CPH687 Homework 4

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1 Problem 1

Question:
Let y_i have mean μ and variance σ^2, which are scalars. Find the expecion of the estimator of σ^2

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2.$$

using matrix notations $\boldsymbol{y_{n\times 1}} = (y_1, y_2, \cdots, y_n)'$ and $\bar{y} = \boldsymbol{1_n'y}/n$
Solution:

$$\begin{aligned}
S^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2 \\
&= \frac{1}{n-1}(\sum_{i=1}^{n} y_i^2 - n(\bar{y})^2) \\
E(S^2) &= \frac{1}{n-1}(\sum_{i=1}^{n} E(y_i^2) - nE(\bar{y}^2)) \\
&= \frac{1}{n-1}(\sum_{i=1}^{n}(Var(y_i) + (E(y_i))^2) - n(Var(\bar{}) + (E(\bar{y}))^2)) \\
&= \frac{1}{n-1}(n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2) \\
&= \sigma^2.
\end{aligned}$$

2 Problem 2

Question:
Let $\boldsymbol{y} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where Σ is p.d. Verify that the MGF of $\boldsymbol{y}$ is

$$M_{\boldsymbol{y}}(\boldsymbol{t}) = exp(\boldsymbol{t'\mu} + \frac{1}{2}\boldsymbol{t'\Sigma t}).$$

Solution:

$$
\begin{aligned}
M_{\boldsymbol{y}}(\boldsymbol{t}) &= E(e^{\boldsymbol{t'y}}) \\
&= C\int_{-\infty}^{\infty} e^{\boldsymbol{t'y}} e^{-\frac{1}{2}\boldsymbol{(y-\mu)'\Sigma^{-1}(y-\mu)}}\, d\boldsymbol{y} \\
C &= \frac{1}{(2\pi)^{-n/2}}|\boldsymbol{\Sigma}|^(-1/2) \\
&= \int_{-\infty}^{\infty} e^{-\frac{1}{2}\boldsymbol{((y-\mu)'\Sigma^{-1}(y-\mu)-2t'y)}}\, d\boldsymbol{y} \\
\boldsymbol{(y-\mu)'\Sigma^{-1}(y-\mu)} - 2\boldsymbol{t'y} &= \boldsymbol{y'\Sigma^{-1}y} - 2\boldsymbol{y'\Sigma^{-1}\mu + \mu'\Sigma^{-1}\mu - 2t'y} \\
&= \boldsymbol{y'\Sigma^{-1}y - 2y'\Sigma^{-1}\mu + \mu'\Sigma^{-1}\mu - 2y'\Sigma^{-1}\Sigma t} \\
&= \boldsymbol{y'\Sigma^{-1}y - 2y'\Sigma^{-1}(\mu+\Sigma t) + \mu'\Sigma^{-1}\mu} \\
&= \boldsymbol{[y-(\mu+\Sigma t)]'\Sigma^{-1}[y-(\mu+\Sigma t)] - (\mu+\Sigma t)'\Sigma^{-1}(\mu+\Sigma t) + \mu'\Sigma^{-1}\mu} \\
&= \boldsymbol{[y-(\mu+\Sigma t)]'\Sigma^{-1}[y-(\mu+\Sigma t)] - (2\mu't + t'\Sigma t)} \\
&= \boldsymbol{[y-(\mu+\Sigma t)]'\Sigma^{-1}[y-(\mu+\Sigma t)] - (2t'mu + t'\Sigma t)} \\
\therefore M_{\boldsymbol{y}}(\boldsymbol{t}) &= C\int_{-\infty}^{\infty} e^{\boldsymbol{([y-(\mu+\Sigma t)]'\Sigma^{-1}[y-(\mu+\Sigma t)]-(2t'mu+t'\Sigma t))}}\, d\boldsymbol{y} \\
&= e^{\boldsymbol{t'\mu}+\frac{1}{2}\boldsymbol{t'\Sigma t}} C\int_{-\infty}^{\infty} e^{-\frac{1}{2}\boldsymbol{([y-(\mu+\Sigma t)]'\Sigma^{-1}[y-(\mu+\Sigma t)])}}\, d\boldsymbol{y} \\
&= e^{\boldsymbol{t'\mu}+\frac{1}{2}\boldsymbol{t'\Sigma t}}.
\end{aligned}
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3 Problem 3

