CPH 687 Homework 2

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1 Problem 1

Question:
Suppose that $\boldsymbol{C}$ is nonsingular, show that the eigenvalues of $\boldsymbol{C^{-1}AC}$ are the same as the eigenvalues of $\boldsymbol{A}$
Proof:

$$
\begin{aligned}
&\text{Assume that the eigenvalues of } \boldsymbol{C^{-1}AC} \text{ are } \lambda \\
&\text{Then } \boldsymbol{C^{-1}ACx} = \lambda \boldsymbol{x} \\
\therefore &\boldsymbol{ACx} = \lambda \boldsymbol{Cx} \\
\therefore &\text{The eigenvalues of } \boldsymbol{A} \text{ are the same as the eigenvalues of } \boldsymbol{C^{-1}AC}.
\end{aligned}
$$

2 Problem 2

Question:
If the eigenvalues of $\boldsymbol{A}$ satisfy $|\lambda_i| < 1$ for all $\boldsymbol{A}$ is diagonable, show that $(\boldsymbol{I}-\boldsymbol{A})^{-1} = \Sigma_{i=0}^{\infty} A^i$
Proof:

$$
\begin{aligned}
\because \boldsymbol{A} is diagonable& \\
\therefore \boldsymbol{A} &= \boldsymbol{T\Lambda T'} \\
\boldsymbol{A^i} &= \boldsymbol{T\Lambda^i T'} \\
\therefore \Sigma_{i=0}^{\infty}\boldsymbol{A^i} &= \boldsymbol{T}\Sigma_{i=0}^{\infty}\boldsymbol{\lambda^i T^i} \\
(\boldsymbol{I}-\boldsymbol{A})^{-1} &= (\boldsymbol{T}(\boldsymbol{I}-\boldsymbol{\Lambda})\boldsymbol{T'})^{-1} \\
&= \boldsymbol{T}(\boldsymbol{I}-\boldsymbol{\Lambda})^{-1}\boldsymbol{T'} \\
(\boldsymbol{I}-\boldsymbol{\Lambda})^{-1} &= (diag(1-\lambda_1, 1-\lambda_2, \cdots, 1-\lambda_n))^{-1} \\
&= diag(\frac{1}{1-\lambda_1}, \frac{1}{1-\lambda_2}, \cdots, \frac{1}{1-\lambda_n}) \\
\Sigma_{i=0}^{\infty}\boldsymbol{\Lambda}^i &= diag(\Sigma_{i=0}^{\infty}\lambda_1^i, \Sigma_{i=0}^{\infty}\lambda_2^i, \cdots, \Sigma_{i=0}^{\infty}\lambda_n^i) \\
&= diag(\frac{1}{1-\lambda_1}, \frac{1}{1-\lambda_2}, \cdots, \frac{1}{1-\lambda_n}) \\
\therefore (\boldsymbol{I}-\boldsymbol{\Lambda})^{-1} &= \Sigma_{i=0}^{\infty}\Lambda^i \\
\therefore (\boldsymbol{I}-\boldsymbol{A})^{-1} &= \Sigma_{i=0}^{\infty}A^i.
\end{aligned}
$$

3 Problem 3

Question:
Let $Q(\boldsymbol{t}) = \frac{t'\boldsymbol{A}t}{t'\boldsymbol{M}t}$, where $\boldsymbol{M_{n\times n}}$ is positive definite and $\boldsymbol{A_{n\times n}}$ is symmetric. Then, show that

$$\max_{\boldsymbol{t}\neq 0} Q(\boldsymbol{t}) = \lambda_1 \text{ and } \min_{\boldsymbol{t}\neq 0} Q(\boldsymbol{t}) = \lambda_n.$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of $\boldsymbol{M^{-1}A}$.
Proof:

$$\begin{aligned}
&\because \boldsymbol{M} \text{ is positive definite}\\
&\therefore \text{There exists invertible matrix } \boldsymbol{C} \text{ such that } \boldsymbol{M} = \boldsymbol{C'C}\\
&\therefore Q(\boldsymbol{t}) = \frac{t'\boldsymbol{A}t}{t'\boldsymbol{C'C}t}\\
&\text{Let } \boldsymbol{y} = \boldsymbol{Ct}\\
&\text{Then } \boldsymbol{t} = \boldsymbol{C'y}\\
&\therefore Q(\boldsymbol{t}) = Q(\boldsymbol{y}) = \frac{\boldsymbol{y'(C^{-1})'AC^{-1}y}}{\boldsymbol{y'y}}\\
&\therefore \max_{\boldsymbol{y}\neq 0} Q(\boldsymbol{y}) = \alpha_1 \text{ and } \min_{\boldsymbol{y}\neq 0} Q(\boldsymbol{y}) = \alpha_n\\
&\text{where } \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \text{ are the eigenvalues of } \boldsymbol{(C^{-1})'AC^{-1}}\\
&\because \text{the eigenvalues of } \boldsymbol{AB} \text{ are equal to the eigenvalues of } \boldsymbol{BA}\\
&\therefore \text{the eigenvalues of } \boldsymbol{(C^{-1})'AC^{-1}} \text{ are equal to the eigenvalues of } \boldsymbol{C^{-1}(C^{-1})'A}\\
&\because \boldsymbol{C^{-1}(C^{-1})'} = \boldsymbol{M^{-1}}\\
&\therefore \lambda_i = \alpha_i (i = 1, 2, \cdots, n)\\
&\therefore \max_{\boldsymbol{t}\neq 0} Q(\boldsymbol{t}) = \lambda_1 \text{ and } \min_{\boldsymbol{t}\neq 0} Q(\boldsymbol{t}) = \lambda_n.
\end{aligned}$$

4 Problem 4

Question:
Given that $\boldsymbol{X_{n\times p}}$ has full rank, show $\boldsymbol{X'X}$ is positive definite.
Proof:

$$\begin{aligned}
&\because \boldsymbol{X} \text{ has full rank}\\
&\therefore \boldsymbol{Xt} \neq 0 (\boldsymbol{t} \neq 0)\\
&\therefore \boldsymbol{t'X'Xt} = (\boldsymbol{Xt})'(\boldsymbol{Xt}) > 0\\
&\therefore \boldsymbol{X'X} \text{ is positive definite.}
\end{aligned}$$