h^2 = -\frac{1}{2}\rho_o g(S - b)^2, \qquad (9.18)$$

Using Eq. (9.18) together with the floating condition (1.53) we find

$$
\begin{aligned}
2h\tau_{xx} &= \frac{1}{2}\rho g h^2 - \frac{1}{2}\rho_o g(S - b)^2 \\
&= \frac{1}{2}\rho g h^2 - \frac{1}{2}\rho_o g h(S - b)\frac{\rho}{\rho_o} \\
&= \frac{1}{2}\rho g h(h - S + b) \\
&= \frac{1}{2}\rho g h(s - S)
\end{aligned}
$$

or

$$\tau_{xx} = \frac{1}{4}\rho g f, \qquad (9.19)$$

at the calving front where $x = x_c$

Comparing the above boundary condition, valid at the calving front, with expression (9.9), valid anywhere within the ice shelf, we find that integration constant $K$ in Eq. (9.9) is equal to zero, and therefore

$$\tau_{xx} = \frac{1}{4}\rho g f,$$

everywhere.

Note that the (horizontal) boundary condition (9.19) is now identical to the expression for horizontal deviatoric stresses valid everywhere within the ice shelf. Physically this implies that at any given location within the ice shelf, the stresses are identical to stresses imposed by the calving-front boundary condition at that location. In other words, if we were to cut off the ice shelf at any given location, thereby forming a new calving front, the stresses at that newly formed calving front will not be affected. In this respect, the ice shelf downstream of a given point is 'passive' and does not affect the flow in upstream direction. In particular, the stresses in the ice shelf downstream from the grounding line do not affect the stresses in the ice shelf at the grounding line.

## 9.2 The SSA as an expression of horizontal force balance.

Inserting the Glen-Steinemann flow law directly into the boundary condition (9.19) gives

$$A^{-1/n}|\partial_x u|^{(1-n)/n}\, \partial_x u = \frac{1}{4}\rho g f$$

Since $(s - S) \geq 0$, both $\tau_{xx}$ and $\partial_x u$ are positive, and Eq. (9.19) can be written on the form

$$\dot{\epsilon}_{xx} = \partial_x u = A(\rho g f/4)^n, \qquad (9.20)$$

at $x = x_c$. Further versions of Eq. (9.19) are

$$2A^{-1/n}h|\partial_x u|^{(1-n)/n}\, \partial_x u = \frac{1}{2}\rho g(1 - \rho/\rho_o)h^2 \qquad (9.21)$$

and

$$8\eta h\partial_x u - \rho g(1-\rho/\rho_o)h^2 = 0, \tag{9.22}$$

at $x = x_c$.

Note that if we differentiate Eq. (9.22) with respect to $x$, we find

$$4\partial_x(\eta h\partial_x u) = \rho g(1-\rho/\rho_o)h\partial_x h. \tag{9.23}$$

which is formally identical to Eq. (9.6) valid for any $x$. We arrived at (9.23) by considering the vertical integrated force balance along the calving front using 1) the fact that the effective stress $\tau_{xx}$ is independent of $z$, and 2) that the vertical stress component $\sigma_{zz}$ is equal to the weight of the ice (Eq. 9.1). By simply making these two assumptions, rather than deriving these facts through scaling analysis as we have done, we would have been able to derive Eq. (9.6) in a fairly simple manner.

## 9.3 Stresses and strains within a one-dimensional plane-strain ice shelf

We now know that

$$\tau_{xx} = \frac{1}{4}\rho g f, \tag{9.24}$$

everywhere within a 1HD ice shelf.

The only non-zero terms of the deviatoric stress tensor are $\tau_{xx}$ and $\tau_{zz}$. From (9.14) we find that

$$\tau = |\tau_{xx}| = \frac{1}{4}\rho g f.$$

and using the Glen-Steinemann flow law that

$$\dot\epsilon_{xx} = \partial_x u = A\,(\rho g f/4)^n, \tag{9.25}$$

or alternatively using (9.5)

$$\dot\epsilon_{xx} = \partial_x u = A\,(\varrho g h/4)^n, \tag{9.26}$$

where

$$\varrho = \rho\,(1-\rho/\rho_o).$$

The horizontal stress component can then be calculated as

$$\sigma_{xx} = 2\tau_{xx} + \sigma_{zz} = \frac{1}{2}\rho g f + (z-s)\rho g \tag{9.27}$$

where we have used Eq. (9.1). As can be seen $\sigma_{xx}$ varies linearly with depth. Along the upper surface where $z = s$, $\sigma_{xx} = \rho g f/2$ and is positive, and along the lower surface, where $z = b$, $\sigma_{xx} = -\rho(1-\rho/\rho_o)gh/2$, and is negative.

The pressure is given by

$$p = \tau_{zz} - \sigma_{zz} = -\frac{1}{4}\rho g f + (s-z)\rho g.$$

Note that the pressure is not hydrostatic.

We write the transverse stress component as

$$\sigma_{yy} = \tau_{yy} - p = \tau_{yy} + \sigma_{zz} - \tau_{zz}.$$

The plane strain conditions implies $\tau_{yy} = 0$ and incompressibility $\tau_{zz} = -\tau_{xx}$, hence

$$\sigma_{yy} = \sigma_{zz} + \tau_{xx} = (z-s)\rho g + \rho g f/4.$$

## 9.4 Shear stress

The shear stress $\tau_{xz}$ is a first-order quantity (but its vertical derivative enters the equilibrium equations at zeroth order). From

$$\partial_x \sigma_{xx} + \partial_z \sigma_{xz} = 0.$$

we find

$$\partial_z \sigma_{xz} = -\partial_x \sigma_{xx} = \frac{1}{2}\rho g \partial_x s, \tag{9.28}$$

which shows that $\sigma_{xz}$ also varies linearly with depth and that

$$\tau_{xz} = \frac{1}{2}\rho g\, \partial_x s\; z + K, \tag{9.29}$$

where $K$ is an integration constant.

The boundary condition at the upper surface ($z = s$) is

$$\tau_{xz}(z = s) = \sigma_{xx}(z = s)\; \partial_x s$$

giving

$$\tau_{xz}(z = s) = \frac{1}{2}\rho g s \partial_x s. \tag{9.30}$$

using (9.27). Similarly the boundary condition at $z = b$ gives,

$$\begin{aligned} \tau_{xz}(z = b) &= \rho g h\, \partial_x b + \sigma_{xx}(z = b)\partial_x b, \\ &= \rho g h\, \partial_x b + \frac{1}{2}\rho g f\, \partial_x b - \rho g h\, \partial_x b, \\ &= \frac{1}{2}\rho g f\, \partial_x b. \end{aligned} \tag{9.31}$$

From Eq. (9.30) we find can now determine $K$ in (9.29) and find

$$\tau_{xz} = \frac{1}{2}\rho g\, \partial_x s\; (z - S). \tag{9.32}$$

It remains to be seen if this expression is consistent with the other boundary condition (9.31) at the lower surface. Inserting $z = b$ into (9.32) gives

$$\tau_{xz} = \frac{1}{2}\rho g\, \partial_x s\; (b - S). \tag{9.33}$$

Using the floating condition one can show that

$$\partial_x s (b - S) = \partial_x b\, (s - S),$$

and therefore that (9.32) fulfils the lower boundary condition also.

From (9.32) we see that $\sigma_{xz}/\tau_{zz}$ is $O(\delta)$ as expected. In contrast to the other stress terms listed above, the shear stress $\tau_{xz}$ is a first-order quantity.

Summarising, the stress tensor in an ice shelf where all transverse gradients are zero ($\partial/\partial y = 0$) and $S = 0$, is given by

$$\boldsymbol{\sigma} = -\rho g \begin{pmatrix} s/2 - z & 0 & -\frac{1}{2} z \partial_x s \\ 0 & 3s/4 - z & 0 \\ -\frac{1}{2} z \partial_x s & 0 & s - z \end{pmatrix}, \tag{9.34}$$

and the pressure by

$$p = \rho g (3s/4 - z),$$

and deviatoric stress tensor

$$\boldsymbol{\tau} = \boldsymbol{\sigma} + \mathbf{1} p,$$

is

$$\boldsymbol{\tau} = \rho g \begin{pmatrix} s/4 & 0 & \frac{1}{2} z \partial_x s \\ 0 & 0 & 0 \\ \frac{1}{2} z \partial_x s & 0 & -s/4 \end{pmatrix}. \tag{9.35}$$

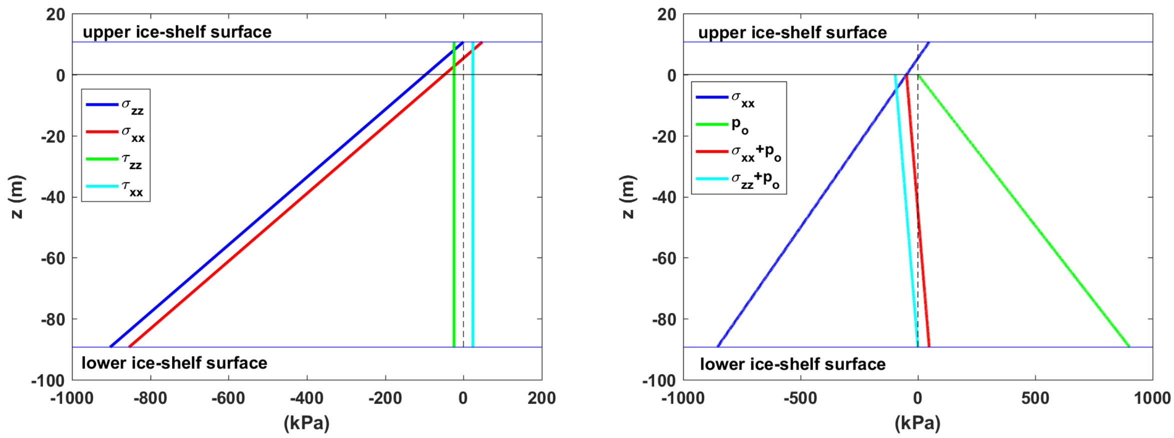



Figure 9.1: Left: Stresses within a one-dimensional ice shelf. Horizontal Cauchy stresses are positive at the surface and negative below $z = d/2$ where $d$ is the is ice shelf draft. Horizontal and vertical deviatoric stresses are independent of depth, and horizontal deviatoric stresses positive while vertical deviatoric stresses are negative. Parameters: $\rho = 910\,\mathrm{kg\,m^{-3}}$, $\rho_o = 1030\,\mathrm{kg\,m^{-3}}$, $h = 100\,\mathrm{m}$.

Note that the sign of the horizontal stress components ($\sigma_{xx}$ and $\sigma_{yy}$) changes with depth. At the surface ($z = s$) horizontal stresses are positive, at the base ($z = b$) they are negative. The longitudinal horizontal stresses ($\sigma_{xx}$) are only positive (compression) for $z > s/2$ (see Fig. 9.1).

The stresses in the shelf given by Eq. (9.34) are valid everywhere within the ice shelf, in particular the stresses in the ice shelf at the grounding line are also given by Eq. (9.34). As is evident from Eq. (9.34) the stresses, and therefore also the strain rates, are at each location functions of local surface slope and local ice thickness only.

Note that it has here been assumed that $S = 0$, in which case $s = f$, so one could replace $s$ with the freeboard ($f$) in the above expressions for the stresses.

The ocean pressure, $p_o$ is

$$p_o = \rho_o(S - z)$$

for $z < S$, and therefore

$$\sigma_{xx} + p_o = \rho g(s/2 - z) + \rho_o z$$

for $S = 0$. For $z = -\frac{1}{2}\frac{\rho}{\rho_o}h$ the sum of horizontal stresses and ocean pressure is zero (Fig. 9.1).

## 9.5 Steady-state geometry of a 1HD plane-strain ice sheet

We will now derive an analytical expressions for steady-state geometry and the velocity of a 1HD plane-strain ice sheet. The surface mass-balance is assumed to be constant. This surface mass balance ($a$) can be thought of as the sum of the mass fluxes along the upper ($a_s$) and the lower surface ($a_b$), i.e. $a = a_s + a_b$.

From Eq. (9.26) we have

$$\partial_x u = A\,(\varrho g h/4)^n. \tag{9.36}$$

In a steady state, mass continuity requires

$$\partial_x(uh) = a, \tag{9.37}$$

where we have used that the ice flux $q$ is $q = uh$. Assuming constant accumulation rate we can integrate Eq. (9.37) giving

$$u(x)h(x) - q_{gl} = a(x - x_{gl}) \tag{9.38}$$

where $x_{gl}$ is the grounding line position, and $q_{gl} = q(x_{gl})$ is the flux at the grounding line. We define the origin of the $x$ coordinates so that $x_{gl} = 0$.

We also have from Eq. (9.37)

$$h\,\partial_x u + u\partial_x h = a \tag{9.39}$$

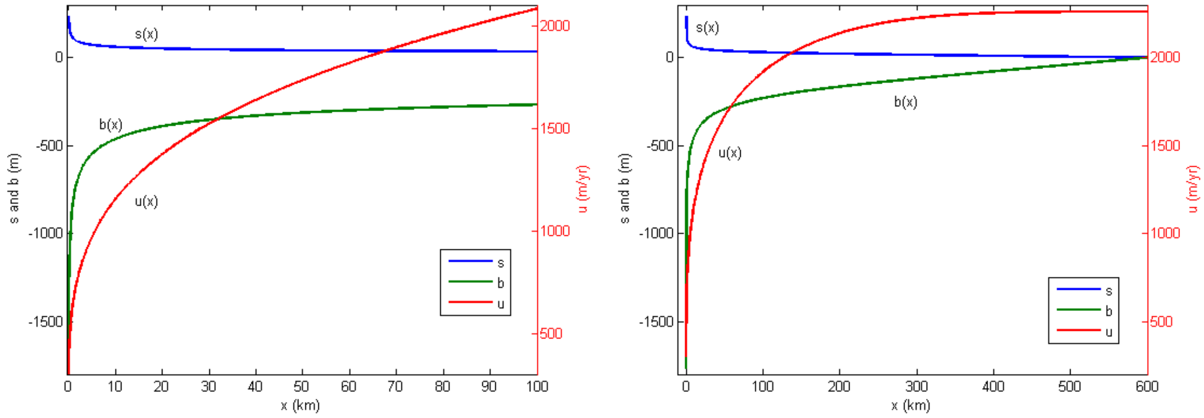



Figure 9.2: Analytical ice shelf profile. The left hand figure is for an accumulation of $a = 0.3\,\mathrm{m\,a^{-1}}$, while the figure on the right was made for $a = -1\,\mathrm{m\,a^{-1}}$. All other parameters are the same in both cases. Parameters: $A = 1.14 \times 10^{-9}\,\mathrm{kPa^{-3}a^{-1}}$, $n = 3$, $h_{gl} = 2000\,\mathrm{m}$, $u_{gl} = 300\,\mathrm{m\,a^{-1}}$, $\rho = 910\,\mathrm{kg\,m^{-3}}$, $\rho_o = 1030\,\mathrm{kg\,m^{-3}}$. The value for $A$ corresponds to an ice temperature of about -10 degrees Celsius.

Replacing $u$ in (9.39) using (9.38) and inserting (9.36) for $\partial_x u$, gives

$$Ah\,(\varrho g h/4)^n + ((ax + q_{gl})/h)\partial_x h = a, \tag{9.40}$$

which we write as

$$\gamma h^{n+2} + (ax + q_{gl})d_x h = ah, \tag{9.41}$$

with

$$\gamma = A(\varrho g/4)^n\,. \tag{9.42}$$

Separating variables

$$\frac{dh}{ah - \gamma h^{n+2}} = \frac{dx}{ax + q_{gl}}, \tag{9.43}$$

integrating both sides and simplifying gives

$$h = \left(\frac{1}{a}\left(\gamma + \frac{K}{(q_{gl} + ax)^{n+1}}\right)\right)^{-1/(n+1)}, \tag{9.44}$$

where $K$ is an integration constant.

We determine $K$ by specifying the thickness at the grounding line, i.e.

$$h(x_{gl} = 0) = h_{gl},$$

and using $q_{gl} = h_{gl} u_{gl}$ which gives

$$K = q_{gl}^{n+1}(a/h_{gl}^{n+1} - \gamma). \tag{9.45}$$

The solution is shown in Fig. 9.2. Possibly the most striking aspect of the solution is how quickly the thickness decreases downstream from the grounding line.

Now that the ice geometry has been determined, the ice velocity can be calculated directly from

$$\begin{aligned} u(x) &= (ax + q_{gl})/h(x) \\ &= \left(\frac{K + \gamma(q_{gl} + ax)^{n+1}}{a}\right)^{1/(n+1)} \end{aligned} \tag{9.46}$$

For $a > 0$, the ice sheet is infinitely long and approaches asymptotically the thickness $h(x \to +\infty) = (a/\gamma)^{1/(n+1)}$. For $a < 0$, the ice shelf has a finite length $l$ given by $l = -q_{gl}/a$.

The special case $a = 0$ is not covered by the above equations. One finds that for $a = 0$ the thickness distribution is given by

$$h = (h_{gl}^{-(n+1)} + \gamma(n+1)x/q_{gl})^{-1/(n+1)}. \tag{9.47}$$

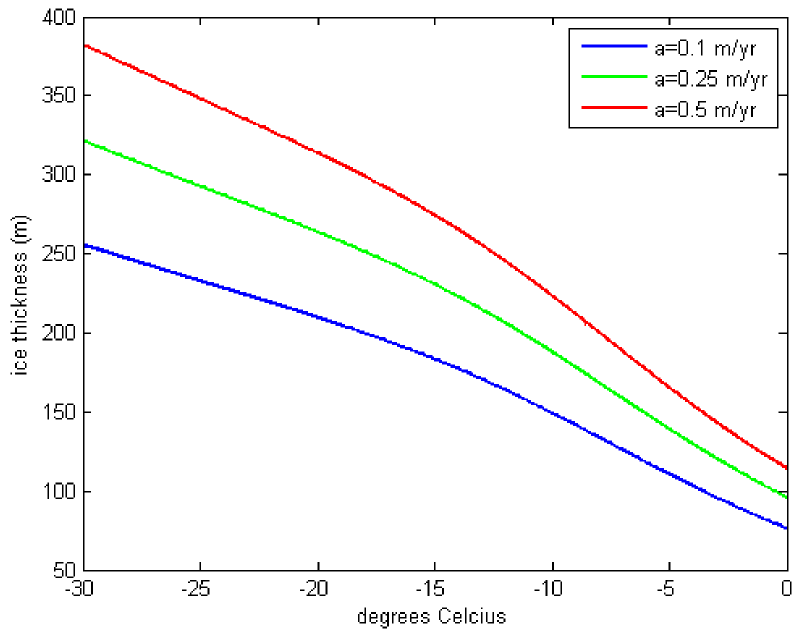



Figure 9.3: Ice shelf thickness as $x \to +\infty$ as a function of englacial temperature and surface accumulation. Parameters: $n = 3$, $\rho = 910\,\mathrm{kg\,m^{-3}}$, $\rho_o = 1030\,\mathrm{kg\,m^{-3}}$.

The above solution describes an ice shelf, with a given ice thickness $h = h_{gl}$ at the grounding line, that spreads out in 1HD without any addition or removal of mass. In the limit $x \to +\infty$ the ice thickness is zero.

Note that in the above analysis we fixed the flux at the grounding line to $q_{gl}$. We then determined the ice geometry and velocities down-stream of the grounding line. We are here not in a position to calculate the flux at the grounding line for a given ice thickness (or as a function of any other aspects of the ice geometry that might affect the flux). For a given ice thickness $h = h_{gl}$, and a given ocean bathymetry ($B(x)$), we can however always determine possible positions of the grounding line from the floating condition $\rho h = \rho_o H$, where $H = S - B$ with $S$ the ocean surface and $B$ the ocean bed.

Also note that there are two further possible solutions to the differential equation (9.41). There is the (trivial) solution $h = 0$ and also the somewhat more interesting solution $h = (a/\gamma)^{1/(n+1)}$. For $a < 0$ this solution can be discarded (negative thickness). However, if $a > 0$ this solution represents an ice shelf with a constant ice thickness that is regenerated through snow fall at the same rate that it spreads out. The ice thickness of such a steady-state ice shelf is shown in Fig. 9.3 as a function of temperature and surface mass balance.

Another interesting fact is that the ice shelf thickness can *increase* with distance downstream from the grounding line. This happens whenever

$$h_{gl} < (a/\gamma)^{1/(n+1)}.$$

This can be seen from an inspection of the solution shown above, or by writing

$$uh = ax + q_{gl},$$

differentiating and inserting Eq. (9.26), giving

$$u\partial_x h + hA(\varrho g h/4)^n = a,$$

and then using $u = (ax + q_{gl})/h$ and solving for $\partial_x h$ to arrive at

$$\partial_x h = \frac{a - \gamma h^{n+1}}{(ax + q_{gl})/h},$$

showing that $\partial_x h$ is positive for $h < (a/\gamma)^{1/(n+1)}$ .

## 9.6 Side-drag dominated ice shelf

We are here interested in the limiting case when all the driving stress is balanced by side drag alone, i.e. the term $\partial_x(h\tau_{xx})$ now dropped. This is the opposite limit to the one we considered above where $\partial_x(h\tau_{xx})$ was the dominating term and was $\partial_y(h\tau_{xy})$ was ignored.

