Úa
Compendium

G. Hilmar Gudmundsson

University of Northumbria

May 31, 2018

Contents

Introduction

Úa is a finite-element ice flow model. This document provides some theoretical background information to *Úa* . It is not a manual. This is a 'life' document, i.e. it is constantly being modified and changed and the current from of the document is in no means final. It contains some general material on glacier mechanics, a bit on the FE method, and lots of some very specific *Úa* related stuff.

Notation and definitions

Typical problem geometry is depicted in Fig. 1 and the main geometrical variables listed in Table 1. Upper and lower glacier surfaces are denoted by s and b, respectively, while the ocean surface and the bedrock are denoted by S and B, respectively. The ice thickness is $h = s - b$ and is always positive. The distance from bedrock (B)) to the ocean surface (S) is $H = S - B$, and this quantity can be either positive or negative, depending on location.

The maximal ice thickness possible without grounding is denoted by h_f and is

$$h_f := (S - B)\rho_o/\rho,$$

where ρ and ρ_o are the ice and the ocean densities, respectively.

The freeboard is

$$f := s - S,$$

and the draft d is defined as

$$d := \begin{cases} S - b, & \text{if } S > b \\ 0, & \text{otherwise} \end{cases}$$

which can also be written as

$$d = \mathcal{H}(H)(S - b),$$

where $\mathcal{H}(x)$ is the Heaviside function, defined as

$$\mathcal{H}(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1/2 & \text{for} \quad x = 1/2 \\ 1 & \text{for} \quad x > 0 \end{cases}$$

The function $\mathcal{H}(h - h_f)$ (equal to one if grounded, zero if afloat), is needed frequently and we define θ as

$$\theta = \mathcal{H}(h - h_f)$$

Hence

$$\theta = \mathcal{H}(h - h_f) = \begin{cases} 0 & \text{over floating areas} \\ 1/2 & \text{at the grounding line} \\ 1 & \text{over grounded areas} \end{cases}$$

The 'positive' ocean depth H_+ is defined as

$$H_+ := \mathcal{H}(H)H, \tag{1}$$

i.e. $H_+ = H$ if $H > 0$ and zero otherwise.

To distinguish between continuous quantities and discrete quantities we use bold face for the latter. In a finite-element context we might write

$$f(x) = f_q \, \phi_q(x)$$

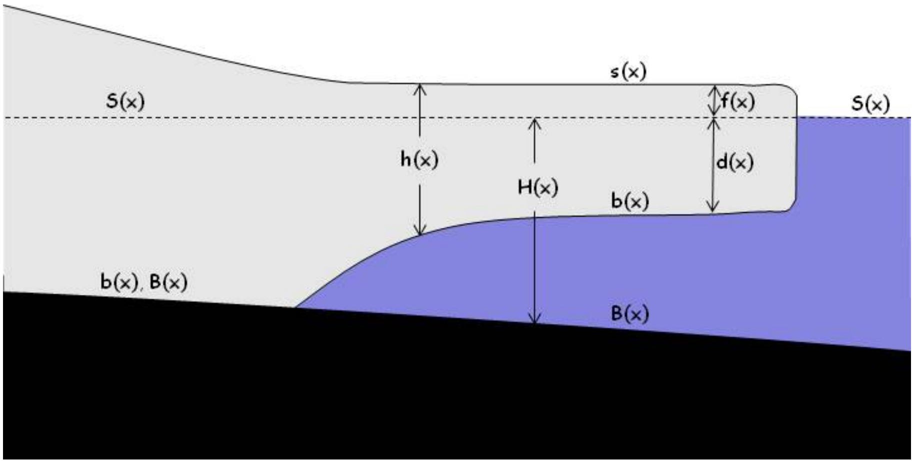


Figure 1: Geometrical variables: Glacier surface (s)), glacier bed (b), ocean surface (S), ocean floor (B), glacier thickness ($h = s - b$)), ocean depth ($H = S - B$), glacier draft ($d = S - b$), glacier freeboard ($f = s - S$).

Here f is a continuous function, f_q the nodal variables, and ϕ_q the shape/form functions. We sometimes group the nodal variables together into a vector writing

$$\boldsymbol{f} = [f_1, f_1, \ldots, f_N]^T$$

and then

$$f(x) = \boldsymbol{f}^T \boldsymbol{\phi}$$

Note that f and f_q very are different quantities. If, for a vector $\boldsymbol{f}$, we need to refer to the q element of the vector, we write $[\boldsymbol{f}]_q$, i.e.

$$f_q = [\boldsymbol{f}]_q$$

The matrix representation of a continuous operator $L : H_1 \to H_2$ is written in bold as $\boldsymbol{L}$. The elements of the matrix are $L_{pq} = [\boldsymbol{L}]_{pq}$.

The L^2 norm is

$$(f, g)_{L^2} = \int f(x)\, g(x)\, dx$$

where f and g are square intergrable functions and the l^2 norm is

$$(\boldsymbol{f}, \boldsymbol{g})_{l^2} = \boldsymbol{f}^T \cdot \boldsymbol{g}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are vectors. The inner products define corresponding L^2 and l^2 norms.

We often need to linearise various quantities, which we do by calculating directional derivatives. The directional derivative $D\boldsymbol{f}$ of a function $\boldsymbol{f}$ of the variable $\boldsymbol{x}$ in the direction $\boldsymbol{v}$ is defined as

$$D\boldsymbol{f}[\boldsymbol{x}; \boldsymbol{v}] = \lim_{\epsilon \to 0} \frac{d}{d\epsilon} \boldsymbol{f}(\boldsymbol{x} + \epsilon \boldsymbol{v})$$

Often we think of $\boldsymbol{v}$ being a small perturbation to $\boldsymbol{x}$ in which case we write

$$D\boldsymbol{f}[\boldsymbol{x}; \Delta \boldsymbol{x}] = \lim_{\epsilon \to 0} \frac{d}{d\epsilon} \boldsymbol{f}(\boldsymbol{x} + \epsilon \Delta \boldsymbol{x})$$

and may then simply write $D\boldsymbol{f}[\boldsymbol{x}]$. Another common notation for the directional derivative of the function $\boldsymbol{f}(\boldsymbol{x})$ in the direction $\boldsymbol{v}$ is $\nabla_{\boldsymbol{v}} \boldsymbol{f}(\boldsymbol{x})$.

Table 1: List of variables

s	upper glacier surface
b	lower glacier surface
S	ocean surface
B	bedrock / ocean floor
$h := s - b$	glacier thickness
$H := S - B$	ocean depth (pos. or neg. depending on location)
$H_+ = \mathcal{H}(H)H$	(positive) ocean depth
$d := \mathcal{H}(H)(S - b)$	glacier draft (positive by definition)
$f := s - S$	freeboard (always positive)
$h_f := \rho_o H/\rho$	maximal ice thickness without grounding
α	tilt of coordinate system
$\theta = \mathcal{H}(h - h_f)$	grounding/floating mask
ρ	ice density
ρ_o	water/ocean density
$\varrho = \rho\,(1 - \rho/\rho_o)$	
(u_b, v_b, w_b)	xyz components of basal velocity
(u_s, v_s, w_s)	xyz components of surface velocity
τ_{xx}, τ_{yy}, τ_{xy} etc.	deviatoric stress components
σ_{xx}, σ_{yy}, σ_{xy} etc.	Cauchy stress components
$\dot\epsilon_{xx}$, $\dot\epsilon_{yy}$, $\dot\epsilon_{xy}$ etc.	strain rates
g	gravitational acceleration

Chapter 1

Equations of ice flow

1.1 Shallow Ice Stream Approximation (SSTREAM/SSA)

The shallow-ice stream (SSTREAM/SSA/Shelfy) equations are

$$\partial_x(h(2\tau_{xx}+\tau_{yy}))+\partial_y(h\tau_{xy})-t_{bx}=(\rho g h\partial_x s+\frac{1}{2}h^2 g\,\partial_x\rho)\cos\alpha-\rho g h\sin\alpha \tag{1.1}$$

$$\partial_y(h(2\tau_{yy}+\tau_{xx}))+\partial_x(h\tau_{xy})-t_{by}=(\rho g h\partial_y s+\frac{1}{2}h^2 g\,\partial_y\rho)\cos\alpha \tag{1.2}$$

were α is the tilt of the coordinate system with respect to the gravity vector. Defining the *resistive stress tensor* as

$$\boldsymbol{R}=\begin{pmatrix}2\tau_{xx}+\tau_{yy} & \tau_{xy}\\ \tau_{xy} & 2\tau_{yy}+\tau_{xx}\end{pmatrix} \tag{1.3}$$

and

$$\nabla_{xy}=(\partial_x,\partial_y)$$

the field equations can be written in a compact form as

$$\nabla_{xy}\cdot(h\,\boldsymbol{R})-\boldsymbol{t}_{bh}=\rho g h\nabla_{xy}\,s+\frac{1}{2}gh^2\nabla_{xy}\,\rho\,, \tag{1.4}$$

for $\alpha=0$, where

$$\boldsymbol{t}_{bh}=\begin{pmatrix}t_{bx}\\ t_{by}\end{pmatrix}$$

is the horizontal part of the basal stress vector (basal traction)

$$\boldsymbol{t}_b=\boldsymbol{\sigma}\hat{\boldsymbol{n}}-(\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}\hat{\boldsymbol{n}})\hat{\boldsymbol{n}},$$

with $\hat{\boldsymbol{n}}$ being a unit normal vector to the bed pointing into the ice.

Note that it is the horizontal component of the basal traction that enters the field equations. We will sometimes just write $\boldsymbol{t}_b$ instead of $\boldsymbol{t}_{bh}$ which is a slight abuse of notation, and strictly speaking incorrect.

1.2 Shallow Ice Shelf (SSHELF/SSA)

The shallow ice shelf approximation is simply the shallow ice stream approximation with the drag term dropped.

Since

$$s=S+(1-h\rho/\rho_o)$$

we have over a floating ice shelf

$$\rho g h \nabla_{xy}\, s = \frac{1}{2} \varrho g \nabla_{xy}\, h^2$$

and the momentum equations have the form

$$\nabla_{xy} \cdot (h\,\boldsymbol{R}) = \frac{1}{2} \varrho g \nabla_{xy}\, h^2 + \frac{1}{2} g h^2 \nabla_{xy}\, \rho\,, \tag{1.5}$$

or if we skip the spatial density gradient

$$\nabla_{xy} \cdot \left(\boldsymbol{R} - \frac{1}{2} \varrho g \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \right) = 0 \tag{1.6}$$

1.3 Shallow Ice Sheet (SSHEET/SIA)

The shallow ice sheet equations for the deformational velocities are

$$(u, v) = -E\, |\nabla_{xy} s|^{n-1} \, \left(h^{n+1} - (s-z)^{n+1} \right) \, (\partial_x s, \partial_y s)$$

where

$$E = \frac{2A}{n+1} (\rho g)^n$$

The vertically integrated flux is

$$q_x = \int_b^s \rho u \, dz,$$

giving

$$(q_x, q_y) = -\rho D\, |\nabla_{xy} s|^{(n-1)} \, h^{n+2} \, (\partial_x s, \partial_y s)$$

where

$$D = \frac{2A}{n+2} (\rho g)^n$$

or

$$(q_x, q_y) = F \;\rho\, h\, (u, v)$$

with

$$F = \frac{n+1}{n+2}$$

where

$$|\nabla_{xy} s| = \sqrt{(\partial_x s)^2 + (\partial_y s)^2}$$

1.4 Equation of mass conservation

The prognostic equation is a vertically integrated expression of mass conservation. The local form of the mass-conservation equation is

$$\nabla \cdot (\rho \boldsymbol{v}) + \partial_t \rho = 0 \tag{1.7}$$

The kinematic boundary conditions at the upper and lower surface are written as

$$\partial_t s + u_s \partial_x s + v_s \partial_x s - w_s = a_s \quad \text{at} \quad z = s \tag{1.8}$$

$$\partial_t b + u_b \partial_x b + v_b \partial_x b - w_b = -a_b \quad \text{at} \quad z = b \tag{1.9}$$

The mass balance distributions, a_b and a_s, are in the units of meters of water equivalent per time. Note the sign convention for a_s and a_b used in (1.8) and (1.9). Mass flux into the ice is defined positive irrespective over which surface it takes place. Surface accumulation is positive, melting always negative.

The horizontal ice flux is defined as

$$\boldsymbol{q}_{xy} = \int_b^s \rho \boldsymbol{v}_{xy}.$$

Focusing on the flow-line case for the moment we find that

$$\begin{aligned}
\partial_x q_x &= \int_b^s \partial_x(\rho u)\, dz \\
&= \int_b^s \partial_x(\rho u)\, dz + \rho u_s \partial_x s - \rho u_b \partial_x b \\
&= -\int_b^s (\partial_z(\rho w) + \partial_t \rho)\, dz + \rho u_s \partial_x s - \rho u_b\, \partial_x b \\
&= -\rho w_s + \rho w_b - h\partial_t \rho + \rho u_s \partial_x s - \rho u_b\, \partial_x b \\
&= -h\partial_t \rho - \rho w_s + \rho u_s \partial_x s + \rho w_b - \rho u_b\, \partial_x b \\
&= -h\partial_t \rho + \rho a_s - \rho \partial_t s + \rho a_b + \rho \partial_t b \\
&= -h\partial_t \rho + \rho a - \rho \partial_t h
\end{aligned}$$

where

$$a = a_s + a_b$$

and therefore, once the y component has been added

$$\rho\, \partial_t h + \nabla_{xy} \cdot \boldsymbol{q}_{xy} + h\, \partial_t \rho = \rho a \tag{1.10}$$

In most modelling work of large ice masses it is assumed that the density ρ is uniform in space and does not vary with time. In *Úa* we relax this condition somewhat and only assume that the density at a given location does not change with time, i.e.

$$\partial_t \rho = 0.$$

but allow the density to vary in the horizontal ($\partial_x \rho$ and $\partial_y \rho$ are not assumed to be equal to zero). Hence the mass conservation equations (1.7) becomes

$$\nabla \cdot (\rho \boldsymbol{v}) = 0.$$

and (1.10)

$$\rho\, \partial_t h + \nabla_{xy} \cdot \boldsymbol{q}_{xy} = \rho a \tag{1.11}$$

or

$$\rho\, \partial_t h + \partial_x(\rho h u) + \partial_y(\rho h v) = \rho a \tag{1.12}$$

for a velocity field that does not change with depth.

Eq. (1.11) is the form of the mass continuity equation used in *Úa*. The effect of horizontal gradients in (vertically integrated) density are also included in the momentum equations (1.4).

Vertical velocities

Note that

$$w_s = u_s\, \partial_x s + w_b - u_b\, \partial_x b - \frac{1}{\rho}\partial_x q_x - \frac{h}{\rho}\partial_t \rho$$

or

$$w_s = u_s\, \partial_x s + \partial_t b + a_b - \frac{1}{\rho}\partial_x q_x - \frac{h}{\rho}\partial_t \rho$$

which can be used to calculate vertical velocities.

If the bed elevation does not change with time and if furthermore $\partial_t \rho = 0$, we have the special case

$$w_s = u_s\, \partial_x s + a_b - \frac{1}{\rho}\partial_x q_x$$

1.5 Sliding law

The power-law type Weertman sliding law is

$$\boldsymbol{T}\boldsymbol{\sigma}\hat{\boldsymbol{n}} + C^{-1/m}|\boldsymbol{T}\boldsymbol{v}|^{1/m-1}\,\boldsymbol{T}\boldsymbol{v} = 0$$

where

$$\boldsymbol{T} = \mathbf{1} - \hat{\boldsymbol{n}} \otimes \hat{\boldsymbol{n}}$$

is the tangential operator. One can also define the normal operator $\boldsymbol{N}$ as

$$\boldsymbol{N} = \hat{\boldsymbol{n}} \otimes \hat{\boldsymbol{n}}$$

and we have

$$\boldsymbol{\sigma} = \boldsymbol{N}\boldsymbol{\sigma} + \boldsymbol{T}\boldsymbol{\sigma}$$

The basal drag only acts over the grounded parts, a fact which we can express

$$\boldsymbol{t}_b = \mathcal{H}(h - h_f)\boldsymbol{f}(\boldsymbol{v}_b)$$

Different formulations are

$$\begin{aligned} \boldsymbol{t}_b &= \mathcal{H}(h-h_f)\; C^{-1/m}|\boldsymbol{v}_b|^{1/m-1}\,\boldsymbol{v}_b \\ \boldsymbol{v}_b &= (\mathcal{H}(h-h_f))^{-m}\; C|\boldsymbol{t}_b|^{m-1}\,\boldsymbol{t}_b \\ |\boldsymbol{v}_b| &= (\mathcal{H}(h-h_f))^{-m}\;\; C|\boldsymbol{t}_b|^{m} \end{aligned}$$

where $\boldsymbol{v}_b$ is the tangential velocity, i.e. the basal sliding velocity

$$\begin{aligned} \boldsymbol{v}_b &= \boldsymbol{v} - (\hat{\boldsymbol{n}}^T \cdot \boldsymbol{v})\hat{\boldsymbol{n}} \\ \boldsymbol{v}_b &= \boldsymbol{T}\boldsymbol{v} \end{aligned}$$

and $\boldsymbol{t}_b$ is the tangential component of the basal traction

$$\boldsymbol{t}_b = -\boldsymbol{T}(\boldsymbol{\sigma}\hat{\boldsymbol{n}})$$

Since

$$\mathcal{H}(x) = \mathcal{H}(x)^m$$

for any power m we do not strictly need to include the stress exponent m in any equations involving $\mathcal{H}$. However if we use an approximation to $\mathcal{H}$ then this stress exponent should be included.

If we write

$$\boldsymbol{t}_b = \beta^2\,\boldsymbol{v}_b$$

then

$$\beta^2 = C^{-1/m}|\boldsymbol{v}_b|^{1/m-1}$$

In Úa the power-law sliding law is given as

$$\begin{pmatrix} t_{bx} \\ t_{by} \end{pmatrix} = \mathcal{H}(h-h_f)\; \beta^2 \begin{pmatrix} u_b \\ v_b \end{pmatrix}$$

where

$$\beta^2 = (C + C_0)^{-1/m}\; (u^2 + v^2 + u_o^2)^{(1-m)/2m}$$

where C_0 and u_0 are some (small) regularisation parameters.

1.6 Ocean drag term

To simulate drag exerted on the ice by the ocean we add an ocean drag term over the floating section of the form

$$\boldsymbol{t}_b^o = \mathcal{H}(h_f - h) C_o^{-1/m_o} |\boldsymbol{v}_b - \boldsymbol{v}_o|^{1/m_o - 1} (\boldsymbol{v}_b - \boldsymbol{v}_o)$$

where $\boldsymbol{v}_0$ is the velocity of the ocean current.

The sea-ice literature suggest $m_0 = 1/2$, i.e.

$$\boldsymbol{t}_b^o = \mathcal{H}(h_f - h) C_o^{-2} |\boldsymbol{v}_b - \boldsymbol{v}_o| (\boldsymbol{v}_b - \boldsymbol{v}_o)$$

and defines

$$D_o = C_o^{-2}$$

where

$$D_o = \rho_o c_o$$

and typically $c_0 = 0.0055$. Hence

$$C_0 = \frac{1}{\sqrt{D_o}} = \frac{1}{\sqrt{\rho_o c_o}} \approx 0.4 \quad [\sqrt{(m/s)/Pa}]$$

The total drag is a sum of that due to basal sliding and ocean currents.

$$\boldsymbol{t}_b = \mathcal{H}(h - h_f)\, \beta^2\, \boldsymbol{v} + \mathcal{H}(h_f - h)\, \beta_o^2\, (\boldsymbol{v} - \boldsymbol{v}_o)$$

1.7 Flow law

Glen's flow law is

$$\dot{\epsilon}_{ij} = A \tau^{n-1} \tau_{ij},$$

where

$$\tau = \sqrt{\tau_{ij}\tau_{ij}/2}$$

The flow law can also be written as

$$\tau_{ij} = A^{-1/n}\, \dot{\epsilon}^{(1-n)/n}\, \dot{\epsilon}_{ij}, \tag{1.13}$$

where

$$\dot{\epsilon} = \sqrt{\dot{\epsilon}_{ij}\dot{\epsilon}_{ij}/2}$$

which in the Shallow Ice Stream Approximation takes the form

$$\dot{\epsilon} = \sqrt{(\dot{\epsilon}_{xx})^2 + (\dot{\epsilon}_{yy})^2 + \dot{\epsilon}_{xx}\, \dot{\epsilon}_{yy} + (\dot{\epsilon}_{xy})^2} \tag{1.14}$$

$$= ((\partial_x u)^2 + (\partial_y v)^2 + \partial_x u\, \partial_y v + (\partial_x v + \partial_y u)^2/4)^{1/2} \tag{1.15}$$

If we write

$$\tau_{ij} = 2\eta\dot{\epsilon}_{ij}$$

then η is the effective viscosity given by

$$\eta = \frac{1}{2} A^{-1/n}\, \dot{\epsilon}^{(1-n)/n}$$

or

$$\eta = \frac{1}{2} A^{-1/n}\, ((\partial_x u)^2 + (\partial_y v)^2 + \partial_x u\, \partial_y v + (\partial_x v + \partial_y u)^2/4)^{(1-n)/2n}$$

1.8 Floating relationships

Where the ice is afloat, we have

$$\rho g h = \rho_o g d.$$

If the ice thickness is greater than $\rho_o H/\rho$ the ice is grounded. For a given bedrock geometry B and ocean surface S the ice is floating provided $h < h_f$ where

$$h_f := \rho_o H/\rho, \tag{1.16}$$

and for $h \geq h_f$ the glacier is grounded.

Where the glacier is afloat, i.e. $h \leq h_f$, the following relations hold:

$$\begin{aligned}
h &= \rho_o d/\rho = \frac{s-S}{1-\rho/\rho_o} = \frac{\rho_o}{\rho}(S-b), \\
b &= \frac{\rho s - \rho_o S}{\rho - \rho_o} = S - \frac{\rho}{\rho_o} h, \\
s &= S + (1-\rho/\rho_o)h = (1-\rho_o/\rho)b + \frac{\rho_o}{\rho} S, \\
f &= (1-\rho/\rho_o)h.
\end{aligned}$$

Furthermore, if $\partial_x S = 0$ the slopes of the upper and the lower boundary are related through

$$b\,\partial_x s - s\,\partial_x b = S\,\partial_x h, \tag{1.17}$$

and also

$$\partial_x s = (1-\rho/\rho_o)\partial_x h.$$

At the grounding line we have:

$$\begin{aligned}
h &= h_f \\
d &= H.
\end{aligned}$$

where h_f is defined by (1.16)

1.9 Expressing geometrical variables in terms of ice thickness

It is advantageous to be able to express geometrical variables such as s, b, and d in terms of ice thickness h.

It is easy to see that

$$s = \mathcal{H}(h-h_f)\,(h+B) + \mathcal{H}(h_f-h)\,(S + (1-\rho/\rho_o)\,h), \tag{1.18}$$

$$b = \mathcal{H}(h-h_f)\,B + \mathcal{H}(h_f-h)\,(S - \rho h/\rho_o), \tag{1.19}$$

and that

$$d = \mathcal{H}(H)\,[\mathcal{H}(h_f-h)\rho h/\rho_o + \mathcal{H}(h-h_f)H], \tag{1.20}$$

i.e.

$$d = \begin{cases} H, & \text{if } h > h_f \text{ and } H > 0 \\ \rho h/\rho_o, & \text{if } h < h_f \text{ and } H > 0 \\ 0, & \text{if } H < 0 \end{cases}$$

The draft is always $0 \leq d \leq \rho h/\rho_o$.

Eq. (1.20) can be simplified a bit further by noticing that if $H > 0$ then $\mathcal{H}(H)\mathcal{H}(h_f - h) = \mathcal{H}(h_f - h)$. On the other hand if $H < 0$ then $\mathcal{H}(H) = 0$ but so is $\mathcal{H}(h_f - h)$ because if $H = S - B < 0$ then $h_f = \frac{\rho_o}{\rho}(S - B) < 0$ and since h is always positive we have $\mathcal{H}(h_f - h) = 0$, i.e.

$$\mathcal{H}(H)\mathcal{H}(h_f - h) = \mathcal{H}(h_f - h)$$

and d can be therefore be written as

$$d = \mathcal{H}(h_f - h)\rho h/\rho_o + \mathcal{H}(H)\mathcal{H}(h - h_f)H. \tag{1.21}$$

or as

$$d = \mathcal{H}(h_f - h)\rho h/\rho_o + \mathcal{H}(h - h_f)H_+. \tag{1.22}$$

using

$$H_+ := \mathcal{H}(H)\, H$$

1.10 Stress boundary conditions at an ice front

We consider the case of an ice front in contact with water of a given depth. The treatment is general and includes the cases of zero water depth, i.e. glacier terminating on land, and a floating ice front, i.e. a glacier terminating in an ocean.

At the calving front the jump condition is

$$\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}_{xy} = -p_o \hat{\boldsymbol{n}}_n.$$

where p_o is the hydrostatic ocean pressure, and

$$\hat{\boldsymbol{n}}_{xy} = (n_x, n_y, 0)^T,$$

is a unit normal pointing horizontally outward from the ice front. The vertically integrated form of this stress condition is

$$\int_b^s (\sigma_{xx} n_x + \sigma_{xy} n_y)\, dz = -\int_b^S p_o n_x\, dz \quad \text{on} \quad \Gamma_2 \tag{1.23}$$

$$\int_b^s (\sigma_{xy} n_x + \sigma_{yy} n_y)\, dz = -\int_b^S p_o n_y\, dz \quad \text{on} \quad \Gamma_2 \tag{1.24}$$

If the draft d at the ice front is zero, i.e. if the ice front is fully grounded, then $S < b$, the right hand sides of (1.23) and (1.24) are to be set to zero.

Using

$$\sigma_{xx} = 2\tau_{xx} + \tau_{yy} + \sigma_{zz},$$

and with

$$\sigma_{zz} = -\rho g(s - z),$$

(where we have set $\alpha = 0$), if follows that

$$\begin{aligned} \int_b^s \sigma_{xx}\, dz &= \int_b^s (2\tau_{xx} + \tau_{yy})\, dz - \int_b^s \rho g(s - z)\, dx \\ &= h(2\tau_{xx} + \tau_{yy}) - \frac{\rho g}{2} h^2. \end{aligned}$$

The x component of the vertically integrated ocean pressure acting on the calving front is

$$\begin{aligned} -\int_b^S p_o n_x \, dz &= -\int_b^S \rho_o g(S-z) n_x \, dz \\ &= -\frac{1}{2}\rho_o g(S-b)^2 \\ &= -\frac{1}{2}\rho_o g d^2. \end{aligned}$$

Boundary conditions (1.23) and (1.24) can therefore be written as

$$h(2\tau_{xx}+\tau_{yy})n_x + h\tau_{xy}n_y = \frac{g}{2}(\rho h^2 - \rho_o d^2)\, n_x, \tag{1.25}$$

$$h(2\tau_{yy}+\tau_{xx})n_y + h\tau_{xy}n_x = \frac{g}{2}(\rho h^2 - \rho_o d^2)\, n_y, \tag{1.26}$$

or more compactly as

$$\boldsymbol{R} \cdot \hat{\boldsymbol{n}}_{xy} = \frac{g}{2h}(\rho h^2 - \rho_o d^2)\hat{\boldsymbol{n}}_{xy}. \tag{1.27}$$

The boundary condition (1.27) is valid for both grounded and floating ice edges.

1.10.1 Floating

In the particular case where the calving front is afloat, $\rho h = \rho_o d$ boundary conditions (1.25) and (1.26) simplify to

$$h(2\tau_{xx}+\tau_{yy})n_x + h\tau_{xy}\, n_y = \frac{1}{2}\varrho g h^2 n_x \tag{1.28}$$

$$h(2\tau_{yy}+\tau_{xx})n_y + h\tau_{xy}\, n_x = \frac{1}{2}\varrho g h^2 n_y \tag{1.29}$$

where

$$\varrho := \rho(1-\rho/\rho_o),$$

Written in terms of the velocity components the boundary conditions along a floating ice front are:

$$\eta h(4\partial_x u + 2\partial_y v)n_x + \eta h(\partial_x v + \partial_y u)n_y = \frac{\varrho g h^2}{2} n_x, \tag{1.30}$$

$$\eta h(\partial_x v + \partial_y u)n_x + \eta h(4\partial_y v + 2\partial_x u)n_y = \frac{\varrho g h^2}{2} n_y. \tag{1.31}$$

1.10.2 Grounded

On the other hand if the ice terminates on land then $d = 0$ and

$$h(2\tau_{xx}+\tau_{yy})n_x + h\tau_{xy}\, n_y = \frac{g}{2}\rho h^2\, n_x, \tag{1.32}$$

$$h(2\tau_{yy}+\tau_{xx})n_y + h\tau_{xy}\, n_x = \frac{g}{2}\rho h^2\, n_y. \tag{1.33}$$

1.11 Boundary condition at a glacier terminus as a natural boundary condition

For solving (1.1) and (1.2) it is advantageous to modify the equations in such a way that the boundary conditions (1.25) and (1.26) become the 'natural' boundary conditions. Furthermore, for an implicit time integration with respect to both velocities, grounding-line position, and ice thickness, it is of advantage to write all evolving geometrical variables (s, b) in terms of the ice thickness h.

The key idea is to rewrite (assuming $\alpha = 0$) the Eqs. (1.1) and (1.2) as

$$\partial_x(h(2\tau_{xx} + \tau_{yy})) + \partial_y(h\tau_{xy}) - t_{bx} = \frac{1}{2}g\partial_x(\rho h^2 - \rho_o d^2) + g(\rho h - \rho_o d)\partial_x b, \tag{1.34}$$

$$\partial_y(h(2\tau_{yy} + \tau_{xx})) + \partial_x(h\tau_{xy}) - t_{by} = \frac{1}{2}g\partial_y(\rho h^2 - \rho_o d^2) + g(\rho h - \rho_o d)\partial_y b, \tag{1.35}$$

(note the d term is not missing a $\mathcal{H}(H)$ because d is automatically zero whenever $\mathcal{H}(H) = 0$.) where it has been used that $\partial_x \rho_o = \partial_y \rho_o = 0$ and $\partial_x S = \partial_y S = 0$. Note that in Eqs. (1.34) and (1.35) the second terms on the right hand sides are automatically zero where the ice is afloat and that this formulation can also be used if the ice density varies in the horizontal.

The equality of the right-hand terms in (1.1) and (1.34) (for $\alpha = 0$) follows from

$$\begin{aligned}
\frac{1}{2}g\partial_x(\rho h^2 - \rho_o d^2) + g(\rho h - \rho_o d)\partial_x b &= \frac{1}{2}gh^2\partial_x\rho + g(\rho h\partial_x h - \rho_o d\partial_x d) + g(\rho h - \rho_o d)\partial_x b \\
&= \frac{1}{2}gh^2\partial_x\rho + g(\rho h\partial_x(s-b) - \rho_o d\partial_x d) + g(\rho h - \rho_o d)\partial_x b \\
&= \frac{1}{2}gh^2\partial_x\rho + g(\rho h\partial_x s - \rho_o d\partial_x d) - g\rho_o d\partial_x b \\
&= \frac{1}{2}gh^2\partial_x\rho + g(\rho h\partial_x s - \rho_o d\partial_x(\mathcal{H}(H)(S-b))) - g\rho_o d\partial_x b \\
&= \frac{1}{2}gh^2\partial_x\rho + g\rho h\partial_x s - g\rho_o d(S-b)\partial_x\mathcal{H}(H)
\end{aligned}$$

and the last term is (in an integrated sense) zero

$$\begin{aligned}
\int g\rho_o d(S-b)\partial_x\mathcal{H}(H)\,dx &= \int g\rho_o d(S-b)\partial_H\mathcal{H}(H)\,\partial_x H\,dx \\
&= \int g\rho_o d(S-b)\delta(H)\,\partial_x H\,dx \\
&= g\rho_o d\,(S-b) \qquad \text{(for } x \text{ where } H = 0) \\
&= 0
\end{aligned}$$

where the last step follows from the fact that where $H = 0$, we have $S = b$, because if $H = 0$, then $h_f = \rho H/\rho_o = 0$, and hence $h \geq h_f$ because h is never negative. Where $H = 0$ the ice is therefore grounded, and $B = b$ and therefore $S - b = S - B = H = 0$, so $S = b$.

Hence

$$g\rho h\partial_x s + \frac{1}{2}gh^2\partial_x\rho = \frac{1}{2}g\partial_x(\rho h^2 - \rho_o d^2) + g(\rho h - \rho_o d)\partial_x b. \tag{1.36}$$

Because $\rho_o d \leq \rho h$, with the equality sign fulfilled where the ice is afloat, the second terms on the right-hand sides of Eqs. (1.34) and (1.35) are positive where the ice is both partly and fully grounded, and zero where it is afloat. Therefore

$$\begin{aligned}
g(\rho h - \rho_o d)\partial_x b &= \mathcal{H}(h - h_f)g(\rho h - \rho_o d)\partial_x b \\
&= \mathcal{H}(h - h_f)g(\rho h - \rho_o d)\partial_x B \\
&= \mathcal{H}(h - h_f)g(\rho h - \rho_o H_+)\partial_x B
\end{aligned}$$

where we used (1.22) and

$$d = \mathcal{H}(h_f - h)\rho h/\rho_o + \mathcal{H}(h - h_f)H_+,$$

and hence

$$\mathcal{H}(h - h_f)d = \mathcal{H}(h - h_f)H_+,$$

in an integrated sense.

