Stat 421: Probability Theory
Fall 2012
Homework 2–Supplement

1. If n balls are placed at random into n cells, find the probability that exactly one cell remains empty.

   Solution: The denominator is n^n because this is the number of ways to place n balls in n cells. The numerator is the number of ways of placing the balls such that exactly one cell is empty. There are n ways to specify the empty cell. There are $n-1$ ways of choosing the cell with two balls. There are $\binom{n}{2}$ ways of picking the 2 balls to go into this cell. And there are $(n-2)!$ ways of placing the remaining $n-2$ balls into the $n-2$ cells, one ball in each cell. The product of these is the numerator $n \times (n-1) \times (n-2)! \times \binom{n}{2} = n!\binom{n}{2}$. The probability is $n!\binom{n}{2}/n^n$.

2. (U) If a multivariate function has continuous partial derivatives, the order in which the derivatives are calculated does not matter. Thus for example, the function $f(x,y)$ of two variables has equal third partials
$$\frac{\partial^3}{\partial x^2 \partial y} f(x,y) = \frac{\partial^3}{\partial y \partial x^2} f(x,y).$$
   Prove that a function of n variables has $\binom{n+k-1}{k}$ k^{th} partial derivatives.

   Solution: Think of the n variables as n bins. Differentiating with respect to one of the variables is equivalent to putting a marker in a bin. Thus there are k identical (indistinguishable) markers to be placed in n bins, and there are $\binom{n+k-1}{k}$ ways to do this. (See lecture notes for the number of ways k markers can be placed in n bins.)

3. A closet contains n pairs of shoes. If $2r$ shoes are chosen at random ($2r < n$), what is the probability that there will be no matching pair in the sample?

   Solution: There are $\binom{2n}{2r}$ ways of choosing $2r$ shoes from a total of $2n$ shoes. Thus there are $\binom{2n}{2r}$ equally likely outcomes in the sample space. The numerator is the number of outcomes for which there will be no matching pair. There are $\binom{n}{2r}$ ways of choosing 2r different shoes style. There are two ways of choosing within a given shoe style (left shoe or right shoe), which gives 2^{2r} ways of arranging each one of the $\binom{n}{2r}$ arrays. The product of this is the numerator $\binom{n}{2r} \times 2^{2r}$. the probability is $\binom{n}{2r} \times 2^{2r}/\binom{2n}{2r}$.

4. (G) Two people each toss a fair coin n times. Find the probability that they will toss the same number of heads.

   Solution: There are 2^n possible outcomes person A and there are 2^n possible outcomes for person B. So altogether there are $2^n \times 2^n = 2^{2n} = 4^n$ possible outcomes in the sample space. There are $\binom{n}{k}$ ways for person A to toss k heads and there are also $\binom{n}{k}$ for person B to toss k heads. So in total there are $\binom{n}{k}\binom{n}{k}$ ways for both A and B to toss k heads each. Therefore,
$$P(\text{“A and B toss same number of heads”}) = \frac{\sum_{k=0}^{n} \binom{n}{k}\binom{n}{k}}{4^n} = \frac{\binom{2n}{n}}{4^n}.$$
   The second equality follows from one of the mini-homework assignments.

5. (G) Prove that

$$\sum_{\kappa} 1 = \binom{n+k-1}{n}$$

where $\kappa = \{(n_1, n_2, \ldots, n_k) : n_i \in \mathbb{N} \text{ and } \sum_{i=1}^{k} n_i = n\}$.

Solution: Notice that $\sum_{\kappa} 1 = |\kappa|$, the number of elements in κ. Each element in κ is one possible way of putting n identical markers in k bins where n_i is the number of markers in bin i. All possible ways of putting n markers in k bins are included in κ. Therefore $|\kappa|$ is the number of ways n markers can be distributed into k bins. That is, $|\kappa| = \binom{n+k-1}{n}$. (See lecture notes for the number of ways of distributing n markers into k bins.)