The shallow-ice stream (SSTREAM/SSA/Shelfy) equations are

$$\partial_x(h(2\tau_{xx}+\tau_{yy}))+\partial_y(h\tau_{xy})-t_{bx}=\rho g h\partial_x s \tag{9.48}$$

$$\partial_y(h(2\tau_{yy}+\tau_{xx}))+\partial_x(h\tau_{xy})-t_{by}=\rho g h\partial_y s \tag{9.49}$$

Over the floating section $t_{bx}=t_{by}=0$. We assume

$$\begin{aligned}\dot\epsilon_{yy}=\tau_{yy}&=0,\\ \partial_y\tau_{xx}&=0,\\ \partial_y s=\partial_y h&=0,\end{aligned}$$

and ignore longitudinal stretching in the momentum balance, hence Eqs. (9.48) and (9.49) become

$$\partial_y(h\tau_{xy})=\varrho g h\partial_x h, \tag{9.50}$$

$$\partial_x(h\tau_{xy})=0. \tag{9.51}$$

We integrate (9.50) with respect to $y$ from the centre-line $y=0$ to the left-hand margin $y=w$, where the total width of the ice shelf is $2w$, i.e.

$$\tau_m=-\varrho g w\partial_x h,$$

where we $\tau_m$ is a (positive) shear stress at the margin, i.e. $\tau_m=-\tau_{xy}(y=w)$, and we have used that $\tau_{xy}(y=0)=0$. We consider the plastic case when $\tau_m$ is a yield stress and independent of ice velocity. This simplification allows us to *calculate the ice thickness directly* as

$$h=h_c-\frac{\tau_m}{\varrho g w}(x-x_c),$$

where $h_c$ is the ice thickness at the calving front $x=x_c$. The ice thickness is now a linear function of distance $x$. For a given location, $x_c$, of the grounding line, the grounding line will simply be located where the ice draft reaches the ocean floor. In this particular case the mass balance upstream of the grounding line has no impact on the position of the grounding line.

Eq. (9.51) remains correct if evaluated along the centre line where $\tau_{xy}=0$. However, it is unclear if and how this equation can be consistent with the assumption of constant side shear stress and variable ice thickness. But if the ice behaves, as we assume, as a plastic material, then this poses no problem.

Similar to the case where a plastic formulation for basal sliding is used, the geometry is determined directly from the yield stress.[1] The steady-state velocity can then be calculated from the ice thickness distribution using

$$h\partial_x u+u\partial_x h=0,$$

giving

$$(\lambda-(x-x_c))\,\partial_x u-u=0,$$

where

$$\lambda=\frac{\varrho g w h_c}{\tau_m}.$$

The length scale $\lambda$ is $\varrho g h_c/\tau_m$ times the half-width of the ice shelf. I'm guessing a reasonable estimate is $O(\varrho g h_c/\tau_m)=1$ so $\lambda$ might be similar to the width.

[1] For a plastic symmetrical ice sheet on a flat bed

$$\rho g h\partial_x h=\tau_b$$

where $\tau_b$ is here the basal yield stress and therefore $\partial_x h^2=\frac{2\tau_b}{\rho g}$ and hence

$$h^2=\frac{2\tau_b}{\rho g}|x-x_c|$$

# Chapter 10

# Simple 1d solution for an ice-stream

## 10.1 Problem definition:

Uniform ice thickness $h$ on a constant sloping bed with slope $\alpha$. The calving front position is at $x = l$. The calving front can be either grounded or floating, and $d$ $0 \le d < \rho h/\rho_o$.

$$4\partial_x(h\eta\partial_x u) - \beta^2\, u = \rho g h \partial_x s$$

with

$$\eta = \frac{1}{2}A^{-1/n}\,|\partial_x u|^{(1-n)/n}$$
$$\beta^2 = C^{-1/m}\|u\|^{(1-m)/m}$$

Boundary conditions:

$$u = C\rho g h\alpha \quad \text{at} \quad x = 0 \tag{10.1}$$
$$\tau_{xx} = \frac{1}{4h}g(\rho h^2 - \rho_o d^2) \quad \text{at} \quad x = l \tag{10.2}$$

Boundary condition (10.2) can also be written as

$$\partial_x u|_{x=l} = A\left(\frac{g(\rho h^2 - \rho_o d^2)}{4h}\right)^n$$

## 10.2 Solution:

The non-linear case is

$$\frac{2hA^{-1/n}}{n}(\partial_x u)^{1/n-1}\,\partial^2_{xx}u - C^{-1/m}u^{1/m} = -\rho g h\alpha$$

which I'm not sure if can be solved.

However the linear case

$$\partial^2_{xx}u - \kappa^2 u = -\frac{A\tau}{2h},$$

has the general solution

$$u = c_1 e^{\kappa x} + c_2 e^{-\kappa x} + C\tau,$$

with

$$\kappa^2 = \frac{A}{2hC},$$

and

$$\tau = \rho g h\alpha$$

BCs (10.1) and (10.2) give

$$c_1 + c_2 + C\,\tau = C\,\tau$$
$$c_1\kappa e^{\kappa l} - c_2\kappa e^{-\kappa l} = K$$

where

$$K = A\frac{g(\rho h^2 - \rho_o d^2)}{4h}.$$

Hence

$$u = C\tau + \frac{K \sinh \kappa x}{\kappa \cosh \kappa l}.$$

# Chapter 11

# Grounding-line dynamics

Here some basic aspects of grounding-line dynamics are summarised. This is all well-known stuff from the literature.

## 11.1 Ice-Shelf Buttressing

Ice-shelf buttressing is defined by the impact of the ice shelf on the stress at the grounding line. If the vertically integrated horizontal stress state is unaffected by the ice shelf — i.e. if removing the ice shelf does not affect the state of stress at the grounding line — the ice-shelf provides no mechanical support to the grounding line beside that of the ocean, and there is no ice-shelf buttressing.

It is sometimes convenient to define a *buttressing parameter* $\theta$ as

$$\theta = \frac{N}{\frac{1}{2}\varrho g h}$$

where

$$N = \hat{\boldsymbol{n}}_h^T \cdot (\boldsymbol{R}\hat{\boldsymbol{n}}_h) \tag{11.1}$$

and

$$\varrho = \rho(1 - \rho/\rho_o),$$

and where $\hat{\boldsymbol{n}}_h$ is a normal vector pointing horizontally outwards from the grounding line. Buttressing is the difference between the normal stress at the grounding line with and without an ice shelf.

In the particular case of a floating ice shelf, the field equations Eq. (1.4) can be written as

$$\nabla_{xy}^T \cdot (h\,\boldsymbol{R}) = \frac{1}{2}\nabla_{xy}^T(\varrho g \rho h^2),$$

Using the divergence theorem we find

$$\oint(\boldsymbol{R}\cdot\hat{\boldsymbol{n}}_h - \frac{1}{2}\varrho g\rho h\,\hat{\boldsymbol{n}}_h)\,d\Gamma = 0 \tag{11.2}$$

The integrand is identical to the (point wise) expression of the force balance (1.72) at the calving front of a freely floating ice shelf. We can split this path integral into a 1) section along the grounding line, 2) section along the margins, and 3) section along the calving front. If the margins do not contribute, the contribution along the grounding line is equal to that of the calving front. Hence, unbuttressed uniformly-wide ice shelves are passive and don't provide any buttressing.

From Eq. (11.2) if follows that $\theta = 1$ implies no ice-shelf buttressing. This can be taken a bit further by defining normal and tangential buttressing numbers, but the principle is the same: If the ice-shelf does not affect the state of stress along the grounding line, the ice-shelf provides no buttressing.

## 11.2 Kinematic expression for GL migration

At the grounding line we have

$$\rho h = \rho_o H.$$

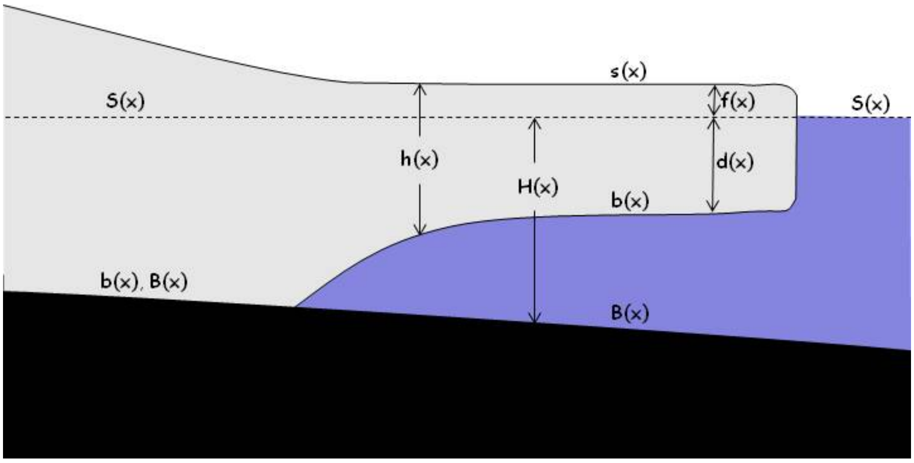



Figure 11.1: Geometrical variables: Glacier surface ($s$), glacier bed ($b$), ocean surface ($S$), ocean floor ($B$), glacier thickness ($h = s - b$)), ocean depth ($H = S - B$), glacier draft ($d = S - b$), glacier freeboard ($f = s - S$).

or simply

$$h = h_f$$

At any given time $t$, this condition must always be fulfilled at the grounding line. Note that $H = S - B$ is independent of time, whereas $h = s - b$ is a function of time and space, i.e.

$$h = h(x,t)$$
$$H = H(x)$$

We consider the rate-of-change as the grounding line is followed, i.e.

$$\left.\frac{dh_{gl}}{dt}\right|_{x=x_{\text{gl}}} = \frac{\rho_o}{\rho} \left.\frac{dH_{gl}}{dt}\right|_{x=x_{\text{gl}}} , \tag{11.3}$$

where

$$h_{gl} = h_{gl}(t, x_{\text{gl}}(t)) ,$$

and

$$H_{gl} = H_{gl}(x_{\text{gl}}(t)) ,$$

and therefore

$$\frac{dh_{gl}}{dt} = \frac{\partial h_{gl}}{\partial t} + \frac{\partial h_{gl}}{\partial x_{\text{gl}}} \frac{\partial x_{\text{gl}}}{\partial t} , \tag{11.4}$$

and

$$\frac{dH_{gl}}{dt} = \frac{\partial H}{\partial x_{\text{gl}}} \frac{\partial x_{\text{gl}}}{\partial t} . \tag{11.5}$$

Inserting (11.4) and (11.5) into (11.3) gives

$$\partial_t h + \dot{x}_{gl}\, \partial_x h = \frac{\rho_o}{\rho} \dot{x}_{gl}\, \partial_x H$$

or

$$\dot{x}_{gl} = \frac{\partial_t h}{\frac{\rho_o}{\rho} \partial_x H - \partial_x h} \tag{11.6}$$

$$= \frac{\partial_t h}{\partial_x (h_f - h)} \tag{11.7}$$

where we now have skipped writing the index $gl$ on the right-hand sides. All derivatives in (11.6) are local derivatives (but it is to be understood that all quantities must be evaluated at the current position of the grounding line).

Eqs. (11.6) and (11.7) are kinematic relationships, and we refer to this equation (11.7) as the *kinematic grounding-line equation.* It contains no additional physics beside that of the floating condition.

At the grounding line $h_f - h = 0$, and with increasing distance directly downstream of the grounding line $h_f - h$ must increase, and hence $\partial_x (h_f - h) \geq 0$. Therefore the grounding line must advance whenever $\partial_t h > 0$ .

One possible issue with using (11.7) might arise if the gradient in thickness ($\partial_x h$) were discontinuous across the grounding line. However, the boundary conditions dictate that $\partial_x u$ approaches the same value at the grounding line from both upstream and downstream directions. Furthermore, if $h$ is continuous across the grounding line, then mass conservation dictates that $\partial_x h$ is also continuous. So under these conditions using (11.7) is justified. As most numerical models do not use the kinematic grounding-line equation directly, the equation can, for example, be used as check on the accuracy of calculated grounding-line migration rates.

## 11.3 Geometrical grounding-line migration

Changing the sea level ($S$) causes a shift in the position of the grounding line. This shift is only related to geometrical factors such as ice thickness ($h = s - b$) and ocean depth bedrock ($H = S - B$). To distinguish this shift in grounding-line position from ice-dynamical effects, we refer to this shift as at 'geometrical grounding-line migration'. This horizontal shift in grounding line position due to time dependent changes in the height of the ocean surface can be determined as follows.

Ocean height ($S$) changes with time as

$$S(t) = \bar{S} + \Delta S(t),$$

but not spatially ( i.e. $\partial_x S(t) = \partial_x \bar{S} = \partial_x \Delta S = 0$)

For $S = \bar{S}$ the grounding line is at $x = x_{\mathrm{gl}}$ and

$$\rho \; (s(x_{\mathrm{gl}}) - b(x_{\mathrm{gl}})) = \rho_o(\bar{S} - B(x_{\mathrm{gl}})). \tag{11.8}$$

For a given perturbation, $\Delta S$, in ocean height, the grounding line moves by some distance $\Delta L$ in either up or down-stream direction. At this new grounding line position the floating condition must again hold, and we have

$$\rho \, (s(x_{\mathrm{gl}} + \Delta L) - b(x_{\mathrm{gl}} + \Delta L)) = \rho_o(\bar{S} + \Delta S - B(x_{\mathrm{gl}} + \Delta L))$$

or

$$\rho(s(x_{\mathrm{gl}}) + \partial_x s \Delta L - b(x_{\mathrm{gl}}) - \partial_x b \Delta L) = \rho_o(\bar{S} + \Delta S - B(x_{\mathrm{gl}}) - \partial_x B \Delta L))$$

(For notational simplicity we have not indicated in the above equation that the derivatives are to be evaluated on both sides of the grounding line and that they are in fact distinct directional derivatives in the up and down-stream directions.) Using (11.8) gives

$$\rho(\partial_x s - \partial_x b)\Delta L = \rho_o(\Delta S - \partial_x B \Delta L))$$

or

$$\Delta L^{+/-} = \frac{\rho_o \Delta S}{\rho(\partial_x s^{+/-} - \partial_x b^{+/-}) + \rho_o \partial_x B^{+/-}} \tag{11.9}$$

Eq. (11.9) is valid even if the derivatives are not constant across the location of grounding line ($x_{\mathrm{gl}}$) at mean tide ($\Delta S = 0$). We have indicated this by adding the superscript $+/-$. Here $\Delta S$ is positive for a high tide, and negative for a low tide. At low tide the gradients downstream of $x_{\mathrm{gl}}$ are to be used (minus sign), and at high tide the gradients upstream of $x_{\mathrm{gl}}$ (positive sign).

If $\partial_x B$ is the same on both sides of the $x_{\mathrm{gl}}$, then it follows from (11.9) that the shifts in grounding line position at high and low tides are only equal in magnitude provided $\partial_x h$ is the same on both sides of $x = x_{\mathrm{gl}}$ as well. In other words, for a constant bed slope, a tidally-induced grounding line migration is symmetrical if, and only if, the thickness gradient does not change across the grounding line. Conversely, if the thickness gradients are not equal on both sides of the grounding line, the grounding line movement will always be asymmetric with respect to the tidal cycle. In the particular case when the ice thickness is constant (and $\partial_x h = 0$) the (horizontal) grounding-line migration is symmetrical with respect to the tide.

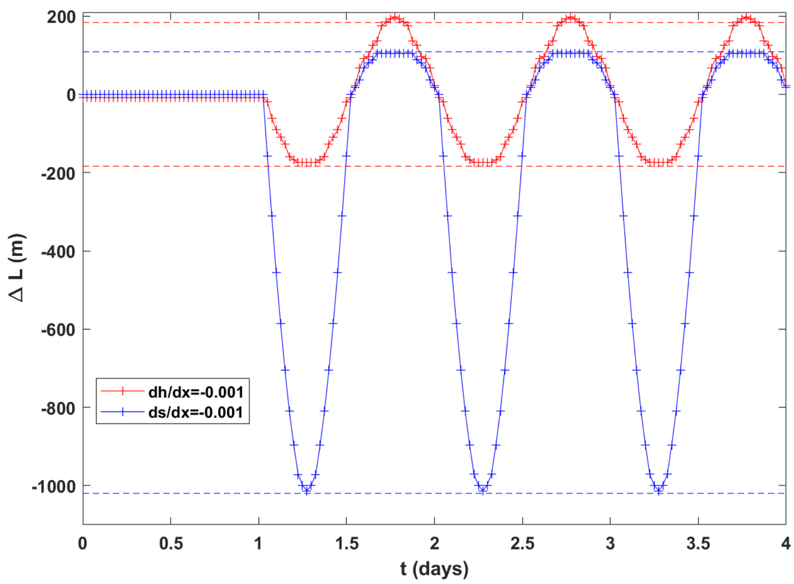



Figure 11.2: Example of grounding line migration in response to tidal forcing using the hydrostatic assumption. The curves were calculated using the flow model Úa which is a vertically integrated flow model that calculates grounding line positions using the hydrostatic assumption. The blue curve was calculated for a constant surface slope of $ds/dx = -0.001$ and red curve for a constant thickness gradient of $dh/dx = -0.001$. In both cases the bedrock gradient as $dB/dx = -0.01$. The tidal amplitude was 2 m and the tidal period 1 day. To suppress the effects of ice flow, the flow parameters were set to values that made the ice effectively rigid and basal sliding was enforced to be close to zero. The dashed lines show the upper and lower extent of horizontal grounding-line migration as calculated by Eq. (11.9).

Such an asymmetrical grounding line migration takes place if, for example, the upper surface gradient $(\partial_x s)$ is constant across the grounding line. In that case the thickness gradient cannot be equal on both sides of $x = x_{\rm gl}$ as well. There is then a break in thickness gradient at $x = x_{\rm gl}$ given by

$$\partial_x h = \begin{cases} \partial_x h^+ = \partial_x s - \partial_x B & \text{for } x \le x_{\rm gl} \\ \partial_x h^- = \partial_x s/(1-\rho/\rho_o) & \text{for } x \ge x_{\rm gl} \end{cases}$$

(There is no need to use the superscripts $+/-$ with $\partial_x s$ and $\partial_x B$ in the above equation because here we are assuming that those derivatives are continuous across the grounding line.) When this expression for the break in thickness gradient is inserted in (11.9) we find that the migration distance $\Delta L^-$ at a low tide (when $S = \bar{S} - \Delta L$), is given by

$$\Delta L^- = -\frac{\Delta S}{\frac{\rho/\rho_o}{1-\rho/\rho_o}\partial_x s + \partial_x B},$$

and at a high tide by

$$\Delta L^+ = \frac{\Delta S/(1-\rho/\rho_o)}{\frac{\rho/\rho_o}{1-\rho/\rho_o}\partial_x s + \partial_x B},$$

and that

$$|\Delta L^-| = (1-\rho/\rho_o)|\Delta L^+|.$$

In the case of a constant surface gradient, the upstream grounding-line shift at high tide is therefore about 9 times as large as the downstream shift at low tide.

In general we expect neither the thickness nor the surface gradient to be constant across the grounding line. Since the migration is only symmetrical in the particular case of a constant thickness gradient across the grounding line, we expect an asymmetrical grounding line migration in response to tides to be the general rule rather than an exception.

An example of transient hydrostatic grounding-line migration in response to tides is shown in Fig. 11.2. The migration was calculated using the flow model Úa. This model, as do most commonly used flow model in glaciology, assumes that the grounding line is always exactly where the hydrostatic floating condition $\rho h = \rho_o(S - b)$ is met. The modelled grounding line displacements are in a good agreement with those calculated using Eq. (11.9). For example, in the case of constant surface slope the modelled values are $\Delta L^- = 105$ m and $\Delta L^+ = -1013$ while those based on Eq. (11.9) are $\Delta L^- = 109$ m and $\Delta L^+ = -1020$. These differences of a few meters are considerably smaller than the spatial dimension of 85 m of the smallest element of the mesh used in this particular run by the FE-model Úa.

## 11.4 Flux at the grounding line

Upstream from the grounding line

$$2\partial_x \left( hA^{-1/n} \, |\partial_x u|^{1/n-1} \, \partial_x u \right) - C^{-1/m} \|u\|^{1/m-1} \, u = \rho g h \, \partial_x s \tag{11.10}$$

In terms of stresses this equation can also be written as

$$2\partial_x(h\tau_{xx}) - t_{bx} = \rho g h \partial_x s \tag{11.11}$$

Boundary conditions at the grounding line where $x = x_{\text{gl}}$ are

$$\partial_x u = A(\varrho g h/4)^n \tag{11.12}$$

$$h = \rho_o H/\rho. \tag{11.13}$$

Note that boundary condition (11.12) can be rearranged as

$$2A^{-1/n}(\partial_x u)^n = \frac{1}{2}\varrho g h$$

If we insert the above expression into (11.10), assuming $\partial_x u > 0$, then we arrive at

$$\frac{1}{2}\varrho g \partial_x h^2 - t_{bx} = \rho g h \partial_x s$$

If we now assume that $\partial_x s \approx \partial_x h$ upstream of the grounding line, for example by setting $s = B + h$ with $\partial_x B = 0$ for $x > x_{\text{gl}}$, we obtain

$$[\frac{1}{2}\varrho g \partial_x h^2] - [t_{bx}] = [\frac{1}{2}\rho g \partial_x h^2] \tag{11.14}$$

where the brackets are used to indicate that we are here simply comparing sizes of terms, and where we have written $h\partial_x h = \frac{1}{2}\partial_x h^2$

Since $\varrho \approx \rho/10$ is it clear that the first term on the left-hand side of (11.14) is small compared to the right-hand side, and that the right-hand side must therefore be approximately balanced by the second term on the left-hand side of (11.14), i.e.

$$[t_{bx}] = [-\frac{1}{2}\rho g \partial_x h^2]. \tag{11.15}$$

Note that the key assumption here is that $\partial_x s \approx \partial_x h$ for $x < x_{\text{gl}}$.

Downstream of the grounding line, flotation implies that $\partial_x s = \varrho \partial_x h/\rho$ (see Eq. 1.52) and if, for example, $\partial_x s \approx \varrho \partial_x h/\rho$, upstream of the grounding line we instead of (11.14) arrive at

$$[\frac{1}{2}\varrho g h^2] - [t_{bx}] = [\frac{1}{2}\varrho g \partial_x h^2] \tag{11.16}$$

and clearly now it is the first term on the left-hand side that balances the right-hand side.

We will now derive an approximation for the flux at the grounding line as a function of (local) ice thickness, and start by making the assumption that $\partial_x s \approx \partial_x h$ in which case as we have seen

$$t_{bx} \approx -\rho g h \partial_x s,$$

i.e. that the second term on the left-hand sides of (11.10) and (11.11) is now approximately balanced by their respective right-hand sides. Hence, using Weertman sliding law we have

$$u = C(-\rho g h \partial_x h)^m, \tag{11.17}$$

where we have anticipated that $\partial_x h$ will be strictly negative.