The basal drag terms are also zero where the ice is afloat and the system can therefore be written as

$$\partial_x(h(2\tau_{xx} + \tau_{yy})) + \partial_y(h\tau_{xy}) - t_{bx} = \frac{1}{2} g\partial_x(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h - h_f)(\rho h - \rho_o H_+)\partial_x B \quad (1.37)$$

$$\partial_y(h(2\tau_{yy} + \tau_{xx})) + \partial_x(h\tau_{xy}) - t_{by} = \frac{1}{2} g\partial_y(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h - h_f)(\rho h - \rho_o H_+)\partial_y B \quad (1.38)$$

This form suggest how we can get boundary condition (1.27) to be the natural boundary condition of our FE formulation. We simply need to take into the boundary integral the first terms on the left and right-hand sides of (1.37) and (1.38) and the details are given in Section 2.1.

Written in terms of the velocity components:

$$\begin{aligned}
&\partial_x(h\eta(4\partial_x u + 2\partial_y v)) + \partial_y(h\eta(\partial_y u + \partial_x v)) - t_{bx} = \\
&\qquad \frac{1}{2} g\partial_x(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h - h_f)(\rho h - \rho_o H_+)\partial_x B \quad (1.39)\\
&\partial_y(h\eta(4\partial_y v + 2\partial_x u)) + \partial_x(h\eta(\partial_x v + \partial_y u) - t_{by} = \\
&\qquad \frac{1}{2} g\partial_y(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h - h_f)(\rho h - \rho_o H_+)\partial_y B \quad (1.40)
\end{aligned}$$

1.12 SSTREAM in 1HD

In one horizontal dimension (1HD), i.e. in the flow-line case, the SSTREAM equation becomes

$$4\partial_x(h\eta\partial_x u) - t_{bx} = \frac{1}{2} g\partial_x(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h - h_f)(\rho h - \rho_o H_+)\partial_x B$$

with

$$\eta = \frac{1}{2} A^{-1/n} \dot{\epsilon}^{(1-n)/n} = \frac{1}{2} A^{-1/n} |\partial_x u|^{(1-n)/n}$$

if we use Glen's flow law, and with

$$t_{bx} = \mathcal{H}(h - h_f)\; C^{-1/m} |u|^{1/m-1}\, u$$

if we use Weertman sliding law.

Inserting Glen's flow law we get

$$2\partial_x(A^{-1/n}\, h\, |\partial_x u|^{(1-n)/n} \partial_x u) - t_{bx} = \frac{1}{2} g\partial_x(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h - h_f)(\rho h - \rho_o H_+)\partial_x B \quad (1.41)$$

and if we can assume that $u > 0$ and $\partial_x u > 0$ then the SSTREAM equation is

$$2\partial_x \left(A^{-1/n}\, h\, (\partial_x u)^{1/n}\right) - \mathcal{H}(h - h_f) C^{-1/m} u^{1/m} = \rho g h \partial_x s + \frac{1}{2} g h \partial_x \rho \quad (1.42)$$

Eq. (1.42) is a fairly common way of writing down the SSTREAM/SSA equation in 1HD.

Chapter 2

Finite-element implementation

2.1 FE formulation of the diagnostic equations

. In the FE method the inner product of the field equations with a test function is formed. The inner product is

$$< \phi, \theta >= \iint_\Omega \phi\, \theta \, dx \, dy$$

where ϕ and θ are some functions. One form of Green's theorem states that

$$\iint_\Omega \phi \, \partial_x \theta \, dx \, dy = - \iint_\Omega \partial_x \phi \, \theta \, dx \, dy + \oint_\Gamma \phi \, \theta \, n_x \, d\Gamma$$

Applying the Green's theorem on the stress terms and the first term on the right-hand side of that of Eq. (1.37), i.e. x direction, leads to

$$\begin{aligned} 0 = \iint_\Omega \Big(& h(2\tau_{xx} + \tau_{yy})\partial_x\phi + h\tau_{xy} \, \partial_y \phi - \frac{1}{2} g(\rho h^2 - \rho_o d^2) \, \partial_x \phi + t_{bx} \, N \\ & + \phi \, g \, \mathcal{H}(h - h_f)(\rho h - \rho_o H_+)\partial_x B \Big) \, dx \, dy \\ & - \oint_\Gamma (h(2\tau_{xx} + \tau_{yy}) \, \phi \, n_x + h\tau_{xy} \, \phi \, n_y - \frac{1}{2} g(\rho h^2 - \rho_o d^2) \, \phi n_x) \, , d\Gamma \end{aligned}$$

Performing the same calculation of the y direction results in boundary terms that are identically equal to zero for the boundary condition (1.27). The natural boundary condition is therefore exactly (1.27) and covers not only the case of a fully floating ice front, but that of a grounded and partially grounded ice fronts as well. Using Eq. (1.20) the draft (d) appearing the equations above can be written in terms of the ice thickness (h). This formulation is therefore well suited as a starting point for a linearisation around h required for a fully implicit solution of transient flow.

Expressing this equation in terms of the velocity components u and v

$$\begin{aligned} 0 = & \iint_\Omega (h\eta(4\partial_x u + 2\partial_y v)\partial_x \phi + h\eta(\partial_y u + \partial_x v)\partial_y \phi + \mathcal{H}(h - h_f)\beta^2 u \, \phi \\ & - \frac{1}{2} g(\rho h^2 - \rho_o d^2)\partial_x \phi + \phi \, g \, \mathcal{H}(h - h_f)(\rho h - \rho_o H_+)\partial_x B) \, dx \, dy \\ & - \oint_\Gamma (h\eta(4\partial_x u + 2\partial_y v) \, \phi \, n_x + h\eta(\partial_y u + \partial_x v) \, \phi \, n_y - \frac{1}{2} g(\rho h^2 - \rho_o d^2) \, N) \, n_x \, d\Gamma \end{aligned} \tag{2.1}$$

$$\begin{aligned} 0 = & \iint_\Omega (h\eta(4\partial_y v + 2\partial_x u)\partial_y \phi + h\eta(\partial_x v + \partial_y u)\partial_x \phi + \mathcal{H}(h - h_f)\beta^2 v \, \phi \\ & - \frac{1}{2} g(\rho h^2 - \rho_o d^2)\partial_y \phi + \phi \, g \, \mathcal{H}(h - h_f)(\rho h - \rho_o H_+)\partial_x B) \, dx \, dy \\ & - \oint_\Gamma (h\eta(4\partial_y v + 2\partial_x u) \, \phi \, n_y + h\eta(\partial_x v + \partial_y u) \, \phi \, n_x - \frac{1}{2} g(\rho h^2 - \rho_o d^2) \, N) \, n_y \, d\Gamma \end{aligned} \tag{2.2}$$

where the corresponding expression in y direction has been added.

2.2 FE formulation of the prognostic equations

Úa allows for fully implicit time integration with respect to both geometry, grounding-line migration, and velocity. This approach is not limited by the CFL condition and is unconditionally stable allowing for arbitrarily large times steps irrespective of spatial discretisation. The time step is only limited by the convergence radius of the Newton-Raphson method.

The recommended option in a transient run is to use a fully implicit Θ *method combined with the consistent streamline-upwind Petrov-Galerkin method (SUPG). This is the default option.*

In *Úa* a semi-implicit approach (implicit with respect to geometry, explicit with respect to velocity) is also implemented. Unless memory is a limiting factor, the fully implicit approach is always preferable to the semi-implicit (staggered) approach.

Experience shows the Θ method to give good results when used in a combination with a fully implicit forward time integration. For a semi-implicit approach a third-order Taylor Galerkin (TG3) is a better approach.

In 2HD both Θ and TG3 have been implemented for both staggered and implicit approach. (The 1HD fully implicit was only done using the Θ method and not using TG3.) Using TG3 in 1HD staggered approach resulted in a great improvement over the Θ method. It appears that in the implicit approach there is no great advantage of using TG3.

There is no separate diffusion term added to the prognostic equations in *Úa* , and no shock-stabilisation term either. Even just using the fully implicit approach without SUPG generally gives good results. But using SUPG is nevertheless recommended, especially for problems involving grounding line migration.

2.2.1 Mass flux equation

The vertically integrated form of the mass conservation equation used in *Úa* is Eq. (1.11).

2.2.2 Θ method or the 'generalized trapezoidal rule'

In the Θ method the left-hand side is approximated by the discrete first-order derivative $\Delta h/\Delta t = (h_1 - h_0)/(t_1 - t_0)$ and the right-hand side is replace by a weighted average of the values at time step $t = t_1$ and $t = t_0$, i.e.

$$\frac{\Delta h}{\Delta t} = \Theta \partial_t h_1 + (1-\Theta)\partial_t h_0$$

where

$$\rho \partial_t h_0 = \rho a_0 - \partial_x q_{x0} - \partial_y q_{y0}$$
$$\rho \partial_t h_1 = \rho a_1 - \partial_x q_{x1} - \partial_y q_{y1}$$

and $0 \leq \Theta \leq 1$. For $\Theta > 0$ the resulting system is implicit with respect to both thickness (h) and velocity (u and v).

2.2.3 Third order implicit Taylor Galerkin (TG3)

This method is also referred to as fourth-order Crank-Nicolson time-stepping.

First expand h at time step 1 and 0 using third order Taylor expansion as

$$h_1 = h_0 + \Delta t\, \partial_t h_0 + \frac{(\Delta t)^2}{2} \partial^2_{tt} h_0 + \frac{(\Delta t)^3}{6} \partial^3_{ttt} h_0,$$

$$h_0 = h_1 - \Delta t\, \partial_t h_1 + \frac{(\Delta t)^2}{2} \partial^2_{tt} h_1 - \frac{(\Delta t)^3}{6} \partial^3_{ttt} h_1,$$

adding and simplifying gives

$$\frac{1}{\Delta t}(h_1 - h_0) = \frac{1}{2}(\partial_t h_0 + \partial_t h_1) + \frac{\Delta t}{4}(\partial^2_{tt} h_0 - \partial^2_{tt} h_1) + \frac{(\Delta t)^2}{12}(\partial^3_{ttt} h_0 + \partial^3_{ttt} h_1) \tag{2.3}$$

Note that including only the first term of the Taylor expansion is equal to using the Θ method with $\Theta = 1/2$.)

Then replace the third-order derivative is expressed through finite differences giving

$$\begin{aligned} \frac{1}{\Delta t}(h_1 - h_0) &= \frac{1}{2}(\partial_t h_0 + \partial_t h_1) + \frac{\Delta t}{4}(\partial^2_{tt} h_0 - \partial^2_{tt} h_1) + \frac{(\Delta t)^2}{12\Delta t}(\partial^2_{tt}(h_1 - h_0) + \partial^2_{tt}(h_1 - h_0)) \\ &= \frac{1}{2}(\partial_t h_0 + \partial_t h_1) + \frac{\Delta t}{4}(\partial^2_{tt} h_0 - \partial^2_{tt} h_1) + \frac{\Delta t}{6}\,\partial^2_{tt}(h_1 - h_0) \end{aligned}$$

i.e.

$$h_1 - h_0 = \frac{\Delta t}{2}(\partial_t h_0 + \partial_t h_1) + \frac{(\Delta t)^2}{12}(\partial^2_{tt} h_0 - \partial^2_{tt} h_1) \tag{2.4}$$

The final step is to replace time derivatives with spatial derivatives through repeated use of the prognostic equation.

The flux $\boldsymbol{q}$ is a time dependent function of both h and $\boldsymbol{v}$, using the prognostic equation the second time derivative of h can be written as

$$\begin{aligned} \rho\partial^2_{tt} h &= \rho\partial_t a - \nabla_{xy} \cdot \partial_t \boldsymbol{q} \\ &= \rho\partial_t a - \nabla_{xy} \cdot (\partial_h \boldsymbol{q}\,\partial_t h + (\nabla_{uv}\boldsymbol{q}) \cdot \partial_t \boldsymbol{v}) \\ &= \rho\partial_t a - \nabla_{xy} \cdot (\partial_h \boldsymbol{q}(\rho a - \nabla_{xy} \cdot \boldsymbol{q})/\rho + (\nabla_{uv}\boldsymbol{q}) \cdot \partial_t \boldsymbol{v}) \end{aligned}$$

or

$$\rho\partial^2_{tt} h = \rho\partial_t a - \nabla_{xy} \cdot (\partial_h \boldsymbol{q}(\rho a - \nabla_{xy} \cdot \boldsymbol{q})/\rho + (\nabla_{uv}\boldsymbol{q}) \cdot \partial_t \boldsymbol{v}) \tag{2.5}$$

where $\nabla_{uv} := (\partial_u, \partial_v)$ is the *horizontal velocity gradient operator.*

Third-order Taylor-Galerkin (TG3) for SSHEET/SSA

Using (2.5) in the SSHEET/SIA approximation where

$$\boldsymbol{q} = \boldsymbol{v}(h),$$

we find that

$$\rho\partial^2_{tt} h = \rho\partial_t a - \nabla_{xy} \cdot (\partial_h \boldsymbol{q}(\rho a - \nabla_{xy} \cdot \boldsymbol{q})).$$

TG3 for SSTREAM/SSA

Using (2.5) in the SSTREAM/SSA approximation where $\boldsymbol{q} = \rho h \boldsymbol{v}$ we find[1]

$$\rho\partial^2_{tt} h = \rho\partial_t a - \nabla_{xy} \cdot (\boldsymbol{v}(\rho a - \nabla_{xy} \cdot \boldsymbol{q}) + \rho h\,\partial_t \boldsymbol{v}) \tag{2.7}$$

[1]This expression can also be derived operating on each component as follows

$$\begin{aligned} \partial^2_{tt} h &= \partial_t a - \partial^2_{tx}(hu) - \partial^2_{ty}(hv) \\ &= \partial_t a - \partial_x(h\partial_t u + u\partial_t h) - \partial_y(h\partial_t v + v\partial_t h) \end{aligned}$$

leading to

$$\partial^2_{tt} h = \partial_t a - \partial_x(h\partial_t u + u(a - \partial_x(hu) - \partial_y(hv))) - \partial_y(h\partial_t v + v(a - \partial_x(hu) - \partial_y(hv))) \tag{2.6}$$

which is identical to Eq. (2.7).

The Third-Order-Taylor-Galerkin (TG3) method is obtained by inserting Eqs. (1.12) and (2.7) into Eq. (2.4) leading to

$$\begin{aligned} < \rho(h_1 - h_0), N > = {} & \frac{\Delta t}{2} (< \rho a_0 - \nabla_{xy} \cdot \boldsymbol{q}_0, N > + < \rho a_1 - \nabla_{xy} \cdot \boldsymbol{q}_1, N >) \\ & + \frac{1}{2} \frac{\Delta t^2}{6} (< \rho a_0 - \nabla_{xy} \cdot \boldsymbol{q}_0, \boldsymbol{v}_0 \cdot \nabla_{xy} N > - < \rho a_1 - \nabla_{xy} \cdot \boldsymbol{q}_1, \boldsymbol{v}_1 \cdot \nabla_{xy} N >) \end{aligned} \tag{2.8}$$

(where a few terms involving $\partial_t u$ and $\partial_t a$ have been omitted as well as the boundary terms, see below). This is from suitable for a fully implicit approach, i.e. where both thickness and velocity is solved for implicitly. Note that the higher-order terms (i.e. those of second and third order) in the implicit TG3 method for $t = t_0$ and $t_1 = t_0 + \Delta t$ have opposite signs. In steady-state they will therefore cancel each other out.

In more detail the TG3 system is as follows (missing ρ in a number of places):

$$\begin{aligned} 0 = {} & \frac{1}{\Delta t}(h_1 - h_0) \\ & - \frac{1}{2}(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y}) + a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})) \\ & - \frac{\Delta t}{12}(\partial_t a_0 - \partial_x(h_0 \partial_t u_0 + u_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y}))) - \partial_y(h_0 \partial_t v_0 + v_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))) \\ & + \frac{\Delta t}{12}(\partial_t a_1 - \partial_x(h_1 \partial_t u_1 + u_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1}))) - \partial_y(h_1 \partial_t v_1 + v_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))) \end{aligned} \tag{2.9}$$

For (2.9) corresponding Galerkin system is

$$\begin{aligned} 0 = {} & \int (h_1 - h_0 - \frac{\Delta t}{2}(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y}) + a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1}))) \, N \, dA \\ & - \frac{\Delta t^2}{12} \int (\partial_t a_0 N + (h_0 \partial_t u_0 + u_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))\partial_x N \\ & \qquad + (h_0 \partial_t v_0 + v_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))\partial_y N) \, dA \\ & + \frac{\Delta t^2}{12} \int (\partial_t a_1 N + (h_1 \partial_t u_1 + u_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))\partial_x N \\ & \qquad + (h_1 \partial_t v_1 + v_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))\partial_y N) \, dA \\ & + \frac{\Delta t^2}{12} \oint ((h_0 \partial_t u_0 + u_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))n_x \\ & \qquad + (h_0 \partial_t v_0 + v_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))n_y) \, N \, d\gamma \\ & - \frac{\Delta t^2}{12} \oint ((h_1 \partial_t u_1 + u_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))n_x \\ & \qquad + (h_1 \partial_t v_1 + v_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))n_y) \, N \, d\gamma \end{aligned} \tag{2.10}$$

where the second order spatial derivatives have be eliminated through partial integration.

The boundary term can be written as

$$
\begin{aligned}
&\frac{\Delta t^2}{12}\oint((h_0\partial_t u_0 + u_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))n_x \\
&\qquad + (h_0\partial_t v_0 + v_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))n_y)\,N\,d\gamma \\
&- \frac{\Delta t^2}{12}\oint((h_1\partial_t u_1 + u_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))n_x \\
&\qquad + (h_1\partial_t v_1 + v_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))n_y)\,N\,d\gamma \\
&= \frac{\Delta t^2}{12}\oint((h_0\partial_t u_0 + u_0\partial_t h_0)n_x + (h_0\partial_t v_0 + v_0\partial_t h_0)n_y)\,N\,d\gamma \\
&- \frac{\Delta t^2}{12}\oint((h_1\partial_t u_1 + u_1\partial_t h_1)n_x + (h_1\partial_t v_1 + v_1\partial_t h_1)n_y)\,N\,d\gamma \\
&= \frac{\Delta t^2}{12}\oint(\partial_t(q_{x0})n_x + \partial_t(q_{0y})n_y)\,N\,d\gamma - \frac{\Delta t^2}{12}\oint(\partial_t(q_{x1})n_x + \partial_t(h_1 v_1)n_y)\,N\,d\gamma \\
&= \frac{\Delta t^2}{12}\oint \partial_t(h_0\boldsymbol{v}_0 - h_1\boldsymbol{v}_1)\cdot\boldsymbol{n}N\,d\gamma
\end{aligned}
$$

showing that it disappears if $\partial_t q = 0$ over the boundary. Experience suggests that this boundary term can be ignored.

2.3 Consistent Streamline-Upwind Petrov-Galerkin (SUPG)

The standard SUPG is on the form

$$
< \rho\partial_t h + \nabla\boldsymbol{q} - \rho a, N + M >= 0 \tag{2.11}
$$

where M is a perturbation to the test-function space. In the literature various forms for M have been suggested. One such form is

$$
M = \tau\,\boldsymbol{v}\cdot\nabla N
$$

where τ is a parameter with the dimension of time. Note that in (2.11) the added term is applied to all terms, including time derivative. This is sometimes referred to as a 'consistent' weighting. The extra terms are interpreted element-wise, as

$$
< \rho\partial_t h + \nabla\boldsymbol{q} - \rho a, N > +\beta\sum_e < \rho\partial_t h + \nabla q - \rho a, \tau\,\boldsymbol{v}\cdot\nabla N >= 0
$$

The extra term, which is considered as a correction term, is zero for an exact solution in the classical sense.

There is no one single accepted/optimal way of selecting τ, and in the literature various definition has been proposed.

The SUPG was initially introduced for equations on the form $\partial_t h + \boldsymbol{v}\cdot\nabla h - \nabla\cdot k\nabla h) + g = 0$ and in this case, and for linear elements and regular grids, the optimal value for τ is

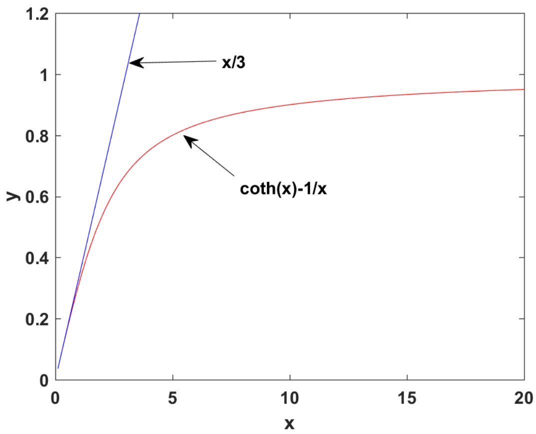
$$
\tau = \frac{l}{2|\boldsymbol{v}|}\left(\coth \mathrm{Pe} - \frac{1}{\mathrm{Pe}}\right). \tag{2.12}
$$

where the Péclet number is

$$
\mathrm{Pe} = |\boldsymbol{v}|l/(2k)
$$

with k the diffusivity and l is a measure of the (local) element size. In the limiting case where $k \to 0$, the equation becomes hyperbolic, $\mathrm{Pe} \to +\infty$, and τ simply becomes

$$
\tau = \frac{l}{2|\boldsymbol{v}|}
$$



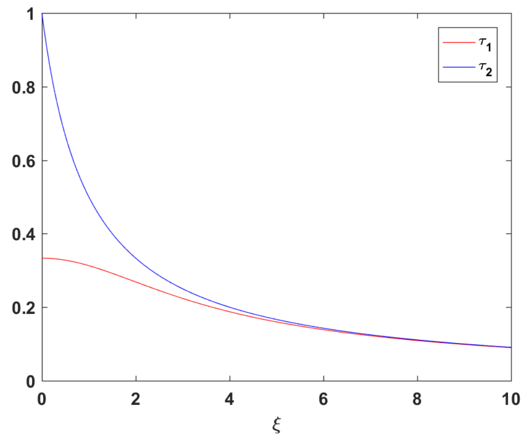


Figure 2.1: SUPG

and perturbation term to the test function has the form

$$M = \frac{l}{2}\frac{\boldsymbol{v}}{|\boldsymbol{v}|} \cdot \nabla N$$

If on the other hand $\mathrm{Pe} \to 0$ then (2.12) leads to

$$\tau = \frac{l}{2|\boldsymbol{v}|}\frac{\mathrm{Pe}}{3} = \frac{l}{2|\boldsymbol{v}|}\frac{|\boldsymbol{v}|l}{6k} = \frac{l^2}{12k}$$

in which case

$$M = \frac{l^2}{12k}\boldsymbol{v} \cdot \nabla N$$

The value of τ given by (2.12) does not depend on Δt. For transient problems with high Péclet number it has been suggested using

$$\tau = \tau_t$$

where

$$\tau_t := \Delta t/2 \tag{2.13}$$

as well as

$$\tau = \tau_s$$

where

$$\tau_s := \frac{l}{2|\boldsymbol{v}|} \tag{2.14}$$

The first definition is a temporal criterion while the second is a spatial criterion. The first form is often used in transient situations where diffusion is small and the problem either hyperbolic or close to being hyperbolic. The second form is often used for convection diffusion problems involving temperature such as $\boldsymbol{v} \cdot \nabla T - \nabla(k\nabla T) + f = 0$ (no time dependency).

A guidance as to how to select τ is looking at some specific limits. The SUPG correction term should vanish if $\Delta t \to 0$, if $l \to 0$, and also if $|\boldsymbol{v}| \to 0$. A smart choice could then be

$$\tau = \tau_1$$

where

$$\tau_1 := \frac{l}{2\,|\boldsymbol{v}|}\,\kappa \tag{2.15}$$

with

$$\kappa = \coth \xi - \frac{1}{\xi} \tag{2.16}$$

where

$$\xi = |\boldsymbol{v}| \, \Delta t / l$$

is the element Courant number and l is a characteristic local element size. For $\xi \ll 1$ we have $\kappa \sim \xi/3$, and

$$\tau = \frac{l}{2\,|\boldsymbol{v}|} \frac{|\boldsymbol{v}|\,\Delta t}{3l} = \Delta t/6$$

hence

$$M = \frac{\Delta t}{6} \boldsymbol{v} \cdot \nabla N$$

showing that the perturbation term goes to zero as either $\Delta t \to 0$ and $|\boldsymbol{v}| \to 0$. If $l \to 0$ then $\kappa \to 1$ and $\tau \to 0$, so all the above listed limits are obtained with τ given by (2.15). In summary, for $\tau = \tau_1$ given by (2.15) we have

$$M = \frac{l\kappa}{2|\boldsymbol{v}|} \, \boldsymbol{v} \cdot \nabla N$$

and the following limits

$$\begin{aligned}
M &\to 0 \quad \text{when} \quad \Delta t \to 0 \\
M &\to 0 \quad \text{when} \quad |\boldsymbol{v}| \to 0 \\
M &\to 0 \quad \text{when} \quad l \to 0 \\
M &\to \frac{l}{2} \frac{\boldsymbol{v}}{|\boldsymbol{v}|} \cdot \nabla N \quad \text{when} \quad \Delta t \to \infty && (2.17) \\
M &\to \frac{l}{2} \frac{\boldsymbol{v}}{|\boldsymbol{v}|} \cdot \nabla N \quad \text{when} \quad |\boldsymbol{v}| \to \infty && (2.18) \\
M &\to \frac{\Delta t}{6} \boldsymbol{v} \cdot \nabla N \quad \text{when} \quad l \to \infty && (2.19)
\end{aligned}$$

In the literature it is shown that (2.17) and (2.18) is the optimal choice in the convective limit, i.e. for large element Courant numbers. Limit (2.19) can be justified using Taylor-Galerkin approach (see below).

The local element size, l, can be defined in a number of similar ways leading to different numerical pre-factors to τ and ξ. The exact functional relationship between κ and ξ is also not uniquely defined.

Another option of creating a smooth transition between the temporal and spatial criteria 2.13 and 2.14 is to select τ as

$$\tau = \tau_2$$

where

$$\tau_2 := \left(\frac{1}{\tau_t} + \frac{1}{\tau_s} \right)^{-1} \tag{2.20}$$

which can also be written as

$$\tau_2 := \frac{1}{2} \frac{\Delta t}{1 + \xi} \tag{2.21}$$

Expression (2.21) gives the limits

$$\begin{aligned}
M &\to 0 \quad \text{when} \quad \Delta t \to 0 \\
M &\to 0 \quad \text{when} \quad |\boldsymbol{v}| \to 0 \\
M &\to 0 \quad \text{when} \quad l \to 0 \\
M &\to \frac{l}{2}\frac{\boldsymbol{v}}{|\boldsymbol{v}|} \cdot \nabla N \quad \text{when} \quad \Delta t \to \infty \\
M &\to \frac{l}{2}\frac{\boldsymbol{v}}{|\boldsymbol{v}|} \cdot \nabla N \quad \text{when} \quad |\boldsymbol{v}| \to \infty \\
M &\to \frac{\Delta t}{2}\boldsymbol{v} \cdot \nabla N \quad \text{when} \quad l \to \infty
\end{aligned} \tag{2.22}$$

Apart from a different numerical factor in the last limit, all limits are the same as obtained using definition (2.15).

These two above listed options for τ can also be written as

$$\tau_1 = \frac{\Delta t}{2}\frac{1}{\xi}(\coth \xi - 1/\xi) \tag{2.23}$$

$$\tau_2 = \frac{\Delta t}{2}\frac{1}{1+\xi} \tag{2.24}$$

and they are shown in Fig. 2.1b as functions of ξ for $\Delta t/2 = 1$. Only for $|\boldsymbol{v}|\Delta t < 2l$ is there any significant difference, and the difference in newer larger than a factor 3 obtained in the limit $\xi \to 0$. It appears unlikely that there will be any significant resulting differences between selecting $\tau = \tau_1$ or $\tau = \tau_2$ (see (2.23) and (2.24)). Currently the SUPG implementation in *Úa* uses $\tau = \tau_1$.

Implementing SUPG implicitly using the Θ method leads to

$$\begin{aligned}
0 = &< \rho(h_1 - h_0)/\Delta t + (1-\Theta)(\nabla_{xy} \cdot \boldsymbol{q}_0 - a_0) + \Theta(\nabla_{xy} \cdot \boldsymbol{q}_1 - a_1), N > \\
&+\beta < \rho(h_1 - h_0)/\Delta t + (1-\Theta)(\nabla_{xy} \cdot q_0 - a_0) + \Theta(\nabla_{xy} \cdot \boldsymbol{q}_1 - a_1), \tau\,((1-\Theta)\boldsymbol{v}_0 + \Theta\boldsymbol{v}_1) \cdot \nabla_{xy} N >
\end{aligned}$$

Here the perturbation to the test-function space is a weighted average over the values at the beginning and the end of the time step. This adds another source of non-linearity to the problem. Experience showed this to reduce the radius of convergence considerably and to increase grumpiness on a personal level. Former can be avoided by using the value of perturbation term at the beginning of the time step, i.e.

$$\begin{aligned}
0 = &< \rho(h_1 - h_0)/\Delta t + (1-\Theta)(\nabla_{xy} \cdot \boldsymbol{q}_0 - a_0) + \Theta(\nabla_{xy} \cdot \boldsymbol{q}_1 - a_1), N > \\
&+ \beta < \rho(h_1 - h_0)/\Delta t + \nabla_{xy} \cdot \boldsymbol{q}_0 - a_0, \tau\, \boldsymbol{v}_0 \cdot \nabla_{xy} N >
\end{aligned}$$

2.4 SIA-motivated diffusion

$$\boldsymbol{q} = \boldsymbol{q}^b + \boldsymbol{q}^d$$

where

$$\boldsymbol{q}^b = \boldsymbol{v} h$$

and

$$\boldsymbol{q}^d = \rho\, D h^{n+2}\, |\nabla_{xy} s|^{n-1}\, \nabla s$$

where

$$D = \frac{2A}{n+2}(\rho g)^n$$

$$s = (h + B)\mathcal{H}(h - h_f) + (1 - \mathcal{H}(h - h_f))(S + (1 - \rho/\rho_o)\, h)$$

and therefore (almost)

$$\partial_x s = (\partial_x h + \partial_x B)\mathcal{H}(h - h_f) + (1 - \mathcal{H}(h - h_f))(S + (1 - \rho/\rho_o)\,\partial_x h)$$

This motivates adding a SIA based diffusion term

$$-D < |\nabla_{xy} s|^{n-1}\, h^{n+2} \nabla s | \nabla_{xy} N >$$

However, this is (currently) not done in Úa.

In 1HD the SIA form of the continuity equation can be written as

$$\rho \partial_t s + \partial_x (k \partial_x s) = \rho a$$

with

$$k := \frac{2\rho A (\rho g)^n}{n+2} |\partial_x s|^{n-1} h^{n+2}$$

suggesting a Peclet number

$$\mathrm{Pe} = \frac{uL}{k}$$

2.5 Connection between third order Taylor-Galerkin (TG3) and streamline-upwind Petrov-Galerkin (SUPG)

In the context of SUPG, the TG3 system given by (2.8) can be thought of as also introducing extra weighting term beside the standard N term. But those additional weighting terms are only applied to the source term (a) and the spatial term, and not to the time-derivative term. Furthermore, as mentioned above in a steady-state these extra weighting terms cancel each other out.

It is instructive to consider as well the case where a second-order forward Taylor expansion

$$h_1 = h_0 + \Delta t\, \partial_t h_0 + \frac{(\Delta t)^2}{2} \partial^2_{tt} h_0 \tag{2.25}$$

is used instead of the centred expansion given by Eq. (2.4). Inserting (1.12) and (2.7) into (2.25) gives

$$0 = h_1 - h_0 - \Delta t\, (a - \nabla_{xy} \cdot \boldsymbol{q}) - \frac{(\Delta t)^2}{2} (\partial_t a - \nabla_{xy} \cdot (\boldsymbol{v}(a - \nabla_{xy} \cdot \boldsymbol{q}) + h\, \partial_t\, \boldsymbol{v}))$$

The Galerkin system is

$$\begin{aligned} < \rho(h_1 - h_0), N > = \Delta t\ & < \rho a - \nabla_{xy} \cdot \boldsymbol{q}, N > \\ & + \frac{(\Delta t)^2}{2} < \rho \partial_t a, N > - \frac{(\Delta t)^2}{2} < \rho\, h\, \partial_t \boldsymbol{v}, N > \\ & + \frac{(\Delta t)^2}{2} < \rho a - \nabla_{xy} \cdot \boldsymbol{q}, \boldsymbol{v} \cdot \nabla_{xy} N > \end{aligned}$$

where a partial integration has been used to get rid of second order derivatives (not writing the boundary terms). The last term is similar to what in some other ad-hoc methods is introduced as a stabilisation term. This term only acts in the direction of flow and is zero transverse to the flow direction. In the above expression all terms are to be evaluated at the beginning of the interval. This approach is usually referred to as the second-order explicit Taylor-Galerkin (TG2e) method. If we evaluate all terms by taking the mean value over the time interval, we get an implicit method, but now the second-order terms do not cancel out in steady-state, and the resulting method is quite similar to the streamline-upwind Petrov-Galerkin.

