1. (2) $\cos\alpha_1=\frac{1}{3}$ $\cos\alpha_2=\frac{2}{3}$ $\cos\alpha_3=\frac{2}{3}$

$\frac{\partial u}{\partial x}\Big|_{(1,0,1)}=\frac{x}{\sqrt{x^2+y^2+z^2}}\Big|_{(1,0,1)}=\frac{\sqrt{2}}{2}$ $\frac{\partial u}{\partial y}\Big|_{(1,0,1)}=\frac{y}{\sqrt{x^2+y^2+z^2}}\Big|_{(1,0,1)}=0$ $\frac{\partial u}{\partial z}\Big|_{(1,0,1)}=\frac{\sqrt{2}}{2}$

∴ 方向导数：$\frac{1}{3}\times\frac{\sqrt{2}}{2}+\frac{2}{3}\times\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}$

(4) $\vec{l}=\overrightarrow{MM_0}=(3,4,12)$ $\Rightarrow$ $\cos\alpha_1=\frac{3}{13}$ $\cos\alpha_2=\frac{4}{13}$ $\cos\alpha_3=\frac{12}{13}$

$\frac{\partial u}{\partial x}\Big|_{(2,1,3)}=y+z\Big|_{(2,1,3)}=4$, $\frac{\partial u}{\partial y}\Big|_{(2,1,3)}=5$ $\frac{\partial u}{\partial z}\Big|_{(2,1,3)}=3$

∴ 方向导数：$\frac{3}{13}\times4+\frac{4}{13}\times5+\frac{12}{13}\times3=\frac{68}{13}$

2. (2) $\nabla u=(6x,-4y,6z)$

(3) $\nabla u=\left(\frac{z^2x}{\sqrt{x^2+y^2}},\frac{2z^2y}{\sqrt{x^2+2y^2}},2z\sqrt{x^2+2y^2}\right)$，在 $M(1,\frac{1}{\sqrt{2}},1)$ 处 $=(\frac{1}{\sqrt{2}},1,2\sqrt{2})$

3. (1) $\nabla u=(2x,2y-2z,-2y)$，在 $M(-1,2,1)$：$(-2,2,-4)$

∴ 沿 $(-2,2,4)$ 方向导数最大，最大为 $\sqrt{4+4+16}=\sqrt{24}=2\sqrt{6}$

即 $(-\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}})$ $(\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}})$

(2) 沿 $(2,-2,-4)$ 方向导数最小，最小为 $-\sqrt{4+4+16}=-2\sqrt{6}$

4. (3) $u_x=ae^{ax}\frac{(y-z)}{a^2+1}+\frac{\partial((y-z)/a^2+1)}{\partial x}e^{ax}$ $e^{ax}(a^2+1)\sin x$

$=\frac{ae^{ax}(y-z)}{a^2+1}+\frac{ae^{ax}}{a^2+1}\cos x+\frac{\sin x\, e^{ax}}{a^2+1}$ 再代入 $=e^{ax}\sin x$

(4) $u_t=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial t}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{1}{t}+\frac{\partial f}{\partial z}\sec^2 t$

5. (2) 令 $u=x^2+y^2$, $v=xy$，则 $z=ue^{\frac{u}{v}}$ $\Rightarrow\frac{\partial z}{\partial u}=(\frac{u}{v}+1)e^{\frac{u}{v}}$；$\frac{\partial z}{\partial v}=-\frac{u^2}{v^2}e^{\frac{u}{v}}$

$\frac{\partial u}{\partial x}=2x$，$\frac{\partial u}{\partial y}=2y$ $\frac{\partial v}{\partial x}=y$ $\frac{\partial v}{\partial y}=x$

$\Rightarrow z_x=z_u\cdot u_x+z_v\cdot v_x=2x(\frac{u}{v}+1)e^{\frac{u}{v}}-y\frac{u^2}{v^2}e^{\frac{u}{v}}$

$=\frac{2(x^2+y^2+xy)}{y}e^{\frac{x^2+y^2}{xy}}-\frac{(x^2+y^2)^2}{x^2y}e^{\frac{x^2+y^2}{xy}}=\left(\frac{x^2}{y}+2x-\frac{y^3}{x^2}\right)e^{\frac{x^2+y^2}{xy}}$