In a steady state

$$\partial_x(uh) = a, \tag{11.18}$$

which allows us to write

$$\partial_x h = (a - h\partial_x u)/u. \tag{11.19}$$

Inserting the boundary condition (11.12) into (11.19) gives

$$\partial_x h = (a - hA(\varrho gh/4)^n)/u, \tag{11.20}$$

and then inserting (11.20) into (11.17) and assuming that $a \ll hA(\varrho gh/4)^n$ gives

$$u = C\left(\rho g h h A \left(\rho g \delta g h/4\right)^n/u\right)^m. \tag{11.21}$$

or

$$u^{m+1} = 4^{-nm}\, C\, A^m (g\rho)^{m+nm} \delta^{nm}\; h^{nm+2m},$$

where $\delta$ is defined as

$$\delta := 1 - \rho/\rho_o$$

The ice flux $q = uh$ at the grounding line is therefore

$$q = \left[4^{-nm}\, C\, A^m (g\rho)^{m+nm}\;\; (1-\rho/\rho_o)^{nm}\right]^{1/(m+1)}\; h^{(nm+3m+1)/(m+1)}, \tag{11.22}$$

where[1] $h$ is the thickness at the grounding line, i.e.

$$h = h_{\mathrm{gl}} = \rho_o H/\rho.$$

The relationship between flux relationship (11.22) is identical to that of Schoof's 'B' model. We arrived at this flux relationship by assuming that velocity at the grounding line follows from SIA, and that $\partial_x u$ at the grounding line is given by the boundary condition (11.12) for horizontal strain rates. In addition we assumed steady-state conditions and that $a \ll h\partial_x u$.

Note that, as Eq. (11.21) shows, it is possible to express the velocity at the grounding line as a function of the ice thickness $h$ alone, i.e. without any reference to the surface slope $\partial_x h$. This is possible because the surface slope at the grounding line is related to thickness through Eq. (11.20). We were able to use the boundary condition (11.12), the mass conservation equation (11.18), and the simplified momentum equation (11.17) to arrive at Eq. (11.21), giving velocity at the grounding-line as a function of thickness alone.

## 11.5 Balance between terms on both side of the grounding line

(What follows is basically a slightly different framing of the argument in section 11.4 used to show that the basal shear traction will balance the driving stress upstream of the grounding line, provided $\partial_x s \approx \partial_x h$.)

Field equation and boundary condition at the grounding line written in terms of velocity are

$$2A^{-1/n}\partial_x\left(h\|\partial_x u\|^{(1-n)/n}\,\partial_x u\right) - \mathcal{H}(h-h_f)\,C^{-1/m}\,\|u\|^{(1-m)/m}u = \rho g h \partial_x s \tag{11.23}$$

$$2A^{-1/n}h\|\partial_x u\|^{(1-n)/n}\,\partial_x u = \frac{1}{2}\varrho g\, h^2 \quad \text{at} \quad x = x_{\mathrm{gl}} \tag{11.24}$$

$$h = h_f \quad \text{at} \quad x = x_{\mathrm{gl}} \tag{11.25}$$

where $\mathcal{H}$ is the Heaviside step function, and where

$$h_f = \rho_o H/\rho$$

and where $\varrho = \rho(1 - \rho/\rho_o)$.

[1] For $q = \rho u h$ and keeping the $\theta$ term we have

$$q = \rho\left(4^{-n}C^{1/m}A(\rho g)^{n+1}\delta^n\right)^{m/(1+m)}\;\theta^{nm/(1+m)}\;h^{(nm+3m+1)/(1+m)}$$

Inserting (11.24) into (11.23) and assuming that over the grounded area $|\partial_x s| \gg |\partial_x b|$, and therefore that $\partial_x h = \partial_x s - \partial_x b \approx \partial_x s$, we arrive at

$$\frac{1}{2}\varrho g \partial_x \left(h^2\right) - \mathcal{H}(h-h_f)\, C^{-1/m}\, \|u\|^{(1-m)/m} u = \rho g h \partial_x h$$

or

$$\frac{1}{2}\varrho g \partial_x h^2 - \mathcal{H}(h-h_f)\, C^{-1/m}\, \|u\|^{(1-m)/m} u = \frac{1}{2}\rho g \partial_x h^2$$

which immediately shows that the first term on the left hand side is $\varrho/\rho = \rho(1-\rho/\rho_o))/\rho = (1-\rho/\rho_o) \approx 0.1$ the size of the right-hand side.

Downstream of the grounding line $s = (1-\rho_o/\rho)h$ and therefore $\partial_x s = \delta \partial_x b \approx 0.1 \partial_x b$, or $\partial_x s \ll \partial_x h$ and this reversal of the relative sizes of the upper and lower slopes ensures that we now also have the right balance downstream of the grounding line, where the first term on the left-hand side now equals the right-hand side.

## 11.6 GL scalings (Schoof)

Field equations written in terms of velocity and stresses, respectively, are

$$2A^{-1/n}\partial_x \left(h\|\partial_x u\|^{(1-n)/n}\, \partial_x u\right) - \mathcal{H}(h-h_f)\, C^{-1/m}\, \|u\|^{(1-m)/m} u = \rho g h \partial_x s \tag{11.1}$$

$$2\partial_x \left(h\tau_{xx}\right) - \tau_b = \rho g h \partial_x s \tag{11.2}$$

where $\mathcal{H}$ is the Heaviside step function. Boundary conditions at $x = x_{\mathrm{gl}}$ are

$$2A^{-1/n} h\|\partial_x u\|^{(1-n)/n}\, \partial_x u = \frac{1}{2}\varrho g\, h^2 \tag{11.3}$$

$$h = h_f \tag{11.4}$$

where

$$h_f = \rho_o H/\rho$$

and where $\varrho = \rho(1-\rho/\rho_o)$. The above model is only valid for $[z]/[x] \ll 1$ and $u_d/u_b \ll 1$, where $u_d$ is the deformational velocity and $u_b$ the sliding velocity.

Scalings: With $[u]$ and $[x]$ as scales for the horizontal velocity and the span of the ice sheet, the kinematic boundary condition suggests

$$\frac{[u][z]}{[x]} = [a] \quad \text{and} \quad [t] = \frac{[x]}{[u]}$$

We set a scale for $C$ by writing

$$[u]^{1/m}/[C]^{1/m} = \rho g [z][z]/[x]$$

i.e. we balance basal shear stress with the driving stress. We introduce

$$\epsilon = \frac{\tau_{xx}}{\sigma_{xx}} = \frac{A^{-1/n}([u]/[x])^{1/n}}{2\rho g [z]} \tag{11.5}$$

and will consider the case $\epsilon \ll 1$, and we also define

$$\delta = 1 - \rho/\rho_o \tag{11.6}$$

where $\delta \approx 0.1$.

Inserting these scalings into field equation (11.1) and the boundary condition (11.3) gives

$$4\epsilon\, \partial_x \left(h\|\partial_x u\|^{(1-n)/n}\, \partial_x u\right) - \|u\|^{(1-m)/m} u = h\, \partial_x s \quad \text{for} \quad x < x_{\mathrm{gl}} \tag{11.7}$$

$$\|\partial_x u\|^{(1-n)/n}\, \partial_x u = \frac{\delta h}{8\epsilon} \quad \text{at} \quad x = x_{\mathrm{gl}} \tag{11.8}$$

Summarising the scaled momentum equations are

$$4\epsilon\, \partial_x \left(h\|\partial_x u\|^{(1-n)/n}\, \partial_x u\right) - \|u\|^{(1-m)/m} u = h\, \partial_x s \quad \text{for} \quad x < x_{\mathrm{gl}} \tag{11.9}$$

$$\|\partial_x u\|^{(1-n)/n}\, \partial_x u = \frac{\delta h}{8\epsilon} \quad \text{at} \quad x = x_{\mathrm{gl}} \tag{11.10}$$

Now we consider the scaled momentum equation (11.7) in the vicinity of the grounding line. As we approach $x \to x_{\mathrm{gl}}$ from the upstream side we expect the first term on the left-hand side of (11.7) to be given by (11.8). Inserting (11.8) into (11.7) gives

$$\delta\, h\, \partial_x h - \|u\|^{(1-m)/m} u = h\, \partial_x s, \tag{11.11}$$

all quantities evaluated at the grounding line.

Now consider the right-hand side of the above equation We always have

$$\partial_x s = \partial_x (h + b),$$

Upstream of the grounding line we do not expect $\partial_x b$ to be related (at least not in some simple way) to $\partial_x h$. Consider the case $\partial_x b = 0$ while $\partial_x h$ takes some finite value, then equation (11.2) is

$$\delta\, h \partial_x h - \|u\|^{(1-m)/m} u = h\, \partial_x h, \tag{11.12}$$

and since $\delta \ll 1$ (in fact $\delta \approx 0.1$) the second term on the left-hand side approximately balances the right-hand side. We therefore have an approximate balance between basal stress and driving stress.

In physical terms the situation down-stream of the grounding line is clear. There the first term on the left-hand side must balance the term on the right hand side. Here our formulation gives the right balance because since $\partial_x b$ and $\partial_x h$ are related by the floating condition

$$\partial_x s = \partial_x (h + b) = \partial_x (h - (1 - \rho/\rho_o) h) = \delta\, \partial_x h,$$

and when inserted into

$$\frac{1}{2} \delta\, \partial_x\, (hh) = \delta\, h\, \partial_x h, \tag{11.13}$$

we arrive, which is the right balance.

Summarising, if $h$ and $b$ are related through the floating condition, the balance is always between the first lhs term and the rhs, but the balance is between the second rhs term and the lhs if

$$|\partial_x b| \ll |\partial_x h|$$

In the particular case $\partial_x b = 0$ the balance is always between the basal shear stress term and the driving stress.

## 11.7 Grounding-line instability

For a steady state with the grounding line located at $x = x_{\mathrm{gl}}$, mass conservation requires

$$\gamma\, x_{\mathrm{gl}} + q(x_{\mathrm{gl}}) = 0$$

where we have assumed that the ice divide is at $x = 0$ and the surface mass balance is $\gamma$. We assume the surface mass balance is spatially constant and positive, i.e. $\gamma > 0$. Clearly if $\gamma < 0$, no ice sheet is possible.

Perturb $x_{\mathrm{gl}}$ by $\Delta x$

$$x'_{\mathrm{gl}} = x_{\mathrm{gl}} + \Delta x$$

If

$$\gamma\, \Delta x - \partial_x q\, \Delta x < 0$$

then more ice flows across the grounding line than is added over the new surface $\Delta x$ upstream of the grounding line. The volume of the (grounded) ice sheet must decrease with time and the grounding line must retreat towards the original steady state. (Note that the derivative of $q$ with respect to $x$ is the derivative of $q$ following the grounding-line position.)

Hence, if

$$\partial_x q > \gamma$$

then the grounding line is stable, but

$$\begin{aligned}
\frac{\partial q}{\partial x} &= \frac{\partial q}{\partial h} \frac{\partial h}{\partial x} \\
&= \frac{\partial q}{\partial h} \frac{\rho_o}{\rho} \frac{\partial (S - B)}{\partial x} \\
&= -\frac{\rho_o}{\rho} \frac{\partial q}{\partial h} \frac{\partial B}{\partial x}
\end{aligned}$$

For a prograde bed, $\partial_x B < 0$, and therefore we must have $\partial_h q > 0$ for the grounding line to be stable. Conversely, for a retrograde bed where $\partial_x B > 0$ we must have $\partial_x q < 0$ for the grounding line to be stable.

# Chapter 12

# Time scales

The *Volume time scale*, $t_V$ is the time it takes to fill the perturbation in steady-state ice volume following a perturbation in, for example, surface mass balance, that is

$$t_V = \frac{\Delta V}{\Delta q}$$

This is a very general concept that can be applied to various situations to provide a lower estimate of the response time $t_R$. The volume time scale is, hence,

$$t_V = \frac{\text{change in volume following a perturbation}}{\text{change in flux into that volume}}$$

## 12.1 Alpine glaciers

Here

$$t_V = \frac{\Delta V}{A_0 \Delta a}$$

where $A_0$ is the original steady state glacier area, $\Delta a$ is the perturbation in surface mass balance, and $\Delta V$ is the change in volume between the initial and final steady states.

In steady state the integrated surface mass balance is zero and therefore to first order

$$\Delta A \, a_t + A_0 \, \Delta a = 0$$

where $a_t$ is the mass balance at the terminus. Again to first order

$$\Delta V = \partial_A V \, \Delta A$$

and therefore

$$\begin{aligned} t_V &:= \frac{\Delta V}{A_0 \Delta a} \\ &= \frac{\partial_A V \, \Delta A}{A_0 \, \Delta a} \\ &= -\frac{\partial_A V \, \Delta A}{\Delta A \, a_t} \\ &= -\frac{\partial_A V}{a_t} \end{aligned}$$

Jóhannesson's estimate for alpine glaciers is

$$\partial_A V = h_{eq}$$

where $h_{eq}$ is the ice thickness as the equilibrium line, gives Jóhannesson's volume time scale

$$t_V = -\frac{h_{eq}}{a_t}$$

He further showed that

$$t_d, t_p \ll t_V$$

where $t_d$ and $t_p$ are diffusion and propagation time scales associated with local mass redistribution of mass and therefore

$$t_r \approx t_V$$

where $t_r$ is the response time to external perturbation.

## 12.2 Marine ice sheets

Here we consider a perturbation to surface mass balance being balanced by changes in flux across the grounding line, and therefore to first order

$$\Delta a\, A_0 + \partial_A q\, \Delta A = 0$$

where $q$ is the vertically integrated flux across the grounding line (for example $q = uhw$ where $u$ is velocity, $h$ ice thickness, and $w$ the width) or if we consider a flow line

$$\Delta a\, l_0 + \partial_x q\, \Delta l = 0$$

Hence, in a flow line situation

$$\begin{aligned} t_V &= -\frac{w\partial_x V\, \Delta l}{w\, \Delta l\, \partial_x q} \\ &= -\frac{\partial_x V}{\partial_x q} \end{aligned}$$

showing that $t_V \to \infty$ when $\partial_x q \to 0$ as expected.

# Chapter 13

# Derivation of the Shallow Ice Stream Approximation (SSTREAM/SSA)

## 13.1 Field equations and boundary conditions

The field equations are

$$\begin{aligned} v_{i,i} &= 0 \quad \text{(mass)} && (13.1)\\ \sigma_{ki,k} + \rho b_i &= 0 \quad \text{(linear momentum)} && (13.2)\\ \sigma_{ij} - \sigma_{ji} &= 0 \quad \text{(angular momentum)} && (13.3) \end{aligned}$$

where $v_i$ are the components of the velocity vector, $\sigma_{ij}$ the components of the full stress tensor (i.e. the Cauchy stress tensor), and $\rho$ the ice density.

In addition we have the kinematic boundary conditions

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} - w = a \tag{13.4}$$

valid at the surface $z = s(x, y)$, where $a$ is the accumulation rate, and we have used $u$, $v$, and $w$ to denote the $x$, $y$, and $z$ components of the velocity vector, respectively. There is a corresponding equation valid at the glacier sole. At the glacier sole both the accumulation rate and the rate of elevation can often be ignored. The kinematic boundary condition is then usually referred to as the 'no-penetration condition'.

The relation between strain rates and stresses is taken to be

$$\dot{\epsilon}_{ij} = A(T)\,\tau^{n-1}\,\tau_{ij}. \tag{13.5}$$

where $A$ is the rate factor and $n$ the stress exponent. Furthermore, $\tau_{ij}$ are the deviatoric stress components

$$\tau_{ij} = \sigma_{ij} - \delta_{ij}\sigma_{kk}/3,$$

$\dot{\epsilon}_{ij}$ are the components of the deformation rate tensor (the stretching tensor), and $\tau$ is *effective stress* (the square root of the (negative of the) second invariant of the deviatoric stress tensor), i.e.

$$\tau = \sqrt{\tau_{ij}\tau_{ij}/2}.$$

Eq. (13.5) is the well-known Glen-Steinemann law (Steinemann, 1954, 1958a,b; Glen, 1955). Outside of glaciology it is better known as the Norton-Hoff rheology model, or simply as power-law rheology. An increasingly popular alternative description of ice rheology can be found in Goldsby and Kohlstedt (2001).

In the situation when the glacier slides over its bed various theoretical arguments show that we have a mixed-type basal boundary condition to consider of the type

$$\boldsymbol{t}_b = \boldsymbol{f}(\boldsymbol{v}_b)$$

where $\boldsymbol{f}$ is some function, $\boldsymbol{t}_b$ is the basal stress vector given by

$$\boldsymbol{t}_b = \boldsymbol{\sigma}\hat{\boldsymbol{n}} - (\hat{\boldsymbol{n}}^T \cdot \boldsymbol{\sigma}\hat{\boldsymbol{n}})\hat{\boldsymbol{n}},$$

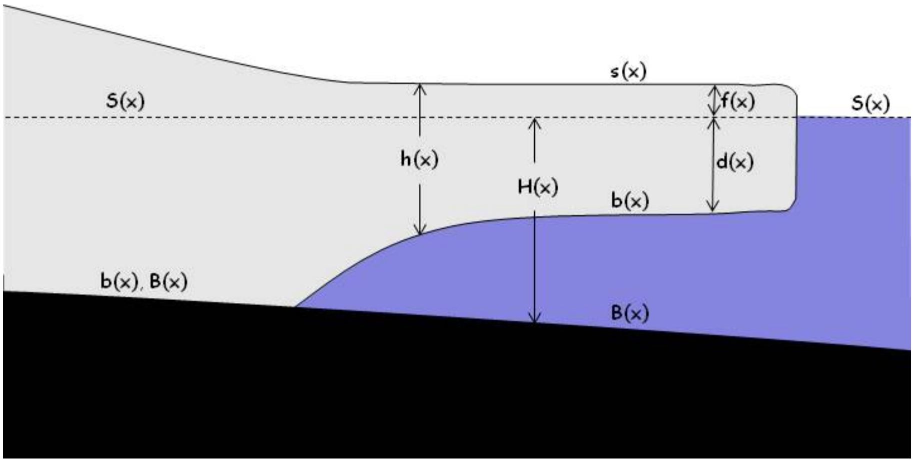



Figure 13.1: **Problem geometry.**

with $\hat{\boldsymbol{n}}$ being a unit normal vector to the bed pointing into the ice, and $\boldsymbol{v}_b$ is the basal sliding velocity

$$\boldsymbol{v}_b = \boldsymbol{v} - (\hat{\boldsymbol{n}}^T \cdot \boldsymbol{v})\hat{\boldsymbol{n}}.$$

Sometimes a power-law type sliding law

$$\boldsymbol{v}_b = c\,|\boldsymbol{t}_b|^{m-1}\boldsymbol{t}_b, \tag{13.6}$$

is used, but the correctness of this assumption is debated. The function $c$ is referred to as the basal slipperiness. The sliding law (13.6) is usually referred to as Weertman sliding law. Alternative forms of Weertman sliding law are

$$|\boldsymbol{v}_b| = c|\boldsymbol{t}_b|^m,$$

and

$$\boldsymbol{t}_b = c^{-1/m}|\boldsymbol{v}_b|^{(1-m)/m}\boldsymbol{v}_b.$$

## 13.2 Shallow Ice Stream Scalings

### 13.2.1 Definition of scales

The shallow ice stream approximation is one of the classical scaling theories in glaciology. It plays a fundamental role in the study of ice stream dynamics. The resulting equations are often referred to as the MacAyeal equations (MacAyeal, 1989) but I will here refer to these equations as the shallow-ice-stream equations (SSTREAM). These equations have been derived numerous times in various papers and doctoral thesis, (e.g. Morland, 1984; Muszynski and Birchfield, 1987; MacAyeal, 1989; Baral, 1999; Schoof, 2006). One will find reading the literature that there are many different ways of deriving the equations.

In the following we will perform scaling analysis of all relevant equations. In general we write for any variable $X$

$$X = [X]X^*$$

where $[X]$ is the scale and $X^*$ the dimensionless counterpart to $X$. We do this to all variables entering the problem. The idea is that the sizes of all dimensionless variables are comparable, and that the relative sizes of two variables can be deduced from comparing their respective scales.

The SSTREAM scalings are motivated by three observations:

First, horizontal span is much larger than thickness. The ratio between ice thickness and horizontal span is therefore small compared to unity.

Second, most of the forward motion of ice streams is due to sliding. It is not uncommon for the slip ratio, defined as the ratio between forward motion due to basal motion to forward motion due to shearing throughout the ice, to be on the order of 100 to 1000. Hence, basal and surface velocities are about equal, and one can use the same scale for both of them.

Third, on ice streams and ice shelves vertical shearing is small compared to longitudinal stretching, and vertical shear stresses small compared to horizontal deviatoric stresses. In particular, one can expect that horizontal strain rates are 'balanced' by the horizontal deviatoric stresses. [1] What is meant by 'balancing' different quantities will become clear in what follows.

Motivated by these observations we now consider the case of an ice stream with horizontal length scale $[x]$ and vertical length scale $[z]$ where the shallow ice approximation $[z]/[x] = \delta \ll 1$ holds, and write

$$(x, y, z) = [x](x^*, y^*, \delta z^*).$$

where the asterisks denote scaled dimensionless variables.

For the mass conservation equation ($v_{i,i} = 0$) to be invariant we scale the velocity as

$$(u, v, w) = [u](u^*, v^*, \delta w^*). \tag{13.7}$$

If we furthermore require the kinematic boundary condition at the surface

$$\partial_t s + u\,\partial_x s + v\,\partial_y s - w = a,$$

where $s$ is the surface to be invariant under the scalings we must have

$$a = \delta[u]a^*,$$

where $a$ is the accumulation rate. Thus the scale for $a$ is $[a] = \delta[u] = [w]$, which seems reasonable as we can expect the vertical velocity to scale with accumulation rate for small surface slopes. We also find, using the same invariant requirement for the surface kinematic boundary condition, that the time must be scaled as

$$t = [x][u]^{-1}t^*.$$

For $[x] \sim 1000\,\mathrm{km}$, and a vertical dimension of 100 m to 1 km, we have $\delta$ in the range of 0.001 to 0.01. Horizontal velocities can be expected to be on the order of a $100\,\mathrm{m\,a^{-1}}$ and $w$ around 0.1 to $1\,\mathrm{m\,a^{-1}}$, giving the same range of $\delta$. The time scale $[t] = [x][u]^{-1}$ is therefore on the order of 1 to 10 ka.