Dropping the time derivatives of a and $\boldsymbol{v}$, we can rewrite the above system as

$$< \rho(h_1 - h_0)/\Delta t, N > = < \rho a - \nabla_{xy} \cdot \boldsymbol{q}, N + \frac{1}{2} \Delta t\ \boldsymbol{v} \cdot \nabla_{xy} N >$$

showing that the TG2e results in an 'inconsistent' weighting with $\tau = \Delta t$ and $\beta = 1/2$.

Comparing 2.8 with (2.11), TG3 can be interpreted as a some sort of Petrov-Galerkin method where only the spatial terms and the source terms are multiplied by a modified test function. In TG3 the modified test function is

$$N + \frac{\Delta t}{6}\, \boldsymbol{v} \cdot \nabla_{xy} N, \tag{2.26}$$

wheres in SUPG it has the form

$$N + \beta\tau\, \boldsymbol{v} \cdot \nabla_{xy} N. \tag{2.27}$$

The weighting is done inconsistently in TG3, i.e. not over the time-derivative. Apart for the inconsistent weighting used in TG3, the SUPG is equal to TG3 provided the two adjustable parameters β and τ are selected as $\beta = 1/6$ and $\tau = \Delta t$.

The TG3 methods follows automatically from a third-order Taylor expansion and involves no adjustable parameters. The SUPG is in essence a heuristic method.

2.6 Implementing fully-implicit

In a fully implicit approach using the Newton-Raphson iteration the unknowns at time-step 1 are written in incremental form as

$$\begin{aligned} u_1^{i+1} &= \Delta u + u_1^i, \\ v_1^{i+1} &= \Delta v + v_1^i, \\ h_1^{i+1} &= \Delta h + h_1^i, \end{aligned}$$

where u_1^{i+1} is the estimate for u_1 at Newton-Raphson iteration step i. (Note that $\Delta h \neq h_1 - h_0$ and that $\Delta h \to 0$ with increasing i.)

2.6.1 First-order fully implicit

Taking only the first-order Taylor terms from (2.10), and only considering the x components for the time being, gives

$$\begin{aligned} 0 = {} & \frac{\rho}{\Delta t}(\Delta h + h_1^i - h_0) \\ & - \frac{1}{2}(\rho(a_0 + a_1) - \partial_x(\rho u_0 h_0) - \partial_x(\rho(\Delta h + h_1^i)(\Delta u + u_1^i))) \end{aligned}$$

If the specific mass balance is a function of thickness, i.e.

$$a = a(h)$$

then an additional term must be added to the matrix on the left-hand side, and the right-hand side terms must be evaluated within the NR loop every time that the thickness is updated. The first-order equation is then

$$\begin{aligned} 0 = {} & \frac{\rho}{\Delta t}(\Delta h + h_1^i - h_0) \\ & - \frac{1}{2}\left(\rho\left(a_0(h_0) + a_1(h_1) + \partial_h a_1|_{h_1}\Delta h\right) - \partial_x(q_{x0}) - \partial_x((\Delta h + h_1^i))(\Delta u + u_1^i)\right) \end{aligned}$$

Ignoring second-order terms and taking the terms involving the unknown Δh to the left-hand side leads to

$$\begin{aligned} & \rho\frac{\Delta h}{\Delta t} + \frac{1}{2}\left(\partial_x(\rho u_1^i\,\Delta h + \rho h_1^i\,\Delta u) + \partial_y(\rho v_1^i\,\Delta h + \rho h_1^i\,\Delta v) - \rho\partial_h a|_{h_1}\Delta h\right) \\ & \quad = \frac{\rho}{2}(a_0 + a_1) - \frac{\rho}{\Delta t}(h_1^i - h_0) - \frac{1}{2}(\partial_x(q_{x0}) + \partial_x(\rho h_1^i u_1^i) + \partial_y(q_{0y}) + \partial_y(\rho h_1^i v_1^i)) \end{aligned} \tag{2.28}$$

where the corresponding y terms have been added.

2.6.2 Fully implicit SSTREAM time integration with the Θ method

$$\mathrm{K}^{uu}\,\Delta\boldsymbol{u} := D\boldsymbol{r}_x(\boldsymbol{u}_1^i, \boldsymbol{v}_1^i, \boldsymbol{h}_1^i)[\Delta\boldsymbol{u}]$$

$$\begin{bmatrix} \mathrm{K}^{uu} & \mathrm{K}^{uv} & \mathrm{K}^{uh} \\ \mathrm{K}^{vu} & \mathrm{K}^{vv} & \mathrm{K}^{vh} \\ \mathrm{K}^{hu} & \mathrm{K}^{hv} & \mathrm{K}^{hh} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{u} \\ \Delta\boldsymbol{v} \\ \Delta\boldsymbol{h} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}_u \\ \boldsymbol{r}_v \\ \boldsymbol{r}_h \end{bmatrix} \tag{2.29}$$

We go from time step $t = t_0$ to $t = t_1$ and solve for the unknown values for $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{h}$, at $t = t_1$ ($\boldsymbol{u}_1$, $\boldsymbol{v}_1$, and $\boldsymbol{h}_1$) given their respective values at $t = t_0$ ($\boldsymbol{u}_0$, $\boldsymbol{v}_0$, and $\boldsymbol{h}_0$).

$$\begin{aligned} \boldsymbol{u}_1^{i+1} &= \boldsymbol{u}_1^i + \Delta\boldsymbol{u} \\ \boldsymbol{v}_1^{i+1} &= \boldsymbol{v}_1^i + \Delta\boldsymbol{v} \\ \boldsymbol{h}_1^{i+1} &= \boldsymbol{h}_1^i + \Delta\boldsymbol{h} \end{aligned}$$

For notational simplicity we omit the i superscript and it is to be understood that the values of η, β^2, h, u, and v are the estimated values at iteration i.

At element level the matrices are

$$\begin{aligned} [\mathrm{K}^{uu}]_{pq} = & \int_\Omega \big\{4\eta h \partial_x N_p\, \partial_x N_q + h\eta \partial_y N_p\, \partial_y N_q + \mathcal{H}(h-h_f)\beta^2 N_p N_q \\ & + h D_{eu}(4\partial_x u + 2\partial_y v)\, \partial_x N_p + h D_{eu}(\partial_x v + \partial_y u)\, \partial_y N_p \\ & + D_b\, u\, u\, N_p\, N_q\big\}\, dx\, dy \end{aligned}$$

$$\begin{aligned} [\mathrm{K}^{vv}]_{pq} = & \int_\Omega \big\{4\eta h \partial_y N_p\, \partial_y N_q + h\eta \partial_x N_p\, \partial_x N_q + \mathcal{H}(h-h_f)\beta^2 N_p N_q \\ & + h D_{ev}(4\partial_y v + 2\partial_x u)\, \partial_y N_p + h D_{ev}(\partial_x v + \partial_y u)\, \partial_x N_p \\ & + D_b\, v\, v\, N_p\, N_q\big\}\, dx\, dy \end{aligned}$$

$$\begin{aligned} [\mathrm{K}^{uv}]_{pq} = & \int_\Omega \big\{h\eta(2\partial_x N_p\, \partial_y N_q + \partial_y N_p\, \partial_x N_q) \\ & + h D_{ev}(4\partial_x u + 2\partial_y v)\, \partial_x N_p + h D_{ev}(\partial_x v + \partial_y u)\, \partial_y N_p \\ & + D_b\, u\, v\, N_p\, N_q\big\}\, dx\, dy \end{aligned}$$

$$\begin{aligned} [\mathrm{K}^{uv}]_{pq} = & \int_\Omega \big\{h\eta(2\partial_y N_p\, \partial_x N_q + \partial_x N_p\, \partial_y N_q) \\ & + h D_{eu}(4\partial_y v + 2\partial_x u)\, \partial_y N_p + h D_{eu}(\partial_x v + \partial_y u)\, \partial_x N_p \\ & + D_b\, u\, v\, N_p\, N_q\big\}\, dx\, dy \end{aligned}$$

(floating only (original version)

$$\begin{aligned} [\mathrm{K}^{xh}]_{pq} = & \int_\Omega \big\{\eta(4\partial_x u + 2\partial_y v)\partial_x N_p\, N_q + \eta(\partial_y u + \partial_x v)\, \partial_y N_p\, N_q \\ & + \delta(h-h_f)\beta^2 u N_p\, N_q \\ & + \rho g(\mathcal{H}(h-h_f) + h\delta(h-h_f))\, ((\rho/\rho_o \partial_x h - \partial_x H)\cos\alpha - \sin\alpha) N_p N_q \\ & + \frac{\rho^2 g}{\rho_o} h\mathcal{H}(h-h_f)\, \cos\alpha\, N_p\, \partial_x N_q \\ & - \varrho g h \cos\alpha\, \partial_x N_p\, N_q\big\}\, dx\, dy \end{aligned}$$

)

(floating only (corrected July 2011)

$$
\begin{aligned}
[\mathrm{K}^{xh}]_{pq} \quad = \quad & \int_\Omega \{\eta(4\partial_x u + 2\partial_y v)\partial_x N_p\, N_q + \eta(\partial_y u + \partial_x v)\, \partial_y N_p\, N_q \\
& + \delta(h - h_f)\beta^2 u N_p\, N_q \\
& + \rho g \mathcal{H}(h - h_f)\, ((\rho/\rho_o \partial_x h - \partial_x H)\cos\alpha - \sin\alpha) N_p N_q \\
& + \frac{\rho^2 g}{\rho_o} h \mathcal{H}(h - h_f)\, \cos\alpha\, N_p\, \partial_x N_q \\
& + \frac{\rho^2}{\rho_o} \delta(h - h_f)\, \cos\alpha\, h\, \partial_x h\, N_p\, N_q \\
& - \varrho g h \cos\alpha\, \partial_x N_p\, N_q\}\, dx\, dy
\end{aligned}
$$

)

(general case

$$
\begin{aligned}
[\mathrm{K}^{uh}]_{pq} = & \int_\Omega \{\eta(4\partial_x u + 2\partial_y v)\partial_x N_p\, N_q + \eta(\partial_y u + \partial_x v)\, \partial_y N_p\, N_q \\
& + \delta(h - h_f)\beta^2 u N_p\, N_q \\
& + \rho g \mathcal{H}(h - h_f)\, \partial_x B\, \cos\alpha\, N_p\, N_q - \rho g \sin\alpha N_p\, N_q \\
& - \rho g h \left(1 - \frac{\rho}{\rho_o} \mathcal{H}(h_f - h)\right) \cos\alpha\, \partial_x N_p\, N_q
\end{aligned}
$$

)

(floating only version:

$$
\begin{aligned}
[\mathrm{K}^{vh}]_{pq} \quad = \quad & \int_\Omega \{\eta(4\partial_y v + 2\partial_x u)\partial_y N_p\, N_q + \eta(\partial_x v + \partial_y u)\, \partial_x N_p\, N_q \\
& + \delta(h - h_f)\beta^2 v N_p\, N_q \\
& + \rho g(\mathcal{H}(h - h_f) + h\delta(h - h_f))\, (\rho/\rho_o \partial_y h - \partial_y H)\cos\alpha\, N_p N_q \\
& + \frac{\rho^2 g}{\rho_o} h \mathcal{H}(h - h_f)\, \cos\alpha\, N_p\, \partial_y N_q \\
& - \varrho g h \cos\alpha\, \partial_y N_p\, N_q\}\, dx\, dy
\end{aligned}
$$

)

(general case

$$
\begin{aligned}
[\mathrm{K}^{vh}]_{pq} \quad = \quad & \int_\Omega \{\eta(4\partial_y v + 2\partial_x u)\partial_y N_p\, N_q + \eta(\partial_x v + \partial_y u)\, \partial_x N_p\, N_q \\
& + \delta(h - h_f)\beta^2 v N_p\, N_q \\
& + \rho g(\mathcal{H}(h - h_f) + h\delta(h - h_f))\, (\rho/\rho_o \partial_y h - \partial_y H)\cos\alpha\, N_p N_q \\
& + \frac{\rho^2 g}{\rho_o} h \mathcal{H}(h - h_f)\, \cos\alpha\, N_p\, \partial_y N_q \\
& - \varrho g h \cos\alpha\, \partial_y N_p\, N_q\}\, dx\, dy
\end{aligned}
$$

)

$$
\begin{aligned}
[\mathrm{K}^{hu}]_{pq} \quad &= \quad \theta(\partial_x h\, N_q + h\, \partial_x N_q)\, N_p \\
[\mathrm{K}^{hv}]_{pq} \quad &= \quad \theta(\partial_y h\, N_q + h\, \partial_y N_q)\, N_p
\end{aligned}
$$

$$[\mathrm{K}^{hh}]_{pq} \quad = \quad (N_q/\Delta t + \theta(\partial_x u\, N_q + u\partial_x N_q + \partial_y v\, N_q + v\partial_y N_q))\, N_p$$

In the equations the quantities D_{eu}, D_{ev}, E, and D_b, which arise because of the linearisation of η and β^2, are given by

$$D_{eu} \quad = \quad E((2\partial_x u + \partial_y v)\partial_x N_q + \frac{1}{2}(\partial_x v + \partial_y u)\partial_y N_q) \tag{2.30}$$

$$D_{ev} \quad = \quad E((2\partial_y v + \partial_x u)\partial_y N_q + \frac{1}{2}(\partial_x v + \partial_y u)\partial_x N_q) \tag{2.31}$$

$$\tag{2.32}$$

$$E = \frac{1-n}{4n} A^{-1/n} \dot{\epsilon}^{(1-3n)/n}$$

$$D_b := (1/m - 1)\, C^{-1/m}\, |\boldsymbol{v}|^{(1-3m)/m}$$

Third-order Taylor-Galerkin fully implicit

The terms in

$$\frac{\Delta t^2}{12} \int (\partial_t a_1 N + (h_1 \partial_t u_1 + u_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))\partial_x N + (h_1 \partial_t v_1 + v_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1})))\partial_y N)\, dA \tag{2.33}$$

from (2.10), need to be linearised. Starting with

$$h_1 \partial_t u_1 + u_1(a_1 - \partial_x(q_{x1}) - \partial_y(q_{y1}))$$

and and inserting $h^{i+1} = \Delta h + h_1^i$ etc. gives

$$(\Delta h + h_1^i)\partial_t(\Delta u + u_1^i) + (\Delta u + u_1^i)(a_1 - \partial_x((\Delta h + h_1^i)(\Delta u + u_1^i)) - \partial_y((\Delta h + h_1^i)(\Delta v + v_1^i))$$

and first ignoring only some second-order terms

$$\partial_t u_1^i\, \Delta h + h_1^i\, \partial_t \Delta u` + h_1^i \partial_t u_1^i + (a_1\, \Delta u + u_1^i a_1) - (\Delta u + u_1^i)(\partial_x(u_1^i\, \Delta h + h_1^i\, \Delta u + h_1^i u_1^i) + \partial_y(v_1\, \Delta h + h_1^i\, \Delta v + h_1^i v_1^i))$$

and then ignoring the remaining second-order terms

$$\begin{aligned} &((u_1^i - u_0)/\Delta t)(\Delta h + h_1^i) + a_1\, \Delta u + u_1^i a_1 \\ &\qquad - u_1^i(\partial_x(u_1^i\, \Delta h + h_1^i\, \Delta u + h_1^i u_1^i) + \partial_y(v_1\, \Delta h + h_1^i\, \Delta v + h_1^i v_1^i)) \\ &\qquad - (\partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i))\, \Delta u \end{aligned}$$

where $\partial_t u_1^i = (u_1^i - u_0)/\Delta t$ and $\partial_t \Delta u$ has been set to zero. Now shifting the unknowns over to the left-hand side

$$\begin{aligned} &\partial_t u_1^i\, \Delta h + a_1\, \Delta u \\ &- u_1^i(\partial_x(u_1^i\, \Delta h + h_1^i\, \Delta u) + \partial_y(v_1\, \Delta h + h_1^i\, \Delta v)) \\ &- (\partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i))\, \Delta u \\ &= u_1^i(\partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i)) - u_1^i a_1 - \partial_t u_1^i\, h_1^i \end{aligned}$$

and adding the y terms

$$
\begin{aligned}
&(\partial_t u_1^i \partial_x N + \partial_t v_1^i \partial_y N)\Delta h + a_1\,(\partial_x N\,\Delta u + \partial_y N\,\Delta v)\\
&- (u_1^i \partial_x N + v_1^i \partial_y N)(\partial_x(u_1^i\,\Delta h + h_1^i\,\Delta u) + \partial_y(v_1\,\Delta h + h_1^i\,\Delta v))\\
&- (\partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i))\,(\partial_x N\,\Delta u + \partial_y N\,\Delta v)\\
=&(\partial_x N\,u_1^i + \partial_y N\,v_1^i)\,(\partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i)) - a_1(\partial_x N\,u_1^i + \partial_y N\,v_1^i) - (\partial_t u_1^i \partial_x N + \partial_t v_1^i \partial_x N)\,h_1^i
\end{aligned}
$$

Then adding the remaining higher-order Taylor terms in (2.10), involving fields from time-step zero, to the right-hand side gives

$$
\begin{aligned}
&(\partial_t u_1^i \partial_x N + \partial_t v_1^i \partial_y N)\,\Delta h\\
&+ (\partial_x N\,\Delta u + \partial_y N\,\Delta v)\,(a_1 - \partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i))\\
&- (\partial_x N\,u_1^i + \partial_y N\,v_1^i)\,(\partial_x(u_1^i\,\Delta h + h_1^i\,\Delta u) + \partial_y(v_1\,\Delta h + h_1^i\,\Delta v))\\
=&\partial_t(a_0 - a_1)N\\
&- (u_1^i\,\partial_x N + v_1^i\,\partial_y N)(a_1 - \partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i)) - h_1^i\,(\partial_t u_1^i\,\partial_x N + \partial_t v_1^i\,\partial_y N)\\
&+ (u_0\,\partial_x N + v_0\,\partial_y N)(a_0 - \partial_x(q_{x0}) + \partial_y(q_{0y})) + h_0\,(\partial_t u_0\,\partial_x N + \partial_t v_0\,\partial_y N)
\end{aligned}
$$

Now adding first-order Taylor terms from (2.28) to the expression above

$$
\begin{aligned}
&\Delta h\,N + \frac{\Delta t}{2}\big(\partial_x(u_1^i\,\Delta h + h_1^i\,\Delta u) + \partial_y(v_1^i\,\Delta h + h_1^i\,\Delta v)\big)\,N\\
&+ \gamma(\partial_t u_1^i \partial_x N + \partial_t v_1^i \partial_y N)\,\Delta h\\
&+ \gamma(\partial_x N\,\Delta u + \partial_y N\,\Delta v)\,(a_1 - \partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i))\\
&- \gamma(\partial_x N\,u_1^i + \partial_y N\,v_1^i)\,(\partial_x(u_1^i\,\Delta h + h_1^i\,\Delta u) + \partial_y(v_1\,\Delta h + h_1^i\,\Delta v))\\
=&\left(h_0 - h_1^i + \frac{\Delta t}{2}(a_0 + a_1) - \frac{\Delta t}{2}(\partial_x(q_{x0}) + \partial_x(h_1^i u_1^i) + \partial_y(q_{0y}) + \partial_y(h_1^i v_1^i))\right)N\\
&+ \gamma\partial_t(a_0 - a_1)N\\
&- \gamma(u_1^i\,\partial_x N + v_1^i\,\partial_y N)(a_1 - \partial_x(h_1^i u_1^i) + \partial_y(h_1^i v_1^i)) - \gamma h_1^i\,(\partial_t u_1^i\,\partial_x N + \partial_t v_1^i\,\partial_y N)\\
&+ \gamma(u_0\,\partial_x N + v_0\,\partial_y N)(a_0 - \partial_x(q_{x0}) + \partial_y(q_{0y})) + \gamma h_0\,(\partial_t u_0\,\partial_x N + \partial_t v_0\,\partial_y N)
\end{aligned}
$$

which can also be written as

$$
\begin{aligned}
&\Delta h\,N + \big(\kappa\,N - \gamma\,(\partial_x N\,u_1^i + \partial_y N\,v_1^i)\big)\,\big(\partial_x(u_1^i\,\Delta h + h_1^i\,\Delta u) + \partial_y(v_1^i\,\Delta h + h_1^i\,\Delta v)\big)\\
&+ \gamma(\partial_t u_1^i \partial_x N + \partial_t v_1^i \partial_y N)\,\Delta h\\
&+ \gamma(\partial_x N\,\Delta u + \partial_y N\,\Delta v)\,(a_1 - \partial_x(h_1^i u_1^i) - \partial_y(h_1^i v_1^i))\\
=&(h_0 - h_1^i)\,N\\
&+ (\kappa\,N + \gamma(u_0\,\partial_x N + v_0\,\partial_y N))\,(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))\\
&+ \big(\kappa\,N - \gamma(u_1^i\,\partial_x N + v_1^i\,\partial_y N)\big)\,\big(a_1 - \partial_x(h_1^i u_1^i) - \partial_y(h_1^i v_1^i))\big)\\
&+ \gamma\partial_t(a_0 - a_1)N\\
&- \gamma h_1^i\,(\partial_t u_1^i\,\partial_x N + \partial_t v_1^i\,\partial_y N)\\
&+ \gamma h_0\,(\partial_t u_0\,\partial_x N + \partial_t v_0\,\partial_y N)
\end{aligned}
$$

where

$$\gamma = \frac{(\Delta t)^2}{12}$$

and

$$\kappa = \frac{\Delta}{2}$$

2.6.3 Semi-implicit: uv explicit, and h implicit

In the semi-implicit approach h_1 is treated as the unknown while h_0, u_1, v_1, h_0, v_0 are assumed to be known.

Taking the unknown h_1 in (2.10) to the left-hand side gives

$$\begin{aligned}
&\int (h_1 + \frac{\Delta t}{2}(\partial_x(q_{x1}) + \partial_y(q_{y1})))\, N\, dA \\
&+ \frac{\Delta t^2}{12} \int ((h_1 \partial_t u_1 - u_1(\partial_x(q_{x1}) + \partial_y(q_{y1})))\partial_x N + (h_1 \partial_t v_1 - v_1(\partial_x(q_{x1}) + \partial_y(q_{y1})))\partial_y N)\, dA \\
&- \frac{\Delta t^2}{12} \oint ((h_1 \partial_t u_1 - u_1(\partial_x(q_{x1}) + \partial_y(q_{y1})))n_x + (h_1 \partial_t v_1 - v_1(\partial_x(q_{x1}) + \partial_y(q_{y1})))n_y)\, N\, d\gamma \\
&= \int (h_0 + \frac{\Delta t}{2}(a_0 + a_1 - \partial_x(q_{x0}) - \partial_y(q_{0y})))\, N\, dA \\
&+ \frac{\Delta t^2}{12} \int (\partial_t(a_0 - a_1)N + (h_0 \partial_t u_0 - u_1 a_1 + u_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))\partial_x N \\
&\qquad\qquad + (h_0 \partial_t v_0 - v_1 a_1 + v_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))\partial_y N)\, dA \\
&- \frac{\Delta t^2}{12} \oint ((h_0 \partial_t u_0 - u_1 a_1 + u_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))n_x \\
&\qquad + (h_0 \partial_t v_0 - v_1 a_1 + v_0(a_0 - \partial_x(q_{x0}) - \partial_y(q_{0y})))n_y)\, N\, d\gamma
\end{aligned}$$

2.7 Transient implicit SSHEET/SIA with the Θ method

An implicit method us used where h_0 and h_1 are the ice thicknesses at the beginning and the end of the time step, respectively. The system is solved using NR, and we write

$$\begin{aligned}
h^{i+1} &= h^i + \Delta h \\
s^{i+1} &= s^i + \Delta h
\end{aligned}$$

where i is the number of the non-linear iteration step. Provided the method converges, Δh goes to zero with increasing i. In the following we simply write h instead of h^i.

$$(h + \Delta h - h_0)/\Delta t = -(1 - \Theta)\nabla_{xy} \cdot \boldsymbol{q}_0 - \Theta \nabla_{xy} \cdot \boldsymbol{q}_1(h + \Delta h)$$

$$\int_\Omega (h + \Delta h - h_0) N \Omega = -(1 - \Theta)\Delta t \int_\Omega N \nabla_{xy} \cdot \boldsymbol{q}_0 - \Theta \Delta t \int_\Omega N \nabla_{xy} \cdot \boldsymbol{q}_1(h + \Delta h) \tag{2.34}$$

To use the above equation we need to know the perturbation in flux due to perturbation in thickness.

The simplest way of perturbing $q(h)$ with respect to h is to find

$$\delta q = \lim_{\dot\epsilon \to 0} \frac{d}{d\dot\epsilon} q(h + \dot\epsilon \Delta h).$$

As using SSHEET for floating ice shelves is somewhat questionable we here only consider the case of grounded ice[2]. Where grounded

$$q_x(h) = -\rho\, D \left((\partial_x h + \partial_x B)^2 + (\partial_y h + \partial_y B)^2\right)^{(n-1)/2}\ h^{n+2}\, \partial_x s,$$

[2] In general the surface is related tho the ice thickness and bed through

$$s = (h + B)\mathcal{H}(h - h_f) + (S + (1 - \rho/\rho_o)h)\,\mathcal{H}(h_f - h).$$

and

$$\Delta s = \Delta h,$$

with

$$D = \frac{2A}{n+2}(\rho g)^n$$

Therefore

$$\begin{aligned}
\delta q_x &= \lim_{\dot\epsilon \to 0} \frac{d}{d\dot\epsilon} q_x(h + \dot\epsilon \Delta h) \\
&= -D \lim_{\dot\epsilon \to 0} \frac{d}{d\dot\epsilon} D \left(\partial_x (s + \dot\epsilon \Delta h)^2 + \partial_y (s + \dot\epsilon \Delta h)^2\right)^{(n-1)/2} \; (h + \dot\epsilon \Delta h)^{n+2} \, \partial_x (s + \dot\epsilon \Delta h) \\
&= -D \, |\nabla_{xy} s|^{n-1} \, h^{n+2} \, \partial_x \Delta h \\
&\quad - D\,(n+2) |\nabla_{xy} s|^{n-1} \; h^{n+1} \, \partial_x s \; \Delta h \\
&\quad - D\,(n-1) |\nabla_{xy} s|^{n-3} \, (\partial_x s \, \partial_x \Delta h + \partial_y s \, \partial_y \Delta h) \;\; h^{n+2} \, \partial_x s
\end{aligned}$$

The first term of the final expression shown above is the perturbation in flux due to increase in slope, the second one the perturbation due to increase in thickness. The third term is non-linear terms that vanishes for $n = 1$. This third term represents changes flux caused by a change in effective viscosity due to perturbations in slope. This third terms shows that a change in slope in y direction gives rise to an increase in flux in x direction.

As expected, for a negative unperturbed surface slope ($\partial_x s < 0$), both a positive perturbation in thickness ($\Delta h > 0$), and an increase in (negative) surface slope ($\partial_x \Delta h < 0$), result in a positive perturbation in flux ($\delta q_x > 0$).

2.7.1 SSHEET with no-flux natural boundary condition

Using a variant of Gauss theorem given by Eq. (C.3) we write (2.34) on the form

$$\int_\Omega N \nabla_{xy} \cdot \boldsymbol{q} \, d\Omega = \oint_{\partial\Omega} N \boldsymbol{q} \cdot \hat{\boldsymbol{n}} \, d\Gamma - \int_\Omega \nabla_{xy} N \cdot \boldsymbol{q} \, d\Omega \tag{2.35}$$

Applying (2.35) on (linearised) (2.34) and assuming that $\boldsymbol{q} \cdot \hat{\boldsymbol{n}} = 0$ on $\partial\Omega$, i.e. ice flux across the boundary is zero (homogeneous Neumann boundary condition), gives

$$\begin{aligned}
\int_\Omega (h + \Delta h - h_0) \, N \, d\Omega =& (1-\Theta)\Delta t \int \boldsymbol{q}_0 \cdot \nabla_{xy} N \, d\Omega \\
&+ \Theta \Delta t \int \boldsymbol{q}_1^i \cdot \nabla_{xy} N \, d\Omega \\
&+ \Theta \Delta t \int \Delta \boldsymbol{q} \cdot \nabla_{xy} N \, d\Omega
\end{aligned}$$

or

$$\begin{aligned}
\int_\Omega & \Delta h \, N \, d\Omega - \Theta \Delta t \int_\Omega \Delta \boldsymbol{q} \cdot \nabla_{xy} N \, d\Omega \\
&= - \int_\Omega (h - h_0) \, N \, d\Omega \\
&\quad + (1-\Theta)\Delta t \int_\Omega \boldsymbol{q}_0 \cdot \nabla_{xy} N \, d\Omega \\
&\quad + \Theta \Delta t \int_\Omega \boldsymbol{q}_1^i \cdot \nabla_{xy} N \, d\Omega
\end{aligned}$$

where $\boldsymbol{q}^{i+1} = \boldsymbol{q}^i + \Delta\boldsymbol{q}$ is the NR iteration, with

$$\begin{aligned}\Delta q_x = &- D(n-1)h^{n+2}|\nabla_{xy}s|^{n-3}\partial_x s(\partial_x s\,\partial_x\Delta h + \partial_y s\,\partial_y\Delta h))\\ &- D(n+2)|\nabla_{xy}s|^{n-1}h^{n+1}\,\partial_x s\;\Delta h\\ &- D|\nabla_{xy}s|^{n-1}\,h^{n+2}\;\partial_x\Delta h\end{aligned}$$

Collecting terms, and writing s and h instead of s^{i+1} and h^{i+1}, respectively, and s_0 and h_0 instead of s^i and h^i, gives

$$\begin{aligned}&\int N_p N_q\;\Delta h_q\;d\Omega\\ +\Theta\Delta t&\int D(n+2)|\nabla_{xy}s|^{n-1}h^{n+1}\,(\partial_x N_p\,\partial_x s + \partial_y N_p\,\partial_y s)\,N_q\;\Delta h_q\;d\Omega\\ +\Theta\Delta t&\int D|\nabla_{xy}s|^{n-1}\,h^{n+2}\;(\partial_x N_p\,\partial_x N_q + \partial_y N_p\,\partial_y N_q)\;\Delta h_q\;d\Omega\\ +\Theta\Delta t&\int D(n-1)h^{n+2}|\nabla_{xy}s|^{n-3}(\partial_x N_p\,\partial_x s + \partial_y N_p\,\partial_y s)(\partial_x s\,\partial_x N_q + \partial_y s\,\partial_y N_q)\;\Delta h_q\;d\Omega\\ =&\int (h_0 - h)N_p\;d\Omega\\ +(1-\Theta)\Delta t&\int (q_{x0}\partial_x N_p + q_{y0}\partial_y N_p)\;d\Omega\\ +\Theta\Delta t&\int (q_{x1}\partial_x N_p + q_{y1}\partial_y N_p)\;d\Omega\end{aligned}$$

where

$$\begin{aligned}q_{x0} &= D\,|\nabla_{xy}s_0|^{(n-1)}\;h_0^{n+2}\;\partial_x s_0\\ q_{y0} &= D\,|\nabla_{xy}s_0|^{(n-1)}\;h_0^{n+2}\;\partial_y s_0\\ q_{x1} &= D\,|\nabla_{xy}s_1|^{(n-1)}\;h_1^{n+2}\;\partial_x s_1\\ q_{y1} &= D\,|\nabla_{xy}s_1|^{(n-1)}\;h_1^{n+2}\;\partial_y s_1\end{aligned}$$

and

$$D = \frac{2A(\rho g)^n}{n+2}$$

2.7.2 Transient SSHEET/SIA with a free-flux natural boundary condition

To arrive at a formulation where free flux is the natural boundary condition we express the flux in terms of the deformational velocity as‘

$$\boldsymbol{q} = F\,h\,\boldsymbol{v}_d$$

For a ‘free-flux’ boundary condition this is the flux at the in and outflow boundaries.
We solve

$$\int_\Omega u_d N\;\Omega = \int_\Omega E\;|\nabla_{xy}s|^{n-1}\;h^{n+1}\;\partial_x s\;d\Omega \tag{2.36}$$

$$\int_\Omega v_d N\;\Omega = \int_\Omega E\;|\nabla_{xy}s|^{n-1}\;h^{n+1}\;\partial_y s\;d\Omega \tag{2.37}$$

$$\int_\Omega (h_1 - h_0)N\;\Omega = -(1-\Theta)\,\Delta t\int_\Omega FN\nabla_{xy}\cdot h_0\boldsymbol{v}_{d0}\;\Omega - \Theta\,\Delta t\int_\Omega FN\nabla_{xy}\cdot h\boldsymbol{v}_{d1}\;d\Omega \tag{2.38}$$

for u_d, v_d, and h as unknowns. Writing this as a coupled system with u_d, v_d, and h all as unknowns gives a system of first order differential equations, rather than one second order equation. This eliminates the need to get rid of a second spatial derivative and the natural boundary conditions now corresponds to a 'free-flux' condition.