$z_y=\left(\frac{y^2}{x}+2y-\frac{x^3}{y^2}\right)e^{\frac{x^2+y^2}{xy}}$

虽然可用 $x^{x^y}=e^{x^y(\ln x)}$ 直接写偏导，但代换后更方便。

(4) $z=x^{x^y}$ 令 $u=x$ $v=x^y$，则 $z=u^v$

$\frac{\partial z}{\partial u}=vu^{v-1}$ $\frac{\partial z}{\partial v}=(\ln u)\cdot u^v$ $\frac{\partial u}{\partial x}=1$ $\frac{\partial u}{\partial y}=0$ $\frac{\partial v}{\partial x}=yx^{y-1}$ $\frac{\partial v}{\partial y}=(\ln x)x^y$

$\frac{\partial z}{\partial x}=vu^{v-1}+(\ln u)u^v yx^{y-1}$
$=x^{x^y-1+y}+y(\ln x)x^{x^y+y-1}$

$\frac{\partial z}{\partial y}=(\ln u)u^v(\ln x)x^y$
$=(\ln x)^2x^{x^y+y}$

6\. (1) $z_x = \frac{\partial f}{\partial(x+y)} + \frac{\partial f}{\partial(x-y)} = f_1' + f_2'$ $\quad z_y = \frac{\partial f}{\partial(x+y)} - \frac{\partial f}{\partial(x-y)} = f_1' - f_2'$

(3) $z_x = y\frac{\partial f}{\partial \frac{y}{x}} \times \left(-\frac{y}{x^2}\right) = -\frac{y^2}{x^2}\frac{\partial f}{\partial \frac{y}{x}}$ $\quad z_y = f\left(\frac{y}{x}\right) + \frac{y}{x}\frac{\partial f}{\partial\left(\frac{y}{x}\right)}$

7\. (1) $\frac{\partial z}{\partial x} = y\frac{df}{d(x^2-y^2)} \cdot 2x = 2xy f'(x^2-y^2)$ $\quad \frac{\partial z}{\partial y} = f(x^2-y^2) - 2y^2 f'(x^2-y^2)$

$\therefore \frac{1}{x}\cdot\frac{\partial z}{\partial x} + \frac{1}{y}\cdot\frac{\partial z}{\partial y} = 2y f'(x^2-y^2) + \frac{z}{y^2} - 2y f'(x^2+y^2) = \frac{z}{y^2}$

(2) $\frac{\partial u}{\partial x} = kx^{k-1} f\left(\frac{z}{x}, \frac{y}{x}\right) + x^k \frac{\partial f}{\partial\left(\frac{z}{x}\right)}\cdot\left(-\frac{z}{x^2}\right) + x^k \frac{\partial f}{\partial\left(\frac{y}{x}\right)}\cdot\left(-\frac{y}{x^2}\right)$

$\frac{\partial u}{\partial y} = x^{k-1}\frac{\partial f}{\partial\left(\frac{y}{x}\right)}$; $\frac{\partial u}{\partial z} = x^{k-1}\frac{\partial f}{\partial\left(\frac{z}{x}\right)}$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \left(ku - zx^{k-1}\frac{\partial f}{\partial\left(\frac{z}{x}\right)} - yx^{k-1}\frac{\partial f}{\partial\left(\frac{y}{x}\right)}\right) + yx^{k-1}\frac{\partial f}{\partial\left(\frac{y}{x}\right)} + zx^{k-1}\frac{\partial f}{\partial\left(\frac{z}{x}\right)}) = ku$

8\. $x = r\cos\theta \quad y = r\sin\theta \quad \frac{\partial x}{\partial r} = \cos\theta \quad \frac{\partial x}{\partial \theta} = -r\sin\theta \quad \frac{\partial y}{\partial r} = \sin\theta \quad \frac{\partial y}{\partial\theta} = r\cos\theta$

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} = \cos\theta\frac{\partial u}{\partial x}$ 而 $\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial\theta} = \frac{\partial v}{\partial y}\, r\cos\theta$

又 $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta}$

$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cdot\cos\theta \quad \frac{\partial u}{\partial\theta} = \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\theta} = \frac{\partial u}{\partial y}\, r\cos\theta$

又 $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta}$

$\frac{\partial v}{\partial y} = \frac{1}{x}\cdot\frac{x^2}{x^2+y^2} = \frac{x}{x^2+y^2}$

9\. (1) $u = \frac{1}{2}\ln(x^2+y^2) \Rightarrow \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \quad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} \quad \frac{\partial v}{\partial x} = -\frac{y}{x^2}\cdot\frac{x^2}{x^2+y^2} = -\frac{y}{x^2+y^2}$