We assume that the velocity is of same order across the whole ice thickness. In particular we assume that the horizontal components of the basal sliding velocity ($u_b$ and $v_b$) are of the same order as the surface velocity, i.e.

$$(u_b, v_b) = [u](u_b^*, v_b^*) \tag{13.8}$$

We are considering a situation where the vertical shear components are small compared to all other stress components. A set of scalings for the stresses which reflects this situation is

$$\begin{aligned}&(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{xz}, \tau_{yz})\\ &\quad = [\sigma](\sigma_{xx}^*, \sigma_{yy}^*, \sigma_{zz}^*, \tau_{xy}^*, \delta\tau_{xz}^*, \delta\tau_{yz}^*).\end{aligned} \tag{13.9}$$

Same scale is used for the pressure, that is $p = [\sigma]p^*$. For the time being, we do not specify how the scale $[\sigma]$ relates to other variables entering the problem.

Note that we are assuming a ratio between vertical and horizontal dimensions equal to that of the vertical and horizontal deviatoric stresses, so for example

$$\delta = \frac{[z]}{[x]} = \frac{[\tau_{xz}]}{[\tau_{xx}]}.$$

In other words, the *aspect ratio*, $[z]/[x]$, is the same as the *stress ratio*, $[\tau_{xz}]/[\tau_{xx}]$.

### 13.2.2 Scaling the equations

The analysis is done in a coordinate system which is tilted forward in $x$ direction by the angle $\alpha$. The equilibrium equations are

$$\begin{aligned}\partial_x\sigma_{xx} + \partial_y\tau_{xy} + \partial_z\tau_{xz} &= -\rho g \sin\alpha,\\ \partial_x\tau_{xy} + \partial_y\sigma_{yy} + \partial_z\sigma_{yz} &= 0,\\ \partial_x\tau_{xz} + \partial_y\sigma_{yz} + \partial_z\sigma_{zz} &= \rho g \cos\alpha.\end{aligned}$$

[1] Note that this situation contracts sharply with what is found on most alpine glaciers, ice sheets and ice caps, where rates of ice deformation due to shearing dominate horizontal strain rates (except for the top most layer). In this case normal deviatoric stresses are small compared to the shear stress and the normal stress field close to being isotropic.

The above listed scalings give

$$[\sigma][x]^{-1}\partial_{x^*}\sigma^*_{xx} + [\sigma][x]^{-1}\partial_{y^*}\tau^*_{xy} + [\sigma]\delta[x]^{-1}\delta^{-1}\partial_{z^*}\tau^*_{xz} = -\rho g \sin\alpha,$$
$$[\sigma][x]^{-1}\partial_{x^*}\tau^*_{xy} + [\sigma][x]^{-1}\partial_{y^*}\sigma^*_{yy} + [\sigma]\delta[x]^{-1}\delta^{-1}\partial_{z^*}\sigma^*_{yz} = 0,$$

$$[\sigma][x]^{-1}\delta\partial_{x^*}\tau^*_{xz} + [\sigma][x]^{-1}\delta\partial_{y^*}\sigma^*_{yz} + [\sigma][x]^{-1}\delta^{-1}\partial_{z^*}\sigma^*_{zz} = \rho g \cos\alpha,$$

which can be written as

$$\begin{aligned}
\partial_{x^*}\sigma^*_{xx} + \partial_{y^*}\tau^*_{xy} + \partial_{z^*}\tau^*_{xz} &= -\rho g[x][\sigma]^{-1}\sin\alpha, & (13.10)\\
\partial_{x^*}\tau^*_{xy} + \partial_{y^*}\sigma^*_{yy} + \partial_{z^*}\sigma^*_{yz} &= 0, & (13.11)\\
\delta^2\partial_{x^*}\tau^*_{xz} + \delta^2\partial_{y^*}\sigma^*_{yz} + \partial_{z^*}\sigma^*_{zz} &= \rho g\delta[x][\sigma]^{-1}\cos\alpha, & (13.12)
\end{aligned}$$

If we now fix the scale $[\sigma]$ for the stresses as

$$[\sigma] = \rho g[z] = \rho g\delta[x], \qquad (13.13)$$

we arrive at

$$\begin{aligned}
\partial_{x^*}\sigma^*_{xx} + \partial_{y^*}\tau^*_{xy} + \partial_{z^*}\tau^*_{xz} &= -\delta^{-1}\sin\alpha, & (13.14)\\
\partial_{x^*}\tau^*_{xy} + \partial_{y^*}\sigma^*_{yy} + \partial_{z^*}\sigma^*_{yz} &= 0, & (13.15)\\
\delta^2\partial_{x^*}\tau^*_{xz} + \delta^2\partial_{y^*}\sigma^*_{yz} + \partial_{z^*}\sigma^*_{zz} &= \cos\alpha. & (13.16)
\end{aligned}$$

For the two terms on the right-hand side of the above set of equations to be of order unity we must furthermore require

$$\alpha = O(\delta),$$

i.e. the tilt angle $\alpha$ of the coordinate system must be small. Hence, the angle $\alpha$ is not arbitrary. (As we will see below, and as is to be expected, the shear stress $\tau_{xz}$ scales with $\rho g[z]\sin\alpha$ so for it to be small in comparison to the stress scale $\rho g[z]$, $\alpha$ must be small.)

Note that the stress scale must be $[\sigma] = \rho g[z] = \rho g\delta[x]$ for the right-hand side term in Eq. (13.12) to be of order unity for $\alpha = 0$. We could have defined this to be the stress scale from the outset, but doing so would have obscured the fact that this stress scale is required for the vertical gradient of the vertical stresses (i.e. $\partial_z\sigma_{zz}$) to be balanced by the vertical component of the body force (i.e. $-\rho g$) for $\alpha = 0$. Furthermore, note that had we defined the stress scale to be the product of thickness and mean slope, i.e.$[\sigma] = \rho g[z][z]/[x] = \rho g\delta[x]\delta[x]/[x] = \rho g[x]\delta^2$, the right-hand term in Eq. (13.16) would have been on the order of $\delta^{-1}$ for $\alpha = 0$, with no term on the left-hand side of that equation to mach that term.

If we only consider terms of zeroth order and drop terms of order $\delta$ and higher, we arrive at a reduced system where the horizontal gradients of the vertical shear stresses are omitted from the equilibrium equations. There are no first-order terms in the scaled equilibrium equations ((13.14) to (13.16)), and the resulting reduced system is therefore correct to second order. Despite no first-order terms appearing in the scaled equilibrium equations, it does of course not follow that none of the quantities entering these equations are of first order. The vertical shear stresses, $\tau_{xz}$ and $\tau_{yz}$, are, for example, of first order.

**Sliding law**

We assume a power-law relationship between basal shear stress

$$\boldsymbol{t}_b = \boldsymbol{\sigma}\,\hat{\boldsymbol{n}} - (\hat{\boldsymbol{n}}^T\cdot\boldsymbol{\sigma}\hat{\boldsymbol{n}})\hat{\boldsymbol{n}},$$

and basal velocity

$$\boldsymbol{v}_b = \boldsymbol{v} - (\hat{\boldsymbol{n}}^T\cdot\boldsymbol{v})\hat{\boldsymbol{n}}.$$

or

$$\boldsymbol{v}_b = c\,\|\boldsymbol{t}_b\|^{m-1}\boldsymbol{t}_b, \qquad (13.17)$$

where $\hat{\boldsymbol{n}}$ being a unit normal vector to the bed pointing into the ice. The scaling of the basal sliding law is done in Appendix A. We find that the components of the scaled basal shear stress vector are given by

$$\begin{aligned}
t_{bx} &= [\sigma](\delta\partial_{x^*}b^*(\sigma^*_{zz}-\sigma^*_{xx}) - \delta\partial_{y^*}b^*\tau^*_{xy} + \delta\tau^*_{xz}) + O(\delta^3), & (13.18)\\
t_{by} &= [\sigma](\delta\partial_{y^*}b^*(\sigma^*_{zz}-\sigma^*_{yy}) - \delta\partial_{x^*}b^*\tau^*_{xy} + \delta\sigma^*_{yz}) + O(\delta^3), & (13.19)\\
t_{bz} &= [\sigma](\delta^2((\sigma^*_{zz}-\sigma^*_{xx})(\partial_{x^*}b^*)^2 + (\sigma^*_{zz}-\sigma^*_{yy})(\partial_{y^*}b^*)^2 & (13.20)\\
&\quad -2\tau^*_{xy}\partial_{x^*}b^*\,\partial_{y^*}b^* + \tau^*_{xz}\partial_{x^*}b^* + \sigma^*_{yz}\partial_{y^*}b^*)) + O(\delta^4).
\end{aligned}$$

We see from the above listed equations that the $x$ and $y$ components of the shear stresses vector are of order $\delta$, and that therefore the length of that vector is also or order $\delta$, i.e.

$$\|\boldsymbol{t}_b\| = O(\delta). \tag{13.21}$$

Hence

$$\begin{aligned}\|\boldsymbol{t}_b\| &= [\|\boldsymbol{t}_b\|]|\boldsymbol{t}_b^*| \\ &= \delta[\sigma]|\boldsymbol{t}_b^*|.\end{aligned}$$

or

$$[\|\boldsymbol{t}_b\|] = \delta[\sigma]$$

The scaled basal sliding velocity ($\boldsymbol{v}_b$) is

$$\boldsymbol{v}_b = [u]\begin{pmatrix} u_b^* + O(\delta^2) \\ v_b^* + O(\delta^2) \\ \delta u_b^*\,\partial_{x^*}b^* + \delta v_b^*\,\partial_{y^*}b^* + O(\delta^3)\end{pmatrix} \tag{13.22}$$

We have not yet specified the scale $[c]$ for the parameter $c$ in the sliding law but we have already introduced scales for the velocity and the stresses, so by inserting (13.7), (13.13), and (13.24) into (13.17) we arrive at

$$\begin{aligned}[u]u_b^* &= c\,\|\boldsymbol{t}_b\|^{m-1}|\boldsymbol{t}_b^*|^{m-1}[t_{bx}]\,t_{bx}^* \\ &= c\,\delta^{m-1}[\sigma]^{m-1}|\boldsymbol{t}_b^*|^{m-1}\,\delta[\sigma]\,t_{bx}^* \\ &= c\,\delta^m[\sigma]^m|\boldsymbol{t}_b^*|^{m-1}\,t_{bx}^*\end{aligned}$$

or

$$u_b^* = c\,\delta^m[\sigma]^m[u]^{-1}\,|\boldsymbol{t}_b^*|^{m-1}\,t_{bx}^*. \tag{13.23}$$

If we want the sliding velocity to be balanced by the basal shear stress, terms on both side of Eq. (13.23) must be of same order, hence

$$c\,\delta^m[\sigma]^m[u]^{-1} = O(1).$$

If we write

$$c = [c]c^*, \tag{13.24}$$

then

$$[c] = \delta^{-m}[u][\sigma]^{-m}. \tag{13.25}$$

Eq. (13.25) shows that $c$ is of the order $\delta^{-m}$. In this sense the slipperiness ($c$) must be 'large' for the theory to be consistent.

The product $c[\sigma]^m$ is the (typical) basal sliding velocity, while $[u]$ is the (typical) surface velocity. Hence, Eq. (13.25) simply reflects the condition that for 'most' of the forward motion to be due to basal sliding, the basal slipperiness $c$ must be 'large'. As an example, if $\partial_x b = \partial_y b = 0$ we find that

$$u^* = c^*\tau_{xz}^{*m}.$$

In principle we could have observed right at the beginning that defining the vertical shear stress components to be of $O(\delta)$ and $u = O(1)$ implies $c = O([u][\sigma]^{-m}\,\delta^{-m})$ for a basal sliding law of the form $u_b = c\tau_b^m$ if $u_b$ is to be of same order as $u$.

For the $z$ component of the sliding law obtain using Eq. (13.20) and Eq. (13.22)

$$\begin{aligned}&\delta u^*\partial_{x^*}b^* + \delta v^*\partial_{y^*}b^* = c\,\|\boldsymbol{t}_b\|^{m-1} \\ &\quad[\sigma](\delta^2((\sigma_{zz}^* - \sigma_{xx}^*)(\partial_{x^*}b^*)^2 + (\sigma_{zz}^* - \sigma_{yy}^*)(\partial_{y^*}b^*)^2 - 2\tau_{xy}^*\partial_{x^*}b^*\,\partial_{y^*}b^* + \tau_{xz}^*\partial_{x^*}b^* + \sigma_{yz}^*\partial_{y^*}b^*)).\end{aligned} \tag{13.26}$$

Note that the sum of the two terms on the left-hand-side as given by Eqs. (13.18) and (13.19) gives the left-hand side of (13.26), so these equations are consistent. Note furthermore that the vertical component $w$ does not enter the sliding law The vertical component must be calculated from the basal kinematic boundary condition ($u\,\partial_x b + v\,\partial_y b - w = 0$).

### Flow law

The flow law can be either written as

$$\dot{\epsilon}_{ij} = A(T)\,\tau^{n-1}\tau_{ij}, \tag{13.27}$$

or alternatively as

$$\tau_{ij} = A^{-1/n}\,\dot{\epsilon}^{(1-n)/n}\,\dot{\epsilon}_{ij},$$

where $\dot{\epsilon} = \sqrt{\dot{\epsilon}_{ij}\dot{\epsilon}_{ij}/2}$ is the *effective strain rate*, and $T$ the englacial temperature.

We have so far not discussed the scalings for the normal deviatoric stresses $\tau^*_{xx}$, $\tau^*_{yy}$, and $\tau^*_{zz}$. These scalings follow directly from the fact that we decided above to scale both the normal stresses ($\sigma_{xx}$, $\sigma_{xx}$ $\sigma_{xx}$) and the pressure with $[\sigma]$. Because $\tau_{xx} = \sigma_{xx} + p = [\sigma](\sigma^*_{xx} + p^*) = [\sigma]\tau^*_{xx}$ and similarly for the other components we have

$$(\tau_{xx}, \tau_{yy}, \tau_{zz}) = [\sigma](\tau^*_{xx}, \tau^*_{yy}, \tau^*_{zz}).$$

The square root of the second invariant of the deviatoric stress tensor, or what glaciologist usually refer to as the *effective stress*, is thus

$$\tau = [\sigma]\sqrt{(\tau^{*2}_{xx} + \tau^{*2}_{yy} + \tau^{*2}_{zz})/2 + \tau^{*2}_{xy} + \delta^2(\tau^{*2}_{xz} + \tau^{*2}_{yz})}, \tag{13.28}$$

and therefore

$$[\tau] = [\sigma] \tag{13.29}$$

and

$$\tau = [\sigma]\tau^*$$

where

$$\tau^* = \sqrt{(\tau^{*2}_{xx} + \tau^{*2}_{yy} + \tau^{*2}_{zz})/2 + \tau^{*2}_{xy} + \delta^2(\tau^{*2}_{xz} + \tau^{*2}_{yz})}.$$

The effective stress $\tau$ is of order unity.

Using the flow law and the incompressibility condition we find

$$0 = v_{i,i} = \dot{\epsilon}_{ii} = \tau_{ii}$$

and

$$\tau^2_{zz} = (\tau_{xx} + \tau_{yy})^2$$

which can be used to eliminate $\tau_{zz}$ from Eq. (9.14).

We now look at the relation between the individual components of the stretching tensor (the strain rates) and the deviatoric stress tensor. We find, for example, that

$$\dot{\epsilon}_{xx} = A\tau^{n-1}\tau_{xx},$$

is in scaled variables

$$[u][x]^{-1}\dot{\epsilon}^*_{xx} = [\sigma]^n A {\tau^*}^{n-1}\tau^*_{xx}.$$

or

$$\dot{\epsilon}^*_{xx} = A[\sigma]^n[x][u]^{-1}{\tau^*}^{n-1}\tau^*_{xx}. \tag{13.30}$$

where

$$\dot{\epsilon}^*_{xx} = \partial_{x^*}u^*$$

If we want the horizontal strain rates ($\dot{\epsilon}_{xx}$, $\dot{\epsilon}_{xy}$, and $\dot{\epsilon}_{yy}$) to be balanced by the corresponding horizontal deviatoric stresses, we must require that both sides of Eq. (13.30) are of same order implying

$$A[\sigma]^n[x][u]^{-1} = O(1).$$

Writing

$$A = [A]A^*,$$

therefore leads to

$$[A] = [u][x]^{-1}[\sigma]^{-n}. \tag{13.31}$$

Next we look at

$$\dot{\epsilon}_{xz} = A\tau^{n-1}\tau_{xz},$$

and find that this gives

$$[u][x]^{-1}(\delta^{-1}\partial_z^* u^* + \delta\partial_{x^*} w^*) = \delta[\sigma]^n A{\tau^*}^{n-1}\tau_{xz}^*,$$

which we can also write as

$$\begin{aligned}\partial_z^* u^* + \delta^2\partial_{x^*} w^* &= \delta^2[x][\sigma]^n[u]^{-1}A{\tau^*}^{n-1}\tau_{xz}^* \\ &= \delta^2 A^*{\tau^*}^{n-1}\tau_{xz}^*,\end{aligned}$$

Note that here we are using the scale for $A$ given by (13.31). We must equate terms of same order, and hence find that

$$\partial_z^* u^* = O(\delta^2),$$

and

$$\partial_{x^*} w^* = [x][\sigma]^n[u]^{-1}A{\tau^*}^{n-1}\tau_{xz}^*.$$

We have now reached the important conclusion that the *horizontal velocity component $u$ is independent of depth to second order.* Same argument shows that the other horizontal component $v$ is also independent of depth. Thus, to second order the horizontal velocity components $u$ and $v$ are both independent of $z$.

We have shown that consistency with the scalings used for stresses requires $\partial_z u$ to be $O(\delta^2)$. Note that we have NOT shown vertical shearing ($\dot\epsilon_{xz}$) to be zero. Both $\partial_x w$ and $\tau_{xz}$ enter the field equations as first order terms.

The incompressibility conditions states that

$$\partial_{x^*} u^* + \partial_{y^*} v^* + \partial_{z^*} w^* = 0.$$

Differentiating with respect to $z$ and assuming that the order of differentiation can be changed gives

$$\partial^2_{x^*z^*} u^* + \partial^2_{y^*z^*} v^* + \partial^2_{z^*z^*} w^* = 0.$$

From which using $\partial_{z^*} u^* = \partial_{z^*} v^* = O(\delta^2)$ it follows that

$$\partial^2_{z^*z^*} w^* = O(\delta^2)$$

Hence, *to second order $\dot\epsilon_{zz}$ is independent of depth and the vertical velocity varies linearly with depth.*

It turns out to be more convenient working with the flow law in the form

$$\tau_{ij} = 2\eta\dot\epsilon_{ij}$$

where $\eta$ is the *effective viscosity* defined as

$$\eta = \frac{1}{2}A^{-1/n}\,\dot\epsilon^{(1-n)/n}.$$

Toward this end we determine the effective strain rate

$$\begin{aligned}\dot\epsilon &= \sqrt{(\dot\epsilon_{xx}^2 + \dot\epsilon_{yy}^2 + \dot\epsilon_{zz}^2)/2 + \dot\epsilon_{xy}^2 + \dot\epsilon_{xz}^2 + \dot\epsilon_{yz}^2} \\ &= \sqrt{(\dot\epsilon_{xx}^2 + \dot\epsilon_{yy}^2 + (\dot\epsilon_{xx} + \dot\epsilon_{yy})^2)/2 + \dot\epsilon_{xy}^2 + \dot\epsilon_{xz}^2 + \dot\epsilon_{yz}^2}.\end{aligned}$$

Inserting $\dot\epsilon_{ij} = (v_{i,j} + v_{j,i})/2$ and using the fact that $\partial_z u = O(\delta^2)$ and $\partial_z v = O(\delta^2)$ we find

$$\begin{aligned}\dot\epsilon &= [u][x]^{-1}((\partial_{x^*}u^*)^2 + (\partial_{y^*}v^*)^2 + \partial_{x^*}u^*\partial_{y^*}v^* + (\partial_{x^*}v^* + \partial_{y^*}u^*)^2/4 \\ &\quad +(\delta\partial_{x^*}w^* + O(\delta^2))^2/4 + (\delta\partial_{y^*}w^* + O(\delta^2))^2/4)^{1/2} \\ &= [u][x]^{-1}\sqrt{(\partial_{x^*}u^*)^2 + (\partial_{y^*}v^*)^2 + \partial_{x^*}u^*\partial_{y^*}v^* + (\partial_{x^*}v^* + \partial_{y^*}u^*)^2/4} + O(\delta^2),\end{aligned}$$

or

$$\dot\epsilon = \sqrt{(\partial_x u)^2 + (\partial_y v)^2 + \partial_x u\,\partial_y v + (\partial_x v + \partial_y u)^2/4} + O(\delta^2). \tag{13.32}$$

### Slip ratio

We can write the horizontal velocity component $u$ as the sum

$$u = u_b + u_d$$

where $u_d$ is the *deformational velocity*, and $u_b$ the basal sliding velocity. The ratio

$$\gamma := \frac{u_b}{u_d}$$

between the basal sliding velocity and the deformational velocity is the *slip ratio*. Using the relationship for the basal velocity for a uniformly inclined slab with ice thickness $h$, i.e.

$$u_d = \frac{2A}{n+1}\tau^{n-1}\,\tau_{xz}\,h,$$

we find

$$\begin{aligned} [u_d] &= [A][\tau]^{n-1}[\tau_{xy}][z] & (13.33)\\ &= [u][x]^{-1}[\sigma]^{-n}\,[\sigma]^{n-1}\delta[\sigma]\,\delta[x] & (13.34)\\ &= \delta^2[u], & (13.35) \end{aligned}$$

where we have used Eqs. (13.21), (13.25), (13.29), and (13.31). Since $[u_b] = [u]$, we have

$$\gamma = \frac{[u_b]}{[u_d]} = O(\delta^{-2}). \tag{13.36}$$

Note that the we did not specify from the outset that the sliding velocity had to be large as compared to the deformational velocity, so (13.36) is a result rather than an assumption. It is worthwhile to think about how we arrived at the conclusion that the slip ratio is of order $\delta^{-2}$. By assuming that the horizontal deviatoric stresses are large compared to vertical shear stresses (see Eq. 13.9), and by balancing the horizontal strain rates with the horizontal deviatoric stresses (see Eq. 13.30), we arrived at a scale for the rate factor $A$ (see Eq. 13.31). We furthermore assumed that the basal sliding velocity was of the same order as the surface velocity (see Eq. 13.8), i.e. of order unity. We then found the basal stress to be of order $\delta$ (see Eq. 13.21), and by requiring a balance between the basal sliding velocity and the basal stress implied by the sliding law, we arrived at a scale for the basal slipperiness $c$ (see Eq. 13.25). It then follows, as shown above, that the slip ratio is of order $\delta^{-2}$.