NR with

$$
\begin{aligned}
u^{i+1} &= u^i + \Delta u \\
v^{i+1} &= v^i + \Delta v \\
h^{i+1} &= h^i + \Delta h \\
s^{i+1} &= s^i + \Delta h
\end{aligned}
$$

and linearising gives

$$
\begin{aligned}
u + \Delta u = E\, |\nabla_{xy} s|^{n-1}\, h^{n+1}\, \partial_x s \\
&+ E(n-1)|\nabla_{xy} s|^{n-3}\, h^{n+1}\, \partial_x s\; (\partial_x s\, \partial_x \Delta h + \partial_y s\, \partial_y \Delta h) \\
&+ E(n+1)|\nabla_{xy} s|^{n-1}\, h^n \partial_x s\; \Delta h \\
&+ E|\nabla_{xy} s|^{n-1}\, h^{n+1}\, \partial_x \Delta h
\end{aligned}
$$

Taking the Δ terms to one side

$$
\begin{aligned}
&- \Delta u_q \\
&+ E(n-1)|\nabla_{xy} s|^{n-3}\, h^{n+1}\, \partial_x s\; (\partial_x s\, \partial_x \Delta h + \partial_y s\, \partial_y \Delta h) \\
&+ E(n+1)|\nabla_{xy} s|^{n-1}\, h^n \partial_x s\; \Delta h \\
&+ E|\nabla_{xy} s|^{n-1}\, h^{n+1}\, \partial_x \Delta h \\
&= u - E\, |\nabla_{xy} s|^{n-1}\, h^{n+1}\, \partial_x s
\end{aligned}
$$

Galerkin, u term:

$$
\begin{aligned}
&- < N_p, N_q > \; \Delta u_q \\
&+ (n-1) < N_p, E|\nabla_{xy} s|^{n-3}\, h^{n+1}\, \partial_x s(\partial_x s\, \partial_x N_q + \partial_y s\, \partial_y N_q) > \; \Delta h_q \\
&+ (n+1) < N_p, E|\nabla_{xy} s|^{n-1}\, h^n \partial_x s N_q > \; \Delta h_q \\
&+ < N_p, E|\nabla_{xy} s|^{n-1}\, h^{n+1} \partial_x N_q > \; \Delta h_q \\
&=< N_p, u - E\, |\nabla_{xy} s|^{n-1}\, h^{n+1}\, \partial_x s >
\end{aligned}
$$

The corresponding v term is obtained by replacing u with v and derivatives with respect to x by derivatives with respect to y,

Galerkin q term:

$$
\begin{aligned}
< N_P, N_q > \Delta h_q + \Delta t\; \Theta F < N_p, \partial_x u\, N_q + u\, \partial_x N_q + \partial_y v\, N_q + v\, \partial_y N_q > \Delta h_q \\
&+ \Delta t\; \Theta F < N_p, \partial_x h\, N_q + h \partial_x N_q > \Delta u_q \\
&+ \Delta t\; \Theta F < N_p, \partial_y h\, N_q + h \partial_y N_q > \Delta v_q \\
=&\Delta t < N_p, (1-\Theta)\, a_0 + \Theta\, a_1 > \\
&- < N_p, (h - h_0) > \\
&- \Delta t\, F < N_p, ((1-\Theta)(\partial_x (h_0 u_0) + \partial_y (h_0 u_0)) + \Theta(\partial_x (hu) + \partial_y (hv))) >
\end{aligned}
$$

2.8 Method of characteristics

Think of

$$\partial_t h + \partial_x (uh) = a$$

as

$$D_t h + h\partial_x u = a$$

with

$$D_t h = \partial_t h + u\partial_x h$$

Along the characteristics I have

$$D_t h = \lim_{\Delta t \to 0} \frac{1}{\Delta t}(h(x,t) - h(x - u\,\Delta t, t - \Delta t))$$

and if I just write

$$h(x - u\,\Delta t, t - \Delta t)) = h(x, t - \Delta t) - d_x h(x, t - \Delta t)\, u\, \Delta t$$

and take the limit I (of course) just get

$$\partial_t h + \partial_x(uh) = a$$

again.

The question seem to me to be about how to approximate the variation along the characteristic. If I write

$$h(x - u\,\Delta t, t - \Delta t)) = h_0 - d_x h_0\, u\, \Delta t + \frac{1}{2} d^2_{xx} h_0 (u\,\Delta t)^2$$

where $h0$ is evaluated at x and at $t - \Delta t$, I get a ‘correction’ term and the discretized version is

$$\frac{1}{\Delta t}(h_1 - h_0 + d_x h_0\, u\, \Delta t + \frac{1}{2} d^2_{xx} h_0\, (u\,\Delta t)^2) + h\, d_x u = a$$

It is a bit unclear in the above expression at what time to evaluate u. I could go for the average value over the time step, but I will only know it at the previous time step anyhow.

One way of interpreting the above equation is

$$\frac{1}{\Delta t}(h_1 - h_0) + d_x h\, u + \frac{1}{2} u^2\, \Delta t d^2_{xx} h + h\, d_x u = a$$

and then evaluate all terms at some time within the time step, i.e. the Θ method. The above equation can be written as

$$\frac{1}{\Delta t}(h_1 - h_0) + \partial_x(hu) + \frac{\Delta t}{2} u^2\, \partial^2_{xx} h = a$$

In combination with the Θ method I get

$$\frac{1}{\Delta t}(h_1 - h_0) = \Theta\left(a_1 - \partial_x(q_{x1}) - \frac{1}{2} u_1\, \Delta t\, \partial^2_{xx}(u_1 h_1)\right) + (1 - \Theta)\left(a_0 - \partial_x(q_{x0}) - \frac{1}{2} u_0\, \Delta t\, \partial^2_{xx}(u_0 h_0)\right)$$

(did not write the a correction term)

2.9 Taylor-Galerkin

We have

$$\partial_t h + \partial_x(uh) = a$$

we write

$$h_1 = h_0 + \Delta t\, \partial_t h_0 + \frac{(\Delta t)^2}{2} \partial^2_{tt} h_0$$

which is a second-order accurate Euler method.

Inserting we get

$$\begin{aligned}
h_1 &= h_0 + \Delta t(a - \partial_x(hu)) + \frac{(\Delta t)^2}{2}\partial_t(a - \partial_x(uh)) \\
&= h_0 + \Delta t(a - \partial_x(hu)) + \frac{(\Delta t)^2}{2}(\partial_t a - \partial^2_{xt}(uh)) \\
&= h_0 + \Delta t(a - \partial_x(hu)) + \frac{(\Delta t)^2}{2}(\partial_t a - \partial_x(\partial_h(uh)\partial_t h)) \\
&= h_0 + \Delta t(a - \partial_x(hu)) + \frac{(\Delta t)^2}{2}(\partial_t a - \partial_x(u\partial_t h)) \\
&= h_0 + \Delta t(a - \partial_x(hu)) + \frac{(\Delta t)^2}{2}(\partial_t a - \partial_x(u(a - \partial_x(uh))))
\end{aligned}$$

or

$$\frac{1}{\Delta t}(h_1 - h_0) = a - \partial_x(hu) - \frac{\Delta t}{2}\partial_x(u(a - \partial_x(uh))) + \frac{\Delta t}{2}\partial_t a \tag{2.39}$$

All the therms of the right-hand side of (2.39) refer to time step 0. I get the implicit theta method if I evaluate the right-hand side at both 1 and 0 and weight with Θ.

After a partial integration we get a correction term

$$< \frac{1}{2}\Delta t\,\partial_x u, a - \partial_x(uh) >$$

2.10 Third order implicit Taylor Galerkin (1HD)

A better justification for evaluation at both time step 1 and 0 comes from writing

$$h_1 = h_0 + \Delta t\,\partial_t h_0 + \frac{(\Delta t)^2}{2}\partial^2_{tt}h_0,$$

$$h_0 = h_1 - \Delta t\,\partial_t h_1 + \frac{(\Delta t)^2}{2}\partial^2_{tt}h_1,$$

adding

$$2(h_1 - h_0) = \Delta t\,(\partial_t h_0 + \partial_t h_1) + \frac{(\Delta t)^2}{2}(\partial^2_{tt}h_0 - \partial^2_{tt}h_1)$$

and simplifying gives

$$\frac{1}{\Delta t}(h_1 - h_0) = \frac{1}{2}(\partial_t h_0 + \partial_t h_1) + \frac{\Delta t}{4}(\partial^2_{tt}h_0 - \partial^2_{tt}h_1) \tag{2.40}$$

Now use

$$\partial_t h = a - \partial_x(hu)$$

and

$$\begin{aligned}
\partial^2_{tt}h &= \partial_t a - \partial^2_{tx}(hu) \\
&= \partial_t a - \partial_t(h\partial_x u + u\partial_x h) \\
&= \partial_t a - (\partial_x h\partial_t u + h\partial^2_{xt}u + \partial_t h\partial_x u + u\partial^2_{xt}h) \\
&= \partial_t a - (\partial_x h\partial_t u + h\partial^2_{xt}u + \partial_x u(a - \partial_x(hu)) + u\partial_x(a - \partial_x(hu))) \\
&= \partial_t a - (\partial_x h\partial_t u + h\partial^2_{xt}u + \partial_x u(a - \partial_x(hu)) + u\partial_x(a - \partial_x(hu))) \\
&= \partial_t a - \partial_x h\partial_t u - h\partial^2_{xt}u - \partial_x(u(a - \partial_x(hu)))
\end{aligned}$$

or

$$\partial^2_{tt}h = \partial_t a - \partial_x(h\partial_t u + ua - u\partial_x(uh)) \tag{2.41}$$

Inserting (2.41) into (2.3) gives

$$
\begin{aligned}
0 &= Lh \\
&= \frac{1}{\Delta t}(h_1 - h_0) \\
&- \frac{1}{2}(a_0 - \partial_x(q_{x0}) + a_1 - \partial_x(q_{x1})) \\
&- \frac{\Delta t}{4}(\partial_t a_0 - \partial_x(h_0\partial_t u_0 + u_0(a_0 - \partial_x(q_{x0}))) - \partial_t a_1 + \partial_x(h_1\partial_t u_1 + u_1(a_1 - \partial_x(q_{x1}))))
\end{aligned}
$$

Galerkin

$$< Lh_1|N_p >= 0$$

with $u_1 = N_q u_q$, etc. I used partial integration to get rid of second order spatial derivatives

$$
\begin{aligned}
0 &= < Lu|N_q > \\
&= \frac{1}{\Delta t}\int (h_1 - h_0)N_q\,dx \\
&- \frac{1}{2}\int (a_0 - \partial_x(q_{x0}) + a_1 - \partial_x(q_{x1}))N_q\,dx \\
&- \frac{\Delta t}{4}\int (\partial_t a_0 - \partial_t a_1)N_q\,dx \\
&- \frac{\Delta t}{4}\int (h_0\partial_t u_0 + u_0 a_0 - u_0\partial_x(q_{x0})) - h_1\partial_t u_1 - u_1 a_1 + u_1\partial_x(q_{x1})))\partial_x N_q\,dx \\
&- \frac{\Delta t}{4}(-h_0\partial_t u_0 - u_0(a_0 - \partial_x(q_{x0})) + h_1\partial_t u_1 + u_1(a_1 - \partial_x(q_{x1})))N_q|_{x_l}^{x_r}
\end{aligned}
$$

The unknown is h_1

$$
\begin{aligned}
&\int (h_1 + \frac{\Delta t}{2}\partial_x(q_{x1}))N_q\,dx \\
&+ \frac{(\Delta t)^2}{4}\int (h_1\partial_t u_1 - u_1\partial_x(q_{x1}))\,\partial_x N_q\,dx + \frac{(\Delta t)^2}{4}(u_1\partial_x(q_{x1}) - h_1\partial_t u_1)N_q|_{x_l}^{x_r} \\
&= \int (h_0 + \frac{\Delta t}{2}(a_1 + a_0 - \partial_x(q_{x0})))N_q\,dx \\
&+ \frac{(\Delta t)^2}{4}\int \partial_t(a_0 + a_1)N_q\,dx \\
&+ \frac{(\Delta t)^2}{4}\int (u_0 a_0 - u_1 a_1 + h_0\partial_t u_0 - u_0\partial_x(q_{x0}))\,\partial_x N_q\,dx \\
&+ \frac{(\Delta t)^2}{4}(u_1 a_1 - u_0 a_0 - h_0\partial_t u_0 + u_0\partial_x(q_{x0}))N_q|_{x_l}^{x_r}
\end{aligned}
$$

Writing out the product terms

$$
\begin{aligned}
&\int (h_1 + \frac{\Delta t}{2}(h_1\partial_x u_1 + u_1\partial_x h_1))N_q \, dx \\
+ \quad & \frac{(\Delta t)^2}{4}\int (\partial_t u_1 h_1 - u_1(h_1\partial_x u_1 + u_1\partial_x h_1))\,\partial_x N_q \, dx \\
+ \quad & \frac{(\Delta t)^2}{4}(u_1(h_1\partial_x u_1 + u_1\partial_x h_1) - h_1\partial_t u_1)N_q|_{x_l}^{x_r} \\
= \quad & \int (h_0 + \frac{\Delta t}{2}(a_1 + a_0 - (h_0\partial_x u_0 + u_0\partial_x h_0)))N_q \, dx \\
+ \quad & \frac{(\Delta t)^2}{4}\int \partial_t(a_0 + a_1)N_q \, dx \\
+ \quad & \frac{(\Delta t)^2}{4}\int (u_0 a_0 - u_1 a_1 + h_0\partial_t u_0 - u_0(h_0\partial_x u_0 + u_0\partial_x h_0))\,\partial_x N_q \, dx \\
+ \quad & \frac{(\Delta t)^2}{4}(u_1 a_1 - u_0 a_0 - h_0\partial_t u_0 + u_0(h_0\partial_x u_0 + u_0\partial_x h_0))N_q|_{x_l}^{x_r}
\end{aligned}
$$

Taking this up to third order

$$
\frac{1}{\Delta t}(h_1 - h_0) = \frac{1}{2}(\partial_t h_0 + \partial_t h_1) + \frac{\Delta t}{4}(\partial_{tt}^2 h_0 - \partial_{tt}^2 h_1) + \frac{(\Delta t)^2}{12}(\partial_{ttt}^3 h_0 + \partial_{ttt}^3 h_1)
$$

is easy, if we simply approximate the time derivative in the third-order term through finite differences.

$$
\begin{aligned}
\frac{1}{\Delta t}(h_1 - h_0) &= \frac{1}{2}(\partial_t h_0 + \partial_t h_1) + \frac{\Delta t}{4}(\partial_{tt}^2 h_0 - \partial_{tt}^2 h_1) + \frac{(\Delta t)^2}{12\Delta t}(\partial_{tt}^2(h_1 - h_0) + \partial_{tt}^2(h_1 - h_0)) \\
&= \frac{1}{2}(\partial_t h_0 + \partial_t h_1) + \frac{\Delta t}{4}(\partial_{tt}^2 h_0 - \partial_{tt}^2 h_1) + \frac{\Delta t}{6}\,\partial_{tt}^2(h_1 - h_0) \\
&= \frac{1}{2}(\partial_t h_0 + \partial_t h_1) + \frac{\Delta t}{12}\partial_{tt}^2 h_0 - \frac{\Delta t}{12}\partial_{tt}^2 h_1
\end{aligned}
$$

The only thing that changes is the numerical factor of the second-order term. However this is now correct to third order.

Chapter 3

Constraints

In *Úa* all essential boundary conditions, and various other constraints on the solution, are enforced using the Lagrange multiplier method. Any multi-linear constraints

$$\boldsymbol{L}\boldsymbol{x} - \boldsymbol{c} = 0$$

where $\boldsymbol{x}$ are the unknowns for some $\boldsymbol{L}$ and $\boldsymbol{c}$ can be described.

3.1 Linear system with multi-linear constraints

For a quadratic minimisation problem

$$\min_{\boldsymbol{x}} I(\boldsymbol{x}) = \boldsymbol{x}^T \mathrm{A} \boldsymbol{x} - \boldsymbol{b}\boldsymbol{x}$$

subject to the linear set seat of constraints

$$\boldsymbol{L}\boldsymbol{x} - \boldsymbol{c} = 0$$

the system to be solved is

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{L}^T \\ \boldsymbol{L} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{c} \end{bmatrix}$$

3.2 Non-linear system with non-linear constraints

If we both have a non-linear minimisation problem

$$\min_{\boldsymbol{x}} I(\boldsymbol{x})$$

and non-linear set of constraints $\boldsymbol{l}(\boldsymbol{x}) = 0$, we minimise

$$I(\boldsymbol{x}) + \boldsymbol{\lambda}^T \boldsymbol{l}(\boldsymbol{x}),$$

A stable equilibrium point is a minimum wither respect to x and a maximum with respect to λ.

Setting the derivatives with respect to $\boldsymbol{x}$ and λ to zero leads to

$$\begin{aligned} \partial_{\boldsymbol{x}} I(\boldsymbol{x}) + \boldsymbol{\lambda}^T \partial_{\boldsymbol{x}} \boldsymbol{l}(\boldsymbol{x}) &= 0 \\ \boldsymbol{l}(\boldsymbol{x}) &= \boldsymbol{0} \end{aligned}$$

If this non-linear system is again solved using Newton-Raphson method then we use first-order Taylor expansion and write

$$\begin{aligned} \partial_{\boldsymbol{x}} I(\boldsymbol{x}) &= \partial_{\boldsymbol{x}} I_0 + \boldsymbol{\lambda}_0^T \, \partial_{\boldsymbol{x}} \boldsymbol{l}_0 + \partial^2_{\boldsymbol{x}\boldsymbol{x}} I_0 \, \Delta \boldsymbol{x} + \boldsymbol{\lambda}_0 \, \partial_{\boldsymbol{x}\boldsymbol{x}} \boldsymbol{l}_0 \, \Delta \boldsymbol{x} + \partial_x \boldsymbol{l}_0^T \, \Delta \boldsymbol{\lambda} \\ \boldsymbol{l}(\boldsymbol{x}) &= \boldsymbol{l}(\boldsymbol{x}_0) + \partial_{\boldsymbol{x}} \boldsymbol{l} \, \Delta \boldsymbol{x} \end{aligned}$$

and repeatedly solve

$$\begin{bmatrix} \boldsymbol{H} + \partial^2_{xx}\boldsymbol{l}\,\boldsymbol{\lambda}_0 & \partial_{\boldsymbol{x}}\boldsymbol{l}^T \\ \partial_{\boldsymbol{x}}\boldsymbol{l} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{x} \\ \Delta\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -\partial_{\boldsymbol{x}}I_0 - \partial_{\boldsymbol{x}}\boldsymbol{l}^T\,\boldsymbol{\lambda}_0 \\ -\boldsymbol{l}_0 \end{bmatrix} \tag{3.1}$$

where again $\boldsymbol{H} = \partial_{\boldsymbol{xx}}I$ is the Hessian matrix.

If the constraints are linear, we can write

$$\boldsymbol{l} = \boldsymbol{Lx} - \boldsymbol{c} = \boldsymbol{0}$$

in which case

$$\partial_{\boldsymbol{x}}\boldsymbol{l} = \boldsymbol{L} \quad \text{and} \quad \partial^2_{\boldsymbol{xx}}\boldsymbol{l} = \boldsymbol{0}$$

and the system to be solved is

$$\begin{bmatrix} \boldsymbol{H} & \boldsymbol{L}^T \\ \boldsymbol{L} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -\partial_{\boldsymbol{x}}I_0 - \partial_{\boldsymbol{x}}\boldsymbol{l}^T\,\boldsymbol{\lambda}_0 \\ -\boldsymbol{Lx}_0 + \boldsymbol{c} \end{bmatrix}$$

3.3 FE formulation of the Newton-Raphson method with multi-linear constraints

The constraints $\boldsymbol{l} = 0$ imply

$$< l(x), \phi_q(x) > 0$$

or if we write

We need to solve

$$< f(u), N_p >= 0 \quad \text{subject to} \quad g_q(u) = 0 \quad \text{where} \quad q = 1 \dots N$$

Assume there is a scalar $I(u)$ such that $< f(u), N_p >$ is derivative of I with respect to u_p, then minimising

$$I(u) + < \lambda, g(u) >$$

gives, with $u = N_p u_p$ and $\lambda = N_p \lambda_p$

$$< f(u), N_q > + < \lambda, \partial g/\partial u\, N_q > = 0 \tag{3.2}$$
$$< N_q, g(u) > = 0 \tag{3.3}$$

In general, the $< f(u), N_q >$ system can be expected to be non-linear. Assuming for the moment that the constraints $g(u) = 0$ are linear ($g_q = A_{qr}u_r - b_q = 0$), and that the resulting system is solved using the Newton-Raphson method, gives

$$< f(u_0), N_q > + < \partial f/\partial u\, \Delta u, N_q > + < \lambda, \partial g/\partial u\, N_q > = 0 \tag{3.4}$$
$$< N_q, \partial g/\partial u\, (u_0 + \Delta u) - b > = 0 \tag{3.5}$$

or

$$< f(u_0), N_q > + < \partial f/\partial u\, N_p, N_q > \Delta u_p + < N_p, \partial g/\partial u N_q > \lambda_p = 0 \tag{3.6}$$
$$< N_q, \partial g/\partial u N_p > u_{0p} + < N_q, \partial g/\partial u N_p > \Delta u_p - < N_q, N_p > b_p = 0 \tag{3.7}$$

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{L}^T \\ \boldsymbol{L} & 0 \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{f}(\boldsymbol{u}_0) \\ \boldsymbol{Sb} - \boldsymbol{Lu}_0 \end{bmatrix}$$

where

$$f_q := < f(u_0), N_q >$$

and

$$S_{qp} := < N_q, N_p >$$

and

$$L_{pq} =< N_p, \partial g/\partial u N_q >$$

or if $\boldsymbol{\lambda} = \Delta\boldsymbol{\lambda} + \boldsymbol{\lambda}_0$,

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{L}^T \\ \boldsymbol{L} & 0 \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{u} \\ \Delta\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{f}(\boldsymbol{u}_0) - \boldsymbol{L}^T\boldsymbol{\lambda}_0 \\ \boldsymbol{S}\boldsymbol{b} - \boldsymbol{L}\boldsymbol{u}_0 \end{bmatrix}$$

If the constraints can be written as $Au - b = 0$ then

$$L_{pq} =< N_p, A_{pq} N_q >$$

(no summation implied). If the form functions are delta functions then, $\boldsymbol{S} = \mathbf{1}$ and $\boldsymbol{L} = \boldsymbol{A}$.

3.4 Thickness-positivity constraint

The thickness constraint is

$$h = h_q N_q \geq 0.$$

In the active set method we identify the nodes where $h_i < 0$. The nodes become members of the 'active set' $\mathcal{A}$. Using the Lagrange method, we introduce as many Lagrange parameters as there are elements ($N_{\mathcal{A}}$) in $\mathcal{A}$ and write

$$\lambda = \lambda_p M_p$$

where $p \in \mathcal{A}$. A thickness constrains is

$$h_q = h^{'},$$

where $h^{'}$ is the min allowed ice thickness, usually set to zero.

The Lagrange term is

$$< \lambda, h_q N_q > - < \lambda, h^{'} N_q >$$

(no summation over q), or

$$< M_P \lambda_p, h_q N_q > - < M_p \lambda_p, h^{'}_q N_q >$$

Note that M_p are the Lagrange shape functions. These could be the same as those used for other fields, but in that case the sum only goes over a sub-set of the FE domain's shape functions. In principle one could use different shape functions for the λ variables, for example delta functions, i.e.

$$\lambda = \sum_{p=1}^{N_p} \lambda_p \, \delta(x_p, y_p)$$

where (x_p, y_p) are the coordinates of the nodes in the active set

Taking the derivative with respect to λ and h, gives

$$\sum_{q=1}^{N_h} < M_p, N_q > h_q = 0 \tag{3.8}$$

$$\sum_{p=1}^{N_{\mathcal{A}}} < M_p, N_q > \lambda_p = 0 \tag{3.9}$$

where

$$\begin{bmatrix} \mathrm{H} & \mathrm{L}^T \\ \mathrm{L} & 0 \end{bmatrix} \begin{bmatrix} \Delta h \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} -f(h_0) - \mathrm{L}\lambda_0 \\ -\mathrm{L}h_0 \end{bmatrix}$$

If $M_p = N_p$, i.e. the Lagrange form-functions are the same as used for other fields, then $\mathrm{L} =< N_q, N_p >$ is a subset of the mass matrix. The Lagrange matrix $\mathrm{L} =< N_q, N_p >$ has as many lines as there are thickness constraints, and as many columns as the total number of thickness nodes. It can most easily be formed from the mass matrix by sub-selecting those lines corresponding to constrained thickness nodes.

3.4.1 Thickness barrier

$$I = \gamma_h \lambda_h e^{-(h-h_{\min})/\lambda_h)}$$
$$\partial_h I = -\gamma_h e^{-(h-h_{\min})/\lambda_h}$$
$$\partial^2_{hh} I = \frac{\gamma_h}{\lambda_h} e^{-(h-h_{\min})/\lambda_h}$$

If the problem were self-adjoint then this amounts to adding a term to the prognostic equations, i.e.

$$\partial_t h + \nabla_{xy} \cdot \boldsymbol{q}_h + \partial_h I = a$$

or

$$\partial_t h + \nabla_{xy} \cdot \boldsymbol{q}_{xy} = a - \partial_h I$$

which shows that the method is equivalent to adding a fictitious mass-balance term. For $\gamma_h = 1$, $\lambda = 1$, $h_{\min} = 0$ and $h = 100$ the numerical value is about 10^{-44} and about 10^{-5} for $h = 1$.

If $a = 0$ and $\boldsymbol{v}_h = 0$

$$\partial_t h = -\partial_h I = \gamma_h e^{-(h-h_{\min})/\lambda_h}$$

and γ_h has the units length per time and can be thought of as the fictitious mass balance at $h = h_{\mathrm{hmin}}$.

Solving

$$\partial_t h + \nabla_{xy} \cdot \boldsymbol{q}_h - \gamma_h e^{-(h-h_{\min})/\lambda_h} = a$$

implicitly using NR with respect to h where

$$h_{n+1} = h^i_{n+1} + \Delta h$$

is h at time step $n+1$ and i is the NR iteration number, gives

$$\frac{1}{\Delta t}(\Delta h + h^i_{n+1} - h_n) - \frac{1}{2}\left(\gamma_h e^{-(h^i_{n+1}-h_{\min})/\lambda_h} - \frac{\gamma_h}{\lambda_h} e^{-(h^i_{n+1}-h_{\min})/\lambda_h}\,\Delta h + \gamma_h e^{-(h_n-h_{\min})/\lambda_h}\right) = 0$$

where I have omitted writing the flux and the accumulation terms, i.e.

$$\left(\frac{1}{\Delta t} + \frac{1}{2}\frac{\gamma_h}{\lambda_h} e^{-(h^i_{n+1}-h_{\min})/\lambda_h}\right)\Delta h = -\frac{1}{\Delta t}(h^i_{n+1} - h_n) + \frac{\gamma_h}{2}\left(e^{-(h^i_{n+1}-h_{\min})/\lambda_h} + \gamma_h e^{-(h_n-h_{\min})/\lambda_h}\right)$$

Chapter 4

Solving the non-linear system

4.1 Convergence criteria

The system to be solved is

$$\boldsymbol{R}(\boldsymbol{x}) = \mathbf{0} \tag{4.1}$$

subject to the (linear) constraints

$$\boldsymbol{l} = \boldsymbol{L}\boldsymbol{x} - \boldsymbol{c} = \mathbf{0} \tag{4.2}$$

Introducing Lagrange parameters we solve (3.1) and (4.2) using the Newton-Raphson method

$$\begin{bmatrix} \boldsymbol{H} & \boldsymbol{L}^T \\ \boldsymbol{L} & 0 \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{x} \\ \Delta\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{R}(\boldsymbol{x}_0) - \boldsymbol{L}^T\boldsymbol{\lambda}_0 \\ \boldsymbol{c} - \boldsymbol{L}\boldsymbol{x}_0 \end{bmatrix} \tag{4.3}$$

If the Newton-Raphson method converges then $\Delta\boldsymbol{x} \to 0$ and $\Delta\boldsymbol{\lambda} \to 0$ and one could define an error criteria in terms of the norm of $\Delta\boldsymbol{x}$ and $\Delta\boldsymbol{\lambda}$. However, this is not a good idea. A much better convergence criteria is the norm of the solution error, rather than the update to the solution.

Writing the residual vector $\boldsymbol{R}$ as

$$\boldsymbol{R} = \boldsymbol{T}(\boldsymbol{x}) - \mathbf{F}$$

where $\boldsymbol{T}$ and $\boldsymbol{F}$ are the internal and the external nodal forces, respectively facilitates the definition of a normalised residual error as

$$r = \sqrt{\frac{(\boldsymbol{R} + \boldsymbol{L}^T\boldsymbol{\lambda})^T(\boldsymbol{R} + \boldsymbol{L}^T\boldsymbol{\lambda}) + (\boldsymbol{c} - \boldsymbol{L}\boldsymbol{x})^T(\boldsymbol{c} - \boldsymbol{L}\boldsymbol{x})}{\boldsymbol{F}^T\boldsymbol{F}}} \tag{4.4}$$

At start of an iteration using as start values $\boldsymbol{x} = \mathbf{0}$ and $\boldsymbol{\lambda} = 0$, the internal forces are all zero, $\boldsymbol{T} = \mathbf{0}$, and hence $r = 1$. The iteration is continued until r drops below a prescribed tolerance. If the boundary conditions are linear then $\boldsymbol{L}\boldsymbol{x} - \boldsymbol{c} = 0$ at any iteration step, and the corresponding term in the residual can be omitted. It is always tested internally that this condition is indeed fulfilled at the end of the non-linear iteration procedure. One can also prescribe tolerances on $\Delta\boldsymbol{x}$, but this is not recommended.

4.2 Line search

For increased robustness the NR method is combined with a line-search. We write

$$\begin{aligned} \boldsymbol{x}_{\text{new}} &= \boldsymbol{x}_{\text{old}} + \gamma\Delta\boldsymbol{x} \\ \boldsymbol{\lambda}_{\text{new}} &= \boldsymbol{\lambda}_{\text{old}} + \gamma\Delta\boldsymbol{\lambda}. \end{aligned}$$

A 'full' Newton step corresponds to $\gamma = 1$. If r, as given by Eq. (4.4), is not reduced in the full Newton step, or if the reduction i r is not considered to be sufficiently large, r is minimised as a function of γ using a line-search algorithm. Using (4.4) one finds that

$$\left.\frac{\partial r}{\partial \gamma}\right|_{\gamma=0} = -r(0)$$

Therefore from $r(0)$ not only the error at the start of the step is know but also the variation of the error with respect to the step γ.

Once the system (4.3) has been solved, the normalised residual square error r can be calculated for $\gamma = 1$. If the reduction in r at the full Newton-Raphson step, i.e. $\gamma = 1$ is not judged to be sufficiently large, for example if $r(1)/r(0) > 0.5$, a parabolic fit to $r(\gamma)$ can be constructed given $r(0)$, $r(1)$, and the slope at $\gamma = 0$. (The cost of calculating r is small as it can be done without having to solve (4.3) again.)

Chapter 5

Inverse modelling

We have a forward model

$$F(u(p), p) = 0$$

where p are model parameters and u the state variable. We consider the problem of minimising an objective function J with respect to p. Typically the objective function J can be thought of as a sum of two terms

$$J(u, p) = I(u) + R(p)$$

where I is a misfit term and R a regularisation term.

When inverting for the (distributed) model parameter p we refer to it as as the control variable to distinguish it from any other model parameters.

5.1 Objective functions

The data misfit is the distance between model output and measurements and it could, for example, be measured as

$$I(f) = \frac{1}{2\mathcal{A}}\|f\|^2 = \frac{1}{2A}\iint f(x)\,\gamma(x, x')\,f(x')\;dx\,dx' \tag{5.1}$$

where $\gamma(x, x')$ is the covariance kernel and $\mathcal{A}$ is the domain area. Defined in this manner, I is dimensionless. In the particular case of uncorrelated fields $\gamma(x, x') = c\,\delta(x - x')$

$$I = \frac{1}{2}\int (f(x)/e(x))^2\;dx \tag{5.2}$$

where $e(x) = 1/\sqrt{c}$ are the data errors.