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2}\frac{\partial z}{\partial u} - \frac{y}{x^2+y^2}\frac{\partial z}{\partial v}$

$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{y}{x^2+y^2}\frac{\partial z}{\partial u} + \frac{x}{x^2+y^2}\frac{\partial z}{\partial v}$

$\therefore 0 = (x+y)\frac{\partial z}{\partial x} - (x-y)\frac{\partial z}{\partial y} = \frac{x(x+y)}{x^2+y^2}\frac{\partial z}{\partial u} - \frac{y(x+y)}{x^2+y^2}\frac{\partial z}{\partial v} - \frac{y(x-y)}{x^2+y^2}\frac{\partial z}{\partial u} - \frac{x(x-y)}{x^2+y^2}\frac{\partial z}{\partial v}$

$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$

(2)漏写，见最后

$\Rightarrow$ 变换后为 $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial v}$

10\. (2) $\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} + \frac{1}{y}\frac{\partial f}{\partial\left(\frac{x}{y}\right)} = f_1 + \frac{1}{y}f_2 \quad \frac{\partial z}{\partial y} = -\frac{x}{y^2}\cdot\frac{\partial f}{\partial\frac{x}{y}}$

$\frac{\partial^2 z}{\partial x^2} = f_{11} + \frac{1}{y}f_{12} + \frac{1}{y}f_{21} + \frac{1}{y^2}f_{22} = f_{11} + \frac{2}{y}f_{12} + \frac{1}{y^2}f_{22}$

$\frac{\partial^2 z}{\partial x\partial y} = -\frac{x}{y^2}f_{12} - \frac{1}{y^2}f_2 \cancel{+ \frac{x}{y^4}f_{22}} - \frac{x}{y^3}f_{22}$

$\frac{\partial^2 z}{\partial y^2} = \frac{2x}{y^3}f_2 + \frac{x^2}{y^4}f_{22}$

(4) $z_x = f_1 + y f_2 + \frac{1}{y} f_3$; $z_y = f_1 + x f_2 - \frac{x}{y^2} f_3$

$\frac{\partial^2 z}{\partial x^2} = f_{11} + y f_{12} + \frac{1}{y} f_{13} + y f_{21} + y^2 f_{22} + f_{23} + \frac{1}{y} f_{31} + f_{32} + \frac{1}{y^2} f_{33}$ 易落

$\frac{\partial^2 z}{\partial x \partial y} = f_{11} + x f_{12} - \frac{x}{y^2} f_{13} + y f_{21} + xy f_{22} - \frac{x}{y} f_{23} + \frac{1}{y} f_{31} + \frac{x}{y} f_{32} - \frac{x}{y^3} f_{33} + \underbrace{f_2 - \frac{f_3}{y^2}}$

$\frac{\partial^2 z}{\partial y^2} = f_{11} + x f_{12} - \frac{x}{y^2} f_{13} + x f_{21} + x^2 f_{22} - \frac{x^2}{y^2} f_{23} - \frac{x}{y^2} f_{31} - \frac{x^2}{y^2} f_{32} + \frac{x^2}{y^4} f_{33} + \frac{2x}{y^3} f_3$

11. (1) $\frac{\partial F}{\partial x} = \int_0^{y\cancel{u}x} f(\cancel{u}x)\, g\left(\frac{t}{\cancel{u}x}\right) dt \Rightarrow \frac{\partial^2 F}{\partial x \partial y} = f(\cancel{u}x)\, g\left(\frac{\cancel{u}x\, y}{\cancel{u}x}\right) \frac{d(\cancel{u}x\, y)}{dy} = \cancel{u}x\, f(\cancel{u}x)\, g(y)$

(2) $\frac{\partial F}{\partial x} = 2xy f(x^2 y, e^{x^2 y}) \Rightarrow \frac{\partial^2 F}{\partial x \partial y} = 2x f(x^2 y, e^{x^2 y}) + 2xy\left(x^2 f_1 + x^2 e^{x^2 y} f_2\right)$

12. $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$ $\quad \frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 u}{\partial x^2}\cos^2\theta + u_{12}\cos\theta\sin\theta + u_{21}\sin\theta\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta$