We also have

$$\frac{[c]}{[A]} = \frac{\delta^{-m}[u][\sigma]^{-m}}{[u][x]^{-1}[\sigma]^{-n}} = \delta^{-m}[x][\sigma]^{n-m}.$$

which puts constrains on the numerical value of $c$ with respect to that of $A$.

### Implications of different balances for the slip ratio

When finding the scale $[u_d]$ (see (13.34) we used the fact that the effective stress ($\tau$) is of order unity (see Eq. 13.29). In the absence of any significant horizontal deviatoric stresses however, the effective stress would be of order $\delta$ and $[u_d]/[u] = O(\delta^{n+1})$. If the ice is not subjected to horizontal deviatoric stresses of order unity, we can still balance horizontal strain rates with the horizontal deviatoric stresses as we did above to arrive at $[A]$. However, in that case it seems more logical to balance the vertical shear strain rates and the vertical shear stresses to arrive at a (different) scale for $A$.

Furthermore, had we not assumed the aspect ratio and the stress ratio to be equal, but instead written

$$\frac{[z]}{[x]} = \varepsilon \quad \text{and} \quad \frac{[\tau_{xz}]}{[\tau_{xx}]} = \delta,$$

with $\varepsilon$ now being the aspect ratio, and $\delta$ as 'stress' ratio, we would have found that

$$\begin{aligned} [u_d] &= [A][\sigma]^{n-1}\delta[\sigma][z]\\ &= [u][\sigma]^{-n}[x]^{-1}[\sigma]^{n-1}\delta[\sigma]\varepsilon[x]\\ &= [u]\delta\varepsilon, \end{aligned}$$

i.e.

$$\frac{[u_d]}{[u]} = \varepsilon\,\delta.$$

Remember that the scaling originally introduced for velocity was in terms of the basal sliding velocity, and that therefore in fact

$$\frac{[u_d]}{[u_b]} = \varepsilon\,\delta,$$

which is the inverse of the slip ratio. We see that for any given aspect ratio $\varepsilon$, the slip ratio $([u_b]/[u_d])$ becomes small as the stress ratio $\delta$ goes to infinity, corresponding to the situation where horizontal deviatoric stresses are small compared to vertical shear stresses.

## 13.3 The SSTREAM (zeroth-order) equations

Now that we have scaled all equations we can collect terms to the desired order and go back to dimensional quantities.

Note that in the field equations and all the boundary conditions, first order terms are all identically equal to zero. Although we only collect zeroth order terms, the first order correction is zero and the theory is therefore correct to second order in $\delta$.

### 13.3.1 Boundary conditions

The stress conditions at the surface $\boldsymbol{\sigma}\hat{\boldsymbol{n}} = \mathbf{0}$ gives to second order

$$-\sigma_{xx}\,\partial_x s - \tau_{xy}\,\partial_y s + \tau_{xz} = 0, \tag{13.37}$$

$$-\tau_{xy}\,\partial_x s - \sigma_{yy}\,\partial_y s + \tau_{yz} = 0, \tag{13.38}$$

$$\sigma_{zz} = 0, \tag{13.39}$$

for $z = s(x, y)$. And

$$u_b = c\,\|\boldsymbol{t}_b\|^{m-1} t_{bx}, \tag{13.40}$$

$$v_b = c\,\|\boldsymbol{t}_b\|^{m-1} t_{by}, \tag{13.41}$$

where

$$t_{bx} = \partial_x b\,(\sigma_{zz} - \sigma_{xx}) - \partial_y b\,\tau_{xy} + \tau_{xz}, \tag{13.42}$$

$$t_{by} = \partial_y b\,(\sigma_{zz} - \sigma_{yy}) - \partial_x b\,\tau_{xy} + \tau_{yz}, \tag{13.43}$$

for $z = b(x, y)$.

We could use the $z$ component of the sliding law to calculate $w_b$. But since the sliding law is fully consistent with the no-penetration condition (see for example Eq. 13.26) , it is easier to determine $w_b$ as a function of $u_b$ and $v_b$ and the bed geometry directly using the no-penetration condition.

### 13.3.2 Field equations

To zeroth order we obtain form Eq. (13.14) to (13.16)

$$\partial_x\sigma_{xx} + \partial_y\tau_{xy} + \partial_z\tau_{xz} = -\rho g \sin\alpha \tag{13.44}$$

$$\partial_x\tau_{xy} + \partial_y\sigma_{yy} + \partial_z\tau_{yz} = 0, \tag{13.45}$$

$$\partial_z\sigma_{zz} = \rho g \cos\alpha. \tag{13.46}$$

Using $\tau_{ij} = 2\,\eta\dot{\epsilon}_{ij}$ and $\sigma_{ij} = \tau_{ij} - p\,\delta_{ij}$ we can also write this system as

$$-\partial_x p + 2\partial_x(\eta\dot{\epsilon}_{xx}) + 2\partial_y(\eta\dot{\epsilon}_{xy}) + 2\partial_z(\eta\dot{\epsilon}_{xz}) = -\rho g \sin\alpha, \tag{13.47}$$

$$-\partial_y p + 2\partial_x(\eta\dot{\epsilon}_{xy}) + 2\partial_y(\eta\dot{\epsilon}_{yy}) + 2\partial_z(\eta\dot{\epsilon}_{yz}) = 0, \tag{13.48}$$

$$-\partial_z p + 2\partial_z(\eta\dot{\epsilon}_{zz}) = \rho g \cos\alpha. \tag{13.49}$$

### 13.3.3 Vertical integration

We start by considering Eq. (13.46). Integrating from $z$ to $z = s(x, y)$ gives

$$\sigma_{zz}(s) - \sigma_{zz}(z) = -(s - z)\rho g \cos\alpha. \tag{13.50}$$

From (13.39) we find $\sigma_{zz}(s) = 0$ so that

$$\sigma_{zz} = (z - s)\rho g \cos\alpha. \tag{13.51}$$

We now integrate Eq. (13.44) over the depth and use Leibniz' rule

$$\partial_x \int_{b(x)}^{s(x)} f(x,z)\,dz = \int_{b(x)}^{s(x)} \partial_x f(x,z)\,dz + f(x,s)\partial_x s - f(x,b)\partial_x b$$

to interchange the order of integration and differentiation, and find

$$\begin{aligned} -\rho g(s-b)\sin\alpha \;=\; & \partial_x \int_b^s \sigma_{xx}\,dz + \partial_y \int_b^s \tau_{xy}\,dz \\ & -\sigma_{xx}(s)\,\partial_x s - \tau_{xy}(s)\,\partial_y s + \tau_{xz}(s) \\ & +\sigma_{xx}(b)\,\partial_x b + \tau_{xy}(b)\,\partial_y b - \tau_{xz}(b). \end{aligned}$$

Note that we did not have to specify how $\tau_{xz}$ varies across the depth. Because of boundary condition (13.37) the second line is equal to zero. Using (13.42) we find that the third line can be written as $-t_{bx} + \partial_x b\,\sigma_{zz}(b)$ so that

$$-\rho g(s-b)\sin\alpha = \partial_x \int_b^s \sigma_{xx}\,dz + \partial_y \int_b^s \tau_{xy}\,dz - t_{bx} + \partial_x b\,\sigma_{zz}(b).$$

Since $p = \tau_{zz} - \sigma_{zz}$ and $\tau_{xx} + \tau_{yy} + \tau_{zz} = 0$ because ice is incompressible, we find that $\sigma_{xx}$ can be written as

$$\begin{aligned} \sigma_{xx} &= \tau_{xx} - p \\ &= \tau_{xx} - \tau_{zz} + \sigma_{zz} \\ &= \tau_{xx} - (-\tau_{xx} - \tau_{yy}) + \sigma_{zz} \\ &= 2\tau_{xx} + \tau_{yy} + \sigma_{zz}. \end{aligned} \tag{13.52}$$

Because $u$ and $v$ are independent of depth it follows that $\tau_{xy}$, $\tau_{xx}$, and $\tau_{yy}$ are also all independent of depth. The corresponding vertical integrals are therefore simple to evaluate and we obtain

$$-\rho g h \sin\alpha = \partial_x \int_b^s \sigma_{zz}\,dz + \partial_x(h(2\tau_{xx} + \tau_{yy})) + \partial_y(h\tau_{xy}) - t_{bx} + \partial_x b\,\sigma_{zz}(b), \tag{13.53}$$

where $h = s - b$ is the ice thickness. We have already determined $\sigma_{zz}$ (see Eq. (13.51)) and find that

$$\begin{aligned} \partial_x \int_b^s \sigma_{zz}\,dz &= \partial_x \int_b^s (z-s)\rho g \cos\alpha\,dz \\ &= \partial_x \left(-\frac{1}{2}(s-b)^2 \rho g \cos\alpha\right) \\ &= -(s-b)\rho g \cos\alpha(\partial_x s - \partial_x b) \\ &= \sigma_{zz}(b)(\partial_x s - \partial_x b), \end{aligned}$$

which when inserted into Eq. (13.53) gives

$$\partial_x(h(2\tau_{xx} + \tau_{yy})) + \partial_y(h\tau_{xy}) - t_{bx} = \partial_x s\, h\rho g \cos\alpha - \rho g h \sin\alpha.$$

We can express this result in terms of the components of the velocity vector using $\tau_{ij} = \eta(v_{i,j} + v_{j,i})$ and find that

$$\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) + \partial_y(h\eta(\partial_x v + \partial_y u)) - t_{bx} \;=\; \rho g h(\partial_x s \cos\alpha - \sin\alpha), \tag{13.54}$$

$$\partial_y(4h\eta\partial_y v + 2h\eta\partial_x u) + \partial_x(h\eta(\partial_y u + \partial_x v)) - t_{by} \;=\; \rho g h \partial_y s \cos\alpha, \tag{13.55}$$

where we have added the results for the $y$ direction which follow in an identical manner.[2] The effective viscosity is

$$\eta = \frac{1}{2} A^{-1/n}\, \dot{\epsilon}^{(1-n)/n},$$

where

$$\dot{\epsilon} = \sqrt{(\partial_x u)^2 + (\partial_y v)^2 + \partial_x u\, \partial_y v + (\partial_x v + \partial_y u)^2/4}.$$

### 13.3.4 Tensor of restive stresses

In most modelling work the coordinate system is not tilted. For $\alpha = 0$ the vertically integrated form of the momentum equation in $x$ and $y$ directions is

$$\partial_x(h(2\tau_{xx} + \tau_{yy})) + \partial_y(h\tau_{xy}) - t_{bx} = \rho g\, h\, \partial_x s\ , \tag{13.58}$$

$$\partial_y(h(2\tau_{xx} + \tau_{yy})) + \partial_x(h\tau_{xy}) - t_{by} = \rho g\, h\, \partial_y s, \tag{13.59}$$

This system can written in a more compact form as

$$\nabla_{xy}^T \cdot (h\, \boldsymbol{R}) - \boldsymbol{t}_{bh} = \rho g h \nabla_{xy}^T\, s, \tag{13.60}$$

where

$$\boldsymbol{R} = \begin{pmatrix} 2\tau_{xx} + \tau_{yy} & \tau_{xy} \\ \tau_{xy} & 2\tau_{yy} + \tau_{xx} \end{pmatrix}, \tag{13.61}$$

is sometimes referred to as the *resistive stress tensor*, and

$$\nabla_{xy} = (\partial_x, \partial_y)^T,$$

and

$$\boldsymbol{t}_{bh} = \begin{pmatrix} t_{bx} \\ t_{by} \end{pmatrix}.$$

Note that as Eq. (13.52) shows

$$2\tau_{xx} + \tau_{yy} = \sigma_{xx} - \sigma_{zz}, \tag{13.62}$$

$$2\tau_{yy} + \tau_{xx} = \sigma_{yy} - \sigma_{zz}, \tag{13.63}$$

and the resistive stress tensor can therefore be written as

$$\boldsymbol{R} = \begin{pmatrix} \sigma_{xx} - \sigma_{zz} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} - \sigma_{zz} \end{pmatrix}. \tag{13.64}$$

The resistive stress tensor is neither equal to the deviatoric stress tensor or the Cauchy stress tensor. In the Shallow Ice Stream approximation, the deviatoric and the resistive stress tensors are both independent of depth, whereas the Cauchy stress tensor is not.

### 13.3.5 Weertman sliding law

If we use Weertman sliding law the components of basal shear stress, $t_{bx}$ and $t_{by}$ can be written in terms of the basal velocity as

$$t_{bx} = c^{-1/m}\, \|\boldsymbol{v}_b\|^{1/m-1}\, u, \tag{13.65}$$

$$t_{by} = c^{-1/m}\, \|\boldsymbol{v}_b\|^{1/m-1}\, v. \tag{13.66}$$

The sliding law is sometimes written as

$$t_{bx} = \beta^2\, u, \tag{13.67}$$

$$t_{by} = \beta^2\, v, \tag{13.68}$$

[2] If $\rho$ is a function of $x$ and $y$ then $\partial_x \int_b^s \sigma_{zz}\, dz = \sigma_{zz}(b)\partial_x h - \frac{1}{2} h^2 g \cos\alpha\, \partial_x \rho$ and we have

$$\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) + \partial_y(h\eta(\partial_x v + \partial_y u)) - t_{bx} = \rho g h(\partial_x s \cos\alpha - \sin\alpha) + \frac{1}{2} h^2 g \cos\alpha\, \partial_x \rho, \tag{13.56}$$

$$\partial_y(4h\eta\partial_y v + 2h\eta\partial_x u) + \partial_x(h\eta(\partial_y u + \partial_x v)) - t_{by} = \rho g h \partial_y s \cos\alpha + \frac{1}{2} h^2 g \cos\alpha\, \partial_y \rho, \tag{13.57}$$

where

$$\beta = c^{-1/2m} \|\boldsymbol{v}_b\|^{\frac{1-m}{2m}}.$$

in which case we can also write the field equations as

$$\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) + \partial_y(h\eta(\partial_x v + \partial_y u)) - \beta^2 u = \rho g h(\partial_x s \cos\alpha - \sin\alpha), \tag{13.69}$$

$$\partial_y(4h\eta\partial_y v + 2h\eta\partial_x u) + \partial_x(h\eta(\partial_y u + \partial_x v)) - \beta^2 v = \rho g h \partial_y s \cos\alpha. \tag{13.70}$$

We now have a system of two partial differential equations that, given appropriate boundary conditions, can be solved for the velocity components $u$ and $v$. Remember that in general both $\eta$ and $\beta^2$ are functions of the strain rates and the velocity, respectively. The system is therefore non-linear and if solved numerically, some sort of appropriate iterative algorithm (e.g. Newton-Raphson) must be used.

## 13.4 The shallow ice shelf approximation (SSHELF)

The scaling analysis shown above applies to ice shelves as well. The only change we have to make is setting the basal shear stress to zero. There is no reason to do the analysis in a tilted coordinate system so we also set $\alpha = 0$. Thus for floating ice shelves we have

$$\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) + \partial_y(h\eta(\partial_x v + \partial_y u)) = \rho g h\, \partial_x s, \tag{13.71}$$

$$\partial_y(4h\eta\partial_y v + 2h\eta\partial_x u) + \partial_x(h\eta(\partial_y u + \partial_x v)) = \rho g h\, \partial_y s, \tag{13.72}$$

No stresses act at the upper boundary $z = s(x, y)$, and therefore the stress boundary conditions at the upper surface are (to second order) simply

$$-\sigma_{xx}\partial_x s - \tau_{xy}\partial_y s + \tau_{xz} = 0, \tag{13.73}$$

$$-\tau_{xy}\partial_x s - \sigma_{yy}\partial_y s + \tau_{yz} = 0, \tag{13.74}$$

$$\sigma_{zz} = 0. \tag{13.75}$$

The stress boundary conditions at the lower boundary $z = b(x, y)$ are

$$-\sigma_{xx}\partial_x b - \tau_{xy}\partial_y b + \tau_{xz} = p_w\, \partial_x b, \tag{13.76}$$

$$-\tau_{xy}\partial_x b - \sigma_{yy}\partial_y b + \tau_{yz} = p_w\, \partial_y b, \tag{13.77}$$

$$\sigma_{zz} = -p_w, \tag{13.78}$$

where

$$p_w = \rho_w g(S - b),$$

is the ocean pressure acting on the lower boundary of the ice shelf, with $\rho_w$ denoting the ocean density.

We know that

$$\sigma_{zz} = \rho g(z - s), \tag{13.79}$$

and the boundary condition (13.78) therefore implies that

$$\rho g(s - b) = \rho_w g(S - b),$$

or

$$\rho g h = \rho_w g d.$$

One can now work out various other floating relationships, and one finds that where the glacier is afloat the following relations hold:

$$h = \rho_w d/\rho = \frac{s - S}{1 - \rho/\rho_w} = \frac{\rho_w}{\rho}(S - b), \tag{13.80}$$

$$b = \frac{\rho s - \rho_w S}{\rho - \rho_w} = S - \frac{\rho}{\rho_w} h, \tag{13.81}$$

$$s = S + (1 - \rho/\rho_w)h = (1 - \rho_w/\rho)b + \frac{\rho_w}{\rho} S, \tag{13.82}$$

$$f = (1 - \rho/\rho_w)h. \tag{13.83}$$

If $\partial_x S = 0$, the slopes of the upper and the lower boundary are related through

$$b\,\partial_x s + s\,\partial_x b = S\,\partial_x h, \tag{13.84}$$

and also

$$\partial_x s = (1 - \rho/\rho_w)\partial_x h. \tag{13.85}$$

The maximum ice thickness that an ice shelf can have without grounding is

$$h_f := \rho_w H/\rho.$$

Where

$$h \geq h_f,$$

the ice is grounded.

Using (13.85) in (13.71) and (13.72) gives

$$\begin{aligned}
\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) + \partial_y(h\eta(\partial_x v + \partial_y u)) &= \rho g(1 - \rho/\rho_w)h\,\partial_x h, & (13.86)\\
\partial_y(4h\eta\partial_y v + 2h\eta\partial_x u) + \partial_x(h\eta(\partial_y u + \partial_x v)) &= \rho g(1 - \rho/\rho_w)h\,\partial_y h. & (13.87)
\end{aligned}$$

Expressed in terms of stresses these equations read

$$\begin{aligned}
\partial_x(h(2\tau_{xx} + \tau_{yy})) + \partial_y(h\tau_{xy}) &= \rho g(1 - \rho/\rho_w)h\,\partial_x h, & (13.88)\\
\partial_y(h(2\tau_{yy} + \tau_{xx})) + \partial_x(h\tau_{xy}) &= \rho g(1 - \rho/\rho_w)h\,\partial_y h. & (13.89)
\end{aligned}$$

Using the definition of the resisting stress tensor (see Eq. 1.3) the equations can be written on a compact form as

$$\nabla_{xy}^T \cdot (h\,\boldsymbol{R}) = \varrho\,g\,h\,\nabla_{xy}\,h,$$

where

$$\varrho = \rho\,(1 - \rho/\rho_w).$$

### 13.4.1 Boundary conditions at the calving front

At the calving front, $\Gamma_s$, we require balance of vertically integrated horizontal stresses, i.e.

$$\int_b^s \boldsymbol{\sigma}\hat{\boldsymbol{n}}_h = -\int_b^S p_w\,\hat{\boldsymbol{n}}_h \quad \text{on} \quad \Gamma_c,$$

where $p_w$ is the hydrostatic ocean pressure, and

$$\hat{\boldsymbol{n}}_h = (n_x, n_y, 0)^T, \tag{13.90}$$

is a unit normal pointing horizontally outward from the ice front. In $x$ and $y$ directions this stress condition is

$$\int_b^s (\sigma_{xx}n_x + \tau_{xy}n_y)\,dz = -\int_b^S p_w n_x\,dz \quad \text{on} \quad \Gamma_c, \tag{13.91}$$

$$\int_b^s (\tau_{xy}n_x + \sigma_{yy}n_y)\,dz = -\int_b^S p_w n_y\,dz \quad \text{on} \quad \Gamma_c. \tag{13.92}$$

If the draft $d$ at the ice front is zero, i.e. if the ice front is fully grounded, then $S < b$, the right-hand sides of (13.91) and (13.92) are to be set to zero.