Table 5.1: Notation used in inverse modelling

$F(u(p), p) = 0$	forward model
p	control variables
p_{prior}	a priori estimates of model parameters
u_{meas}	estimates of the state variable (measurements)
J	objective function ($J = I + R$)
R	regularisation term
I	misfit term
u	state variable
K	covariance matrix

If $\tilde{u}$ denotes estimates of u then a typical misfit term might be on the form

$$I = \frac{1}{2\mathcal{A}}\|u-\tilde{u}\|^2 = \frac{1}{2A}\int ((u-\tilde{u})/e_u)^2 \, dA$$

where e_u are measurement errors.

In Bayesian context the regularisation term has the same form as $I(f)$ and is a measure of the distance between the system state and the a prior. In a discrete form the misfit term could, for example, be written as

$$R = (\boldsymbol{p}-\hat{\boldsymbol{p}})^T \boldsymbol{K}^{-1}(\boldsymbol{p}-\hat{\boldsymbol{p}})$$

where $\boldsymbol{K}$ is a covariance matrix, $\boldsymbol{p}$ the model parameters, and $\hat{\boldsymbol{p}}$ the a prior estimates of those model parameters. Apart from often having only a very limited knowledge of the covariance matrix $\boldsymbol{K}$, problems with this formulation can, for example, arise if the inverse of $\boldsymbol{K}$ is not sparse. Practical reasons often influence the form of regularisation term. A pragmatic approach is to select a differential operator having a sparse representation. Preferable such a sparse operator is also the inverse of a 'reasonable' covariance matrix.

As a regularisation term we consider

$$R(f) = \|Lf\|^2$$

where L is a differential operator related to the inverse of the a prior covariance for p. For example using the Helmholtz equation leads to

$$R = \frac{\gamma_1^2}{2\mathcal{A}}\int \left((\nabla(p-\hat{p}))^2 + \gamma_2^2\,(p-\hat{p})^2 \right) \, dA \tag{5.3}$$

where p are the model parameters for which we are inverting for (system state, model parameters) and $\hat{p}$ the prior. The corresponding covariance kernel is the real part of

$$g(r) = \frac{i}{2\pi\gamma_1^2} H_0^{(1)}(i\gamma_2|r|) \tag{5.4}$$

where $H^{(1)}$ is a Hankel function, which is a monotonically decreasing function. For R to be dimensionless

$$\begin{aligned} [\gamma_1^2\gamma_2^2] &= [p]^{-2} \\ [\gamma_1^2] &= [p]^{-2}[l] \end{aligned}$$

where l denotes length, so for example

$$\begin{aligned} [\gamma_1] &= [l]\,[p]^{-1} \\ [\gamma_2] &= [l]^{-1} \end{aligned}$$

5.2 Misfit functions in Úa

Currently the misfit function I has the form

$$I = I_u + I_v$$

or

$$\begin{aligned} I =& \frac{1}{2\mathcal{A}}\int ((u-u_{\mathrm{meas}})/u_{\mathrm{error}})^2 \, dA \\ &+ \frac{1}{2\mathcal{A}}\int ((v-v_{\mathrm{meas}})/v_{\mathrm{error}})^2 \, dA \end{aligned}$$

where u and v are the horizontal velocity components, and

$$\mathcal{A} = \int dA$$

is the total area.

5.2.1 ...not yet, but soon, was there, then disappeared, may come again...

Currently the rate-of-elevation change in not included in the cost function. This may soon be added, in which case

$$I = I_u + I_v + I_{\dot{h}}$$

or

$$\begin{aligned} I =& \frac{1}{2\mathcal{A}} \int ((u - u_{\text{meas}})/u_{\text{error}})^2 \, dA \\ &+ \frac{1}{2\mathcal{A}} \int ((v - v_{\text{meas}})/v_{\text{error}})^2 \, dA \\ &+ \frac{1}{2\mathcal{A}} \int ((\dot{h} - \dot{h}_{\text{meas}})/\dot{h}_{\text{error}})^2 \, dA \end{aligned}$$

where u and v are the horizontal velocity components, and $\dot{h}$ is the rate of thickness change.

The rate of elevation change is calculated as

$$\dot{h} = -(a - \partial_x(uh) - \partial_y(vh))$$

This will be solved in diagnostic mode, and in the adjoint method the gradient of the cost function with respect to the state variable $\mathbf{v}$ acquires an additional term:

$$2I = \|u - \tilde{u}\|^2 + \|\dot{h}_d - (a - \partial_x(uh))\|$$

or

$$I_{\dot{h}} = \frac{1}{2\mathcal{A}} \int (a - \partial_x(uh) - \partial_y(vh) - \dot{h}_{\text{meas}})^2 \, dx \, dy$$

and

$$\delta_u I_{\dot{h}} = \frac{1}{\mathcal{A}} \int (a - \partial_x(uh) - \partial_y(vh) - \dot{h}_{\text{meas}}) \partial_x(h\delta u) \, dx \, dy$$

5.3 Regularisation in *Úa*

In *Úa* the regularisation can be done either using a Bayesian or Tikhonov regularisation.

Using the Bayesian motivated approach the regularisation term has the form

$$R = (\boldsymbol{p} - \hat{\boldsymbol{p}})^T \boldsymbol{K}_{pp}^{-1} (\boldsymbol{p} - \hat{\boldsymbol{p}}) \tag{5.5}$$

where $\boldsymbol{K}$ is the a priori covariance. The Bayesian approach has a clear statistical interpretation, but requires an inversion of the covariance matrix $\boldsymbol{K}$ which can be impractical for large problems.

Alternatively, one can use Tikhonov regularisation where

$$R = \frac{1}{2\mathcal{A}} \int \left(\gamma_s^2 \left(\nabla (p - \hat{p}) \right)^2 + \gamma_a^2 \left(p - \hat{p} \right)^2 \right) \, dA \tag{5.6}$$

with the s and a subscripts being mnemonics for slope and amplitude, respectively.[1]

The units of γ_a and γ_s are

$$[\gamma_a] = \frac{1}{[p]}$$
$$[\gamma_s] = \frac{[l]}{[p]}$$

The inversion can be done directly with respect to the variable p, or with respect to the logarithm of the variable, i.e. $\log_{10} p$. If done with respect to the logarithm of p, the Tikhonov regularisation term has the form

$$\begin{aligned} R &= \frac{1}{2\mathcal{A}} \int \left(\gamma_s^2 \left(\nabla \left(\log_{10}(p) - \log_{10}(\hat{p})\right)\right)^2 + \gamma_a^2 \left(\log_{10}(p) - \log_{10}(\hat{p})\right)^2 \right) dA \\ &= \frac{1}{2\mathcal{A}} \int \left(\gamma_s^2 \left(\nabla \log_{10}(p/\hat{p})\right)^2 + \gamma_a^2 \log_{10}^2(p/\hat{p}) \right) dA \end{aligned} \tag{5.7}$$

Now γ_a is dimensionless and the dimension of γ_s is length, i.e.

$$[\gamma_a] = []$$
$$[\gamma_s] = [l]$$

The Tikhonov is a commonly used and a practical approach. As shown in section (5.5) the Tikhonov approach leads to

$$R = \frac{1}{2}(\boldsymbol{p} - \hat{\boldsymbol{p}})^T \left(\gamma_a^2 \boldsymbol{M} + \gamma_s^2(\boldsymbol{D}_x + \boldsymbol{D}_y)\right) (\boldsymbol{p} - \hat{\boldsymbol{p}}) \tag{5.8}$$

$$= \frac{1}{2}(\boldsymbol{p} - \hat{\boldsymbol{p}})^T \boldsymbol{K}_{pp}^{-1} (\boldsymbol{p} - \hat{\boldsymbol{p}}) \tag{5.9}$$

where

$$\boldsymbol{K}_{pp}^{-1} = \frac{1}{2}\left(\gamma_a^2 \boldsymbol{M} + \gamma_s^2(\boldsymbol{D}_x + \boldsymbol{D}_y)\right) \tag{5.10}$$

and $\boldsymbol{M}$ is the mass matrix and $\boldsymbol{D}_x$ and $\boldsymbol{D}_y$ the stiffness matrices. Hence, with the Tikhonov approach we are calculating directly the inverse of the covariance matrix $\boldsymbol{K}_{pp}$ given by (5.10) and this inverse will be sparse.

5.4 Calculation of the gradient of the objective function with the adjoint method

The adjoint method is a simple trick to speed up the calculation of gradients of the objective function with respect to the control variables.

Assume I want to solve the minimisation problem

$$\min_{p \in P, u \in U} J(u(p))$$

subject to forward model

$$F(u(p), p) = 0.$$

[1] If we think of (5.6) in terms of (5.3) and (5.4) then

$$\gamma_s = \gamma_1$$
$$\gamma_a = \gamma_1 \gamma_2$$

The objective function is

$$J : U \times P \to \mathbb{R}$$

and the forward model, i.e. the state equation

$$F : U \times P \to \mathbb{W}$$

the model control parameter space P (also referred to as control space), the state space U and the image space W are Banach spaces.

We want to determine the sensitivity of cost function J with respect to the (distributed) control parameter p.

We define the Lagrange function

$$\mathcal{L} = U \times V \times W^* \to \mathbb{R}$$

as

$$\mathcal{L}(u(p), p, \lambda) = J(u(p), p) + < \lambda \mid F(u(p), p) >_{W^*,W} \tag{5.11}$$

The adjoint variable λ is in the dual of the image space $\mathbb{W}$.

Equation (5.11) provides no constraints on the adjoint variable λ because the second term is always equal to zero for any value of p. Therefore

$$\mathcal{L}(u(p), p, \lambda) = J(u(p), p)$$

and

$$D\mathcal{L}(p)[\phi] = DJ(p)[\phi] = < \nabla_p \mathcal{L} \mid \phi >$$

Also note that

$$d_p F = \partial_p F + \partial_u F \, d_p u = 0.$$

Introducing

$$j(u) = J(u(p), p) = \mathcal{L}(u(p), p, \lambda)$$

we have

$$j'(p) = (\partial u/\partial p)^* \partial \mathcal{L}(u(p), p, \lambda)/\partial u + \partial \mathcal{L}(u(p), p, \lambda)/\partial p \tag{5.12}$$

Now we chose λ such that

$$\partial \mathcal{L}(u(p), p, \lambda)/\partial u = 0$$

Hence λ must be a solution to

$$\partial J(u(p), p)/\partial u + (\partial F/\partial u)^* \lambda = 0$$

and then the direction derivative $j'(p)$ is

$$j'(p) = \partial J(u(p), p)/\partial p + (\partial F/\partial p)^* \lambda \tag{5.13}$$

Eq. (5.12) can also be written as

$$< j'(p), \phi >_{P^*,P} = < \partial_u \mathcal{L}, \partial_p u \, \phi >_{U^*,U} + < \partial_p \mathcal{L}, \phi >_{P^*,P}$$

and the directional derivative is then

$$< \partial_u \mathcal{L}, \partial_p u \, \phi >_{U^*,U} = 0$$

for all ϕ.

Another approach: Differentiating Eq. (5.11) with respect to the control variable p we obtain

$$\begin{aligned} DJ(p)[\phi] = D\mathcal{L}(p)[\phi] &= < \partial_u J \mid d_p u \, \phi > + < \partial_p J, \phi > + < \lambda \mid \partial_u F \, d_p u \, \phi > + < \lambda \mid \partial_p F \, \phi > + < d_p \lambda \mid F > \\ &= < \partial_u J \mid d_p u \, \phi > + < (\partial_u F)^* \lambda \mid d_p u \, \phi > + < \lambda \mid \partial_p F \, \phi > + < \partial_p J, \phi > \\ &= < \partial_u J + (\partial_u F)^* \lambda \mid d_p u \, \phi > + < \lambda \mid \partial_p F \, \phi > + < \partial_p J, \phi > \end{aligned}$$

We now use the freedom that λ has not been specified and now determine λ by specifying

$$< \partial_u J + (\partial_u F)^* \lambda \mid \phi >= 0$$

and therefore

$$DJ(p)[\phi] =< (\partial_p F)^* \lambda + \partial_p J \mid \phi > \tag{5.14}$$

which is identical to (5.13).

This now gives us three-step method for calculating the directional gradient of object function J with respect to p:

1. Solve the state equation, i.e. the forward problem

   $$< F(u(p), p)|\phi >_{W^*,W}= 0$$

   for the state variable u.

   This, in general, is a non-linear problem that can be solved iterativly using the Newton-Raphson system, i.e.

   $$< \partial_u F \, \Delta u \mid \phi >= - < F(u) \mid \phi >$$

   and can be written in discrete form as

   $$\boldsymbol{K} \boldsymbol{\Delta u} = \boldsymbol{b}$$

   where

   $$[\boldsymbol{K}]_{pq} =< \partial_{u_p} F, \phi_q >$$

   and

   $$[\boldsymbol{b}]_q = - < F(u), \phi_q >$$

2. Solve the adjoint problem for $< \partial_u J + (\partial_u F)^* \lambda \mid \phi >= 0$ for $\lambda \in W^*$, i.e.

   $$< (\partial_u F)^* \lambda \mid \phi >_{U^*,U}= - < \partial_u J \mid \phi >_{U^*,U}$$

   for the adjoint variable λ. If the forward tangential model $(\partial_u F)$ is self adjoint, this involves solving

   $$\boldsymbol{K} \boldsymbol{\lambda} = \boldsymbol{b}$$

3. Calculate the directional derivative of J as

   $$< j'(p), \phi >_{P^*,P}=< (\partial_p F)^* \lambda + \partial_p J \mid \phi >_{P^*,P}$$

   $$DJ(p)[\phi] =< (\partial_p F)^* \lambda + \partial_p J \mid \phi >_{P^*,P}$$

   In discrete form the derivative can be evaluated as

   $$j'(p) = \boldsymbol{P} \boldsymbol{\lambda} + \boldsymbol{Q}$$

   where

   $$[\boldsymbol{P}]_{rs} =< \partial_{p_r} F, \phi_s >$$

   and

   $$[\boldsymbol{Q}]_r =< \partial_p J, \phi_r >$$

   However, $< (\partial_p F)^* \lambda >$ can usually be evaluated directly within the assembly look without the need of ever forming the matrix $\boldsymbol{P}$. Furthermore, the forward model is solved in a weak form and this often involves some manipulations of the $< \lambda, F >$ term in Eq. (5.11).

Usually only the regularisation term is an explicit function of the control variable, i.e.

$$J(F(p, u(p), p) = I(F(p, u(p))) + R(p)$$

and therefore

$$\partial_p J = \partial_p R$$

The adjoint variable λ is the gradient of the objective function with respect to the state variable u

$$\lambda = \nabla_u J$$

In the adjoint method we need to calculate a number of derivatives. These are:

1. The derivative of the forward model with respect to the state variable, i.e $\partial_u F$
2. The derivative of the objective function with respect to the state variable, i.e. $\partial_u J$
3. The derivative of the forward model with respect to the control variable, i.e. $\partial_p F$.

5.5 Evaluating objective functions and their directional derivatives

If ϕ_i are the basis functions then

$$[\boldsymbol{M}]_{pq} =< \phi_p, \phi_q >$$

is the mass matrix (also known as the Gramian matrix), and

$$\begin{aligned}[\boldsymbol{D}_x]_{pq} &=< \nabla_x \phi_p, \nabla_x \phi_q > \\ [\boldsymbol{D}_y]_{pq} &=< \nabla_y \phi_p, \nabla_y \phi_q >\end{aligned}$$

the stiffness matrices.

For

$$I = \frac{1}{2}\|f\|^2 = \frac{1}{2}\int f(x, y)\, f(x, y)\; dxdy \tag{5.15}$$

and

$$f(x, y) = f_i \phi_i(x, y)$$

we find

$$\begin{aligned}I &= \frac{1}{2}\|f\|^2 \\ &= \frac{1}{2} < f, f > \\ &= \frac{1}{2} < f_p \phi_p, f_q \phi_q > \\ &= \frac{1}{2} f_p < \phi_p, \phi_q > f_q \\ &= \frac{1}{2} f_p M_{pq} f_q\end{aligned}$$

or

$$I = \frac{1}{2}\boldsymbol{f}\boldsymbol{M}\boldsymbol{f}$$

where

$$[\boldsymbol{f}]_i = f_i$$

The directional derivative is

$$
\begin{aligned}
DI(f,\phi_q) &= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}I(f+\epsilon\phi_q)\\
&= \frac{1}{2}\frac{d}{d\epsilon} < f+\epsilon\phi_q, f+\epsilon\phi_q >\\
&=< f,\phi_q >\\
&=< f_p\phi_p,\phi_q >\\
&=< \phi_q,\phi_p > f_p\\
&= M_{qp}f_p\\
&= \boldsymbol{M}\boldsymbol{f}
\end{aligned}
$$

or

$$DI(f,\phi_q) = [\boldsymbol{M}\boldsymbol{f}]_q$$

The p component of the directional derivative represents the (linear) rate-of-change in I as the value of f is perturbed by $\epsilon\phi_p$.

We can also write

$$DI(f,\phi_q) =< f_p\phi_p,\phi_q >$$

and therefore by the definition of a gradient as

$$\frac{d}{d\epsilon}J(f+\epsilon\delta f)|_{\epsilon=0} =< \mathrm{grad}J(f),\delta f >$$

the gradient of I in Eq. (5.15) is f (as it of course should be).

Similarly if

$$I = \frac{1}{2} < \nabla f,\nabla f >= \frac{1}{2}(< \partial_x f,\partial_x f > + < \partial_y f,\partial_y f >)$$

then

$$
\begin{aligned}
I_x &= \frac{1}{2} < \partial_x f,\partial_x f >\\
&= \frac{1}{2} < f_p\partial_x\phi_p, f_q\partial_x f_q >\\
&= \frac{1}{2}f_p < \partial_x\phi_p,\partial_x\phi_q > f_q\\
&= \frac{1}{2}\boldsymbol{f}\boldsymbol{D}_x\boldsymbol{f}
\end{aligned}
$$

and

$$
\begin{aligned}
DI_x(f,\phi_p) &= \frac{1}{2}\lim_{\epsilon\to 0}\frac{d}{d\epsilon} < \partial_x(f+\epsilon\phi_p),\partial_x(f+\epsilon\phi_p) >\\
&=< \partial_x f,\partial_x\phi_p >\\
&=< f_q\partial_x\phi_q,\partial_x\phi_p >\\
&=< \partial_x\phi_p,\partial_x\phi_q > f_q\\
&= \boldsymbol{D}_x\boldsymbol{f}
\end{aligned}
$$

or

$$DI_x(f,\phi_p) = \boldsymbol{D}_x\boldsymbol{f}$$

Summarising, if we have a regularisation term on the form

$$R = \frac{1}{2}\|f\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla f\|^2_{L^2(\Omega)}$$

it can be evaluated knowing the mass and the stiffness matrices as

$$\begin{aligned} R &= \frac{1}{2}\|f\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla f\|^2_{L^2(\Omega)} \\ &= \frac{1}{2}\boldsymbol{f}\boldsymbol{M}\boldsymbol{f} + \frac{1}{2}\boldsymbol{f}(\boldsymbol{D}_x + \boldsymbol{D}_y)\boldsymbol{f} \end{aligned}$$

and direction derivative is

$$\frac{d}{d\epsilon} I(f + \epsilon\phi_p) = [\boldsymbol{M}\boldsymbol{f} + (\boldsymbol{D}_x + \boldsymbol{D}_y)\boldsymbol{f}]_p$$

And the regularisation term (5.3) can be evaluated similarly as

$$\begin{aligned} R &= \frac{\gamma_1^2}{2\mathcal{A}} \int \left((\nabla(p-\hat{p}))^2 + \gamma_2^2\,(p-\hat{p})^2 \right) \, dA \\ &= \frac{\gamma_1^2}{2\mathcal{A}} \left((\boldsymbol{p}-\hat{\boldsymbol{p}})^T \boldsymbol{D}\,(\boldsymbol{p}-\hat{\boldsymbol{p}}) + \gamma_2^2 (\boldsymbol{p}-\hat{\boldsymbol{p}})^T \boldsymbol{M} (\boldsymbol{p}-\hat{\boldsymbol{p}}) \right) \\ &= \frac{\gamma_1^2}{2\mathcal{A}} \left((\boldsymbol{p}-\hat{\boldsymbol{p}})^T (\boldsymbol{D} + \gamma_2^2 \boldsymbol{M})(\boldsymbol{p}-\hat{\boldsymbol{p}}) \right) \end{aligned}$$

and the directional derivative is

$$d_{\boldsymbol{p}} R = \frac{\gamma_1^2}{\mathcal{A}} (\boldsymbol{D} + \gamma_2^2 \boldsymbol{M})(\boldsymbol{p}-\hat{\boldsymbol{p}})$$

5.6 Gradients of objective functions with respect to model parameters

5.6.1 Gradient calculation in 1HD with respect to C

As an example we consider the calculation of the gradient of the objective function J with respect to slipperiness.

$$\boldsymbol{t}_b = \mathcal{H}(h - h_f)\; C^{-1/m} |\boldsymbol{v}_b|^{1/m-1}\, \boldsymbol{v}_b$$

We need to evaluate

$$\begin{aligned} DJ(C)[\phi] &=< (\partial_C F)^* \lambda + \partial_C J, \phi > \\ &=< (\partial_C \boldsymbol{t}_b)^* \lambda + \partial_C J, \phi > \end{aligned}$$

giving

$$DJ(C)[\phi] =< \frac{1}{m} \mathcal{H}(h - h_f) C^{-1/m-1} |\boldsymbol{v}_b|^{1/m-1}\, \boldsymbol{v}_b \lambda, \phi > + < \partial_C J, \phi >$$

In the above listed expression one needs to form a sum between $\boldsymbol{v}_b$ and λ for each value of C. The adjoint variable λ is a solution of the adjoint equation and can be considered as a vector variables with x and y components similarly to $\boldsymbol{v}$.

5.6.2 Gradient calculation in 1HD with respect to A

The directional derivative can be calculated (see Eq. 5.13) as

$$j'(A) = \partial J(u(p), p)/\partial A + (\partial F/\partial A)^* \lambda \tag{5.16}$$

Focusing on the second term

$$\begin{aligned} (\partial_A F)^* \lambda &=< \partial_A \left(2\partial_x (A^{-1/n}\, h\, |\partial_x u|^{(1-n)/n} \partial_x u) - t_{bx} - \frac{1}{2} g \partial_x (\rho h^2 - \rho_o d^2) + g\, \mathcal{H}(h - h_f)(\rho h - \rho_o H_+) \partial_x B \right), \lambda > \\ &= - < 2\partial_A \left(A^{-1/n}\, h\, |\partial_x u|^{(1-n)/n} \partial_x u \right), \partial_x \lambda > \\ &= - < \frac{2}{n} A^{-1/n-1}\, h\, |\partial_x u|^{(1-n)/n} \partial_x u\, \delta A, \partial_x \lambda > \end{aligned}$$

where we have omitted writing the boundary term assuming that λ is set to zero along the boundary (or periodic boundary conditions applied for periodic domains.)

If both A and λ are expanded in the same way. i.e.

$$A = A_P \, \phi_p(x, y)$$
$$\lambda = \lambda_q \, \phi_q(x, y)$$

then

$$\begin{aligned}(\partial FpA)^* \lambda &=< \frac{2}{n} A^{-1/n-1} \, h \, |\partial_x u|^{(1-n)/n} \partial_x u \, \delta A, \partial_x \lambda > \\ &=< \frac{2}{n} A^{-1/n-1} \, h \, |\partial_x u|^{(1-n)/n} \partial_x u \, \phi_p, \lambda_q \partial_x \phi_q > \\ &=< \frac{2}{n} A^{-1/n-1} \, h \, |\partial_x u|^{(1-n)/n} \partial_x u \, \phi_p, \partial_x \phi_q > \lambda_q\end{aligned}$$

or

$$(\partial F / \partial A)^* \lambda = \boldsymbol{K \lambda}$$

where

$$K = (\partial F / \partial A)^*$$

is

$$K_{pq} =< \frac{2}{n} A^{-1/n-1} \, h \, |\partial_x u|^{(1-n)/n} \partial_x u \, \phi_p, \lambda_q \partial_x \phi_q > \lambda_q$$

however it is more efficient to calculate this matrix-vector product directly without ever forming the matrix as

$$\boldsymbol{K \lambda} =< \frac{2}{n} A^{-1/n-1} \, h \, |\partial_x u|^{(1-n)/n} \partial_x u \, \phi_p, \partial_x \lambda >$$

5.7 Inverting for $\log p$

To avoid issues with a strictly positive parameter p becoming negative in the course of the inversion we can make a change of variables writing

$$p = 10^\gamma = e^{\gamma \ln(10)}$$

or

$$\log_{10} p = \gamma$$

We have

$$\begin{aligned}\frac{\partial J}{\partial \gamma} &= \frac{\partial J}{\partial p} \frac{\partial C}{\partial \gamma} \\ &= \frac{\partial J}{\partial p} \ln(10) \, 10^\gamma \\ &= \ln(10) \, p \frac{\partial J}{\partial p}\end{aligned}$$

showing that the change of variables causes a rescaling of the gradient,

5.8 The form of the adjoint equations for Bayesian approach using Gaussian statistics

We anticipate using a Bayesian approach assuming Gaussian statistics and therefore that the cost function might be on the form

$$\begin{aligned}\min_p I(u(p)) &=< u - \hat{u} \mid K_u^{-1} \mid u - \hat{u} > + < p - \hat{p} \mid K_P^{-1} \mid p - \hat{p} > \\ &=< K_u^{-T/2}(u - \hat{u}) \mid K_u^{-1/2}(u - \hat{u}) > + < K_p^{-T/2}(p - \hat{p}) \mid K_p^{-1/2}(p - \hat{p}) >\end{aligned}$$

where K is a covariance matrix (and therefore positive definite).

Therefore

$$\begin{aligned}d_p I = d_p \mathcal{L} &=< K_u^{-1/2}(u - \hat{u}) \mid d_p u > + < \lambda \mid \partial_u F d_p u + \partial_p F > + < \partial_p \lambda \mid F > \\ &=< K_u^{-1/2}(u - \hat{u}) \mid d_p u > + < \lambda \mid \partial_u F d_p u + \partial_p F > \\ &=< K_u^{-1/2}(u - \hat{u}) + \lambda(\partial_u r)^* \mid d_p u > + < \lambda \mid \partial_p F >\end{aligned}$$

where we have omitted the $< p - \hat{p} \mid K_P^{-1} \mid p - \hat{p} >$ term for the time being. We now use the flexibility of λ not having been specified and require that

$$< K_u^{-1/2}(u - \hat{u}) + \lambda(\partial_u r)^* \mid \delta p >= 0$$

and therefore

$$d_p I =< \lambda \mid \partial_p F >$$

For a cost function on the form

$$\min_p I(u(p)) =< u - \hat{u} \mid K_u^{-1} \mid u - \hat{u} > + < p - \hat{p} \mid K_P^{-1} \mid p - \hat{p} >$$

we would arrive at

$$d_p I =< \lambda \mid \partial_p F > + < K_p^{-1/2}(p - \hat{p}) \mid \delta p >$$

One might as why we don't just calculate the cost gradient as

$$d_p =< K_u^{-1/2}(u - \hat{u}) \mid d_p u >$$

But this would require calculating $d_p u$ which is a pain in the neck.

5.9 Adjoint equations (Bayesian case with constraints on vertical velocity)

We want to minimise a cost function $\tilde{J}$ on the form

$$\tilde{J}(u, v, w, p) = I(u, v, w) + F(p)$$

where I is a data discrepancy functional, and R a regularisation term

As a misfit function we use

$$I = I_u + I_v + I_o,$$

where each term has the form

$$I_u =< C_{uu}^{-1/2} u - \hat{(u)}, C_{uu}^{-1/2} u - \hat{(u)} >$$

with $C_{uu}^{-1/2}$ being an error covariance matrix.

The regularisation term has the form

$$R =< C_{pp}^{-1/2}p, C_{pp}^{-1/2}p >$$

We minimise $\tilde{J}$ subject to the conditions

$$F(u(p), v(p), p) = 0$$

and

$$w_s = f(u, v, h, b)$$

where r are the diagnostic equations, f is a function giving the vertical surface velocity w_o as a function of the variables of the diagnostic equations, and where p stands for some control variable (distributed model parameter) such as the basal slipperiness C or the rate factor A. We therefore consider the extended cost function

$$J(u, v, w, \lambda, \mu, p) = I(u, v, w) + F(p)+ < \lambda \mid F(u(p), v(p), p) > + < \mu \mid w - f(u, v, h, b) >$$

where λ and μ are Lagrange multipliers.

