CPH 687 Homework 1

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1 Problem 1

Question:
Proof the following. If $\boldsymbol{A}$ is an $n \times p$ matrix and $\boldsymbol{B}$ a $p \times q$ matrix, then the product $\boldsymbol{C} = \boldsymbol{AB}$ has the following properties:
(a) Every column of $\boldsymbol{C}$ is a linear combination of columns of $\boldsymbol{A}$
(b) Every row of $\boldsymbol{C}$ is a linear combination of rows of $\boldsymbol{B}$
Proof:
(a)

$$\begin{aligned}
&\text{Let } \boldsymbol{e_j} \text{ be a } q \times 1 \text{ vector that the } jth \text{ position will be 1 the other places will be zero} \\
\text{Then } \boldsymbol{c_j} &= \boldsymbol{C} \times \boldsymbol{e_j} \\
&= \boldsymbol{AB} \times \boldsymbol{e_j} \\
&= \boldsymbol{A} \times \boldsymbol{b_j} \\
&= (a_1, a_2, \cdots, a_p)\boldsymbol{b_j} \\
&= \Sigma_{i=1}^{p} a_i b_{ij} \\
&\text{Every column of } \boldsymbol{C} \text{ is a linear combination of columns of } \boldsymbol{A}.
\end{aligned}$$

(b)

$$\begin{aligned}
&\text{Let } \boldsymbol{e_i} \text{ be a } 1 \times n \text{ vector that the } ith \text{ position will be 1 and the other places will be zero} \\
\text{Then } \boldsymbol{c_i} &= \boldsymbol{e_i} \times \boldsymbol{C} \\
&= \boldsymbol{e_i} \times \boldsymbol{AB} \\
&= \boldsymbol{a_i} \times \boldsymbol{B} \\
&= \boldsymbol{a_i}(b_1, b_2, \cdots, b_p)' \\
&= \Sigma_{i=1}^{p} a_{ij} b_j \\
&\text{Every row of } \boldsymbol{C} \text{ is a linear combination of rows of } \boldsymbol{B}.
\end{aligned}$$

2 Problem 2

Question:
Show directly that if $\boldsymbol{A}$ is an $n \times p$ matrix and $\boldsymbol{B}$ is $p \times n$, then

$$(\boldsymbol{I_n}-\boldsymbol{AB})^{-1}=\boldsymbol{I_n}+\boldsymbol{A}(\boldsymbol{I_p}-\boldsymbol{BA})^{-1}\boldsymbol{B}.$$

provided the inverses exist. (This verifies a simplified version of the Woodbury binomial inverse theorem.)
Proof:

$$\begin{aligned}(\boldsymbol{I_n}-(\boldsymbol{AB}))(\boldsymbol{I_n}+\boldsymbol{A}(\boldsymbol{I_p}-\boldsymbol{BA})^{-1}\boldsymbol{B}) &= \boldsymbol{I_n}+\boldsymbol{A}(\boldsymbol{I_p}-\boldsymbol{BA})^{-1}\boldsymbol{B}-\boldsymbol{AB}-\boldsymbol{ABA}(\boldsymbol{I_p}-\boldsymbol{BA})^{-1}B\\ &= \boldsymbol{I_n}-\boldsymbol{AB}+\boldsymbol{A}[(\boldsymbol{I_p}-\boldsymbol{BA})^{-1}-\boldsymbol{BA}(\boldsymbol{I_p}-\boldsymbol{BA})^{-1}]\boldsymbol{B}\\ &= \boldsymbol{I_n}-\boldsymbol{AB}+\boldsymbol{A}(\boldsymbol{I_p}-\boldsymbol{BA})(\boldsymbol{I_p}-\boldsymbol{BA})^{-1}\boldsymbol{B}\\ &= \boldsymbol{I_n}-\boldsymbol{AB}+\boldsymbol{AB}=\boldsymbol{I_n}\\ \therefore(\boldsymbol{I_n}-\boldsymbol{AB})^{-1} &= \boldsymbol{I_n}+\boldsymbol{A}(\boldsymbol{I_p}-\boldsymbol{BA})^{-1}\boldsymbol{B}.\end{aligned}$$

3 Problem 3

Question:
Verify that the Moore-Penrose inverse $\boldsymbol{A}^+$ of a symmetric matrix $\boldsymbol{A}$ is symmetric. (Thus, $\boldsymbol{A}^+$ is also a Moore-Penrose inverse of $\boldsymbol{A}$)
Proof:

$$\begin{aligned}&\boldsymbol{AA}^+\boldsymbol{A}=A\Rightarrow\boldsymbol{AA}^{+'}\boldsymbol{A}=\boldsymbol{A}\\ &\boldsymbol{A}^+\boldsymbol{AA}^+=\boldsymbol{A}^+\Rightarrow\boldsymbol{A}^{+'}\boldsymbol{AA}^{+'}=\boldsymbol{A}^{+'}\\ &(\boldsymbol{AA}^+)'=\boldsymbol{AA}^+\Rightarrow\boldsymbol{A}^{+'}\boldsymbol{A}=(\boldsymbol{A}^{+'}\boldsymbol{A})'\\ &(\boldsymbol{A}^+\boldsymbol{A})'=\boldsymbol{A}^+\boldsymbol{A}\Rightarrow\boldsymbol{AA}^{+'}=(\boldsymbol{AA}^{+'})'.\end{aligned}$$

4 Problem 4

Question:
Use the fact that $r(\boldsymbol{AB})\leq\min(r(\boldsymbol{A}),r(\boldsymbol{B}))$, show that

$$r(\boldsymbol{A})=r(\boldsymbol{PA}),r(\boldsymbol{A})=r(\boldsymbol{AQ}).$$

if $\boldsymbol{P}$ and $\boldsymbol{Q}$ are invertible matrices.
Proof:

$$\begin{aligned}&r(\boldsymbol{PA})\leq\min(r(\boldsymbol{A}),r(\boldsymbol{P}))\leq r(\boldsymbol{A})\\ &\because\boldsymbol{A}=(\boldsymbol{P})^{-1}\boldsymbol{PA}\\ &\therefore r(\boldsymbol{A})=r((\boldsymbol{P})^{-1}\boldsymbol{PA})\leq\min(r((\boldsymbol{P})^{-1}),r(\boldsymbol{PA}))\leq r(\boldsymbol{PA})\\ &\therefore r(\boldsymbol{A})=r(\boldsymbol{PA})\text{ if }\boldsymbol{P}\text{ is invertible matrix.}\end{aligned}$$

$r(\boldsymbol{AQ}) \leq \min(r(\boldsymbol{A}), r(\boldsymbol{Q})) \leq r(\boldsymbol{A})$
$\because \boldsymbol{A} = \boldsymbol{AQ}(\boldsymbol{Q})^{-1}$
$\therefore r(\boldsymbol{A}) = r(\boldsymbol{AQ}(\boldsymbol{Q})^{-1}) \leq \min(r((\boldsymbol{Q})^{-1}), r(\boldsymbol{AQ})) \leq r(\boldsymbol{AQ})$
$\therefore r(\boldsymbol{A}) = r(\boldsymbol{AQ})$ if $\boldsymbol{Q}$ is invertible matrix.