Because $\sigma_{xx}$ can be written as

$$\sigma_{xx} = 2\tau_{xx} + \tau_{yy} + \sigma_{zz},$$

(see Eq. 13.52), and

$$\sigma_{zz} = -\rho g(s - z),$$

within the ice, we find that

$$\begin{aligned}
\int_b^s \sigma_{xx}\,dz &= \int_b^s (2\tau_{xx} + \tau_{yy})\,dz - \int_b^s \rho g(s - z)\,dx\\
&= h(2\tau_{xx} + \tau_{yy}) - \frac{\rho g}{2}h^2
\end{aligned}$$

The $x$ component of the vertically integrated ocean pressure acting on the calving front is

$$\begin{aligned} -\int_b^S p_w n_x \, dz &= -\int_b^S \rho_w g(S-z) n_x \, dz \\ &= -\frac{1}{2}\rho_w g(S-b)^2 \\ &= -\frac{1}{2}\rho_w g d^2 \end{aligned}$$

Boundary conditions (1.68) and (1.69) can therefore be written as

$$h(2\tau_{xx}+\tau_{yy})n_x + h\tau_{xy}n_y = \frac{g}{2}(\rho h^2 - \rho_w d^2)\, n_x \tag{13.93}$$

$$h(2\tau_{yy}+\tau_{xx})n_y + h\tau_{xy}n_x = \frac{g}{2}(\rho h^2 - \rho_w d^2)\, n_y \tag{13.94}$$

or more compactly as

$$\boldsymbol{R}\,\hat{\boldsymbol{n}}_c = \frac{g}{2h}(\rho h^2 - \rho_w d^2)\,\hat{\boldsymbol{n}}_c \tag{13.95}$$

where

$$\hat{\boldsymbol{n}}_c = (n_x, n_y)^T, \tag{13.96}$$

is a unit normal to the calving front.

In arriving at (13.93) and (13.94) we have not specified any particular relationship between ice shelf thickness ($h$) and ice shelf draft ($d$). These boundary conditions therefore apply to both grounded and floating ice edges.

If the ice at the calving front is afloat, then $h$ and $d$ are related through the floating condition $\rho h = \rho_w d$. In that case boundary conditions (13.93) and (13.94) take the form

$$h(2\tau_{xx}+\tau_{yy})n_x + h\tau_{xy}\, n_y = \frac{1}{2}\varrho g h^2 n_x, \tag{13.97}$$

$$h(2\tau_{yy}+\tau_{xx})n_y + h\tau_{xy}\, n_x = \frac{1}{2}\varrho g h^2 n_y, \tag{13.98}$$

where

$$\varrho := \rho(1-\rho/\rho_w),$$

or again more compactly using the resistive stress tensor as

$$\boldsymbol{R}\,\hat{\boldsymbol{n}}_c = \frac{1}{2}\varrho g h\,\hat{\boldsymbol{n}}_c. \tag{13.99}$$

On the other hand if the ice terminates on land then $d=0$ and

$$h(2\tau_{xx}+\tau_{yy})n_x + h\tau_{xy}\, n_y = \frac{g}{2}\rho h^2\, n_x, \tag{13.100}$$

$$h(2\tau_{yy}+\tau_{xx})n_y + h\tau_{xy}\, n_x = \frac{g}{2}\rho h^2\, n_y. \tag{13.101}$$

or

$$\boldsymbol{R}\,\hat{\boldsymbol{n}}_c = \frac{1}{2}\rho g h\,\hat{\boldsymbol{n}}_c. \tag{13.102}$$

Written in terms of the velocity components the boundary conditions along a floating ice front are:

$$\eta h(4\partial_x u + 2\partial_y v)n_x + \eta h(\partial_x v + \partial_y u)n_y = \frac{\varrho g h^2}{2} n_x, \tag{13.103}$$

$$\eta h(\partial_x v + \partial_y u)n_x + \eta h(4\partial_y v + 2\partial_x u)n_y = \frac{\varrho g h^2}{2} n_y. \tag{13.104}$$

### 13.4.2 Ice Shelf Buttressing

The vertically integrated condition on stresses at the calving front is

$$\int_b^s \boldsymbol{\sigma}_h \hat{\boldsymbol{n}}_c \, dz = -\int_b^S p_w\,\hat{\boldsymbol{n}}_c \, dz \quad \text{on} \quad \Gamma_c, \tag{13.105}$$

where the subscript $h$ on the Cauchy stress tensor implies that we are only considering the horizontal stress components, that is

$$\boldsymbol{\sigma}_h = \begin{pmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{pmatrix}. \tag{13.106}$$

We denote the left-hand by $\boldsymbol{t}_i$ (traction on ice side) and those of the right-hand side by $\boldsymbol{t}_o$ (traction on ocean side), that is

$$\boldsymbol{t}_i = \int_b^s \boldsymbol{\sigma}_h \hat{\boldsymbol{n}}_c \, dz \quad \text{on} \quad \Gamma_c$$

and

$$\boldsymbol{t}_o = -\int_b^S p_w \, \hat{\boldsymbol{n}}_c \, dz \quad \text{on} \quad \Gamma_c,$$

and find as shown above that

$$\boldsymbol{t}_i = h\boldsymbol{R}\,\hat{\boldsymbol{n}}_c - \frac{1}{2}\rho g h^2 \, \hat{\boldsymbol{n}}_c$$

and

$$\boldsymbol{t}_o = -\frac{1}{2}\rho_w d^2 \, \hat{\boldsymbol{n}}_c = -\frac{1}{2}\frac{\rho}{\rho_w}\rho g h^2 \, \hat{\boldsymbol{n}}_c$$

For (13.105) to hold, i.e.

$$\boldsymbol{t}_i = \boldsymbol{t}_o$$

it follows that

$$\boldsymbol{R}\,\hat{\boldsymbol{n}}_c = \frac{1}{2}\varrho g h \, \hat{\boldsymbol{n}}_c \tag{13.107}$$

Eq. (13.107) is a boundary condition for the resistive stress tensor valid along a floating calving front. Within an ice shelf, and along the grounding line, the resistive stress tensor will in general not fulfil this condition.

We can define ice-shelf buttressing as the impact of the ice shelf on the stress regime along the grounding line. Note that in the absence of an ice shelf, and assuming that the ice front at the grounding line is exactly at flotation, the ice front will be in a direct contact with the ocean. To quantify the impact of the ice shelf on stresses at the grounding line we must therefore compare the stresses along the grounding line to those caused by the ocean pressure. More specifically, if we denote the vertically integrated horizontal traction along the grounding with $\boldsymbol{t}_{\mathrm{gl}}$, i.e.

$$\boldsymbol{t}_{\mathrm{gl}} = \int_b^s \boldsymbol{\sigma}_h \hat{\boldsymbol{n}}_c \, dz \quad \text{on} \quad \Gamma_{\mathrm{gl}} \tag{13.108}$$

the buttressing, $\boldsymbol{B}$, is by definition

$$\boldsymbol{B} = \boldsymbol{t}_o - \boldsymbol{t}_{\mathrm{gl}}. \tag{13.109}$$

The normal and tangential components of the buttressing vector $\boldsymbol{B}$ are

$$B_n = \hat{\boldsymbol{n}}_{\mathrm{gl}}^T \cdot (\boldsymbol{t}_o - \boldsymbol{t}_{\mathrm{gl}}) \tag{13.110}$$
$$= -\frac{1}{2}\frac{\rho}{\rho_w}\rho g h^2 - h\hat{\boldsymbol{n}}_{\mathrm{gl}} \cdot (\boldsymbol{R}\,\hat{\boldsymbol{n}}_c) + \frac{1}{2}\rho g h^2 \tag{13.111}$$
$$= \frac{1}{2}\varrho g h^2 - h\hat{\boldsymbol{n}}_{\mathrm{gl}}^T \cdot \boldsymbol{R}\,\hat{\boldsymbol{n}}_{\mathrm{gl}} \tag{13.112}$$

and

$$B_t = \hat{\boldsymbol{m}}_{\mathrm{gl}}^T \cdot (\boldsymbol{t}_o - \boldsymbol{t}_{\mathrm{gl}}) \tag{13.113}$$
$$= 0 - h\,\hat{\boldsymbol{m}}_{\mathrm{gl}}^T \cdot (\boldsymbol{R}\,\hat{\boldsymbol{n}}_{\mathrm{gl}}) + 0 \tag{13.114}$$
$$= h\,\hat{\boldsymbol{m}}_{\mathrm{gl}}^T \cdot (\boldsymbol{R}\,\hat{\boldsymbol{n}}_{\mathrm{gl}}) \tag{13.115}$$

where $\hat{\boldsymbol{m}}_{\mathrm{gl}}$ is a unit vector in the horizontal plane tangential to the grounding line and $\hat{\boldsymbol{m}}_{\mathrm{gl}}^T \cdot \hat{\boldsymbol{n}}_{\mathrm{gl}} = 0$.

If we want to non-dimentionalise expressions (13.112) and (13.115) then we can do so in a number of different ways. We could for example normalise with the vertically integrated ocean pressure $|\boldsymbol{t}_o| = \frac{1}{2}\rho_w g d^2$, or we could normalise using the magnitude of vertically integrated resistive stresses at a calving front, i.e. $h\|\boldsymbol{R}\hat{\boldsymbol{n}}_c\| = \frac{1}{2}\varrho g h^2$. In ice shelves, and along grounding lines, horizontal deviatoric stresses are typically on the order of $\varrho g h$ and much smaller than $\rho g d$ and we therefore opt for the second option and define dimensionless normal and tangential buttressing numbers as

$$K_N = \frac{\hat{\boldsymbol{n}}_{\mathrm{gl}}^T \cdot (\boldsymbol{t}_o - \boldsymbol{t}_{\mathrm{gl}})}{h\hat{\boldsymbol{n}}_c^T \cdot \boldsymbol{R}\hat{\boldsymbol{n}}_c} \tag{13.116}$$

$$= \frac{\frac{1}{2}\varrho g h - \hat{\boldsymbol{n}}_{\mathrm{gl}}^T \cdot \boldsymbol{R}\,\hat{\boldsymbol{n}}_{\mathrm{gl}}}{\frac{1}{2}\varrho g h} \tag{13.117}$$

$$= 1 - \frac{2\hat{\boldsymbol{n}}_{\mathrm{gl}}^T \cdot \boldsymbol{R}\,\hat{\boldsymbol{n}}_{\mathrm{gl}}}{\varrho g h} \tag{13.118}$$

and

$$K_T = \frac{\hat{\boldsymbol{m}}_{\mathrm{gl}}^T \cdot (\boldsymbol{t}_o - \boldsymbol{t}_{\mathrm{gl}})}{h\hat{\boldsymbol{n}}_c^T \cdot \boldsymbol{R}\hat{\boldsymbol{n}}_c} \tag{13.119}$$

$$= \frac{2\hat{\boldsymbol{m}}_{\mathrm{gl}}^T \cdot \boldsymbol{R}\,\hat{\boldsymbol{n}}_{\mathrm{gl}}}{\varrho g h}, \tag{13.120}$$

which agrees with Gudmundsson 2013.[3]

## 13.5 Scaling the sliding law

We start by scaling the unit normal and find

$$\begin{aligned}
\hat{\boldsymbol{n}} &= \frac{1}{\sqrt{1+(\partial_x b)^2+(\partial_y b)^2}} \begin{pmatrix} -\partial_x b \\ -\partial_y b \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{1+\delta^2(\partial_{x^*} b^*)^2+\delta^2(\partial_{y^*} b^*)^2}} \begin{pmatrix} -\delta\partial_{x^*} b^* \\ -\delta\partial_{y^*} b^* \\ 1 \end{pmatrix} \\
&= \left(1-\delta^2((\partial_{x^*} b^*)^2+(\partial_{y^*} b^*)^2)/2+O(\delta^4)\right) \begin{pmatrix} -\delta\partial_{x^*} b^* \\ -\delta\partial_{y^*} b^* \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -\delta\partial_{x^*} b^* + O(\delta^3) \\ -\delta\partial_{y^*} b^* + O(\delta^3) \\ 1-\delta^2((\partial_{x^*} b^*)^2+(\partial_{y^*} b^*)^2)/2+O(\delta^4) \end{pmatrix}.
\end{aligned} \tag{13.121}$$

The stress tensor is

$$\boldsymbol{\sigma} = [\sigma] \begin{pmatrix} \sigma_{xx}^* & \tau_{xy}^* & \delta\tau_{xz}^* \\ \tau_{xy}^* & \sigma_{yy}^* & \delta\tau_{yz}^* \\ \delta\tau_{xz}^* & \delta\tau_{yz}^* & \sigma_{zz}^* \end{pmatrix},$$

and the product $\boldsymbol{\sigma}\hat{\boldsymbol{n}}$ is therefore

$$\begin{aligned}
\boldsymbol{\sigma}\hat{\boldsymbol{n}} &= [\sigma] \begin{pmatrix} \sigma_{xx}^* & \tau_{xy}^* & \delta\tau_{xz}^* \\ \tau_{xy}^* & \sigma_{yy}^* & \delta\tau_{yz}^* \\ \delta\tau_{xz}^* & \delta\tau_{yz}^* & \sigma_{zz}^* \end{pmatrix} \begin{pmatrix} -\delta\partial_{x^*} b^* + O(\delta^3) \\ -\delta\partial_{y^*} b^* + O(\delta^3) \\ 1-\delta^2((\partial_{x^*} b^*)^2+(\partial_{y^*} b^*)^2)/2+O(\delta^4) \end{pmatrix} \\
&= [\sigma] \begin{pmatrix} -\delta\sigma_{xx}^*\,\partial_{x^*} b^* - \delta\tau_{xy}^*\,\partial_{y^*} b^* + \delta\tau_{xz}^* + O(\delta^3), \\ -\delta\tau_{xy}^*\,\partial_{x^*} b^* - \delta\sigma_{yy}^*\,\partial_{y^*} b^* + \delta\tau_{yz}^* + O(\delta^3), \\ -\delta^2\tau_{xz}^*\,\partial_{x^*} b^* - \delta^2\tau_{yz}^*\,\partial_{y^*} b^* + \sigma_{zz}^*(1-\frac{1}{2}\delta^2((\partial_{x^*} b^*)^2+(\partial_{y^*} b^*)^2)+O(\delta^4) \\ , \end{pmatrix},
\end{aligned} \tag{13.122}$$

[3]If we normalise with the (absolute value) of the vertically integrated ocean pressure

$$T_N' = \frac{\frac{1}{2}\varrho g h^2 - h\hat{\boldsymbol{n}}_c^T \boldsymbol{R}\,\hat{\boldsymbol{n}}_c}{\frac{1}{2}\rho_w g d^2},$$

then $T_N' = -1$ for a fully grounded calving front, whereas $T+N = -1/(\rho_w/\rho - 1)$. In general

$$T_N' = (\rho_w/\rho - 1)T_N,$$

so $T_N$ is about 10 times larger than $T_N'$.

so that $\hat{\boldsymbol{n}}^T \cdot \boldsymbol{\sigma}^* \hat{\boldsymbol{n}}$ where $\boldsymbol{\sigma} = [\sigma]\boldsymbol{\sigma}^*$ is given by

$$\begin{aligned}\hat{\boldsymbol{n}}^T \cdot \boldsymbol{\sigma}^* \hat{\boldsymbol{n}} =& \delta^2 \partial_{x^*} b^* (\sigma^*_{xx}\, \partial_{x^*} b^* + \tau^*_{xy}\, \partial_{y^*} b^* - \tau^*_{xz}) + \delta^2 \partial_{y^*} b^* (\tau^*_{xy}\, \partial_{x^*} b^* + \sigma^*_{yy}\, \partial_{y^*} b^* - \tau^*_{yz}) \\ & -\delta^2 \tau^*_{xz}\, \partial_{x^*} b^* - \delta^2 \tau^*_{yz}\, \partial_{y^*} b^* + \sigma^*_{zz}(1 - \frac{1}{2}\delta^2((\partial_{x^*} b^*)^2 + (\partial_{y^*} b^*)^2) + O(\delta^4),\end{aligned}$$

which, if we sort this according to order in $\delta$, is

$$\begin{aligned}&\hat{\boldsymbol{n}}^T \cdot \boldsymbol{\sigma}^* \hat{\boldsymbol{n}} = \sigma^*_{zz} \\ &+\delta^2 \left\{ (\sigma^*_{xx} - \sigma^*_{zz})(\partial_{x^*} b^*)^2 + (\sigma^*_{yy} - \sigma^*_{zz})(\partial_{y^*} b^*)^2 + 2\tau^*_{xy} \partial_{x^*} b^*\, \partial_{y^*} b^* - 2\tau^*_{xz} \partial_{x^*} b^* - 2\tau^*_{yz} \partial_{y^*} b^* \right\} \\ &+O(\delta^4).\end{aligned}$$

It follows that the normal stress vector on the bed, $(\hat{\boldsymbol{n}}^T \cdot \boldsymbol{\sigma} \hat{\boldsymbol{n}})\hat{\boldsymbol{n}}$, is

$$(\hat{\boldsymbol{n}}^T \cdot \boldsymbol{\sigma}^* \hat{\boldsymbol{n}})\hat{\boldsymbol{n}} = \begin{pmatrix} -\delta \sigma^*_{zz} \partial_{x^*} b^* + O(\delta^3) \\ -\delta \sigma^*_{zz} \partial_{y^*} b^* + O(\delta^3) \\ \sigma^*_{zz}(1 - \delta^2((\partial_{x^*} b)^2 + (\partial_{y^*} b)^2)/2) + \delta^2((\sigma^*_{xx} - \sigma^*_{zz})(\partial_{x^*} b^*)^2 + (\sigma^*_{yy} - \sigma^*_{zz})(\partial_{y^*} b^*)^2 + 2\tau_{xy} \partial_{x^*} b^*\, \partial_{y^*} b^* - 2\tau_{xz} \partial_{x^*} b^* - 2\tau_{yz} \partial_{y^*} b^*) + O(\delta^4) \end{pmatrix}.$$

Predictably our insistence on keeping things up to third order is making things look a bit messy.

The shear stress vector ($\boldsymbol{t}_b$) can now easily be calculated and is found to be

$$\begin{aligned}\boldsymbol{t}_b =& \boldsymbol{\sigma} \hat{\boldsymbol{n}} - (\hat{\boldsymbol{n}}^T \cdot \boldsymbol{\sigma} \hat{\boldsymbol{n}})\hat{\boldsymbol{n}} \\ =& [\sigma] \begin{pmatrix} \delta \partial_{x^*} b^* (\sigma^*_{zz} - \sigma^*_{xx}) - \delta \partial_{y^*} b^* \tau^*_{xy} + \delta \tau^*_{xz} + O(\delta^3) \\ \delta \partial_{y^*} b^* (\sigma^*_{zz} - \sigma^*_{yy}) - \delta \partial_{x^*} b^* \tau^*_{xy} + \delta \tau^*_{yz} + O(\delta^3) \\ \delta^2((\sigma^*_{zz} - \sigma^*_{xx})(\partial_{x^*} b^*)^2 + (\sigma^*_{zz} - \sigma^*_{yy})(\partial_{y^*} b^*)^2 - 2\tau^*_{xy} \partial_{x^*} b^*\, \partial_{y^*} b^* + \tau^*_{xz} \partial_{x^*} b^* + \tau^*_{yz} \partial_{y^*} b^*) + O(\delta^4) \end{pmatrix}.\end{aligned}$$

Hence, the components of the basal shear stress vector are $O(\delta)$ or less.

We now scale the basal velocity vector

$$\boldsymbol{v}_b = \boldsymbol{v} - (\hat{\boldsymbol{n}}^T \cdot \boldsymbol{v})\hat{\boldsymbol{n}}$$

and find

$$\begin{aligned}\boldsymbol{v}_b =& [u] \begin{pmatrix} u^* \\ v^* \\ \delta w^* \end{pmatrix} \\ & -[u] \left( \begin{pmatrix} -\delta \partial_{x^*} b^* + O(\delta^3) \\ -\delta \partial_{y^*} b^* + O(\delta^3) \\ 1 - \delta^2((\partial_{x^*} b^*)^2 + (\partial_{y^*} b^*)^2)/2 + O(\delta^4) \end{pmatrix}^T \cdot \begin{pmatrix} u^* \\ v^* \\ \delta w^* \end{pmatrix} \right) \begin{pmatrix} -\delta \partial_{x^*} b^* + O(\delta^3) \\ -\delta \partial_{y^*} b^* + O(\delta^3) \\ 1 - \delta^2((\partial_{x^*} b^*)^2 + (\partial_{y^*} b^*)^2)/2 + O(\delta^4) \end{pmatrix} \\ =& [u] \begin{pmatrix} u^* \\ v^* \\ \delta w^* \end{pmatrix} - [u] \begin{pmatrix} \delta^2 u^* (\partial_{x^*} b^*)^2 + \delta^2 v^* \partial_{y^*} b^*\, \partial_{x^*} b^* - \delta^2 \partial_{x^*}\, w^* + O(\delta^3) \\ \delta^2 u^* \partial_{x^*} b^*\, \partial_{y^*} b^* + \delta^2 v^* (\partial_{y^*} b^*)^2 - \delta^2 \partial_{y^*} b^*\, w^* + O(\delta^3) \\ -\delta u^* \partial_{x^*} b^* - \delta v^* \partial_{x^*} b^* + \delta w^* + O(\delta^3) \end{pmatrix} \\ =& [u] \begin{pmatrix} u^* + O(\delta^2) \\ v^* + O(\delta^2) \\ \delta u^* \partial_{x^*} b^* + \delta v^* \partial_{y^*} b^* + O(\delta^3). \end{pmatrix}.\end{aligned}$$

Hence, to second order

$$\boldsymbol{v}_b = [u] \begin{pmatrix} u^* \\ v^* \\ \delta u^* \partial_{x^*} b^* + \delta v^* \partial_{y^*} b^* \end{pmatrix}.$$

# Chapter 14

# Perturbation solutions of the SSTREAM/SSA

## 14.1 Problem definition

We perform a small-amplitude perturbation analysis of the shallow ice stream (SSTREAM) equations. The discussion is limited to 1d along a flow line in which case the SSTREAM equations are

$$4\partial_x(h\eta\partial_x u) - (u/c)^{1/m} = \rho g h \partial_x s \cos\alpha - \rho g h \sin\alpha, \tag{14.1}$$

The horizontal velocity component ($u$) is constant across the depth, and the vertical velocity component ($w$) varies linearly with depth. In these equation $s$ is the surface, $h$ is ice thickness, $\eta$ is the effective ice viscosity, and $c$ is the basal slipperiness. The parameter $m$ and the basal slipperiness $c$ are parameters in the sliding law, We write the basal sliding law on the form

$$\boldsymbol{u}_b = c(x,y)\|\boldsymbol{T}_b\|^{m-1}\boldsymbol{T}_b. \tag{14.2}$$

where $\boldsymbol{T}_b$ is the basal stress vector given by $\boldsymbol{T}_b = \sigma\hat{\boldsymbol{n}} - (\hat{\boldsymbol{n}}^T \cdot \sigma\hat{\boldsymbol{n}})\hat{\boldsymbol{n}}$, with $\hat{\boldsymbol{n}}$ being a unit normal vector to the bed pointing into the ice. The function $c(x)$ is referred to as the basal slipperiness.