The directional derivative of J with respect to λ in the direction of $\delta\lambda$ is defined as

$$\frac{d}{d\epsilon} J(\lambda + \epsilon\, \delta\lambda) \mid_{\epsilon=0}$$

and is denoted by $\delta J(\lambda, \delta\lambda)$

The directional derivatives of J are

$$\delta J(\lambda, \delta\lambda) = \delta_\lambda J =< \delta\lambda, F(u, v, p) >$$
$$J(\mu, \delta\mu) =< \delta\mu, w - f(u, v, h, b) >$$
$$J(u, \delta u) =< C_{uu}^{-1/2}(u - \hat{u}, C^{-1/2}\delta u > + < \lambda, \nabla_u r\, \delta u > - < \mu, \nabla_u f\, \delta u >$$

5.10 Prognostic equations are formally self-adjoint

Define the inner product

$$r =< f_x, \lambda > + < f_y, \mu >$$

$$\begin{aligned} f_x =&\partial_x(h\eta(4\partial_x u + 2\partial_y v)) + \partial_y(h\eta(\partial_y u + \partial_x v)) - \mathcal{H}(h - h_f)t_{bx} \\ &- \frac{1}{2}g\partial_x(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h - h_f)(\rho h - \rho_o H_+)\partial_x B \end{aligned} \tag{5.17}$$

$$\begin{aligned} f_y =&\partial_y(h\eta(4\partial_y v + 2\partial_x u)) + \partial_x(h\eta(\partial_x v + \partial_y u) - \mathcal{H}(h - h_f)t_{by} \\ &- \frac{1}{2}g\partial_y(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h - h_f)(\rho h - \rho_o H_+)\partial_y B \end{aligned} \tag{5.18}$$

or

$$\begin{aligned} r =& \iint \big\{(\partial_x(h\eta(4\partial_x u + 2\partial_y v)) + \partial_y(h\eta(\partial_y u + \partial_x v))) - \mathcal{H}(h - h_f)t_{bx} \\ &- \frac{1}{2}g\partial_x(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h - h_f)(\rho h - \rho_o H_+)\partial_x B\big\}\,\lambda\, dx\, dy \\ &+ \iint \big\{(\partial_y(h\eta(4\partial_y v + 2\partial_x u)) + \partial_x(h\eta(\partial_x v + \partial_y u)) - \mathcal{H}(h - h_f)t_{by} \\ &- \frac{1}{2}g\partial_y(\rho h^2 - \rho_o d^2) + g\,\mathcal{H}(h - h_f)(\rho h - \rho_o H_+)\partial_y B\big\}\mu\, dx\, dy \end{aligned}$$

The use of Green's theorem gives

$$\begin{aligned}
r = &- \iint_\Omega \big\{ h\eta(4\partial_x u + 2\partial_y v)\partial_x\lambda + h\eta(\partial_y u + \partial_x v)\partial_y\lambda + \mathcal{H}(h-h_f)\beta^2 u\,\lambda \\
&- \frac{1}{2}g(\rho h^2 - \rho_o d^2)\partial_x\lambda + \lambda\, g\,\mathcal{H}(h-h_f)(\rho h - \rho_o H_+)\partial_x B \big\}\, dx\, dy \\
&+ \oint_\Gamma (h\eta(4\partial_x u + 2\partial_y v)\,\lambda\, n_x + h\eta(\partial_y u + \partial_x v)\,\lambda\, n_y - \frac{1}{2}g(\rho h^2 - \rho_o d^2)\,\lambda n_x)\; d\Gamma \\
&- \iint_\Omega \big\{ h\eta(4\partial_y v + 2\partial_x u)\partial_y\mu + h\eta(\partial_x v + \partial_y u)\partial_x\mu + \mathcal{H}(h-h_f)\beta^2 v \\
&- \frac{1}{2}g(\rho h^2 - \rho_o d^2)\partial_y\mu + \mu\, g\,\mathcal{H}(h-h_f)(\rho h - \rho_o H_+)\partial_x B \big\}\, dx\, dy \\
&+ \oint_\Gamma (h\eta(4\partial_y v + 2\partial_x u)\,\mu\, n_y + h\eta(\partial_x v + \partial_y u)\,\mu\, n_x - \frac{1}{2}g(\rho h^2 - \rho_o d^2)\,\mu n_y)\; d\Gamma
\end{aligned}$$

and a second use of Green's theorem gives after some rearrangements

$$\begin{aligned}
r = &\iint_\Omega \big\{ \partial_x(h\eta(4\partial_x\lambda + 2\partial_y\mu))\,u + \partial_y(h\eta(\partial_y\lambda + \partial_x\mu))\,u - \mathcal{H}(h-h_f)\beta^2 u\,\lambda \\
&+ \frac{1}{2}g(\rho h^2 - \rho_o d^2)\partial_x\lambda - \lambda\, g\,\mathcal{H}(h-h_f)(\rho h - \rho_o H_+)\partial_x B \big\}\, dx\, dy \\
&+ \oint_\Gamma (h\eta(4\partial_x u + 2\partial_y v)\,\lambda\, n_x + h\eta(\partial_y u + \partial_x v)\,\lambda\, n_y - \frac{1}{2}g(\rho h^2 - \rho_o d^2)\,\lambda n_x)\; d\Gamma \\
&+ \iint_\Omega \big\{ \partial_y(h\eta(4\partial_x\mu + 2\partial_y\lambda))\,v + \partial_x(h\eta(\partial_x\mu + \partial_y\lambda))\,v - \mathcal{H}(h-h_f)\beta^2 u\,\lambda \\
&+ \frac{1}{2}cdg(\rho h^2 - \rho_o d^2)\partial_y\mu - \mu\, g\,\mathcal{H}(h-h_f)(\rho h - \rho_o H_+)\partial_x B \big\}\, dx\, dy \\
&+ \oint_\Gamma (h\eta(4\partial_y v + 2\partial_x u)\,\mu\, n_y + h\eta(\partial_x v + \partial_y u)\,\mu\, n_x - \frac{1}{2}g(\rho h^2 - \rho_o d^2)\,\mu n_y)\; d\Gamma \\
&- \oint_\Omega (uh\eta(4\partial_x\lambda + 2\partial_y\mu)n_x + uh\eta(\partial_y\lambda + \partial_x\mu)n_y)\, d\Gamma \\
&- \oint_\Omega (vh\eta(4\partial_y\mu + 2\partial_y\lambda)n_y + vh\eta(\partial_x\mu + \partial_y\lambda)n_x)\, d\Gamma
\end{aligned}$$

The adjoint approach is based on the use of the adjoint of the linearised/tangent forward model around the converged solution of non-linear forward model. The non-linear forward model is

$$F(u) = \mathbf{0}$$

and the tangent model is the directional derivative of the forward model in the direction $\delta\boldsymbol{u}$.

$$K = D\boldsymbol{F}(\boldsymbol{u})[\delta\boldsymbol{u}]$$

Often this is simply be written as

$$K = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{u}}$$

Define

$$< \boldsymbol{L}\,\delta U, \Lambda > = < U, \boldsymbol{L}\,\Lambda >$$

where $U = (\delta u, \delta v)^T$ and $\boldsymbol{L}$ is the operator acting on U as given by the system (1.39) and (1.40).

$$
\begin{aligned}
< \boldsymbol{L}\,\delta u, \Lambda >= &\iint_\Omega \big\{(\partial_x(h\eta(4\partial_x\delta u + 2\partial_y\delta v)) + \partial_y(h\eta(\partial_y\delta u + \partial_x\delta v)))\,\lambda \\
&\quad +(\partial_y(h\eta(4\partial_y\delta v + 2\partial_x\delta u)) + \partial_x(h\eta(\partial_x\delta v + \partial_y\delta u)))\,\mu\big\}\,dx\,dy \\
= &-\iint_\Omega \big\{h\eta(4\partial_x\delta u + 2\partial_y\delta v)\partial_x\lambda + h\eta(\partial_y\delta u + \partial_x\delta v)\partial_y\lambda \\
&\quad h\eta(4\partial_y\delta v + 2\partial_x\delta u)\partial_y\mu + h\eta(\partial_x\delta v + \partial_y\delta u)\partial_x\mu\big\}\,dx\,dy \\
&+\oint_\Gamma \big\{h\eta(4\partial_x\delta u + 2\partial_y\delta v)\,\lambda\,n_x + h\eta(\partial_y\delta u + \partial_x\delta v)\,\lambda\,n_y) \\
&\quad +h\eta(4\partial_y\delta v + 2\partial_x\delta u)\,\mu\,n_y + h\eta(\partial_x\delta v + \partial_y\delta u)\,\mu\,n_x\big\}\,d\Gamma \\
= &\iint_\Omega \big\{(\partial_x(h\eta(4\partial_x\lambda + 2\partial_y\mu)) + \partial_y(h\eta(\partial_y\lambda + \partial_x\mu)))\,\delta u \\
&\quad (\partial_y(h\eta(4\partial_x\mu + 2\partial_y\lambda)) + \partial_x(h\eta(\partial_x\mu + \partial_y\lambda)))\,\delta v\big\}\,dx\,dy \\
&+\oint_\Gamma \big\{(h\eta(4\partial_x\delta u + 2\partial_y\delta v)\,n_x + h\eta(\partial_y\delta u + \partial_x\delta v)\,n_y)\lambda \\
&\quad +(h\eta(4\partial_y\delta v + 2\partial_x\delta u)\,n_y + h\eta(\partial_x\delta v + \partial_y\delta u)\,n_x)\mu\big\}\,d\Gamma \\
&-\oint_\Gamma (h\eta(4\partial_x\lambda + 2\partial_y\mu)n_x + h\eta(\partial_y\lambda + \partial_x\mu)n_y)\delta u\,d\Gamma \\
&-\oint_\Gamma (h\eta(4\partial_y\mu + 2\partial_y\lambda)n_y + h\eta(\partial_x\mu + \partial_y\lambda)n_x)\delta v\,d\Gamma
\end{aligned}
$$

Once the forward model has been solved the velocity field fulfills given the BCs to a high degree of accuracy. The boundary conditions on the δ fields follow from above

5.11 Covariance kernels

$$F(f) = \int\int f(x)\,\kappa(x,x')\,f(x')\;dx\;dx'$$

where κ is the covariance kernel.

Multipole expansion. Isotropic, stationary, and translation invariance

$$\kappa(x,x') = \kappa(|x-x'|)$$

$$e^{-|x-x_i|^2/4T} = \sum_{n_1,n_2=0}^{\infty} \Theta_{n1n_2}(x-c)...$$

Chapter 6

Further technical implementation details

6.1 Only the (fully) floating condition as a natural boundary condition

Here I am ignoring possible gradients in density and the treatment of the boundary term only includes the fully floated case as a natural condition.

Note that

$$\begin{aligned}\rho g h \, \partial_x s &= \rho g h \, \partial_x s + \frac{1}{2} \varrho g \, \partial_x h^2 - \rho g h (1 - \rho/\rho_o) \, \partial_x h \\ &= \frac{1}{2} \varrho g \, \partial_x h^2 + \rho g h \, \partial_x (s - S - (1 - \rho/\rho_o) \, h)\end{aligned}$$

hence

$$\rho g h \, \partial_x s = \frac{1}{2} \varrho g \, \partial_x h^2 + \rho g h \, \partial_x s^{'} \tag{6.1}$$

with

$$s^{'} := s - S - (1 - \rho/\rho_o) h$$

and

$$\varrho = \rho(1 - \rho/\rho_o),$$

The field equations can therefore also be written as

$$\begin{aligned}\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) + \partial_y(h\eta(\partial_x v + \partial_y u)) - \beta^2 u &= \rho g h(\partial_x s^{'} \cos\alpha - \sin\alpha) + \frac{1}{2} \varrho g \cos\alpha \, \partial_x h^2, \\ \partial_y(4h\eta\partial_y v + 2h\eta\partial_x u) + \partial_x(h\eta(\partial_y u + \partial_x v)) - \beta^2 v &= \rho g h \partial_y s^{'} \cos\alpha + \frac{1}{2} \varrho g \cos\alpha \, \partial_y h^2,\end{aligned}$$

6.1.1 Remark

To see that the right-hands sides of (1.36) and (6.1) i.e.

$$\begin{aligned}\rho g h \, \partial_x s &= \frac{1}{2} \varrho g \, \partial_x h^2 + \rho g h \, \partial_x s^{'} \\ &= \frac{1}{2} g \partial_x (\rho h^2 - \rho_o d^2) + g(\rho h - \rho_o d) \, \partial_x b\end{aligned}$$

are equal (ignoring spatial gradients in density) we consider the three cases:

1. Fully floating: In that case $s^{'} = 0$ and $\rho h = \rho_o d$ and both sides are equal.

2. Fully grounded: We have $d = 0$

$$\begin{aligned}\frac{1}{2}g\partial_x(\rho h^2 - \rho_o d^2) + g(\rho h - \rho_o d)\,\partial_x b &= \frac{1}{2}g\rho\partial_x h^2 + g\rho h\,\partial_x b \\ &= \rho g h \partial_x s\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\varrho g\,\partial_x h^2 + \rho g h\,\partial_x s' &= \frac{1}{2}\varrho g\,\partial_x h^2 + \rho g h\,\partial_x(s - S - (1-\rho/\rho_o)h) \\ &= \rho g h \partial_x s\end{aligned}$$

3. Partly floating:

$$\begin{aligned}\frac{1}{2}g\partial_x(\rho h^2 - \rho_o d^2) + g(\rho h - \rho_o d)\,\partial_x b &= \rho g h\,\partial_x h - \rho_o g d\,\partial_x d + g(\rho h - \rho_o d)\partial_x b \\ &= \rho g h\,\partial_x s - \rho_o g d\,\partial_x d - \rho_o g d\,\partial_x b \\ &= \rho g h\,\partial_x s - \rho_o g d(\partial_x d + \partial_x b) \\ &= \rho g h\,\partial_x s - \rho_o g d(\partial_x S - \partial_x b + \partial_x b) \\ &= \rho g h\,\partial_x s\end{aligned}$$

6.1.2 FE formulation

x direction

$$\int_\Omega (\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) - \frac{1}{2}\varrho g\cos\alpha\,\partial_x h^2 + \partial_y(h\eta(\partial_x v + \partial_y u)) - t_{bx} - \rho g h(\partial_x s'\cos\alpha - \sin\alpha))N\,dx\,dy = 0$$

with Weertman type Neumann BC on Γ_2

$$(4h\eta\partial_x u + 2h\eta\partial_y v)n_x + \eta h(\partial_x v + \partial_y u)n_y = \frac{1}{2}\rho(1-\rho/\rho_o)gh^2 n_x$$

Green's theorem used to get rid of second derivatives gives

$$\begin{aligned}&-\int_\Omega ((4h\eta\partial_x u + 2h\eta\partial_y v)\partial_x N - \frac{1}{2}\varrho g\cos\alpha\,h^2\partial_x N + h\eta(\partial_x v + \partial_y u)\partial_y N)\,dx\,dy \\ &\quad - \int_\Omega (t_{bx} + \rho g h(\partial_x s'\cos\alpha - \sin\alpha)N\,dx\,dy \\ &\quad + \int_\Gamma ((4h\eta\partial_x u + 2h\eta\partial_y v - \frac{1}{2}\varrho g\cos\alpha h^2)n_x + h\eta(\partial_x v + \partial_y u)n_y)N\,d\Gamma = 0\end{aligned}$$

If the von Neumann boundary condition is of Weertman type, the boundary integral along Γ_2 is equal to zero (for $\alpha = 0$), and zero on the remaining part of the boundary if we set the weight functions to zero and determine the values of the unknowns using Dirichlet boundary conditions.

We are left with

$$\begin{aligned}&-\int_\Omega (h\eta(4\partial_x u + 2\partial_y v)\partial_x N + h\eta(\partial_x v + \partial_y u)\partial_y N)\,dx\,dy - \int_\Omega t_{bx}N\,dx\,dy \\ &\quad = \rho g\int_\Omega h((\partial_x s - (1-\rho/\rho_o)\partial_x h)\cos\alpha - \sin\alpha)N\,dx\,dy - \frac{1}{2}\varrho g\cos\alpha\int_\Omega h^2\,\partial_x N\,dx\,dy\end{aligned}$$

y **direction**

$$\int_{\Omega} \partial_y((4h\eta\partial_y v + 2h\eta\partial_x u) - \frac{1}{2}\varrho g \cos\alpha\, \partial_y h^2 + \partial_x(h\eta(\partial_y u + \partial_x v)) - t_{by} - \rho g h \partial_y s^{'} \cos\alpha) N \, dx\, dy$$

with Weertman type boundary condition

$$\eta h(\partial_x v + \partial_y u) n_x + (4\eta h \partial_y v + 2\eta h \partial_x u) n_y = \frac{1}{2}\rho(1 - \rho/\rho_o) g h^2 n_y$$

We have

$$\begin{aligned} &- \int_{\Omega} ((4h\eta\partial_y v + 2h\eta\partial_x u)\partial_y N - \frac{1}{2}\varrho g \cos\alpha\, h^2 \partial_y N + h\eta(\partial_y u + \partial_x v)\partial_x N) \, dx\, dy \\ &\quad - \int_{\Omega} (t_{by} + \rho g h \partial_y s^{'} \cos\alpha) N \, dx\, dy \\ &\quad + \int_{\Gamma} ((4h\eta\partial_y v + 2h\eta\partial_x u - \frac{1}{2}\varrho g \cos\alpha\, h^2) n_y + \eta h(\partial_y u + \partial_x v) n_x) N \, d\Gamma = 0 \end{aligned} \tag{6.2}$$

Again we can ignore the boundary integral as it is identically equal to zero.

$$\begin{aligned} &- \int_{\Omega} ((4h\eta\partial_y v + 2h\eta\partial_x u)\partial_y N + h\eta(\partial_y u + \partial_x v)\partial_x N) \, dx\, dy - \int_{\Omega} t_{by} N \, dx\, dy \\ &= \rho g \cos\alpha \int_{\Omega} h(\partial_y s - (1 - \rho/\rho_o)\partial_y h) N \, dx\, dy - \frac{1}{2}\varrho g \cos\alpha \int_{\Omega} h^2 \partial_y N \, dx\, dy \end{aligned} \tag{6.3}$$

6.1.3 2HD FE diagnostic equation written in terms of h (suitable for fully coupled approach)

Where the ice is afloat, $s - S = (1 - \rho/\rho_o)\, h$ and $s^{'} = 0$, hence

$$s^{'} = \begin{cases} s - S - (1 - \rho/\rho_o)\, h, & \text{if}\, h > h_f \\ 0, & \text{if}\, h \le h_f \end{cases}$$

i.e.

$$s^{'} := \mathcal{H}(h - h_f)(s - S - (1 - \rho/\rho_o) h)$$

where

$$h_f := (S - B)\rho_o/\rho$$

We can also write $s^{'}$ as

$$\begin{aligned} s^{'} &= \mathcal{H}(h - h_f)(s - S - (1 - \rho/\rho_o) h) \\ &= \mathcal{H}(h - h_f)(h\rho/\rho_o + s - S - h) \\ &= \mathcal{H}(h - h_f)(h\rho/\rho_o + h + b - S - h) \\ &= \mathcal{H}(h - h_f)(h\rho/\rho_o + h + B - S - h) \\ &= \mathcal{H}(h - h_f)(h\rho/\rho_o + B - S) \\ &= \mathcal{H}(h - h_f)(h\rho/\rho_o - H) \end{aligned}$$

i.e.

$$s^{'}(x) = \mathcal{H}(h - h_f)(\rho/\rho_o h - H) \tag{6.4}$$

where we used the fact that $b = B$ whenever $\mathcal{H}(h - h_f) = 1$. This expression for $s^{'}$ is needed for linearisation around h.

The field equations can therefore be written as before, i.e. as

$$\begin{aligned}
\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) + \partial_y(h\eta(\partial_x v + \partial_y u)) - \beta^2 u &= \rho g h(\partial_x s^{'} \cos\alpha - \sin\alpha) + \frac{1}{2}\varrho g \cos\alpha\, \partial_x h^2, \\
\partial_y(4h\eta\partial_y v + 2h\eta\partial_x u) + \partial_x(h\eta(\partial_y u + \partial_x v)) - \beta^2 v &= \rho g h \partial_y s^{'} \cos\alpha + \frac{1}{2}\varrho g \cos\alpha\, \partial_y h^2,
\end{aligned}$$

but the linearisation with respect to h needed in a fully coupled approach requires (6.4).

The FE formulation for the prognostic equation is the θ method, i.e.

$$R_p^h = \int_\Omega \{ \frac{1}{\Delta t}(h_1 - h_0) + \theta\partial_x(u_1 h_1) + (1-\theta)\partial_x(u_0 h_0) + \theta\partial_y(v_1 h_1) + (1-\theta)\partial_y(u_0 h_0) - a\} N_p \, dx\, dy = 0 \tag{6.5}$$

where $0 \leq \theta \leq 1$.

$$a := a_s + a_b$$

6.2 Element integrals

$$x = x_p N_P(\xi, \eta)$$

For 3-node element:
Nodal u displacement vector of a particular element

$$u_e := \begin{pmatrix} u1 \\ u2 \\ u3 \end{pmatrix}$$

$$\text{fun} := \begin{pmatrix} N_1(\xi,\eta) \\ N_2(\xi,\eta) \\ N_3(\xi,\eta) \end{pmatrix}$$

$$u(x,y) = u_e^T \text{ fun}$$

$$\text{der} = \begin{pmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} \end{pmatrix}$$

$$\text{coo} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_4 \end{pmatrix}$$

$$\boldsymbol{J} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

$$\boldsymbol{J}^{-1} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial y} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} = \boldsymbol{J} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \boldsymbol{J}^{-1} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix}$$

$$\boldsymbol{J} = \begin{pmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_4 \end{pmatrix}$$

$$\boldsymbol{J} = \text{der coo}$$

$$\boldsymbol{D} = \begin{pmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \end{pmatrix} = \boldsymbol{J}^{-1} \begin{pmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} \end{pmatrix}$$

$$\boldsymbol{D} = \boldsymbol{J}^{-1} \text{ der}$$

Change of integral, example:

$$\int_{A_e} u_p N_p(x,y) N_q(x,y)\, dx\, dy = \int_{\Delta} u_p N_p(\xi,\eta) N_q(\xi,\eta) \text{ det}\, \boldsymbol{J}\, d\xi\, d\eta$$

Example:

$$\int \partial_x u\, \partial_x N\, dx\, dy = (\int D_{1p} D_{1q}\, |\boldsymbol{J}|\, d\eta\, d\xi)\, u_p$$

6.3 Edge integrals

We have integrals on the form

$$\int_\Gamma u(x,y) N(x,y)\, n_x\, d\Gamma$$

with

$$\boldsymbol{n} = \begin{pmatrix} n_x \\ n_y \end{pmatrix}$$

If the boundary is parameterised such that $(x,y) = (x(\gamma), y(\gamma))$ as γ goes form 0 to 1 then

$$\boldsymbol{n} = \frac{1}{\sqrt{(\partial_\gamma x)^2 + (\partial_\gamma y)^2}} \begin{pmatrix} -\partial_\gamma y \\ \partial_\gamma x \end{pmatrix}$$

and

$$d\Gamma = \sqrt{(\partial_\gamma x)^2 + (\partial_\gamma y)^2}\; d\gamma$$

and therefore

$$\boldsymbol{n}\, d\Gamma = \begin{pmatrix} -\partial_\gamma y \\ \partial_\gamma x \end{pmatrix} d\gamma$$

Hence

$$\int_\Gamma u(x,y) N(x,y)\, n_x\, d\Gamma = -\int_0^1 u(x(\gamma), y(\gamma))\, N(x(\gamma), y(\gamma))\, \partial_\gamma y\, d\gamma$$

6.3.1 Edge 12

For edge 12, $\eta = 0$. I parameterise it as $(\xi, \eta) = (1-\gamma, 0)$ as this takes me from node 1 to node 2 in clockwise direction (that is how I order the nodes, most FE do it the other way around)

$$x = x_p N_p(1-\gamma, 0) \quad \text{and} \quad y = y_P N_P(1-\gamma, 0)$$

The normal is

$$\boldsymbol{n}\, d\Gamma = \begin{pmatrix} -\partial_\gamma y \\ \partial_\gamma x \end{pmatrix} d\gamma$$

and

$$\begin{aligned} \frac{1}{\partial \gamma} &= \frac{1}{\partial \xi}\frac{\partial \xi}{\partial \gamma} + \frac{1}{\partial \eta}\frac{\partial \eta}{\partial \gamma} \\ &= -\frac{1}{\partial \xi} \end{aligned}$$

and therefore

$$\partial_\gamma y = -\partial_\xi y = -J_{12}$$

and

$$\boldsymbol{n}\, d\Gamma = \begin{pmatrix} \partial_\xi y \\ -\partial_\xi x \end{pmatrix} d\gamma = \begin{pmatrix} J_{12} \\ -J_{11} \end{pmatrix} d\gamma$$

or simply

$$\boldsymbol{n}\, d\Gamma = \begin{pmatrix} y_q\, \partial_\xi N_q(\xi, 0) \\ -x_q\, \partial_\xi N_q(\xi, 0) \end{pmatrix} d\gamma$$

For the linear triangle, for example, I get

$$\begin{aligned} x &= x_1 \xi + x_2(1-\xi) + x_3\, \eta \\ &= x_1(1-\gamma) + x_2(1-(1-\gamma)) + x_3\, 0 \\ &= x_1(1-\gamma) + x_2\, \gamma \end{aligned}$$

$$y = y_1(1-\gamma) + y_2\, \gamma$$

and

$$\partial_\gamma x = -x_1 + x_2$$

$$\partial_\gamma y = -y_1 + y_2$$

and a normal

$$\boldsymbol{n}\, d\Gamma = \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix} d\gamma$$

6.3.2 Edge 23

For edge 23, $\xi = 0$. I parameterise the edge as $(\xi, \eta) = (0, \gamma)$, this takes me from node 2 to 3 as γ varies from 0 to 1.

$$\boldsymbol{n}\, d\Gamma = \begin{pmatrix} -\partial_\gamma y \\ \partial_\gamma x \end{pmatrix} d\gamma$$

$$\partial_\gamma = \partial_\eta$$

and therefore

$$\boldsymbol{n}\, d\Gamma = \begin{pmatrix} -\partial_\eta y \\ \partial_\eta x \end{pmatrix} = \begin{pmatrix} -J_{22} \\ J_{21} \end{pmatrix} d\gamma$$

For the linear triangle, for example, I get

$$x = x_1 0 + x_2(1-\gamma) + x_3 \gamma$$

$$y = y_1 0 + y_2(1-\gamma) + y_3 \gamma$$

and

$$\partial_\gamma x = -x_2 + x_3$$

$$\partial_\gamma y = -y_2 + y_3$$

and a normal

$$\boldsymbol{n}\, d\Gamma = \begin{pmatrix} y_2 - y_3 \\ x_3 - x_2 \end{pmatrix} d\gamma$$

6.3.3 Edge 32

For edge 32 is parameterised as $(\xi, \eta) = (\gamma, 1-\gamma)$, and

$$\boldsymbol{n}\, d\Gamma = \begin{pmatrix} -\partial_\gamma y \\ \partial_\gamma x \end{pmatrix} d\gamma$$

and

$$\begin{aligned} \frac{1}{\partial\gamma} &= \frac{1}{\partial\xi}\frac{\partial\xi}{\partial\gamma} + \frac{1}{\partial\eta}\frac{\partial\eta}{\partial\gamma} \\ &= \frac{1}{\partial\xi} - \frac{1}{\partial\eta} \end{aligned}$$

and therefore

$$\boldsymbol{n}\, d\Gamma = \begin{pmatrix} -y_q(\partial_\xi N_q - \partial_\eta N_q) \\ x_q(\partial_\xi N_q - \partial_\eta N_q) \end{pmatrix} d\gamma = \begin{pmatrix} J_{22} - J_{12} \\ J_{11} - J_{21} \end{pmatrix} d\gamma$$

For the linear triangle, for example, I get

$$x = x_1\gamma + x_2(1-\gamma-(1-\gamma)) + x_3(1-\gamma)$$

or

$$\begin{aligned} x &= x_1\gamma + x_3(1-\gamma) \\ y &= y_1\gamma + y_3(1-\gamma) \end{aligned}$$

and

$$\begin{aligned} \partial_\gamma x &= x_1 - x_3 \\ \partial_\gamma y &= y_1 - y_3 \end{aligned}$$

and a normal

$$\boldsymbol{n}\, d\Gamma = \begin{pmatrix} y_3 - y_1 \\ x_1 - x_3 \end{pmatrix} d\gamma$$

6.4 Various directional derivatives

6.4.1 Directional derivative of draft with respect to ice thickness

For implicit forward time integration with respect to h using the NR method, various directional derivatives with respect to h must be calculated.

Using Eq. (1.21) we find that the directional derivative of the draft d with respect to h is

$$\begin{aligned} Dd(h)[\Delta h] &= \lim_{\epsilon\to 0}\frac{d}{d\epsilon} d(h+\epsilon\Delta h) \\ &= \mathcal{H}(h_f - h)\rho\,\Delta h\,/\rho_o - \rho h\,\delta(h_f-h)\,\Delta h\,/\rho_o + \mathcal{H}(H)H\,\delta(h-h_f)\,\Delta h \\ &= \frac{\rho}{\rho_o}\mathcal{H}(h_f-h)\,\Delta h + \delta(h-h_f)(H - \frac{\rho}{\rho_o}h)\,\Delta h \end{aligned}$$

When integrated the second term in the above expression integrates to zero, because where $h = h_f$ we have $H = \rho h/\rho_o$, hence[1]

$$Dd(h)[\Delta h] = \frac{\rho}{\rho_o}\mathcal{H}(h_f - h)\,\Delta h. \tag{6.6}$$

Using Eq. (6.6) the directional derivative of

$$D(\frac{1}{2}g(\rho h^2 - \rho_o d^2))[\Delta h]$$

[1]This argument does not hold if the Heaviside function and the Dirac delta functions are approximated. In that case the full expression must be used. Important for getting quadratic convergence in NR.

with respect to perturbation in h is found to be

$$\begin{aligned} D\left(\frac{1}{2}g(\rho h^2-\rho_o d^2)\right)[\Delta h] &= g(\rho(h\,\Delta h-\rho_o d\frac{\rho}{\rho_o}\mathcal{H}(h_f-h)\,\Delta h) \\ &= \rho g\,(h-\mathcal{H}(h_f-h)\,d)\,\Delta h \end{aligned}$$

The directional derivative of $g(\rho h-\rho_o d)\partial_x b$ with respect to h is found to be

$$D\,(g(\rho h-\rho_o d)\partial_x b)\,[\Delta h]=\rho g\,\mathcal{H}(h-h_f)\,\partial_x B\,\Delta h. \tag{6.7}$$

To see this first notice that using Eq. (1.21)

$$\begin{aligned} g(\rho h-\rho_o d)\partial_x b &= g(\rho h-\mathcal{H}(h_f-h)\rho h-\rho_o\mathcal{H}(H)\mathcal{H}(h-h_f)H)\partial_x b \\ &= g(\rho h-(1-\mathcal{H}(h-h_f))\rho h-\rho_o\mathcal{H}(H)\mathcal{H}(h-h_f)H)\partial_x b \\ &= g(\rho h+\mathcal{H}(h-h_f)(\rho h-\rho_o H_+)-\rho h)\partial_x b \\ &= g\mathcal{H}(h-h_f)(\rho h-\rho_o H_+)\partial_x b \\ &= g\mathcal{H}(h-h_f)(\rho h-\rho_o H_+)\partial_x B. \end{aligned}$$

Using the above expression we now calculate the directional derivative of $g(\rho h-\rho_o d)\partial_x b$ with respect to h and find

$$\begin{aligned} D\,(g(\rho h-\rho_o d)\partial_x b)\,[\Delta h] &= \mathcal{H}(h_f-h)\rho g\,\partial_x B\,\Delta h+g\,\delta(h-h_f)(\rho h-\rho_o H_+)\partial_x B\,\Delta h \\ &= (\rho\mathcal{H}(h-h_f)+\delta(h-h_f)(\rho h-\rho_o H_+))\;g\,\partial_x B\,\Delta h \\ &= \rho g\,\mathcal{H}(h-h_f)\,\partial_x B\,\Delta h \end{aligned}$$

where the last step is correct when the expression is evaluated under an integral, thus demonstrating the correctness of Eq. (6.7).

(Not sure where to put this, but keep it here for the time being.) The lower ice surface is related to thickness through

$$b=\mathcal{H}(h-h_f)B+\mathcal{H}(h_f-h)(S-\rho h/\rho_o)$$

and therefore

$$\partial_h b=\delta(h-h_f)B-\delta(h_f-h)(S-\rho h/\rho_o)-\mathcal{H}(h_f-h)\rho/\rho_o$$

and

$$\partial_x b=\partial_h b\,\partial_x h=\delta(h-h_f)B\,\partial_x h-\delta(h_f-h)(S-\rho h/\rho_o)\,\partial_x h-\mathcal{H}(h_f-h)\rho\,\partial_x h/\rho_o$$

and assuming

$$\frac{\partial^2 b}{\partial h\partial x}=\frac{\partial^2 b}{\partial x\partial h}$$

If $f(x)$ is a test function

$$\partial_x\int\delta(x)\,\partial_x f(x)\,dx=-\partial_x\int\mathcal{H}(x)f(x)\,dx=-f(0)-\int\mathcal{H}(x)\,\partial_x f(x)\,dx$$

6.4.2 Linearisation of the 2HD forward problem needed for the adjoint method

For the adjoint method we need

$$\mathrm{K}^{xC}\,\Delta C_q:=-D\boldsymbol{F}_x(\boldsymbol{u}_1^i,\boldsymbol{v}_1^i,\boldsymbol{h}_1^i)[\Delta C_q]$$

Here ΔC_q is the nodal value itself, the perturbation in C is $\Delta C_q N_q$ (no summation).

$$[\mathrm{K}^{xC}]_{pq} = \frac{1}{m} \int_\Omega \mathcal{H}(h-h_f) C^{-1/m-1} \left| \boldsymbol{v}_{xy} \right|^{1/m-1} u N_p \, N_q \, dxdy \tag{6.8}$$

and

$$[\mathrm{K}^{yC}]_{pq} = \frac{1}{m} \int_\Omega \mathcal{H}(h-h_f) C^{-1/m-1} \left| \boldsymbol{v}_{xy} \right|^{1/m-1} v N_p \, N_q \, dxdy \tag{6.9}$$

where

$$\boldsymbol{v}_{xy} = \begin{pmatrix} u \\ v \end{pmatrix}$$

and

$$\beta^2 = C^{-1/m} |\boldsymbol{v}_{xy}|^{1/m-1}$$

$$\mathrm{K}^C = \begin{bmatrix} \mathrm{K}^{xC} \\ \mathrm{K}^{yC} \end{bmatrix}$$

is $2N \times n$ where N are degrees of freedom.

If the cost function I is calculated using FE type inner product , then the gradient of the cost function is then given by

$$\begin{aligned} \frac{\partial I}{\partial C_q} &= [\mathrm{K}^C]_{pq} \lambda_p \\ &= \frac{1}{m} \int_\Omega \mathcal{H}(h-h_f) C^{-1/m-1} \left| \boldsymbol{v}_{xy} \right|^{1/m-1} (u\lambda + v\mu) \, N_q \, dxdy \end{aligned}$$

If I is calculated as a discrete sum over values then

$$\frac{\partial I}{\partial C_q} = \frac{1}{m} \mathcal{H}(h-h_f) C^{-1/m-1} \left| \boldsymbol{v}_{xy} \right|^{1/m-1} (u\lambda + v\mu)$$

Note: Perturbing on particular nodal value in $C = C_r N_r$ can be written as

$$(C_r + \Delta C \delta_{rp}) N_r$$

$C = C_r N_r$, $\Delta C = \Delta \hat{C} N_q$ with $\Delta \hat{C} = \Delta C_q$

$$\begin{aligned} (C + \Delta C)^m &= (C_r N_r + \Delta \hat{C} N_q)^m \\ &= (C_r N_r + \Delta \hat{C} N_q (C_j N_j)/(C_i N_i))^m \\ &= (C_r N_r)^m (1 + \Delta \hat{C} N_q/(C_i N_i))^m \\ &\approx (C_r N_r)^m (1 + m \Delta \hat{C} N_q/(C_i N_i)) \\ &= (C_s N_s)^m + m (C_s N_s)^{m-1} \Delta \hat{C} N_q \\ &= (C_s N_s)^m + m (C_s N_s)^{m-1} \Delta C \end{aligned}$$

I can write the perturbation as

$$K \Delta \hat{C} = m (C_s N_s)^{m-1} N_q \Delta \hat{C}$$

where

$$K = m (C_s N_s)^{m-1} N_q$$

6.5 FE formulation and linearisation for the 1HD Problem

6.5.1 Field equations and boundary conditions (1HD)

$$4\partial_x(h\eta\partial_x u) - \beta^2 u = \rho g h(\partial_x s \cos\alpha - \sin\alpha)$$

$$\beta^2 = C^{-1/m}|u|^{1/m-1}$$

with the sliding law written on the form

$$u = C|t_{bx}|^{m-1}\,t_{bx}$$

We have

$$t_{bx} = C^{-1/m}|u|^{1/m-1}\,u.$$

Glen's flow law is

$$\dot{\epsilon}_{ij} = A\tau^{n-1}\tau_{ij},$$

where

$$\tau = \sqrt{\tau_{ij}\tau_{ij}/2}$$

The flow law can also be written as

$$\tau_{ij} = A^{-1/n}\,\dot{\epsilon}^{(1-n)/n}\,\dot{\epsilon}_{ij},$$

where

$$\dot{\epsilon} = \sqrt{(\partial_x u)^2} = |\partial_x u|$$

If we write

$$\tau_{ij} = 2\eta\dot{\epsilon}_{ij}$$

then η is the effective viscosity given by

$$\eta = \frac{1}{2}A^{-1/n}\,\dot{\epsilon}^{(1-n)/n} = \frac{1}{2}A^{-1/n}\,|\partial_x u|^{(1-n)/n}$$

6.6 Linearisation of field equations (1HD)

In the non-linear case using Newton's method I need to linearise the equation.