Question:
Let $\boldsymbol{A_{r\times n}}$ be a matrix of constants. Verify that $\boldsymbol{Ay} \sim N(\boldsymbol{A\mu}, \boldsymbol{A\Sigma A'})$.
Solution:

$$
\begin{aligned}
E(\boldsymbol{Ay}) &= \boldsymbol{A}E(\boldsymbol{y}) = \boldsymbol{A\mu} \\
Var(\boldsymbol{Ay}) &= \boldsymbol{A}Var(\boldsymbol{y})\boldsymbol{A'} = \boldsymbol{A\Sigma A'} \\
f(\boldsymbol{y}|\boldsymbol{\mu},\boldsymbol{\Sigma}) &= \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{-1/2}} exp[-\frac{1}{2}\boldsymbol{(y-\mu)'\Sigma^{-1}(y-\mu)}] \\
\boldsymbol{x} = \boldsymbol{Ay} &\rightarrow \boldsymbol{y} = \boldsymbol{A^{-1}x} \\
f(\boldsymbol{A^{-1}x}|\boldsymbol{\mu},\boldsymbol{\Sigma}) &= \frac{1}{(2\pi)^{n/2}|\boldsymbol{A\Sigma A'}|^{-1/2}} exp[-\frac{1}{2}\boldsymbol{(A^{-1}x-\mu)'\Sigma^{-1}(A^{-1}x-\mu)}] \\
&= \frac{1}{(2\pi)^{n/2}|\boldsymbol{A\Sigma A'}|^{-1/2}} exp[-\frac{1}{2}\boldsymbol{(A^{-1}x-A^{-1}A\mu)'\Sigma^{-1}(A^{-1}x-A^{-1}A\mu)}] \\
&= \frac{1}{(2\pi)^{n/2}|\boldsymbol{A\Sigma A'}|^{-1/2}} exp[-\frac{1}{2}\boldsymbol{(A^{-1}(x-A\mu))'\Sigma^{-1}A^{-1}(x-A\mu)}] \\
&= \frac{1}{(2\pi)^{n/2}}|\boldsymbol{A\Sigma A'}|^{-1/2} exp[-\frac{1}{2}\boldsymbol{(x-A\mu)'(A\Sigma A')^{-1}(x-A\mu)}] \\
&\therefore \boldsymbol{Ay} \sim N(\boldsymbol{A\mu}, \boldsymbol{A\Sigma A'}).
\end{aligned}
$$

4 Problem 4

Question:
If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are n-dimensional vectors with independent multivariate normal distributions, prove that $a\boldsymbol{X} + b\boldsymbol{Y}$ is also multivariate normal.
Solution:

$$
\begin{aligned}
M_{a\boldsymbol{X}+b\boldsymbol{Y}}(\boldsymbol{t}) &= E(e^{\boldsymbol{t'}(a\boldsymbol{X}+b\boldsymbol{Y})}) \\
&= E(e^{a\boldsymbol{t'X}}e^{b\boldsymbol{t'Y}}) \\
\because &\boldsymbol{X} \text{ and } \boldsymbol{Y} \text{ are n-dimensional vectors with independent multivariate normal distributions} \\
\therefore M_{a\boldsymbol{X}+b\boldsymbol{Y}}(\boldsymbol{t}) &= E(e^{a\boldsymbol{t'X}}e^{b\boldsymbol{t'Y}}) \\
&= E(e^{a\boldsymbol{t'X}})E(e^{b\boldsymbol{t'Y}}) \\
&= exp(a\boldsymbol{t'}\boldsymbol{\mu_X} + \frac{1}{2}a^2\boldsymbol{t'}\boldsymbol{\Sigma_X}\boldsymbol{t})exp(b\boldsymbol{t'}\boldsymbol{\mu_Y} + \frac{1}{2}b^2\boldsymbol{t'}\boldsymbol{\Sigma_Y}\boldsymbol{t}) \\
&= exp(\boldsymbol{t'}(a\boldsymbol{\mu_X} + b\boldsymbol{\mu_Y}) + \frac{1}{2}\boldsymbol{t'}(a^2\boldsymbol{\Sigma_X} + b^2\boldsymbol{\Sigma_Y})) \\
\therefore &a\boldsymbol{X} + b\boldsymbol{Y} \text{ is also multivariate normal with mean } a\boldsymbol{\mu_X} + b\boldsymbol{\mu_Y} \text{ and variance } a^2\boldsymbol{\Sigma_X} + b^2\boldsymbol{\Sigma_Y}.
\end{aligned}
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5 Problem 5

Question:
If $\boldsymbol{Y} \sim N_2(\boldsymbol{0}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma} = (\sigma_{ij})$, prove that

$$
(\boldsymbol{Y'\Sigma^{-1}Y} - \frac{Y_1^2}{\sigma_{11}}) \sim \chi_1^2.
$$

Solution:

$$
\begin{aligned}
\boldsymbol{Y'\Sigma^{-1}Y} &= \boldsymbol{Y'\Sigma^{-1/2}\Sigma^{-1/2}Y} \\
&= \boldsymbol{(\Sigma^{-1/2}Y)'\Sigma^{-1/2}Y} \\
\text{Let } \boldsymbol{Z} &= \boldsymbol{\Sigma^{-1/2}Y} \\
Var(\boldsymbol{Z}) &= I_2 \\
\text{From the Problem 3, we know that } \boldsymbol{Z} &\sim N_2(\boldsymbol{0}, \boldsymbol{I}) \\
\therefore (\boldsymbol{Z'Z} - \frac{Y_1^2}{\sigma_{11}}) &\sim \chi_1^2.
\end{aligned}
$$