For a linear viscous media ($n = 1$) and a non-linear sliding law ($m$ arbitrary) this equation can be linearised and solved analytically using standard methods as follows. We write $f = \bar{f} + \Delta f$, where $f$ stands for some relevant variable entering the problem, and look for a zeroth-order solution where $\bar{f}$ is independent of $x$ and $y$ and time $t$, while the first-order field $\Delta f$ is small but can be a function of space and time.

The perturbations in bedrock ($\Delta b$) and slipperiness ($\Delta c$) are step functions of time. They are applied at $t = 0$, i.e. for $t < 0$ we have $\Delta b = 0$ and $\Delta c = 0$ and for $t \geq 0$ both $\Delta b$ and $\Delta c$ are some constants. Using this history definition the solutions for the velocity field and the surface geometry become functions of time.

### 14.1.1 Bedrock perturbations

We start by considering the response to small perturbation in basal topography ($\Delta b$). Writing $h = \bar{h} + \Delta h$, $s = \bar{s} + \Delta s$, $b = \bar{b} + \Delta b$, where $h$ is ice thickness, $s$ surface topography, and $b$ bedrock topography, and furthermore $u = \bar{u} + \Delta u$, $w = \Delta w$, where $u$ and $w$ are the $x$ and $z$ components of the velocity vector, and $c = \bar{c}$ where $c$ is the basal slipperiness (see Eq. (14.2) and inserting into (14.1) and solving gives the zeroth-order solution

$$\bar{u} = \bar{c}\rho g \bar{h} \sin\alpha. \tag{14.3}$$

The zeroth-order solution represents a plug flow down an uniformly inclined plane.

The first-order field equations are

$$4\eta\bar{h}\partial^2_{xx}\Delta u - \gamma\Delta u = \rho g \bar{h}\cos\alpha\partial_x\Delta s - \rho g \sin\alpha\Delta h, \tag{14.4}$$

where

$$\gamma = \frac{\tau_d^{1-m}}{m\bar{c}}, \tag{14.5}$$

and

$$\tau_d = \rho g \bar{h} \sin \alpha, \tag{14.6}$$

is the driving stress.

The domain of the first-order solution is transformed to that of the zeroth-order problem. Let $f(z)$ be some function of the vertical coordinate $z$. We have

$$f = \bar{f} + \Delta f$$

where $\bar{f}$ is the zeroth order approximation and $\Delta f$ the first order perturbation. For $z = \bar{z} + \Delta z$ we write

$$\begin{aligned} f(z) =& \bar{f}(z) + \Delta f(z) \\ =& \bar{f}(\bar{z}) + \partial_z \bar{f}|_{z=\bar{z}}(\bar{z}) \Delta z + \Delta f(\bar{z}) \end{aligned}$$

where terms of second order have been ignored.

For the kinematic boundary condition at the surface

$$\partial_t s + u \, \partial_x s - w = 0$$

we, for example, get

$$\partial_t(\bar{s} + \Delta s) + (\bar{u} + \Delta u + \partial_z \bar{u}|_{z=\bar{s}} \, \Delta u) \partial_x (\bar{s} + \Delta s) - (\bar{w} + \Delta w + \partial_z \bar{w}|_{z=\bar{s}} \, \Delta s) = 0.$$

We have $\partial_z \bar{u} = 0$ and for the particular zeroth-order solution we are using (plug flow) we have $\partial_z \bar{w} = 0$. It follows that to first order the upper and lower boundary kinematic conditions are

$$\partial_t \Delta s + \bar{u} \, \partial_x \Delta s - \Delta w = 0, \tag{14.7}$$

and

$$\bar{u} \, \partial_x \Delta b - \Delta w = 0, \tag{14.8}$$

respectively. In (14.7) the surface mass-balance perturbation has been set to zero. The jump conditions for the stresses have already been using in the derivation of (14.1) and do not need to be considered further.

This system of equations is solved using standard Fourier and Laplace transform methods. All variables are Fourier transformed with respect to the spatial variables $x$ and $y$ and Laplace transformed with respect to the time variable $t$. The forward Fourier transform $f(k)$ of a function $f(x)$ is

$$f(k) = \int_{-\infty}^{+\infty} f(x) e^{\mathrm{i}kx} \, dx, \tag{14.9}$$

where i is the imaginary unit. The forward Laplace transform $f(r)$ of a function $f(t)$ is

$$f(r) = \int_{0^+}^{+\infty} f(t) \, e^{-rt} \, dt. \tag{14.10}$$

The Fourier and Laplace transforms of the first-order field Eq. (14.4) is

$$4\eta \bar{h} k^2 \, \Delta u + \gamma \Delta u = \rho g \sin \alpha (\Delta s - \Delta b) + \mathrm{i} k \rho g \cos \alpha \bar{h} \Delta s, \tag{14.11}$$

The Fourier transformed mass-conservation equation is

$$-\mathrm{i}k \, \Delta u + \partial_z \Delta w = 0. \tag{14.12}$$

The Fourier-Laplace transformed linearised kinematic boundary condition at the upper boundary (Eq. (14.7)) is

$$-\mathrm{i}k\bar{u} \, \Delta s + r \, \Delta s - \Delta s(t=0) - \Delta w = 0, \tag{14.13}$$

and the one at the lower boundary is

$$-\mathrm{i}k\bar{u} \, \Delta b - w = 0. \tag{14.14}$$

In addition we know that $u$ is constant over the depth, so that

$$\Delta u = \Delta u(k, t) \tag{14.15}$$

and not $u = u(k, z, t)$. Furthermore, since $w$ is a linear function of depth we have

$$\partial_z \Delta w = F(k, t) \tag{14.16}$$

where $F$ is some function of $k$ and $t$ (but not of $z$).

Eqs. (14.11), (14.12), (14.13), (14.14) together with (14.15) and (14.16) can now be solved for $\Delta u$, $\Delta w$, and $\Delta s$ as a function of $\Delta b$ for $\Delta s(t = 0) = 0$. There are various ways of doing this. One can, for example, start by integrating Eq. (14.16) with respect to $z$ giving

$$\Delta w(z) = (z - b)\partial_z \Delta w - \mathrm{i}k\bar{u}\Delta b \tag{14.17}$$

where lower kinematic boundary condition (14.14) has been used to determine the integration constant. Then $\Delta u$ can be eliminated from Eq. (14.12) using Eq. (14.11). Inserting the resulting expression for $\partial_z \Delta w$ into (14.17) and setting $z = \bar{s}$ gives $\Delta w$ as a linear function of $\Delta s$ and $\Delta b$. The upper boundary kinematic condition (14.13) can then be used to get rid of $\Delta w$ giving $\Delta s$ as a (linear) function of $\Delta b$. Doing these algebraic manipulations, one finds that the (complex) ratio between surface and bedrock amplitude $T_{sb} = \Delta s/\Delta b$ is given by

$$T_{sb}(k, r) = -\frac{\mathrm{i}k(\bar{u} + \tau_d/\xi)}{r(r - p)}, \tag{14.18}$$

where

$$p = \mathrm{i}/t_p - 1/t_r, \tag{14.19}$$

and the two timescales $t_p$ (phase time scale) and $t_r$ (relaxation time scale) are given by

$$t_p^{-1} = k(\bar{u} + \tau_d/\xi), \tag{14.20}$$

and

$$t_r^{-1} = \xi^{-1}k^2\tau_d\bar{h}\cot\alpha, \tag{14.21}$$

and furthermore

$$\xi = \gamma + 4\bar{h}k^2\eta. \tag{14.22}$$

The inverse Laplace transform is calculated using the Bromwich integral

$$f(t) = \frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{rt} f(r)\, dr. \tag{14.23}$$

We see from Eq. (14.18) that the $T_{sb}(k, r)$ transfer function has two poles, one at $r = 0$ and another one at $r = p$. The second pole is always in the left-half complex plane and the inverse Laplace transform can be evaluated by contour integration over a semicircle in the left hand plane. We find that

$$T_{sb}(k, l, t) = \frac{\mathrm{i}k(\bar{u}\,\xi + \tau_d)}{p\,\xi}(e^{pt} - 1). \tag{14.24}$$

This transfer function describes the relation between surface and bedrock geometry, where $\Delta s(k, l, t) = T_{sb}(k, l, t)\Delta b(k, l)$. Transfer functions giving the perturbation in velocity can be calculated in the same manner.

**Exercise:** Calculate $T_{ss_0} := s(k, t)/s(k, t = 0)$ for $\Delta b = 0$ and show that

$$T_{ss_0} = e^{pt}. \tag{14.25}$$

### 14.1.2 Slipperiness perturbations

We now consider the response to small perturbation in basal slipperiness. We write the slipperiness perturbation on the form

$$c(x) = \bar{c}(1 + \Delta c(x))$$

where $\Delta c$ is the fractional perturbation in basal slipperiness. The total perturbation is $\bar{c}\Delta c$. As before we write $h = \bar{h} + \Delta h$, $s = \bar{s} + \Delta s$, $u = \bar{u} + \Delta u$, and $w = \Delta w$. Since there is no perturbation in basal topography we have $b = \bar{b}$ and $h = \bar{h} + \Delta s$.

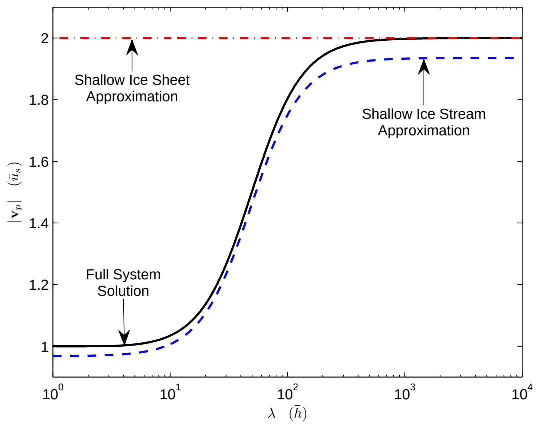



Figure 14.1: The phase speed ($\|\boldsymbol{v}_p\|$) as a function of wavelength for $\theta$=0. The dashed-dotted curve is based on the shallow-ice-sheet (SSHEET) approximation, the dashed one is based on the shallow-ice-stream (SSTREAM) approximation, and the solid one is a full-system (FS) solution. The surface slope is $\alpha$=0.005 and slip ratio $\bar{C}$=30 and $n = m = 1$. The unit on the $y$ axis is the mean surface velocity of the full-system solution ($\bar{u}$=$\bar{C}$+1=31).

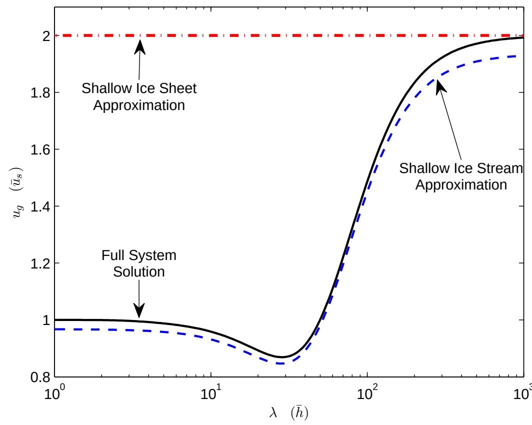



Figure 14.2: The $x$ component of the group velocity ($u_g$) as a function of wavelength for $\theta$=0. Values of mean surface slope and slip ratio are 0.005 and 30, respectively, and $m = n = 1$.

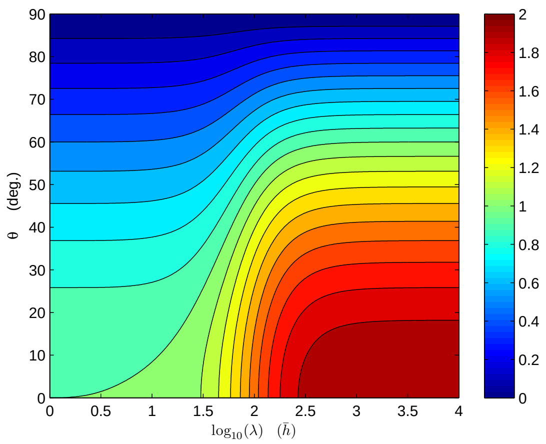


Figure 14.3: The phase speed ($\|\boldsymbol{v}_p\|$) of the full-system solution as a function of wavelength $\lambda$ and orientation $\theta$ of the sinusoidal perturbations with respect to mean flow direction. The mean surface slope is $\alpha$=0.002 and the slip ratio is $\bar{C}$=100, and $n = m = 1$. The plot has been normalised with the non-dimensional surface velocity $\bar{u}$=$\bar{C}$+1=101 of the full-system solution.

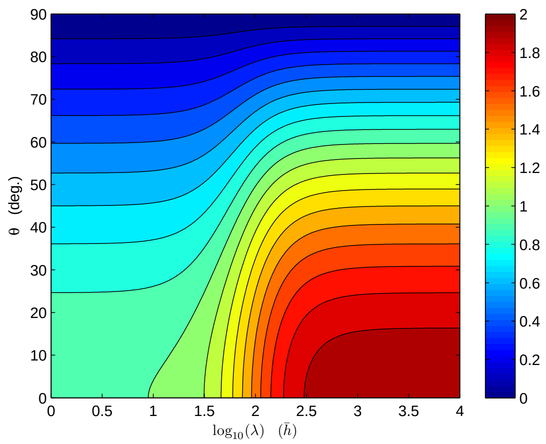


Figure 14.4: The shallow-ice-stream phase speed as a function of wavelength $\lambda$ and orientation $\theta$. As in Fig. 14.3a the mean surface slope is $\alpha$=0.002 and the slip ratio is $\bar{C}$=100, $n = m = 1$, and the plot has been normalised with the non-dimensional surface velocity $\bar{u}$=$\bar{C}$+1=101 of the full-system solution.

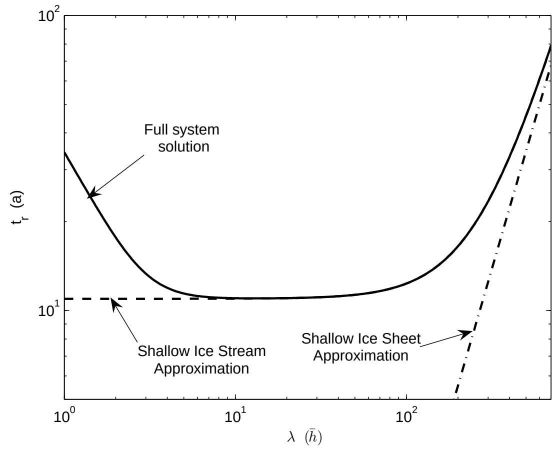



Figure 14.5: The relaxation time scale ($t_r$) as a function of wavelength $\lambda$. The wavelength is given in units of mean ice thickness ($\bar{h}$) and $t_r$ is given in years. The mean surface slope is $\alpha$=0.002, the slip ratio is $\bar{C}$=999, and $n = m = 1$. For these values $t_r$ is on the order of 10 years for a fairly wide range of wavelengths. Lowering the slip ratio will reduce the value of $t_r$. It follows that ice streams will react to sudden changes in basal properties or surface profile by a characteristic time scale of a few years.

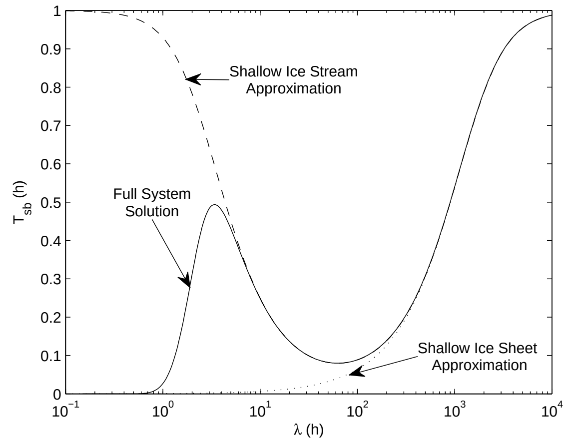



Figure 14.6: Steady-state response of surface topography ($\Delta s$) to a perturbation in bed topography ($\Delta b$). The surface slope is 0.002, the mean slip ratio $\bar{C}$=100, and $n = m = 1$. Transfer functions based on the shallow-ice-stream approximation (dashed line, Eq. 14.24), the shallow-ice-sheet approximation (dotted line, and a full system solution (solid line) are shown.

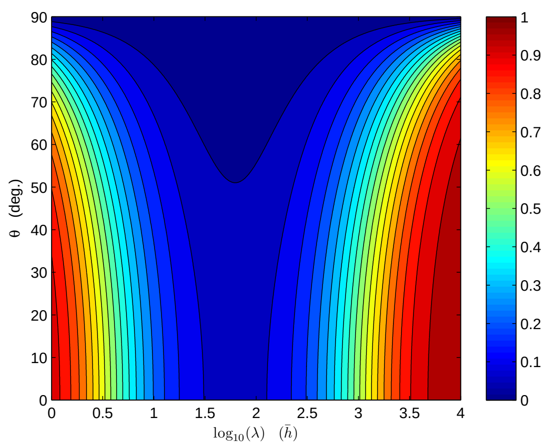



Figure 14.7: **(a)** The SSTREAM amplitude ratio ($|T_{sb}|$) between surface and bed topography (Eq. 14.24). Surface slope is 0.002, the slip ratio $\bar{C}$=99, and $n = m = 1$. $\lambda$ is the wavelength of the sinusoidal bed topography perturbation and $\theta$ is the angle with respect to the x axis, with $\theta$=0 and $\theta$=90 corresponding to transverse and longitudinal undulations in bed topography, respectively.

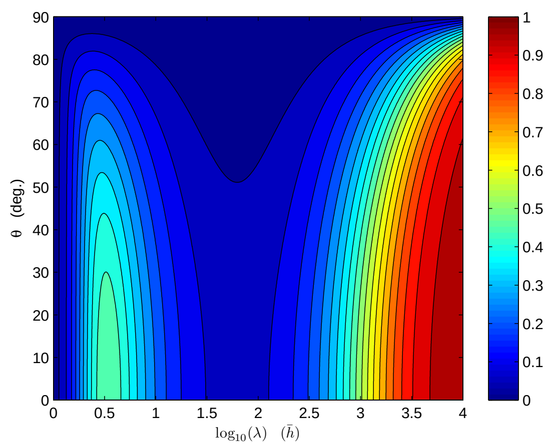



Figure 14.7: **(b)** The FS amplitude ratio between surface and bed topography ($|T_{sb}|$). The shape of the same transfer function for the same set of parameters based on the SSTREAM approximation is shown in Fig. 14.7a.

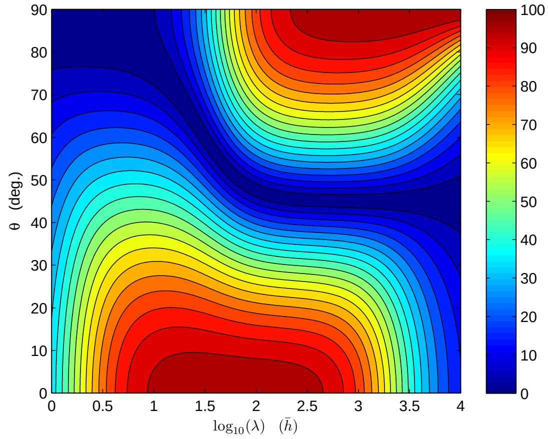



Figure 14.8: **(a)** The steady-state amplitude ratio ($|T_{ub}|$) between longitudinal surface velocity ($\Delta u$) and bed topography ($\Delta b$) in the shallow-ice-stream approximation. Surface slope is 0.002, the slip ratio is 99, and $n = m = 1$.

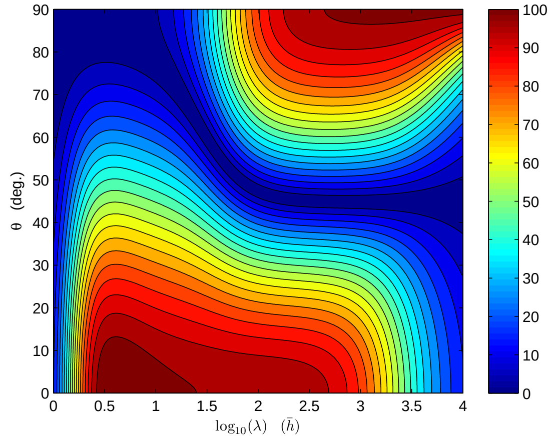



Figure 14.8: **(b)** The steady-state amplitude ratio ($|T_{ub}|$) between longitudinal surface velocity ($\Delta u$) and bed topography ($\Delta b$). The shape of the same transfer function for the same set of parameters, but based on the shallow-ice-stream approximation, is shown in Fig. 14.8a.

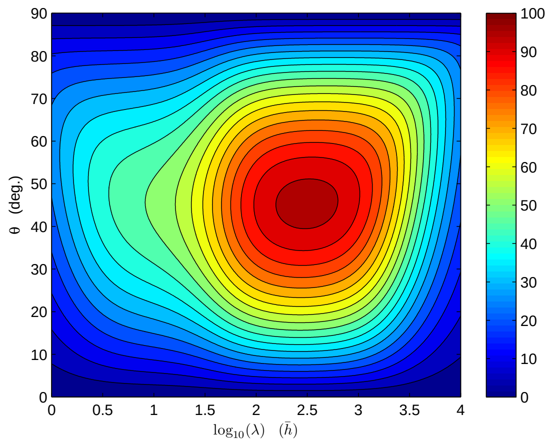



Figure 14.9: **(a)** The steady-state amplitude ratio ($|T_{vb}|$) between transverse velocity ($\Delta v$) and bed topography ($\Delta b$) in the shallow-ice-stream approximation. Surface slope is 0.002, the slip ratio is 99 and $n = m = 1$.