Linearised with respect to u by writing $u = \bar{u} + \Delta u$, $\eta = \bar{\eta} + \partial_u\eta\Delta u$, and $\beta^2 = \bar{\beta}^2 + \partial_u(\beta^2)\Delta u$

$$4\partial_x(h(\bar{\eta} + \partial_u\eta\Delta u)\partial_x(\bar{u} + \Delta u)) - (\bar{\beta}^2 + D\beta^2(u)[\Delta u])(\bar{u} + \Delta u) = \rho g h(\partial_x s \cos\alpha - \sin\alpha)$$

or

$$4\partial_x(h(\bar{\eta}\partial_x\bar{u} + \partial_x\bar{u}\,D\eta(u)[\Delta u] + \bar{\eta}\partial_x\Delta u)) - \bar{\beta}^2\bar{u} - \bar{\beta}^2\Delta u - \bar{u}\,D\beta^2(u)[\Delta u] = \rho g h(\partial_x s\cos\alpha - \sin\alpha) \tag{6.10}$$

where

$$\eta = \frac{1}{2}A^{-1/n}\,\dot{\epsilon}^{(1-n)/n} = \frac{1}{2}A^{-1/n}\,|\partial_x u|^{(1-n)/n}$$

$$\begin{aligned}
D\eta(u)[\Delta u] &= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\eta(u+\epsilon\Delta u) &(6.11)\\
&= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\frac{1}{2}A^{-1/n}\,|\partial_x(u+\epsilon\Delta u)|^{(1-n)/n}\\
&\underset{\partial_x u>0}{=} \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\frac{1}{2}A^{-1/n}\,(\partial_x u+\epsilon\partial_x\Delta u))^{(1-n)/n}\\
&= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\frac{1}{2}A^{-1/n}\,((\partial_x u)^{(1-n)/n}+(\partial_x u)^{(1-2n)/n}\frac{1-n}{n}\epsilon\partial_x\Delta u+\ldots))\\
&= \frac{1}{2}A^{-1/n}\,((\partial_x u)^{(1-2n)/n}\frac{1-n}{n}\partial_x\Delta u)\\
&= \frac{1-n}{2n}A^{-1/n}\,(\partial_x u)^{(1-2n)/n}\partial_x\Delta u
\end{aligned}$$

(6.12)

Doing the same for $\partial_x u<0$ shows that

$$D\eta(u)[\Delta u]=\frac{1-n}{2n}A^{-1/n}\,|\partial_x u|^{(1-2n)/n}\mathrm{sign}(\partial_x u)\,\partial_x\Delta u$$

(old result)

$$\partial_u\eta=\frac{1-n}{2n}\,A^{-1/n}|\partial_x u|^{1/n-2}\,\mathrm{sign}(\partial_x u)$$

The directional derivative of β^2 is

$$\begin{aligned}
D\beta(u)[\Delta u] &= (1/m-1)\,C^{-1/m}|u|^{(1-3m)/m}\,u\,\Delta u\\
&= (1/m-1)\,C^{-1/m}|u|^{(1-2m)/m}\,\mathrm{sign}(u)\,\Delta u
\end{aligned}$$

old result

$$\partial_u\beta^2=(1/m-1)\,C^{-1/m}|u|^{1/m-2}\,\mathrm{sign}(u)$$

Inserting into (6.10) gives

$$4\partial_x(h(\bar\eta\partial_x\bar u+\bar uE\partial_x\Delta u+\bar\eta\partial_x\Delta u))-\bar\beta^2\bar u-\bar\beta^2\Delta u-\bar uB\Delta u=\rho gh(\partial_x s\cos\alpha-\sin\alpha)$$

where

$$B=(1/m-1)\,C^{-1/m}|u|^{(1-2m)/m}\,\mathrm{sign}(u)$$

and

$$E=\frac{1-n}{2n}A^{-1/n}\,|\partial_x u|^{(1-2n)/n}\mathrm{sign}(\partial_x u)$$

which can be written on the form

$$4\partial_x(h(\bar\eta+\bar uE)\partial_x\Delta u)-(\bar\beta^2+\bar uB)\Delta u=\rho gh(\partial_x s\cos\alpha-\sin\alpha)-4\partial_x(h\bar\eta\partial_x\bar u)+\bar\beta^2\bar u$$

6.6.1 Newton Rapson

$$4\partial_x(\eta h\partial_x u)-\mathcal{H}(h-h_f)\beta^2 u=\frac{1}{2}g\cos\alpha\rho(1-\rho/\rho_o)\partial_x h^2+\rho gh\cos\alpha\,\mathcal{H}(h-h_f)\partial_x s^{'}-\rho gh\sin\alpha,\quad(6.13)$$

$$s^{'}(x)=\mathcal{H}(h-h_f)(\rho/\rho_o h-H)$$

where

$$H=S-B,$$

or as

$$4\partial_x(\eta h \partial_x u) - \mathcal{H}(h-h_f)\beta^2 u = \frac{\varrho g}{2}\partial_x h^2 \cos\alpha + \rho g h \mathcal{H}(h-h_f)(\rho/\rho_o \partial_x h - \partial_x H)\cos\alpha - \rho g h \sin\alpha \tag{6.14}$$

The FE formulation is

$$\begin{aligned} R_p^u &= \int \{4\eta h \partial_x u \partial_x N_p + \mathcal{H}(h-h_f)\beta^2 u N_p \\ &\quad -\frac{\varrho g}{2} h^2 \partial_x N \cos\alpha + \rho g h \mathcal{H}(h-h_f)(\rho/\rho_o \partial_x h - \partial_x H) N_p \cos\alpha - \rho g h N_p \sin\alpha\}\, dx \\ &\quad -h(4\eta\partial_x u - \varrho g h/2) u N_p|_{x_0}^{x_l} = 0 \end{aligned} \tag{6.15}$$

where

$$u = N_p u_p$$

If all von Neumann BC are of Weertman type, the boundary term is zero because at the calving front

$$8\eta \partial_x u = \varrho g h.$$

I also have

$$\partial_t h + \partial_x(uh) = a \tag{6.16}$$

where

$$a = a_s + a_b$$

The FE formulation used is the θ method, i.e.

$$R_p^h = \int \{\frac{1}{\Delta t}(h_1 - h_0) + \theta \partial_x(u_1 h_1) + (1-\theta)\partial_x(u_0 h_0) - a_s - a_b\} N_p \, dx = 0 \tag{6.17}$$

where $0 \leq \theta \leq 1$.

I go from time $t = t_0$ to time $t = t_1$ where $t_1 > t_0$, and I assume that the values for u at h at $t = t_0$ (i.e. u_0 and h_0) are known. I iteratively solve for corrections to the values at time step $t = t_1$

$$\begin{aligned} u_1 &= \bar{u}_1 + \Delta u \\ h_1 &= \bar{h}_1 + \Delta h \end{aligned}$$

using the Newton-Raphson method.

For Newton-Raphson I need to take the directional derivative of this equation with respect to u and h,

$$K(\Delta u, \Delta v)^T = \lim_{\epsilon \to 0} \frac{d}{d\epsilon} \boldsymbol{R}(\boldsymbol{v} + \epsilon \Delta \boldsymbol{v}, \boldsymbol{h} + \epsilon \Delta \boldsymbol{h})$$

where I have now ordered the discrete values of u and h into a vector

$$D(\mathcal{H}(h-h_f)\beta^2(u))[\Delta h] = \delta(\bar{h} - h_f)\beta^2(\bar{u})\, \Delta h$$

Directional derivative of right-hand term with respect to h.

$$\begin{aligned} D\{\rho g h \partial_x s'\}[\Delta h] =& D\{\rho g h \partial_x(\mathcal{H}(h-h_f)(\rho h/\rho_o - H))\}[\Delta h] \\ =& \lim_{\epsilon\to 0} \partial_\epsilon \left(\rho g (h + \epsilon\, \Delta h)\, \partial_x(\mathcal{H}(h + \epsilon\, \Delta h - h_f)(\rho(h+\epsilon\,\Delta h)/\rho_o - H))\right) \\ =& \rho g \partial_x(\mathcal{H}(h-h_f)(\rho h/\rho_o - H))\, \Delta h \\ & + \rho g h \partial_x(\mathcal{H}(h-h_f)\rho \Delta h/\rho_o) \\ & + \rho g h \partial_x(\delta(h-h_f)\Delta h\,(\rho h/\rho_o - H)) \quad (=0) \\ =& \rho g \mathcal{H}(h-h_f)(\rho \partial_x h/\rho_o - \partial_x H)\, \Delta h \\ & + \frac{\rho^2}{\rho_o} g h \mathcal{H}(h-h_f)\, \partial_x \Delta h \\ & + \frac{\rho^2}{\rho_o} \delta(h-h_f)\, h\, \partial_x h\, \Delta h \end{aligned}$$

$$
\begin{aligned}
[\mathrm{K}uh]_{pq}\Delta h_q = DR_p^u(u,h)[\Delta h_q] \quad = \quad & \int \{4\bar{\eta}\partial_x\bar{u}\partial_x N_p \\
& +\delta(\bar{h}-h_f)\bar{\beta}^2\bar{u}N_p \\
& -\varrho g\bar{h}\partial_x N_p \cos\alpha \\
& +\rho g\mathcal{H}(\bar{h}-h_f)(\rho/\rho_o\partial_x\bar{h}-\partial_x H)N_p\cos\alpha \\
& +\rho g\bar{h}\delta(\bar{h}-h_f)(\rho/\rho_o\partial_x\bar{h}-\partial_x H)N_p\cos\alpha \\
& +\rho g\bar{h}\mathcal{H}(\bar{h}-h_f)\rho/\rho_o N_p\cos\alpha\,\partial_x \\
& -\rho g N_p\sin\alpha\}\,\Delta h_q\,dx
\end{aligned}
$$

$$
\begin{aligned}
[\mathrm{K}uu]_{pq}\Delta u_q = DR_p^u(u,h)[\Delta u_q] \quad = \quad & \int \{4\bar{h}\partial_u\eta(\bar{u})\,\partial_x\bar{u}\,\partial_x N_p\partial_x \\
& +4\eta(\bar{u})\bar{h}\partial_x N_p\partial_x \\
& +\mathcal{H}(h-h_f)\partial_u\beta^2(\bar{u})N_p \\
& +\mathcal{H}(h-h_f)\beta^2(\bar{u})N_p\}\,\Delta u_q\,dx
\end{aligned}
$$

Linearising (6.17) gives

$$
\int\{(\Delta h+\bar{h}-h_0)/\Delta t+\theta\partial_x((\bar{u}+\Delta u)(\bar{h}+\Delta h))+(1-\theta)\partial_x(u_0h_0)-a_s-a_b\}N_p=0
$$

or

$$
\begin{aligned}
0 = & \int\{\Delta h/\Delta t+\theta\partial_x((\bar{u}\Delta h+\bar{h}\Delta u))\}N_p\,dx+ \\
& \int\{(\bar{h}-h_0)/\Delta t+\theta\partial_x(\bar{u}\bar{h})+(1-\theta)\partial_x(u_0h_0)-a_s-a_b\}N_p\,dx
\end{aligned} \tag{6.18}
$$

or

$$
[\mathrm{K}hu]_{pq}\Delta u_q \quad = \quad \theta(\partial_x\bar{h}+\bar{h}\partial_x)\Delta u_q\,N_p
$$

$$
[\mathrm{K}hh]_{pq} \quad = \quad (1/\Delta t+\theta(\partial_x\bar{u}\,N_q+\bar{u}\partial_x N_q))\,N_p
$$

$$
\begin{bmatrix} \mathrm{K}uu & \mathrm{K}uh \\ \mathrm{K}hu & \mathrm{K}hh \end{bmatrix}
\begin{bmatrix} \Delta\boldsymbol{v} \\ \Delta\boldsymbol{h} \end{bmatrix}
=
\begin{bmatrix} \boldsymbol{R}^u \\ \boldsymbol{R}^h \end{bmatrix} \tag{6.19}
$$

where

$$
\boldsymbol{r}^h = \boldsymbol{T}^h - \boldsymbol{F}^h
$$

where $\boldsymbol{T}$ and $\boldsymbol{F}$ are the internal and external nodal forces, respectively, and $\boldsymbol{R}$ is the residual or out-of-balance nodal forces.

$$
F^h = -\int\{a_s+a_b-(\bar{h}-h_0)/\Delta t\}N_p\,dx
$$

and

$$
T^h = \int\{\theta\partial_x(\bar{u}\bar{h})+(1-\theta)\partial_x(u_0h_0)\}N_p\,dx
$$

$$
T_p^u = \int\{4\eta h\partial_x u\partial_x N_p+\mathcal{H}(h-h_f)\beta^2uN_p)N_p\}\,dx
$$

$$
F_p^u = \int\{\frac{\varrho g}{2}h^2\partial_x N+\rho gh\mathcal{H}(h-h_f)(\rho/\rho_o\partial_x h-\partial_x H)N_p\}\,dx
$$

6.6.2 Connection to Picard iteration

If instead of writing

$$4\partial_x(h(\bar{\eta}\partial_x\bar{u} + \partial_u\eta\partial_x\bar{u}\Delta u + \bar{\eta}\partial_x\Delta u)) - (\bar{\beta^2}\bar{u} + (\bar{\beta^2} + \partial_u\beta^2\bar{u})\Delta u) = \rho gh(\partial_x s\cos\alpha - \sin\alpha)$$

we ignore the dependency of η and β^2 on u we get

$$4\partial_x(h(\bar{\eta}\partial_x\bar{u} + \bar{\eta}\partial_x\Delta u)) - (\bar{\beta^2}\bar{u} + \bar{\beta^2}\Delta u) = \rho gh(\partial_x s\cos\alpha - \sin\alpha)$$

or simply

$$4\partial_x(h\bar{\eta}\partial_x(\bar{u} + \Delta u)) - \bar{\beta^2}(\bar{u} + \Delta u) = \rho gh(\partial_x s\cos\alpha - \sin\alpha)$$

which can be solved directly for $\bar{u} + \Delta u$. This is the Picard iteration, i.e. an incomplete Newton iteration.

6.7 Linearisation in 2HD

6.7.1 Drag-term linearisation (2HD)

Basal drag is generally a function of basal sliding velocity and flotation conditions, i.e.

$$\boldsymbol{t}_b = f(h, h_f, \boldsymbol{v})$$

Using Weertman sliding law, basal drag is

$$\boldsymbol{t}_b = \mathcal{H}(h - h_f)\beta^2\,\boldsymbol{v}_b$$

or

$$\begin{pmatrix} t_{xb} \\ t_{xy} \end{pmatrix} = \mathcal{H}(h - h_f)\,\beta^2 \begin{pmatrix} u \\ v \end{pmatrix} \tag{6.20}$$

with

$$\beta^2 = C^{-1/m}|\boldsymbol{v}_b|^{1/m-1}$$

therefore

$$\tau_b = \tau_b(u, v, h)$$

We therefore need to linearise $\boldsymbol{t}_b$ with respect to u, v, and h.

We start by linearising β^2 with respect to u and v obtaining[2]

$$\begin{aligned}
\beta(u+\epsilon\Delta u, v+\epsilon\Delta v) &= C^{-1/m}((u+\epsilon\Delta u)^2 + (v+\epsilon\Delta v)^2)^{(1/m-1)/2} \\
&= C^{-1/m}((u^2+v^2+2\epsilon(u\,\Delta u + v\,\Delta v))^{(1/m-1)/2} \\
&= C^{-1/m}((u^2+v^2)^{(1/m-1)/2} + (1/m-1)(u^2+v^2)^{(1/m-1)/2-1}\,\epsilon(u\,\Delta u + v\,\Delta v)) \\
&= \beta^2(u,v) + C^{-1/m}(1/m-1)(u^2+v^2)^{(1/m-3)/2}\,\epsilon(u\,\Delta u + v\,\Delta v))
\end{aligned}$$

The directional derivative is defined as

$$D\beta(\boldsymbol{v})[\Delta\boldsymbol{v}_{xy}] = \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\beta^2(u+\epsilon\Delta u, v+\epsilon\Delta v)$$

and we arrive at[3]

$$D\beta(\boldsymbol{v})[\Delta u, \Delta v] = (1/m-1)\,C^{-1/m}|\boldsymbol{v}|^{(1-3m)/m}\,(u\,\Delta u, v\,\Delta v)$$

[2] If $y \ll x$ then $(x+y)^m \approx x^n + m\,x^{m-1}y$.

[3] In 1d we get

$$\begin{aligned}
D\beta(\boldsymbol{v})[\Delta\boldsymbol{v}] &= (1/m-1)\,C^{-1/m}|u|^{(1-3m)/m}\,u\,\Delta u \\
&= (1/m-1)\,C^{-1/m}|u|^{(1-2m)/m}\,\mathrm{sign}(u)\,\Delta u
\end{aligned}$$

and the directional derivatives of $\boldsymbol{t}_b$ with respect to u and v are therefore

$$\begin{aligned}
D\boldsymbol{t}_{xb}[\Delta u, \Delta v] &= \beta^2 \Delta u + D\beta^2[\Delta u]\, u + D\beta^2[\Delta v]\, u \\
&= \beta^2 \Delta u + (1/m-1)\, C^{-1/m} |\boldsymbol{v}|^{(1-3m)/m} \,(u^2 \Delta u + uv\, \Delta v) \\
D\boldsymbol{t}_{yb}[\Delta u, \Delta v] &= \beta^2 \Delta v + (1/m-1)\, C^{-1/m} |\boldsymbol{v}|^{(1-3m)/m} \,(v^2 \Delta v + uv\, \Delta u)
\end{aligned}$$

which can also be written on the form

$$\begin{pmatrix} \Delta t_{xb} \\ \Delta t_{xy} \end{pmatrix} = \mathcal{H}(h-h_f) \begin{pmatrix} \beta^2 + \mathcal{D}\, u^2 & \mathcal{D}\, uv \\ \mathcal{D}\, uv & \beta^2 + \mathcal{D}\, v^2 \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} \tag{6.21}$$

where we have now added back the $\mathcal{H}(h-h_f)$ factor, and where

$$\mathcal{D} := (1/m-1)\, C^{-1/m} |\boldsymbol{v}|^{(1-3m)/m}$$

Note that because of the non-linearity of the sliding law, the basal drag term in x direction depends on both u and v and this is reflected in the directional derivatives above.

In a fully implicit treatment we also must include the effect of grounding/un-grounding the basal drag term, or

$$\begin{aligned}
D\boldsymbol{t}_{xb}[\Delta h] &= \lim_{\epsilon \to 0} \frac{d}{d\epsilon} \boldsymbol{t}_{xb}(h + \epsilon \Delta h) \\
&= \lim_{\epsilon \to 0} \frac{d}{d\epsilon} \big(\mathcal{H}(h + \epsilon \Delta h - h_f)\, \beta^2\, u\big) \\
&= \lim_{\epsilon \to 0} \delta(h + \epsilon \Delta h - h_f) \Delta h\, \beta^2\, u \\
&= \delta(h - h_f)\, \beta^2\, u\, \Delta h
\end{aligned}$$

and therefore

$$\begin{pmatrix} \Delta t_{bx} \\ \Delta t_{by} \end{pmatrix} = \begin{pmatrix} \mathcal{H}(h-h_f)(\beta^2 + \mathcal{D}\, u^2) & \mathcal{H}(h-h_f)\mathcal{D}\, uv & \delta(h-h_f)\, \beta^2\, u \\ \mathcal{H}(h-h_f)\mathcal{D}\, uv & \mathcal{H}(h-h_f)(\beta^2 + \mathcal{D}\, v^2) & \delta(h-h_f)\, \beta^2\, v \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta h \end{pmatrix} \tag{6.22}$$

where again

$$\mathcal{D} := (1/m-1)\, C^{-1/m} |\boldsymbol{v}|^{(1-3m)/m}$$

Ocean drag term

We add an ocean drag term over the floating section of the form

$$\boldsymbol{t}_b^o = \mathcal{H}(h_f - h) \beta_o^2 \,(\boldsymbol{v}_b - \boldsymbol{v}_o)$$

with

$$\beta_o^2(u,v) = C_o^{-1/m_o} |\boldsymbol{v}_b - \boldsymbol{v}_o|^{1/m_o - 1}$$

The total drag is a sum of that due to basal sliding and ocean currents.

$$\boldsymbol{t}_b = \mathcal{H}(h-h_f)\, \beta^2\, \boldsymbol{v} + \mathcal{H}(h_f - h)\, \beta_o^2\, (\boldsymbol{v} - \boldsymbol{v}_o)$$

So

$$\begin{aligned}
t_{bx}^o &= \mathcal{H}(h_f - h) C_o^{-1/m_o} \left((u-u_o)^2 + (v - v_0)^2\right)^{(1-m)/2m} (u - u_o) \\
t_{yx}^o &= \mathcal{H}(h_f - h) C_o^{-1/m_o} \left((u-u_o)^2 + (v - v_0)^2\right)^{(1-m)/2m} (v - v_o)
\end{aligned}$$

and hence

$$
\begin{aligned}
D\boldsymbol{t}^o_{bx}[\Delta u] &= \mathcal{H}(h_f - h) C_o^{-1/m_o} \left(|\boldsymbol{v}_b - \boldsymbol{v}_o|^{(1-m_o)/m_o} + (1/m - 1) |\boldsymbol{v}_b - \boldsymbol{v}_o|^{(1-3m_o)/m_o} (u - u_o)^2 \right) \Delta u \\
D\boldsymbol{t}^o_{bx}[\Delta v] &= \mathcal{H}(h_f - h) C_o^{-1/m_o} (1/m - 1) |\boldsymbol{v}_b - \boldsymbol{v}_o|^{(1-3m_o)/m_o} (v - v_o)(u - u_o) \, \Delta v \\
D\boldsymbol{t}^o_{by}[\Delta u] &= \mathcal{H}(h_f - h) C_o^{-1/m_o} (1/m - 1) |\boldsymbol{v}_b - \boldsymbol{v}_o|^{(1-3m_o)/m_o} (v - v_o)(u - u_o) \, \Delta u \\
D\boldsymbol{t}^o_{by}[\Delta v] &= \mathcal{H}(h_f - h) C_o^{-1/m_o} \left(|\boldsymbol{v}_b - \boldsymbol{v}_o|^{(1-m_o)/m_o} + (1/m - 1) |\boldsymbol{v}_b - \boldsymbol{v}_o|^{(1-3m_o)/m_o} (v - v_o)^2 \right) \Delta v \\
D\boldsymbol{t}^o_{bx}[\Delta h] &= -\delta(h_f - h) C_o^{-1/m_o} |\boldsymbol{v}_b - \boldsymbol{v}_o|^{(1-m_o)/m_o} (u - u_o) \Delta h \\
D\boldsymbol{t}^o_{by}[\Delta h] &= -\delta(h_f - h) C_o^{-1/m_o} |\boldsymbol{v}_b - \boldsymbol{v}_o|^{(1-m_o)/m_o} (v - v_o) \Delta h
\end{aligned}
$$

6.7.2 Flow law linearisation (2HD)

Using Glen's flow law, deviatoric stresses are related to strain rates through Eq. (1.13), i.e.

$$\tau_{ij} = A^{-1/n} \, \dot{\epsilon}^{(1-n)/n} \, \dot{\epsilon}_{ij},$$

where

$$\dot{\epsilon} = \sqrt{\dot{\epsilon}_{ij} \dot{\epsilon}_{ij}/2}$$

which in the Shallow Ice Stream Approximation takes the form

$$
\begin{aligned}
\dot{\epsilon} &= \sqrt{(\dot{\epsilon}_{xx})^2 + (\dot{\epsilon}_{yy})^2 + \dot{\epsilon}_{xx} \, \dot{\epsilon}_{yy} + (\dot{\epsilon}_{xy})^2} && (6.23) \\
&= ((\partial_x u)^2 + (\partial_y v)^2 + \partial_x u \, \partial_y v + (\partial_x v + \partial_y u)^2/4)^{1/2}. && (6.24)
\end{aligned}
$$

If we write

$$\tau_{ij} = 2\eta \dot{\epsilon}_{ij}$$

where η is the effective viscosity given by

$$\eta = \frac{1}{2} A^{-1/n} \, \dot{\epsilon}^{(1-n)/n}$$

then, in the Shallow Ice Stream Approximation, η is

$$\eta = \frac{1}{2} A^{-1/n} \, ((\partial_x u)^2 + (\partial_y v)^2 + \partial_x u \, \partial_y v + (\partial_x v + \partial_y u)^2/4)^{(1-n)/2n}.$$

More generally we can express the stresses as a function of velocities as

$$\tau_{ij} = f(u_q).$$

We now need to linearise each of the stress components with respect to the unknown velocity components u and v velocities. We start with τ_{xx} where we have

$$
\begin{aligned}
\tau_{xx} &= A^{-1/n} \, \dot{\epsilon}^{(1-n)/n} \, \dot{\epsilon}_{xx} && (6.25) \\
&= A^{-1/n} \, \dot{\epsilon}^{(1-n)/n} \, \partial_x u && (6.26)
\end{aligned}
$$

and we linearise

$$
\begin{aligned}
D\tau_{xx}[u; \Delta u] &= \lim_{\epsilon \to 0} \frac{d}{d\epsilon} \tau_{xx}(u; \Delta u) && (6.27) \\
&= \lim_{\epsilon \to 0} \frac{d}{d\epsilon} \left(2\eta \, \partial_x u \right) && (6.28) \\
&= 2\eta \, \partial_x \Delta u + (2 D\eta[\Delta u]) \, \partial_x u && (6.29)
\end{aligned}
$$

$$\eta = \frac{1}{2} A^{-1/n} \left((\partial_x u)^2 + (\partial_y v)^2 + \partial_x u \, \partial_y v + (\partial_x v + \partial_y u)^2/4 \right)^{(1-n)/2n}$$

$$\begin{aligned}
&(\partial_x(u+\Delta u))^2 + (\partial_y(v+\Delta v))^2 + \partial_x(u+\Delta u)\,\partial_y(v+\Delta v) + (\partial_x(v+\Delta v) + \partial_y(u+\Delta u))^2/4 \\
&\quad = (\partial_x u)^2 + 2\partial_x u \partial_x \Delta u \\
&\quad +(\partial_y v)^2 + 2\partial_y v \partial_y \Delta v \\
&\quad +(\partial_x u + \partial_x \Delta u)(\partial_y v + \partial_y \Delta v) \\
&\quad +(\partial_x v + \partial_x \Delta v + \partial_y u + \partial_y \Delta u)^2/4 \\
&\quad = (\partial_x u)^2 + 2\partial_x u \partial_x \Delta u \\
&\quad +(\partial_y v)^2 + 2\partial_y v \partial_y \Delta v \\
&\quad +\partial_x u \partial_y v + \partial_x u \, \partial_y \Delta v + \partial_y v \, \partial_x \Delta u \\
&\quad +(\partial_x v + \partial_y u)/4 + (\partial_x v + \partial_y u)(\partial_x \Delta v + \partial_y \Delta u)/2 \\
&\quad = e^2 + \delta d e^2
\end{aligned}$$

where

$$\begin{aligned}
e^2 &= (\partial_x u)^2 + (\partial_y v)^2 + \partial_x u \partial_y v + (\partial_x v + \partial_y u)/4 \\
&= \dot\epsilon_{xx}^2 + \dot\epsilon_{yy}^2 + \dot\epsilon_{xx}\dot\epsilon_{yy} + \dot\epsilon_{xy}^2
\end{aligned}$$

$$\delta e^2 = 2\partial_x u \partial_x \Delta u + 2\partial_y v \partial_y \Delta v + \partial_x u \, \partial_y \Delta v + \partial_y v \, \partial_x \Delta u + (\partial_x v + \partial_y u)(\partial_x \Delta v + \partial_y \Delta u)/2$$

or

$$\begin{aligned}
\delta e^2 &= (2\partial_x u + \partial_y v)\, \partial_x \Delta u + (2\partial_y v + \partial_x u)\, \partial_y \Delta v + \frac{1}{2}(\partial_x v + \partial_y u)\,(\partial_x \Delta v + \partial_y \Delta u) \\
&= (2\dot\epsilon_{xx} + \dot\epsilon_{yy})\, \partial_x \Delta u + (2\dot\epsilon_{yy} + \dot\epsilon_{xx})\, \partial_y \Delta v + \dot\epsilon_{xy}\,(\partial_x \Delta v + \partial_y \Delta u)
\end{aligned}$$

The directional derivative of η is

$$\begin{aligned}
D\eta(u,v)[\Delta u, \Delta v] &= E\left((2\dot\epsilon_{xx} + \dot\epsilon_{yy})\, \partial_x \Delta u + (2\dot\epsilon_{yy} + \dot\epsilon_{xx})\, \partial_y \Delta v + \dot\epsilon_{xy}\,(\partial_x \Delta v + \partial_y \Delta u)\right) \\
&= E\left((2\partial_x u + \partial_y v)\, \partial_x \Delta u + (2\partial_y v + \partial_x u)\, \partial_y \Delta v + \frac{1}{2}(\partial_x v + \partial_y u)\,(\partial_x \Delta v + \partial_y \Delta u)\right)
\end{aligned}$$

where

$$E := \frac{1-n}{4n} A^{-1/n} e^{(1-3n)/n}$$

which I can also write as

$$D\eta(u,v)[\Delta u, \Delta v] = d_u \eta \, \Delta u + d_v \eta \, \Delta v \tag{6.30}$$

where

$$d_u \eta = E\left((2\dot\epsilon_{xx} + \dot\epsilon_{yy})\, \partial_x + \dot\epsilon_{xy}\, \partial_y\right)$$

and

$$d_v \eta = E\left((2\dot\epsilon_{yy} + \dot\epsilon_{xx})\, \partial_y + \dot\epsilon_{xy}\, \partial_x\right)$$

where

$$E := \frac{1-n}{4n} A^{-1/n} e^{(1-3n)/n}$$

or as

$$D\eta(u,v)[\Delta u, \Delta v] = d_{xu} \partial_x \Delta u + d_{yu} \partial_y \Delta u + d_{yv} \partial_y \Delta v + d_{xv} \partial_x \Delta v \tag{6.31}$$

where

$$d_{xu} = E\,(2\dot\epsilon_{xx} + \dot\epsilon_{yy})$$

and

$$d_{yu} = E\,\dot\epsilon_{xy}$$

and

$$d_{yv} = E\,(2\dot\epsilon_{yy} + \dot\epsilon_{xx})$$

and

$$d_{xv} = E\,\dot\epsilon_{xy}$$

$$\begin{pmatrix} D\eta_x \\ D\eta_y \end{pmatrix} = \begin{pmatrix} E(2\dot\epsilon_{xx} + \dot\epsilon_{yy})\partial_x & E\dot\epsilon_{xy}\partial_y & 0 \\ E\dot\epsilon_{xy}\partial_x & E(\dot\epsilon_{xx} + 2\dot\epsilon_{yy})\partial_y & 0 \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta h \end{pmatrix} \tag{6.32}$$

In the 1d case we get

$$\frac{1-n}{4n} A^{-1/n} |\partial_x u|^{(1-3n)/n}\, 2\partial_x u\, \partial_x \Delta u = \frac{1-n}{2n} A^{-1/n} |\partial_x u|^{(1-2n)/n} \mathrm{sign}(\partial_x u)\, \partial_x \Delta u$$

6.7.3 Field-equation linearisation

linearising

$$\begin{aligned} &\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) + \partial_y(h\eta(\partial_x v + \partial_y u)) - \beta^2 u \\ &= \frac{1}{2}\varrho g \cos\alpha\, \partial_x h^2 + \rho g h(\partial_x s^{'} \cos\alpha - \sin\alpha) \end{aligned}$$

gives

$$\begin{aligned} &\partial_x(4h(\eta\partial_x u + D\eta\,\partial_x u + \eta\,\partial_x \Delta u) + 2h(\eta\partial_y v + D\eta\,\partial_y v + \eta\,\partial_y \Delta v)) \\ &\quad + \partial_y(h(\eta\partial_x v + D\eta\,\partial_x v + \eta\,\partial_x \Delta v) + h(\eta\partial_y u + D\eta\,\partial_y u + \eta\,\partial_y \Delta u)) \\ &\quad - (\beta^2 u + D\beta^2\, u + \beta^2\, \Delta u) \\ &\quad = \frac{1}{2}\varrho g \cos\alpha\, \partial_x h^2 + \rho g h(\partial_x s^{'} \cos\alpha - \sin\alpha) \end{aligned}$$

or

$$\begin{aligned} &\partial_x(4h(D\eta\,\partial_x u + \eta\,\partial_x \Delta u) + 2h(D\eta\,\partial_y v + \eta\,\partial_y \Delta v)) \\ &\quad + \partial_y(h(D\eta\,\partial_x v + \eta\,\partial_x \Delta v) + h(D\eta\,\partial_y u + \eta\,\partial_y \Delta u)) \\ &\quad - (D\beta^2\, u + \beta^2\, \Delta u) \\ &\quad = \frac{1}{2}\varrho g \cos\alpha\, \partial_x h^2 + \rho g h(\partial_x s^{'} \cos\alpha - \sin\alpha) - \partial_x(4h\eta(\partial_x u + \partial_y v)) - \partial_y(h\eta(\partial_x v + \partial_y u)) + \beta^2 u \end{aligned}$$

after having done the partial integration I get within the integral

$$(4h(D\eta\,\partial_x u + \eta\,\partial_x \Delta u) + 2h(D\eta\,\partial_y v + \eta\,\partial_y \Delta v))\,\partial_x N_i \tag{6.33}$$

$$\begin{aligned} &\quad + h((D\eta\,\partial_x v + \eta\,\partial_x \Delta v) + h(D\eta\,\partial_y u + \eta\,\partial_y \Delta u)\,\partial_y N_i \\ &\quad + (D\beta^2\, u + \beta^2\, \Delta u) N_i \\ &\quad = +\frac{1}{2}\varrho g \cos\alpha\, \partial_x h^2 \partial_x N_i - \rho g h(\partial_x s^{'} \cos\alpha - \sin\alpha) N_i - 4h\eta(\partial_x u + 2\partial_y v)\partial_x N_i - h\eta(\partial_x v + \partial_y u)\partial_y N_i - \beta^2 u N_i \end{aligned} \tag{6.34}$$

Inserting (6.30)

$$\begin{aligned} &(4h(\eta\,\partial_x \Delta u + \partial_x u(d_u\eta\,\Delta u + d_v\eta\,\Delta v)) + 2h(\eta\,\partial_y \Delta v + \partial_y v(d_u\eta\,\Delta u + d_v\eta\,\Delta v)))\,\partial_x N_i \\ &\quad + (h(\eta\,\partial_x \Delta v + \partial_x v(d_u\eta\,\Delta u + d_v\eta\,\Delta v)) + h(\eta\,\partial_y \Delta u + \partial_y u(d_u\eta\,\Delta u + d_v\eta\,\Delta v)))\,\partial_y N_i \\ &\quad + (\beta^2\, \Delta u + u\,(d_u\beta^2\ \Delta u + d_v\beta^2\, \Delta v)) N_i \\ &\quad = +\frac{1}{2}\varrho g \cos\alpha\, \partial_x h^2 \partial_x N_i - \rho g h(\partial_x s^{'} \cos\alpha - \sin\alpha) N_i - 4h\eta(\partial_x u + 2\partial_y v)\partial_x N_i - h\eta(\partial_x v + \partial_y u)\partial_y N_i - \beta^2 u N_i \end{aligned}$$

I take all coefficients in front of Δu and put them in the 11 part of the matrix, and everything in front of Δv and put that in the 12 part of the matrix.