$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y} r\cos\theta$ $\quad \frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} r^2\sin^2\theta - \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x}\right) r^2 \sin\theta\cos\theta + \frac{\partial^2 u}{\partial y^2} r^2\cos^2\theta$

$- \left(\frac{\partial u}{\partial x} r\cos\theta + \frac{\partial u}{\partial y} r\sin\theta\right)$

$\therefore \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = u_{11}\cos^2\theta + u_{12}\sin\theta\cos\theta + u_{21}\sin\theta\cos\theta + u_{22}\sin^2\theta$

$+ \frac{1}{r} u_1 \cos\theta + \frac{1}{r} u_2 \sin\theta$

$+ u_{11}\sin^2\theta - (u_{21} + u_{12})\sin\theta\cos\theta + u_{22}\cos^2\theta - \frac{1}{r} u_1\cos\theta - \frac{1}{r} u_2 \sin\theta$

$= u_{11} + u_{22} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

13. $\frac{\partial z}{\partial x} = f' \sin y\, e^x \Rightarrow \frac{\partial^2 z}{\partial x^2} = f''(\sin y\, e^x)^2 + f' \sin y\, e^x$

$\frac{\partial z}{\partial y} = f' e^x \cos y \Rightarrow \frac{\partial^2 z}{\partial x^2} = f''(\cos y\, e^x)^2 - f' \sin y\, e^x$

$\Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = f'' e^{2x}$

$\Rightarrow f'' = z = f. \Rightarrow f(u) = f''(u).$ 令 $g(u) = f'(u)$

而 $f'' = \frac{dg}{du} = \frac{dg}{df/g} = g\frac{dg}{df} \Rightarrow f = g\frac{dg}{df} \Rightarrow f\,df = g\,dg$

$\Rightarrow f^2 = g^2$ 即 $f = g$ 或 $f = -g$

① $f = g$ 即 $f = \frac{df}{du} \Rightarrow f = C_1 e^u$

② $f = -g$ 即 $f = -\frac{df}{du} \Rightarrow f = C_2 e^{-u}$

$\Rightarrow f = C_1 e^u + C_2 e^{-u}$

14. (1) $\frac{\partial x}{\partial u}=(\cos v)e^u \quad \frac{\partial x}{\partial v}=-(\sin v)e^u \quad \frac{\partial y}{\partial u}=(\sin v)e^u \quad \frac{\partial y}{\partial v}=(\cos v)e^u$

$$\frac{\partial z}{\partial u}=z_x x_u+z_y\cdot y_u=(\cos v)e^u\cdot z_x+(\sin v)e^u z_y$$

$$\Rightarrow \frac{\partial^2 z}{\partial u^2}=(\cos v)e^u\cdot z_x+\cos v\, e^u x_u z_{xx}+\cos e^u y_u z_{xy}$$
$$+(\sin v)e^u z_y+\sin v\, e^u x_u z_{yx}+\sin v\, e^u y_u z_{yy}$$

$$\frac{\partial z}{\partial v}=z_x x_v+z_y y_v=-\sin v\, e^u z_x+(\cos v)e^u z_y$$

$$\Rightarrow \frac{\partial^2 z}{\partial v^2}=-\cos v\, e^u z_x-\sin v\, e^u(x_v z_{xx}+y_v z_{xy})$$
$$-\sin v\, e^u z_y+\cos v\, e^u(x_v z_{yx}+y_v z_{yy})$$

$$\Rightarrow \frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=(\sin^2 v+\cos^2 v)e^{2u}z_{xx}+(\sin^2 v+\cos^2 v)e^{2u}z_{yy}=e^{2u}(z_{xx}+z_{yy})$$

由题，$z_{xx}+z_{yy}+m^2 z \Rightarrow \frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}+e^{2u}m^2 z=0$

(2) $u_x=y \quad u_y=x \quad v_x=\frac{1}{y} \quad v_y=-\frac{x}{y^2}$ 而 $u_{xx}=u_{yy}=v_{xx}=0$, $v_{yy}=\frac{2x}{y^3}$

$z_x=z_u u_x+z_v v_x$ $(+u_x v_x z_{uv})$

$z_{xx}=z_u u_{xx}+(u_x)^2 z_{uu}+z_v v_{xx}+(v_x)^2 z_{vv}+u_x v_x z_{vu}$

$z_y=z_u u_y+z_v v_y$

$z_{yy}=z_{uu}(u_y)^2+z_{uv}u_y v_y+z_u u_{yy}+z_{vv}(v_y)^2+z_{vu}v_y u_y+z_v v_{yy}$