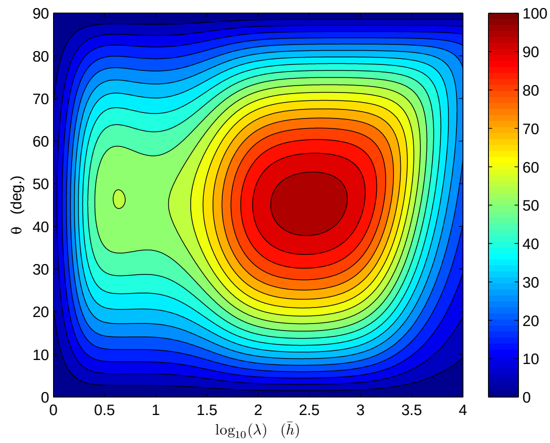



Figure 14.8: **(b)** The steady-state amplitude ratio ($|T_{vb}|$ between transverse velocity ($\Delta v$) and bed topography ($\Delta b$). The shape of the same transfer function for the same set of parameters, but based on the shallow-ice-stream approximation, is shown in Fig. 14.9a.

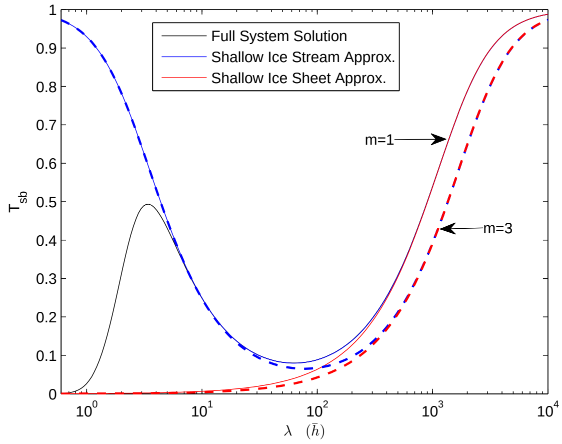



Figure 14.9: Steady-state response of surface topography to a perturbation in bed topography for linear and non-linear sliding. All curves are for linear medium ($n$=1). The solid lines are calculated for linear sliding ($m$=1) and the dashed lines for non-linear sliding ($m$=3). The red lines are SSHEET solutions, the blue ones are SSTREAM solutions, and the black line is a FS solution which is only available for $m$=1. Mean surface slope is 0.002 and slip ratio is equal to 100.

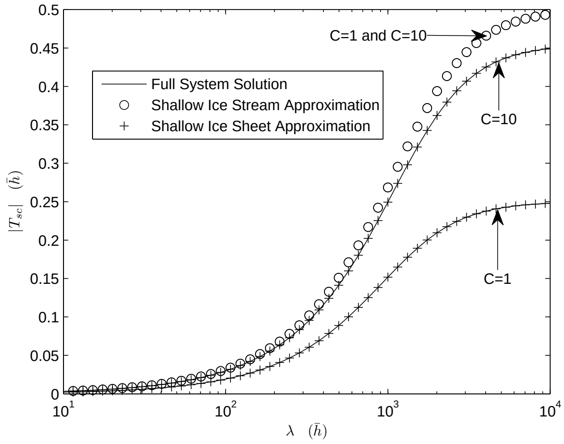



Figure 14.9: Steady-state response of surface topography to a basal slipperiness perturbation. Shown are FS (solid line), SSTREAM (circles), and SSHEET (crosses) transfer amplitudes for both $\bar{C}$=1 and $\bar{C}$=10. In the plot the SSTREAM curves for $\bar{C}$=1 and $\bar{C}$=10 are too similar to be distinguished. The surface slope is 0.002.

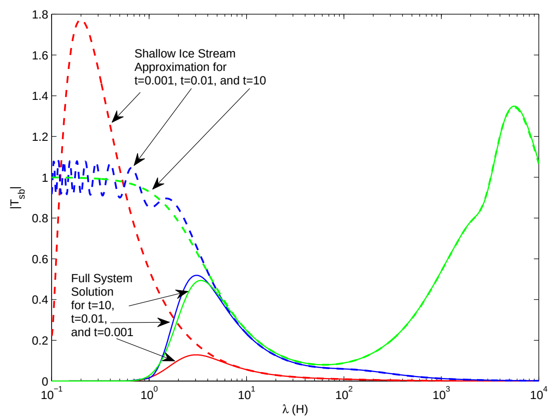



Figure 14.10: Transient surface topography response to a sinusoidal perturbation in bed topography applied at $t$=0. Shown are the amplitude ratios between surface and bed topography ($|T_{sb}|$) as a function of wavelength for $\alpha$=0.002, $\theta$=0, $\bar{C}$=100, and $n = m = 1$ for $t$=0.001 (red), $t$=0.01 (blue), and $t$=10 (green).

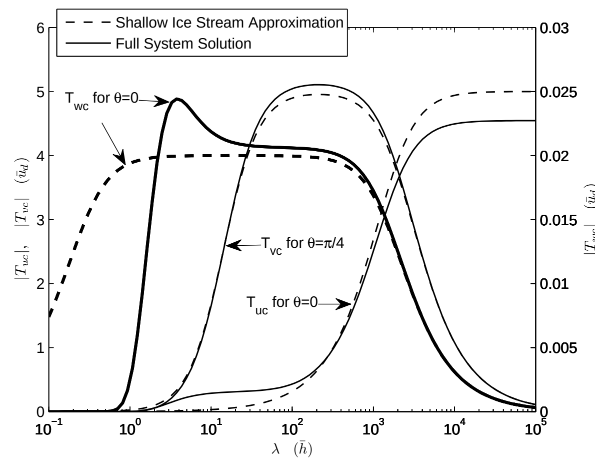



Figure 14.10: Steady-state response of surface longitudinal ($u$), transverse ($v$), and vertical ($w$) velocity components to a basal slipperiness perturbation. The surface slope is 0.002 and the slip ratio $\bar{C}$=10. The $T_{uc}$ and $T_{wc}$ amplitudes are calculated for slipperiness perturbations aligned transversely to the flow direction ($\theta$=0). For $T_{vc}$, $\theta$=45 degrees. Of the two y axis the scale to the left is for the horizontal velocity components ($T_{uc}$ and $T_{wc}$), and the one to the right is the scale for $T_{uc}$.

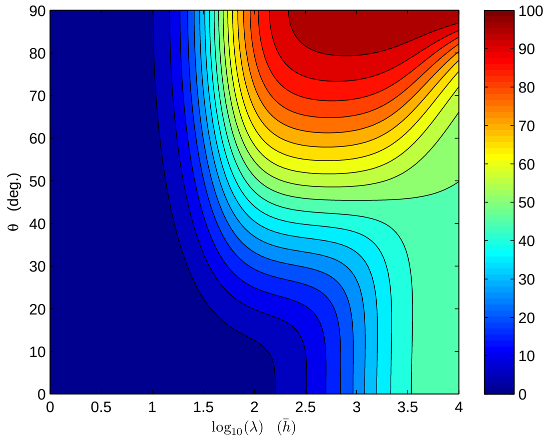


Figure 14.11: Steady-state response of the surface longitudinal ($\Delta u$) velocity component to a basal slipperiness perturbation in the shallow-ice-stream approximation. The surface slope is 0.002 and the slip ratio $\bar{C}$=99.

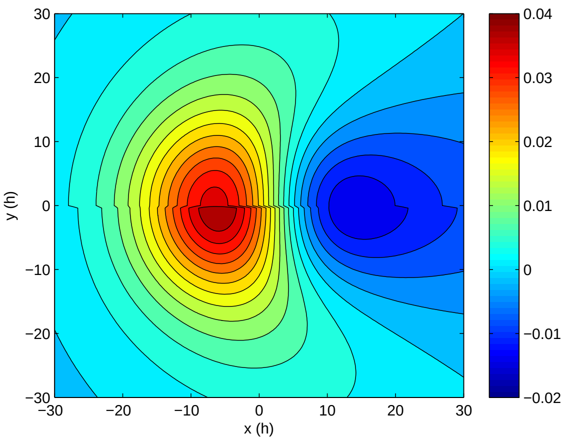


Figure 14.12: **(a)** Surface topography response to a flow over a Gaussian-shaped bedrock disturbance as given by a FS (lower half of figure) and a SSTREAM solution (upper half of figure). The mean flow direction is from left to right. Surface slope is 3 degrees and mean basal velocity equal to mean deformational velocity ($\bar{C}$=1). The spatial unit is one mean ice thickness ($\bar{h}$). The Gaussian-shaped bedrock disturbance has a width of 10 $\bar{h}$ and it's amplitude is 0.1 $\bar{h}$. The problem definition is symmetrical about the x axis ($y$=0) and any deviations in the figure from this symmetry are due to differences in the FS and the SSTREAM solutions.

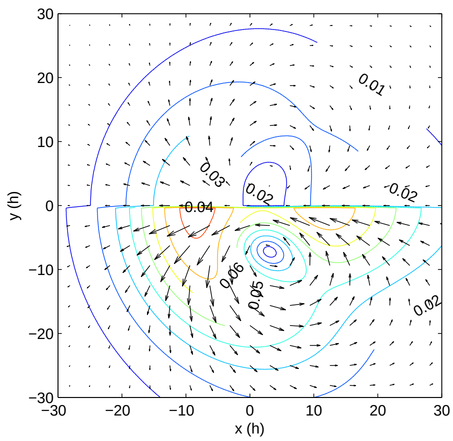


Figure 14.13: **(b)** Response in surface velocity to a Gaussian-shaped bedrock perturbation. All parameters are equal to those in Fig. 14.12a. The contour lines give horizontal speed and the vectors the horizontal velocities. The velocity unit is mean-deformational velocity ($\bar{u}_d$). The slip ratio is equal to one, and the mean surface velocity is $2\bar{u}_d$. The upper half of the figure is the SSTREAM solution and the lower half the corresponding FS solution.

For the zeroth-order problem we get the plug-flow solution as before

$$\bar{u} = \bar{c}\rho g\bar{h}\sin\alpha. \tag{14.26}$$

$$\partial_t \Delta s + \bar{u}\,\partial_x \Delta s - \Delta w = 0, \tag{14.27}$$

and

$$\bar{u}\,\partial_x \Delta b - \Delta w = 0, \tag{14.28}$$

To first order the upper and lower boundary kinematic conditions are

$$\partial_t \Delta s + \bar{u}\,\partial_x \Delta s - \Delta w = 0, \tag{14.29}$$

as before, while the basal boundary conditions are

$$\Delta w = 0, \tag{14.30}$$

In the field equation (14.1) we have, among other terms, the term $(u/c)^{1/m}$. For $u = \bar{u} + \Delta u$ and $c = \bar{c}(1 + \Delta c)$ we find

$$\left(\frac{u}{c}\right)^{1/m} = \left(\frac{\bar{u} + \Delta u}{\bar{c}(1 + \Delta c)}\right)^{1/m} = \bar{c}^{1/m}((\bar{u} + \Delta u)(1 - \Delta c))^{1/m} = \left(\frac{\bar{u}}{\bar{c}}\right)^{1/m} - \gamma\bar{u}\Delta c + \gamma\Delta u$$

where

$$\gamma = \frac{1}{\bar{u}m}\left(\frac{\bar{u}}{\bar{c}}\right)^{1/m} = \frac{\tau_d^{1-m}}{m\bar{c}}$$

where

$$\gamma = \frac{\tau_d^{1-m}}{m\bar{c}}, \tag{14.31}$$

and

$$\tau_d = \rho g\bar{h}\sin\alpha, \tag{14.32}$$

is the driving stress.

We then find that the first-order field equations are

$$4\eta\bar{h}\partial^2_{xx}\Delta u - \gamma\Delta u = \rho g\bar{h}\cos\alpha\partial_x\Delta s - \rho g\sin\alpha\Delta h, \tag{14.33}$$

and these can be solved using standard Fourier and Laplace transformation methods as done above for the case of bedrock perturbation.

# Part III

# Appendices

# Appendix A

# Calculating vertical surface velocity

The sign convention for upper- and lower-surface mass balance is such that the kinematic boundary conditions at the upper and lower surfaces read, respectively,

$$\partial_t s + u\partial_x s + v\partial_y s - w_s = a_s, \tag{A.1}$$

$$\partial_t b + u\partial_x b + v\partial_y b - w_b = -a_b, \tag{A.2}$$

i.e. adding ice is always defined as positive surface mass balance.

Subtracting (A.2) from (A.1) gives

$$\partial_t h + u\,\partial_x h + v\,\partial_y h - w_s + w_b = a_s + a_b,$$

where it has been used that $u$ does not change with depth. Now using (A.5) gives

$$\partial_t h + u\,\partial_x h + v\,\partial_y h + h(\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy}) = a_s + a_b,$$

or

$$\partial_t h + \partial_x(uh) + \partial_y(vh) = a_s + a_b, \tag{A.3}$$

hence, in the flux-conservation equation both upper and lower mass balance terms have the same sign.

## A.1 grounded part

On the grounded part $\partial_t s = \partial_t h$ and the kinematic boundary condition gives

$$w_s = \partial_t h + u\,\partial_x s + v\,\partial_y s - a_s, \tag{A.4}$$

but this requires $\partial_t h$ to be known before we can calculate $w_s$. An alternative approach is to integrate the vertical strain rate $\dot{\epsilon}_{zz}$ over the thickness, use the mass conservation equation and the fact that horizontal strain rates do not change across the thickness, to arrive at

$$w_s = w_b - h\,(\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy}). \tag{A.5}$$

We now calculate $w_b$ from the kinematic boundary condition at the lower surface as

$$w_b = a_b + u\partial_x b + v\partial_y b,$$

and insert into (A.5) arriving at

$$w_s = a_b + u\,\partial_x b + v\,\partial_y b - h\,(\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy}), \tag{A.6}$$

which represents a convenient way of calculation $w_s$ once the horizontal velocity field has been determined.

## A.2 floating part

Where the ice is afloat

$$s = S + (1 - \rho/\rho_o)\, h$$

i.e.

$$\partial_t s = (1 - \rho/\rho_o)\, \partial_t h$$

The kinematic boundary condition at the surface gives

$$w_s = \partial_t s + u\, \partial_x s + v\, \partial_y s - a_s$$

and therefore

$$w_s = (1 - \rho/\rho_o)\, \partial_t h + u\, \partial_x s + v\, \partial_y s - a_s \tag{A.7}$$

If $\partial_t h$ is known (A.7) can be used to calculate $w_s$, otherwise we use (A.3) and find that

$$w_s = (1 - \rho/\rho_o)\, (a_s + a_b - \partial_x q_x - \partial_y q_y) + u\, \partial_x s + v\, \partial_y s - a_s \tag{A.8}$$

An alternative way of calculating $w_s$ is to insert the floating condition

$$s = (1 - \rho_o/\rho) b$$

into (A.1) and to use (A.2) to get rid of $\partial_t b$

$$(1 - \rho_o/\rho)(w_b - a_b - u\partial_x b - vp_y b) + (1 - \rho_o/\rho)(u\partial_x b + v\partial_y b) - w_s = a_s \tag{A.9}$$

to arrive at the simple and intuitive expression

$$(1 - \rho_o/\rho)(w_b - a_b) = a_s + w_s \tag{A.10}$$

and then to use the (A.5) to get rid of $w_b$ leading to

$$w_s = -(1 - \rho/\rho_o)\, (h(\dot\epsilon_{xx} + \dot\epsilon_{yy}) - a_b) - a_s\, \rho/\rho_o \tag{A.11}$$

The above expression shows that adding ice to the surface ($a_s > 0$) over a floating ice shelf gives rise to neg. vertical surface velocity, as does horizontal divergent flow ($\dot\epsilon_{xx} + \dot\epsilon_{yy} > 0$), and basal melting ($a_b < 0$).

# Appendix B

# Integral theorem

If $f$ and $g$ are scalar functions then in $x$ and $y$ directions we have

$$\int_{\Omega} f\,\partial_x g\;d\Omega = -\int_{\Omega} \partial_x f\,g\;d\Omega + \oint_{\partial\Omega} f\,g\,n_x\;d\Gamma \tag{B.1}$$

$$\int_{\Omega} f\,\partial_y g\;d\Omega = -\int_{\Omega} \partial_y f\,g\;d\Omega + \oint_{\partial\Omega} f\,g\,n_y\;d\Gamma \tag{B.2}$$

If we write $g = g_x$ in the upper one and $g = g_y$ in the lower one, add them together and define $\boldsymbol{g} = (g_x, g_y)^T$ and $\hat{\boldsymbol{n}} = (n_x, n_y)^T$ then we arrive at

$$\int_{\Omega} f\,\nabla_{xy}\cdot\boldsymbol{g}\;d\Omega = -\int_{\Omega} \nabla_{xy} f\cdot\boldsymbol{g}\;d\Omega + \oint_{\partial\Omega} f\,\boldsymbol{g}\cdot\hat{\boldsymbol{n}}\;d\Gamma \tag{B.3}$$

# Appendix C

# Definition of gradients in terms of directional derivatives and inner products

Sensitivities are directional derivatives. The directional derivative of the scalar function $J(p)$ in the direction $\phi$ is denoted by $Df(p)[\phi]$ and defined as

$$
\begin{aligned}
Df(p)[\phi] &= \lim_{\epsilon \to 0} \frac{d}{d\epsilon} f(p + \epsilon\phi) \\
&= \lim_{\epsilon \to 0} \frac{f(p + \epsilon\phi) - f(p)}{\epsilon}
\end{aligned}
$$

The directional derivative is sometimes written as $\delta f(p, \phi)$ or as $f'(p, \phi)$ i.e.

$$Df(p)[\phi] = f'(p, \phi) = \delta f(p, \phi)$$

are just different ways of writing the directional derivative.

We define the gradient through

$$Df(p)[\phi] =< \nabla J(p), \phi >_H$$

where $H$ is a Hilbert space and $f : H \to \mathbb{R}$. Here $\nabla J(p)$ is the gradient of $J$, and the expression above *defines* the gradient in terms of the (directional) derivative for a given inner product. In other words, for a function $f : H \to \mathbb{R}$ the gradient is defined as the Riez-representation for the directional derivative $Df(p)[\phi]$ through

$$< \nabla f(p), \phi >_H = Df(p)[\phi]$$

The directional derivative depends on the inner product $<, >_H$ and the gradient is not defined without specifying the inner product.

Example: Consider the case $H = \mathbb{R}^n$ with the inner product

$$< \boldsymbol{x}, \boldsymbol{y} >= \boldsymbol{x}^T \boldsymbol{M} \boldsymbol{y}$$

where $M$ is symmetric and positive definite (for example the mass matrix or any covariance matrix.)

The directional derivative is

$$Df(p)[\phi] = \frac{\partial f}{\partial p_q} \phi_q =< M^{-1} \partial f / \partial p_q, \phi_q >$$

and therefore

$$[\nabla f]_p = [M^{-1}]_{pq} \, \partial f / \partial \phi_q$$

Had we instead used the Euclidean inner product as $< \boldsymbol{x}, \boldsymbol{y} >= \boldsymbol{x}^T \boldsymbol{y}$ the corresponding Euclidean gradient would be

$$[\nabla_E f]_p = \partial f / \partial \phi_p \tag{C.1}$$

or

$$\nabla f = M^{-1}\,\nabla_E f \tag{C.2}$$

where the subscript $E$ denotes the Euclidean gradient. This distinction becomes important in the application of the adjoint method where we obtain a gradient that is dependent on the natural FE inner product

$$\begin{aligned} < f, g > &= \int f\, g\, dA \\ &= \int f_p\, \phi_p\, g_q\, \phi_q dA \\ &= \boldsymbol{f} M \boldsymbol{g} \end{aligned}$$

Hence in FE the dual pairing is

$$< f, g >= \boldsymbol{f}^T \boldsymbol{M} \boldsymbol{g}$$

where $f$.

The adjoint $L^*$ of a given operator $L$ is defined as

$$< L^* f, g >=< f, Lg >$$

for any $f$ and $g$.

If we are working in $H_1 = \mathbb{R}^{d_1}$ and the dual space is $H_2 = \mathbb{R}^{d_2}$ and

$$< f, g >_{H_1,H_2}= \boldsymbol{f}^T \boldsymbol{g}$$

and

$$< L^* f, g >_{H_1,H_2}=< f, Lg >_{H_1,H_2}$$

and we denote by $\boldsymbol{L}$ and $\boldsymbol{L}^*$the matrix representations of the continuous linear operators $L$ and $L^*$, respectively, then

$$\boldsymbol{L}^* = \boldsymbol{L}^T$$

If, on the other hand, we have the dual pairings

$$< f, g >_{H_1,H_2}= \boldsymbol{f}^T \boldsymbol{M} \boldsymbol{g}$$

where $\boldsymbol{M}$ is a positive definite matrix, then

$$\boldsymbol{L}^* = \boldsymbol{M}^{-T} \boldsymbol{L}^T \boldsymbol{M}^T$$

as can be seen as follows

$$\begin{aligned} < M^* f, g > &=< f, Lg > \\ &= \boldsymbol{f}^T \boldsymbol{M}(\boldsymbol{L}\boldsymbol{g}) \\ &= \boldsymbol{f}^T \boldsymbol{M}\boldsymbol{L}\boldsymbol{M}^{-1}\boldsymbol{M}\boldsymbol{g} \\ &= (\boldsymbol{M}^{-T}\boldsymbol{L}^T\boldsymbol{M}^T\boldsymbol{f})^T\boldsymbol{M}\boldsymbol{g} \\ &=< \boldsymbol{M}^{-T}\boldsymbol{L}^T\boldsymbol{M}^T\boldsymbol{f}, \boldsymbol{g}) \end{aligned}$$

We can generalise this a bit further and consider the case where the dual space has a different dimension with

$$\begin{aligned} < f, f >_{H_1 H_1} &= \boldsymbol{f}^T \boldsymbol{M}_1\, \boldsymbol{f} \\ < g, g >_{H_2 H_2} &= \boldsymbol{g}^T \boldsymbol{M}_2\, \boldsymbol{g} \end{aligned}$$

and find that

$$\boldsymbol{L}^* = \boldsymbol{M}_1^{-T} \boldsymbol{L}^T \boldsymbol{M}_2^T$$

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