To make this clear insert (6.31) into (6.34)

$$\begin{aligned}
& 4h(\eta\,\partial_x\Delta u + \partial_x u(d_{xu}\partial_x\Delta u + d_{yu}\partial_y\Delta u + d_{yv}\partial_y\Delta v + d_{xv}\partial_x\Delta v))\partial_x N_i \\
+ \quad & 2h(\eta\,\partial_y\Delta v + \partial_y v(d_{xu}\partial_x\Delta u + d_{yu}\partial_y\Delta u + d_{yv}\partial_y\Delta v + d_{xv}\partial_x\Delta v))\,\partial_x N_i \\
+ \quad & h(\eta\,\partial_x\Delta v + \partial_x v(d_{xu}\partial_x\Delta u + d_{yu}\partial_y\Delta u + d_{yv}\partial_y\Delta v + d_{xv}\partial_x\Delta v))\partial_y N_i \\
+ \quad & h(\eta\,\partial_y\Delta u + \partial_y u(d_{xu}\partial_x\Delta u + d_{yu}\partial_y\Delta u + d_{yv}\partial_y\Delta v + d_{xv}\partial_x\Delta v))\partial_y N_i \\
+ \quad & (\beta^2\,\Delta u + u\,(d_u\beta^2\ \Delta u + d_v\beta^2\,\Delta v))N_i \\
& = +\frac{1}{2}\varrho g\cos\alpha\,\partial_x h^2\partial_x N_i - \rho g h(\partial_x s^{'}\cos\alpha - \sin\alpha)N_i - 4h\eta(\partial_x u + 2\partial_y v)\partial_x N_i - h\eta(\partial_x v + \partial_y u)\partial_y N_i - \beta^2 u N_i
\end{aligned}$$

We have $\Delta u = N_j\Delta u_j$ and $\Delta v = N_j\delta v_j$

$$\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} \Delta u_j \\ \Delta v_j \end{pmatrix}$$

$$\begin{aligned}
[K_{12}]_{ij} \quad = \quad & h\eta\,\partial_x N_j\partial_y N_i \\
& +4h\partial_x u\,d_{yv}\,\partial_y N_j\partial_x N_i + 4h\partial_x u\,d_{xv}\,\partial_x N_j\partial_x N_i \\
& +2h\partial_y v\,d_{yv}\,\partial_y N_j\partial_x N_i + 2h\partial_y v\,d_{xv}\,\partial_x N_j\partial_x N_i \\
& +h\partial_x v\,d_{yv}\,\partial_y N_j\partial_y N_j + h\partial_x v\,d_{xv}\,\partial_x N_j\partial_y N_i \\
& +h\partial_y u\,d_{yv}\,\partial_y N_j\partial_y N_i + h\partial_y u\,d_{xv}\,\partial_x N_j\partial_y N_i \\
= \quad & h\eta\,\partial_x N_j\partial_y N_i \\
& +(4h\partial_x u\,d_{xv} + 2h\partial_y v\,d_{xv})\,\partial_x N_j\partial_x N_i \\
& +(h\partial_x v\,d_{yv} + h\partial_y u\,d_{yv})\,\partial_y N_j\partial_y N_i \\
& +(h\partial_x v\,d_{xv} + h\partial_y u\,d_{xv})\,\partial_x N_j\partial_y N_i \\
& +(4h\partial_x u\,d_{yv} + 2h\partial_y v\,d_{yv})\,\partial_y N_j\partial_x N_i \\
= \quad & h\eta\,\partial_x N_j\partial_y N_i \\
& +2Eh(2\dot\epsilon_{xx} + \dot\epsilon_{yy})\dot\epsilon_{xy}\,\partial_x N_j\partial_x N_i \\
& +Eh2\dot\epsilon_{xy}(2\dot\epsilon_{yy} + \dot\epsilon_{xx})\,\partial_y N_j\partial_y N_i \\
& +Eh2\dot\epsilon_{xy}\dot\epsilon_{xy}\,\partial_x N_j\partial_y N_i \\
& +2Eh(2\dot\epsilon_{xx} + \dot\epsilon_{yy})(2\dot\epsilon_{yy} + \dot\epsilon_{xx})\,\partial_y N_j\partial_x N_i
\end{aligned}$$

If we swap u and v and x and y and then i and j (transpose) we get the same matrix, hence

$$K_{12} = K_{21}'$$

$$
\begin{aligned}
[K_{11}]_{ij} &= 4h(\eta\,\partial_x N_j + \partial_x u(d_{xu}\partial_x N_j + d_{yu}\partial_y N_j))\,\partial_x N_i \\
&\quad +2h\partial_y v(d_{xu}\partial_x N_j + d_{yu}\partial_y N_j)\,\partial_x N_i \\
&\quad +h(\partial_x v(d_{xu}\partial_x N_j + d_{yu}\partial_y N_j)\,\partial_y N_i \\
&\quad +h(\eta\,\partial_y N_j + \partial_y u(d_{xu}\partial_x N_j + d_{yu}\partial_y N_j))\,\partial_y N_i \\
&= h(4\eta + (4\partial_x u + 2\partial_y v)d_{xu})\partial_x N_j\,\partial_x N_i \\
&\quad +h(\eta + (\partial_y u + \partial_x v)d_{yu})\partial_y N_j \partial_y N_i \\
&\quad +h(4\partial_x u + 2\partial_y v)d_{yu}\partial_y N_j \partial_x N_i \\
&\quad +h(\partial_x v + \partial_y u)d_{xu}\partial_x N_j \partial_y N_i \\
&= h(4\eta + E2(2\dot\epsilon_{xx} + \dot\epsilon_{yy})(2\dot\epsilon_{xx} + \dot\epsilon_{yy}))\partial_x N_j\,\partial_x N_i \\
&\quad +h(\eta + 2E\dot\epsilon_{xy}\dot\epsilon_{xy})\partial_y N_j \partial_y N_i \\
&\quad +Eh2(2\dot\epsilon_{xx} + \dot\epsilon_{yy})\dot\epsilon_{xy}\partial_y N_j \partial_x N_i \\
&\quad +Eh2\dot\epsilon_{xy}(2\dot\epsilon_{xx} + \dot\epsilon_{yy})\partial_x N_j \partial_y N_i \\
&= 4h\eta\partial_x N_j\,\partial_x N_i + h\eta\partial_y N_j \partial_y N_i \\
&\quad +2hE(2\dot\epsilon_{xx} + \dot\epsilon_{yy})^2\partial_x N_j\,\partial_x N_i \\
&\quad +2Eh\dot\epsilon_{xy}^2\partial_y N_j \partial_y N_i \\
&\quad +2Eh(2\dot\epsilon_{xx} + \dot\epsilon_{yy})\dot\epsilon_{xy}(\partial_y N_j \partial_x N_i + \partial_x N_j \partial_y N_i)
\end{aligned}
$$

so K_{11} and K_{22} are symmetrical.

One might expect that the $ud_u\beta^2\,\Delta v$ makes the matrix unsymmetrical, but in fact

$$u\,d_v\beta^2 = u(1/m-1)C^{-1/m}|\boldsymbol{v}|^{(1-3m)/m}v = v(1/m-1)C^{-1/m}|\boldsymbol{v}|^{(1-3m)/m}u = v\,d_u\beta^2$$

so the contributions to 12 and 21 are equal, and hence this term does not give rise to an unsymmetrical matrix.

The tangent matrix K is symmetrical for non-linear flow including both the non-linear effects of β^2 and η.

Note: In 1D I get

$$
\begin{aligned}
&4h(D\eta\,\partial_x u + \eta\,\partial_x \Delta u)\partial_x N_i + (D\beta^2\,u + \beta^2\,\Delta u)N_i \\
&\quad = +\frac{1}{2}\varrho g\cos\alpha\,\partial_x h^2\,\partial_x N_i - \rho g h(\partial_x s'\cos\alpha - \sin\alpha)\,N_i - 4h\eta\partial_x u\partial_x N - \beta^2 u\,N
\end{aligned}
$$

6.8 Weak form

x **direction**

$$\int_\Omega (\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) + \partial_y(h\eta(\partial_x v + \partial_y u)) - t_{bx} - \rho g h(\partial_x s\cos\alpha - \sin\alpha))N\,dx\,dy = 0$$

with von Neumann BC on Γ_2

$$(4h\eta\partial_x u + 2h\eta\partial_y v)n_x + \eta h(\partial_x v + \partial_y u)n_y = \frac{1}{2}\rho g h(h-H)n_x$$

and

$$\eta h(\partial_x v + \partial_y u)n_x + (4\eta h\partial_y v + 2\eta h\partial_x u)n_y = \frac{1}{2}\rho g h(h-H)n_y$$

Green's theorem used to get rid of second derivatives gives

$$
\begin{aligned}
&-\int_\Omega ((4h\eta\partial_x u + 2h\eta\partial_y v)\partial_x N + h\eta(\partial_x v + \partial_y u)\partial_y w)\,dx\,dy \\
&\quad -\int_\Omega (t_{bx} + \rho g h(\partial_x s\cos\alpha - \sin\alpha)N\,dx\,dy + \int_\Gamma ((4h\eta\partial_x u + 2h\eta\partial_y v)n_x + h\eta(\partial_x v + \partial_y u)n_y)N\,d\Gamma = 0
\end{aligned}
$$

Using the BC we have

$$
\begin{aligned}
-\int_\Omega &((4h\eta\partial_x u + 2h\eta\partial_y v)\partial_x N + h\eta(\partial_x v + \partial_y u)\partial_y w)\,dx\,dy \\
&- \int_\Omega (t_{bx} + \rho g h(\partial_x s\cos\alpha - \sin\alpha))N\,dx\,dy + \int_{\Gamma_2} \frac{1}{2} g\rho(1-\rho/\rho_o)h^2 n_x N\,d\Gamma = 0
\end{aligned}
$$

y direction

$$
\int_\Omega \partial_y((4h\eta\partial_y v + 2h\eta\partial_x u) + \partial_x(h\eta(\partial_y u + \partial_x v)) - t_{by} - \rho g h\partial_y s\cos\alpha)N\,dx\,dy
$$

Green's

$$
\begin{aligned}
-\int_\Omega &((4h\eta\partial_y v + 2h\eta\partial_x u)\partial_y N + h\eta(\partial_y u + \partial_x v)\partial_x w)\,dx\,dy \\
&- \int_\Omega (t_{by} + \rho g h\partial_y s\cos\alpha)N\,dx\,dy \\
&+ \int_\Gamma ((4h\eta\partial_y v + 2h\eta\partial_x u)n_y + \eta h(\partial_y u + \partial_x v)n_x)N\,d\Gamma
\end{aligned}
\tag{6.35}
$$

the von Neumann BC is

$$
\eta h(\partial_x v + \partial_y u)n_x + (4h\eta\partial_y v + 2h\eta\partial_x u)n_y = \frac{1}{2}\rho g h(h-H)n_y
$$

hence

$$
\begin{aligned}
-\int_\Omega &((4h\eta\partial_y v + 2h\eta\partial_x u)\partial_y N + h\eta(\partial_y u + \partial_x v)\partial_x N)\,dx\,dy \\
&- \int_\Omega (t_{by} + \rho g h\partial_y s\cos\alpha)N\,dx\,dy + \int_{\Gamma_2} \frac{1}{2} g\rho(1-\rho/\rho_o)h^2 n_y w\,d\Gamma = 0
\end{aligned}
$$

Ice shelf

$$
\int_\Omega (\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) + \partial_y(h\eta(\partial_x v + \partial_y u)) - t_{bx} - \frac{1}{2}\rho(1-\rho/\rho_o)g\partial_x h^2 N\,dx\,dy = 0
$$

On Γ_2 we write the von Neumann BC as

$$
(4\eta h\partial_x u + 2\eta h\partial_y v)n_x + \eta h(\partial_x v + \partial_y u)n_y = \frac{1}{2}\rho(1-\rho/\rho_o)gh^2 n_x
$$

and

$$
\eta(\partial_x v + \partial_y u)n_x + (4\eta\partial_y v + 2\eta\partial_x u)n_y = \frac{1}{2}\rho(1-\rho/\rho_o)gh^2 n_y
$$

we consider the term

$$
\int_\Omega (\partial_x(4h\eta\partial_x u + 2h\eta\partial_y v) + \partial_y(h\eta(\partial_x v + \partial_y u)) - \frac{1}{2}\rho(1-\rho/\rho_o)g\partial_x h^2)N\,dx\,dy = \tag{6.36}
$$

$$
-\int_\Omega (4h\eta\partial_x u + 2h\eta\partial_y v + h\eta(\partial_x v + \partial_y u) - \frac{1}{2}\rho(1-\rho/\rho_o)gh^2)\partial_x N\,dx\,dy \tag{6.37}
$$

$$
+\int_\Gamma ((4h\eta\partial_x u + 2h\eta\partial_y v)n_x + h\eta(\partial_x v + \partial_y u)n_y - \frac{1}{2}\rho(1-\rho/\rho_o)gh^2 n_x)N\,d\Gamma \tag{6.38}
$$

Along Γ_2, the path integral disappears and along Γ_1 we set $w^* = 0$, hence

$$
-\int_\Omega (4h\eta\partial_x u + 2h\eta\partial_y v + h\eta(\partial_x v + \partial_y u) - \frac{1}{2}\rho(1-\rho/\rho_o)gh^2)\partial_x w - t_{bx}N)\,dx\,dy = 0 \tag{6.39}
$$

6.9 Thoughts about ice shelf von Neumann BC

6.9.1 1d case

Field equation:

$$4\partial_x(\eta h \partial_x u) - t_x - \rho g h \partial_x s \cos\alpha + \rho g h \sin\alpha = 0$$

Boundary condition

$$4\eta\partial_x u = \frac{1}{2}\rho g(1-\rho/\rho_o)h \tag{6.40}$$

which for $\eta = \frac{1}{2}A^{-1/n}|\partial_x u|^{(1-n)/n}$ can also be written as

$$\partial_x u = A(\rho g f/4)^n$$

where $f = (1-\rho/\rho_o)h(x_c)$, and x_c is the location of the calving front. We write the field equation as

$$4\partial_x(\eta h \partial_x u) - \beta^2 u - \rho g h \partial_x(s' + (1-\rho/\rho_o)h)\cos\alpha + \rho g h \sin\alpha = 0$$

with

$$s' := f - (1-\rho/\rho_o)h = s - S - (1-\rho/\rho_o)h$$

or as

$$4\partial_x(\eta h \partial_x u) - \frac{1}{2}g\cos\alpha\rho(1-\rho/\rho_o)\partial_x h^2 - \rho g h \cos\alpha\partial_x s' - \beta^2 u + \rho g h \sin\alpha = 0,$$

using $\partial_x S = 0$. Here S is the elevation of sea level (usually the coordinate system would be defined so that $S = 0$), and s the surface elevation of the upper ice surface.

When deriving the weak form we do integration by terms on the first two terms

$$\begin{aligned}
&\int (4\partial_x(\eta h \partial_x u) - \frac{1}{2}g\cos\alpha\rho(1-\rho/\rho_o)\partial_x h^2 - \rho g h\cos\alpha\partial_x s' - t_x + \rho g h \sin\alpha)N\,dx \\
&= (4\eta h\partial_x u - \frac{1}{2}g\cos\alpha\rho(1-\rho/\rho_o)h^2)N|_{x_1}^{x_2} \\
&- \int (4\eta h\partial_x u - \frac{1}{2}g\cos\alpha\rho(1-\rho/\rho_o)h^2)\partial_x N\,dx \\
&- \int (\rho g\cos\alpha h\partial_x s' + t_x - \rho g h\sin\alpha)N\,dx
\end{aligned}$$

The neat thing about this formulation is that for the usual BC at the ice-shelf edge, the 'boundary integral term' is zero.

The quantity s' is the difference between the actual surface altitude above sea level, and the surface altitude above sea level if floating. On a floating ice shelf s' is equal to zero everywhere.

If all von Neumann boundary conditions are of the type (6.40) we only have to solve

$$\int ((-4\eta h\partial_x u + \frac{1}{2}g\cos\alpha\rho(1-\rho/\rho_o)h^2)\partial_x w - (\rho g\cos\alpha h\partial_x s' + t_x - \rho g h\sin\alpha)N)\,dx = 0$$

with

$$s' = s - S - (1-\rho/\rho_o)h$$

or

$$-\int 4\eta h\partial_x u\,\partial_x N\,dx - \int t_x\,N\,dx = \rho g\cos\alpha\int h\,\partial_x s' N\,dx - \frac{1}{2}\rho g\cos\alpha(1-\rho/\rho_o)\int h^2\partial_x N\,dx - \rho g\sin\alpha\int hN\,dx$$

Appendix A

Calculating vertical surface velocity

The sign convention for upper- and lower-surface mass balance is such that the kinematic boundary conditions at the upper and lower surfaces read, respectively,

$$\partial_t s + u\partial_x s + v\partial_y s - w_s = a_s, \tag{A.1}$$

$$\partial_t b + u\partial_x b + v\partial_y b - w_b = -a_b, \tag{A.2}$$

i.e. adding ice is always defined as positive surface mass balance.

Subtracting (A.2) from (A.1) gives

$$\partial_t h + u\,\partial_x h + v\,\partial_y h - w_s + w_b = a_s + a_b,$$

where it has been used that u does not change with depth. Now using (A.5) gives

$$\partial_t h + u\,\partial_x h + v\,\partial_y h + h(\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy}) = a_s + a_b,$$

or

$$\partial_t h + \partial_x(uh) + \partial_y(vh) = a_s + a_b, \tag{A.3}$$

hence, in the flux-conservation equation both upper and lower mass balance terms have the same sign.

A.1 grounded part

On the grounded part $\partial_t s = \partial_t h$ and the kinematic boundary condition gives

$$w_s = \partial_t h + u\,\partial_x s + v\,\partial_y s - a_s, \tag{A.4}$$

but this requires $\partial_t h$ to be known before we can calculate w_s. An alternative approach is to integrate the vertical strain rate $\dot{\epsilon}_{zz}$ over the thickness, use the mass conservation equation and the fact that horizontal strain rates do not change across the thickness, to arrive at

$$w_s = w_b - h\,(\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy}). \tag{A.5}$$

We now calculate w_b from the kinematic boundary condition at the lower surface as

$$w_b = a_b + u\partial_x b + v\partial_y b,$$

and insert into (A.5) arriving at

$$w_s = a_b + u\,\partial_x b + v\,\partial_y b - h\,(\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy}), \tag{A.6}$$

which represents a convenient way of calculation w_s once the horizontal velocity field has been determined.

A.2 floating part

Where the ice is afloat

$$s = S + (1 - \rho/\rho_o)\, h$$

i.e.

$$\partial_t s = (1 - \rho/\rho_o)\, \partial_t h$$

The kinematic boundary condition at the surface gives

$$w_s = \partial_t s + u\, \partial_x s + v\, \partial_y s - a_s$$

and therefore

$$w_s = (1 - \rho/\rho_o)\, \partial_t h + u\, \partial_x s + v\, \partial_y s - a_s \tag{A.7}$$

If $\partial_t h$ is known (A.7) can be used to calculate w_s, otherwise we use (A.3) and find that

$$w_s = (1 - \rho/\rho_o)\, (a_s + a_b - \partial_x q_x - \partial_y q_y) + u\, \partial_x s + v\, \partial_y s - a_s \tag{A.8}$$

An alternative way of calculating w_s is to insert the floating condition

$$s = (1 - \rho_o/\rho) b$$

into (A.1) and to use (A.2) to get rid of $\partial_t b$

$$(1 - \rho_o/\rho)(w_b - a_b - u\partial_x b - v p_y b) + (1 - \rho_o/\rho)(u\partial_x b + v\partial_y b) - w_s = a_s \tag{A.9}$$

to arrive at the simple and intuitive expression

$$(1 - \rho_o/\rho)(w_b - a_b) = a_s + w_s \tag{A.10}$$

and then to use the (A.5) to get rid of w_b leading to

$$w_s = -(1 - \rho/\rho_o)\ (h(\dot\epsilon_{xx} + \dot\epsilon_{yy}) - a_b) - a_s\, \rho/\rho_o \tag{A.11}$$

The above expression shows that adding ice to the surface ($a_s > 0$) causes neg. vertical surface velocity, as does horizontal divergent flow ($\dot\epsilon_{xx} + \dot\epsilon_{yy} > 0$), and basal melting ($a_b < 0$).

Appendix B

Simple 1d solution for an icestream

B.1 Problem definition:

Uniform ice thickness h on a constant sloping bed with slope α. The calving front position is at $x = l$. The calving front can be either grounded or floating, and d $0 \leq d < \rho h/\rho_o$.

$$4\partial_x(h\eta\partial_x u) - \beta^2\, u = \rho g h \partial_x s$$

with

$$\eta = \frac{1}{2} A^{-1/n}\, |\partial_x u|^{(1-n)/n}$$
$$\beta^2 = C^{-1/m} |u|^{(1-m)/m}$$

Boundary conditions:

$$u = C\rho g h \alpha \quad \text{at} \quad x = 0 \tag{B.1}$$
$$\tau_{xx} = \frac{1}{4h} g(\rho h^2 - \rho_o d^2) \quad \text{at} \quad x = l \tag{B.2}$$

Boundary condition (B.2) can also be written as

$$\partial_x u|_{x=l} = A\left(\frac{g(\rho h^2 - \rho_o d^2)}{4h}\right)^n$$

B.2 Solution:

The non-linear case is

$$\frac{2hA^{-1/n}}{n}(\partial_x u)^{1/n-1}\,\partial^2_{xx} u - C^{-1/m} u^{1/m} = -\rho g h \alpha$$

which I'm not sure if can be solved.

However the linear case

$$\partial^2_{xx} u - \kappa^2 u = -\frac{A\tau}{2h},$$

has the general solution

$$u = c_1 e^{\kappa x} + c_2 e^{-\kappa x} + C\tau,$$

with

$$\kappa^2 = \frac{A}{2hC},$$

and

$$\tau = \rho g h \alpha$$

BCs (B.1) and (B.2) give

$$c_1 + c_2 + C\,\tau = C\,\tau$$
$$c_1 \kappa e^{\kappa l} - c_2 \kappa e^{-\kappa l} = K$$

where

$$K = A\frac{g(\rho h^2 - \rho_o d^2)}{4h}.$$

Hence

$$u = C\tau + \frac{K \sinh \kappa x}{\kappa \cosh \kappa l}.$$

Appendix C

Integral theorem

If f and g are scalar functions then in x and y directions we have

$$\int_{\Omega} f\,\partial_x g\; d\Omega = -\int_{\Omega} \partial_x f\; g\; d\Omega + \oint_{\partial\Omega} f\,g\,n_x\; d\Gamma \tag{C.1}$$

$$\int_{\Omega} f\,\partial_y g\; d\Omega = -\int_{\Omega} \partial_y f\; g\; d\Omega + \oint_{\partial\Omega} f\,g\,n_y\; d\Gamma \tag{C.2}$$

If we write $g = g_x$ in the upper one and $g = g_y$ in the lower one, add them together and define $\boldsymbol{g} = (g_x, g_y)^T$ and $\hat{\boldsymbol{n}} = (n_x, n_y)^T$ then we arrive at

$$\int_{\Omega} f\,\nabla_{xy} \cdot \boldsymbol{g}\; d\Omega = -\int_{\Omega} \nabla_{xy} f \cdot \boldsymbol{g}\; d\Omega + \oint_{\partial\Omega} f\,\boldsymbol{g} \cdot \hat{\boldsymbol{n}}\; d\Gamma \tag{C.3}$$

Appendix D

Buttressing

It is sometimes convenient to define a *buttressing parameter* θ as

$$\theta = \frac{N}{\frac{1}{2}\varrho g h}$$

where

$$N = \hat{\boldsymbol{n}}_h^T \cdot (\boldsymbol{R}\hat{\boldsymbol{n}}_h) \tag{D.1}$$

and

$$\varrho = \rho(1 - \rho/\rho_o),$$

and where $\hat{\boldsymbol{n}}_h$ is a normal vector pointing horizontally outwards from the grounding line. Buttressing is the difference between the normal stress at the grounding line with and without an iceshelf.

In the particular case of a floating ice shelf, the field equations Eq. (1.4) can be written as

$$\nabla_{xy}^T \cdot (h\,\boldsymbol{R}) = \frac{1}{2}\nabla_{xy}^T(\varrho g \rho h^2),$$

Using the divergence theorem we find

$$\oint (\boldsymbol{R} \cdot \hat{\boldsymbol{n}}_h - \frac{1}{2}\varrho g \rho h\,\hat{\boldsymbol{n}}_h)\,d\Gamma = 0$$

The integrand is identical to the (point wise) expression of the force balance (1.27) at the calving front of a freely floating ice shelf. We can split this path integral into a 1) section along the grounding line, 2) section along the margins, and 3) section along the calving front. If the margins do not contribute, the contribution along the grounding line is equal to that of the calving front. Hence, unbuttressed uniformly wide ice shelves are passive and don't provide any buttressing.

Appendix E

Definition of gradients in terms of directional derivatives and inner products

Sensitivities are directional derivatives. The directional derivative of the scalar function $J(p)$ in the direction ϕ is denoted by $Df(p)[\phi]$ and defined as

$$\begin{aligned} Df(p)[\phi] &= \lim_{\epsilon \to 0} \frac{d}{d\epsilon} f(p + \epsilon\phi) \\ &= \lim_{\epsilon \to 0} \frac{f(p + \epsilon\phi) - f(p)}{\epsilon} \end{aligned}$$

The directional derivative is sometimes written as $\delta f(p, \phi)$ or as $f'(p, \phi)$ i.e.

$$Df(p)[\phi] = f'(p, \phi) = \delta f(p, \phi)$$

are just different ways of writing the directional derivative.

We define the gradient through

$$Df(p)[\phi] =< \nabla J(p), \phi >_H$$

where H is a Hilbert space and $f : H \to \mathbb{R}$. Here $\nabla J(p)$ is the gradient of J, and the expression above *defines* the gradient in terms of the (directional) derivative for a given inner product.

In other words, for a function $f : H \to \mathbb{R}$ the gradient is defined as the Riez-representation for the directional derivative $Df(p)[\phi]$ through

$$< \nabla f(p), \phi >_H = Df(p)[\phi]$$

The directional derivative depends on the inner product $<, >_H$ and the gradient is not defined without specifying the inner product.

Example: Consider the case $H = \mathbb{R}^n$ with the inner product

$$< \boldsymbol{x}, \boldsymbol{y} >= \boldsymbol{x}^T \boldsymbol{M} \boldsymbol{y}$$

where M is symmetric and positive definite (for example the mass matrix or any covariance matrix.)

The directional derivative is

$$Df(p)[\phi] = \frac{\partial f}{\partial p_q} \phi_q =< M^{-1} \partial f / \partial p_q, \phi_q >$$

and therefore

$$\nabla f = [M^{-1}]_{pq}\, \partial f/\partial \phi_q$$

Had we instead used the Euclidean inner product as $< \boldsymbol{x}, \boldsymbol{y} >= \boldsymbol{x}^T \boldsymbol{y}$ the corresponding Euclidean gradient would be

$$[\nabla_E f]_p = \partial f/\partial \phi_q$$

or

$$\nabla f = M^{-1}\, \nabla_E f$$

where the subscript E denotes the Euclidean gradient. This distinction becomes important in the application of the adjoint method where we obtain a gradient that is dependent on the natural FE inner product

$$\begin{aligned} < f, g > &= \int f\, g\, dA \\ &= \int f_p\, \phi_p\, g_q\, \phi_q dA \\ &= \boldsymbol{f} M \boldsymbol{g} \end{aligned}$$

Hence in FE the dual pairing is

$$< f, g >= \boldsymbol{f}^T \boldsymbol{M} \boldsymbol{g}$$

where f.

The adjoint L^* of a given operator L is defined as

$$< L^* f, g >=< f, Lg >$$

for any f and g.

If we are working in $H_1 = \mathbb{R}^{d_1}$ and the dual space is $H_2 = \mathbb{R}^{d_2}$ and

$$< f, g >_{H_1,H_2}= \boldsymbol{f}^T \boldsymbol{g}$$

and

$$< L^* f, g >_{H_1,H_2}=< f, Lg >_{H_1,H_2}$$

and we denote by $\boldsymbol{L}$ and $\boldsymbol{L}^*$the matrix representations of the continuous linear operators L and L^*, respectively, then

$$\boldsymbol{L}^* = \boldsymbol{L}^T$$

If, on the other hand, we have the dual pairings

$$< f, g >_{H_1,H_2}= \boldsymbol{f}^T \boldsymbol{M} \boldsymbol{g}$$

where $\boldsymbol{M}$ is a positive definite matrix, then

$$\boldsymbol{L}^* = \boldsymbol{M}^{-T} \boldsymbol{L}^T \boldsymbol{M}^T$$

as can be seen as follows

$$\begin{aligned} < M^* f, g > &=< f, Lg > \\ &= \boldsymbol{f}^T \boldsymbol{M} (\boldsymbol{L} \boldsymbol{g}) \\ &= \boldsymbol{f}^T \boldsymbol{M} \boldsymbol{L} \boldsymbol{M}^{-1} \boldsymbol{M} \boldsymbol{g} \\ &= (\boldsymbol{M}^{-T} \boldsymbol{L}^T \boldsymbol{M}^T \boldsymbol{f})^T \boldsymbol{M} \boldsymbol{g} \\ &=< \boldsymbol{M}^{-T} \boldsymbol{L}^T \boldsymbol{M}^T \boldsymbol{f}, \boldsymbol{g}) \end{aligned}$$

We can generalise this a bit further and consider the case where the dual space has a different dimension with

$$< f, f >_{H_1 H_1} = \boldsymbol{f}^T \boldsymbol{M}_1 \, \boldsymbol{f}$$
$$< g, g >_{H_2 H_2} = \boldsymbol{g}^T \boldsymbol{M}_2 \, \boldsymbol{g}$$

and find that

$$\boldsymbol{L}^* = \boldsymbol{M}_1^{-T} \boldsymbol{L}^T \boldsymbol{M}_2^T$$