$\Rightarrow z_{xx}=y^2 z_{uu}+\frac{1}{y^2}z_{vv}+z_{uv}+z_{vu}$

$z_{yy}=x^2 z_{uu}+\frac{x^2}{y^4}z_{vv}+\frac{2x}{y^3}z_v-\frac{x^2}{y^2}(z_{uv}+z_{vu})$

由题意，$x^2 z_{xx}-y^2 z_{yy}=0$

~~$\therefore x^2y^2 z_{uu}+\frac{x^2}{y^2}z_{vv}-x^2y^2 z_{uu}-\frac{x^2}{y^2}z_{vv}-\frac{2x}{y}z_v=0$~~

~~$\therefore \frac{2x}{y}z_v=0$~~

$x^2y^2 z_{uu}+\frac{x^2}{y^2}z_{vv}+x^2(z_{uv}+z_{vu})-x^2y^2 z_{uu}-\frac{x^2}{y^2}z_{vv}-\frac{2x}{y}z_v+x^2(z_{uv}+z_{vu})=0$

$\Rightarrow 2x^2(z_{uv}+z_{vu})=\frac{2x}{y}z_v \Rightarrow xy(z_{uv}+z_{vu})=z_v$

即 $u\left(\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v\partial u}\right)=\frac{\partial z}{\partial v}$

15. (1) $z_x = z_u u_x + z_v v_x$

$z_{xx} = z_{uv} u_x v_x + z_{uu}(u_x)^2 + z_u u_{xx} + z_{vu} v_x u_x + z_{vv}(v_x)^2 + z_v v_{xx}$

$z_y = z_u u_y + z_v v_y$

$z_{yy} = z_{uv} u_y v_y + z_{uu}(u_y)^2 + z_u u_{yy} + z_{vu} u_y v_y + z_{vv}(v_y)^2 + z_v v_{yy}$.

$u_x = 1 \quad u_y = a \quad v_x = 1 \quad v_y = b$ 而 $u_{xx} = u_{yy} = v_{xx} = v_{yy} = 0$

$\Rightarrow z_{xx} = z_{uv} + z_{uu} + z_{vu} + z_{vv}$

$z_{yy} = ab z_{uv} + a^2 z_{uu} + ab z_{vu} + b^2 z_{vv}$

$z_{xy} = z_{uu} u_x u_y + z_{uv} u_x v_y + z_u u_{xy} + z_{vv} v_x v_y + z_{vu} v_x u_y + z_v v_{xy}$

$= a z_{uu} + b z_{uv} + b z_{vv} + a z_{vu}$

$z_{xx} + 4 z_{xy} + 3 z_{yy} = 0$

$\Rightarrow z_{uu}(1+3a^2+4a) + z_{vv}(1+3b^2+4b) + z_{uv}(1+3ab+4b) + z_{vu}(1+3ab+4a) = 0$

Q: 没说明 $z_{uv} = z_{vu}$.

要使得可化为 $z_{uv} = 0$，则 $\begin{cases} 1+3a^2+4a = 0 \\ 1+3b^2+4b = 0 \end{cases} \Rightarrow \begin{cases} a = -\frac{1}{3} \\ b = -1 \end{cases}$ 或 $\begin{cases} a = -1 \\ b = -\frac{1}{3} \end{cases}$

(2) 同上，$z_{xx} = z_{uv} + z_{uu} + z_{vu} + z_{vv}$

$z_{yy} = -2a z_{uv} + 4 z_{uu} - 2a z_{vu} + a^2 z_{vv}$

$z_{xy} = a z_{uv} - 2 z_{uu} - 2 z_{vu} + a z_{vv}$

$0 = 6 z_{xx} + z_{xy} - z_{yy}$

$= (6+3a) z_{uv} + (4+2a) z_{vu} + (6+a-a^2) z_{vv}$

$\Rightarrow \begin{cases} 6+3a \neq 0 \\ 6+a-a^2 = 0 \end{cases} \Rightarrow a = 3$

16. $z(x,y) = \int_0^{xy} f(t)(xy-t)\,dt + \int_1^{xy} f(t)(xy-t)\,dt$

$= xy \int_0^{xy} f(t)\,dt - \int_0^{xy} f(t)\,t\,dt + xy \int_1^{xy} f(t)\,dt - \int_1^{xy} f(t)\,t\,dt$

☆ 必须把 x 从 ∫ 内部提出来才能求导

$\frac{\partial z}{\partial x} = y \int_0^{xy} f(t)\,dt + 2xy^2 f(xy) - 2xy^2 f(xy) + y \int_1^{xy} f(t)\,dt$

$\frac{\partial^2 z}{\partial x^2} = y^2 f(xy) + y^2 f(xy) = 2y^2 f(xy)$

17. ① $u(x,y)=f(x)\cdot g(y)$

$$\Rightarrow \frac{\partial u}{\partial x}=g(y)\frac{\partial f}{\partial x} \Rightarrow \frac{\partial^2 u}{\partial x\partial y}=\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}$$

$$\frac{\partial u}{\partial y}=f(x)\frac{\partial g}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}=f(x)g(y)\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}=f(x)g(y)\frac{\partial^2 u}{\partial x\partial y}=u(x,y)\frac{\partial^2 u}{\partial x\partial y}$$

$$\Rightarrow u\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}$$

② $u\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}$

~~左右求偏导:~~ $\cancel{\frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial x\partial y}+u\frac{\partial^3 u}{\partial x\partial y\partial x}=\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial u}{\partial y}+\frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial y\partial x}}$

$$\cancel{\frac{\partial u}{\partial y}\cdot\frac{\partial^2 u}{\partial x\partial y}+u\frac{\partial^3 u}{\partial x\partial y\partial y}=\frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x\partial y}\cdot\frac{\partial u}{\partial y}}$$

$$\cancel{\Rightarrow \begin{cases} u\frac{\partial^3 u}{\partial x\partial y\partial x}=\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial u}{\partial y} \\ u\frac{\partial^3 u}{\partial x\partial y\partial y}=\frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial y}\end{cases}}$$

令 $\frac{\partial u}{\partial x}=v$. 则 $u\frac{\partial v}{\partial y}=v\frac{\partial u}{\partial y} \Rightarrow u\frac{\partial v}{\partial y}-v\frac{\partial u}{\partial y}=0$

而 $\frac{\partial(\frac{v}{u})}{\partial y}=\frac{u\frac{\partial v}{\partial y}-v\frac{\partial u}{\partial y}}{u^2}=0$ ⇒ $\frac{v}{u}$ 是个与y无关的式子.只与x有关: $\varphi_1(x)$

$\Rightarrow \frac{\partial u}{\partial x}=u\varphi_1(x)$ 而 $\frac{1}{u}=\frac{\partial(\ln u)}{\partial u} \Rightarrow \frac{\partial(\ln u)}{\partial x}=\varphi_1(x) \Rightarrow \ln u=\int\varphi_1(x)dx+\varphi_2(y)$

$$\Rightarrow u=e^{\int\varphi_1(x)dx}\times e^{\varphi_2(y)}=f(x)g(y)$$

18. $u_x(x,y)=xy^2+2xy+C_y$ 而 $u(x,0)=xe^x$

$\Rightarrow u_x(x,y)=xy^2+2xy+xe^x$

$\Rightarrow u(x,y)=\frac{1}{2}x^2y^2+x^2y+(x-1)e^x+C_x$

而 $u(0,y)=\cos y$ ⇒ 代入上式有 $u(0,y)=C_x-1$

$\Rightarrow C_x=\cos y+1$

$\Rightarrow u(x,y)=\frac{1}{2}x^2y^2+x^2y+(x-1)e^x+\cos y+1$

9 (2) $w_x=\frac{1}{z}z_x-1$, $w_y=\frac{1}{z}z_y-1$ $\quad u_x=2x$ $u_y=2y$ $v_x=-\frac{1}{x^2}$ $v_y=-\frac{1}{y^2}$

而 $w_x=w_u u_x+w_v v_x$ $\quad w_y=w_u u_y+w_v v_y$

$=2x\,w_u-\frac{1}{x^2}w_v$ $\quad =2y\,w_u-\frac{1}{y^2}w_v$

⇒ 变换 $yz_x-xz_y=(y-x)z$ 得

$y(zw_x+z)-x(zw_y+z)=yz-xz \Rightarrow yw_x-xw_y=0$

$\Rightarrow 2xyw_u-\frac{y}{x^2}w_v-2xyw_u+\frac{x}{y^2}w_v=0 \Rightarrow (x^3-y^3)w_v=0 \Rightarrow w_v=0$