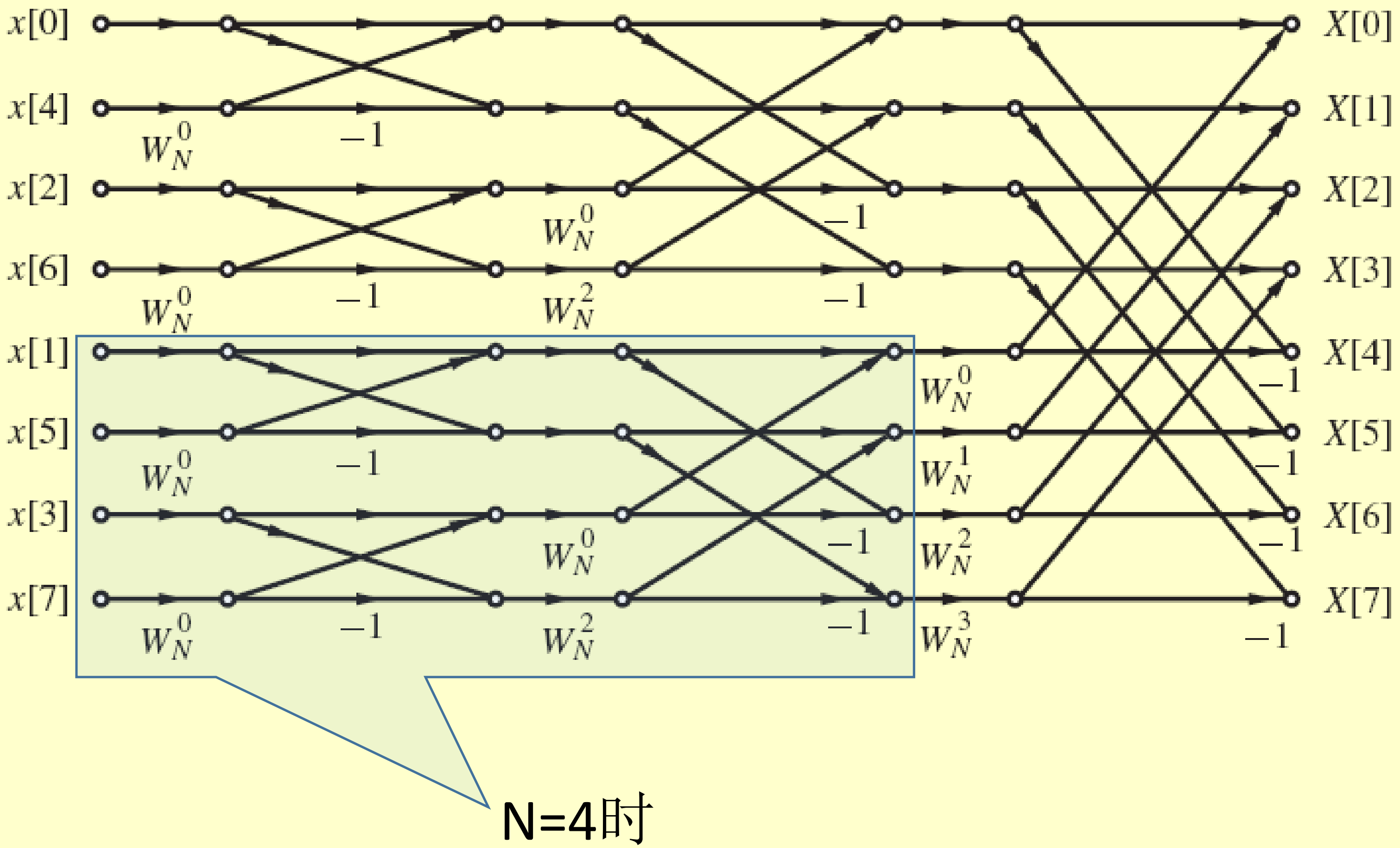
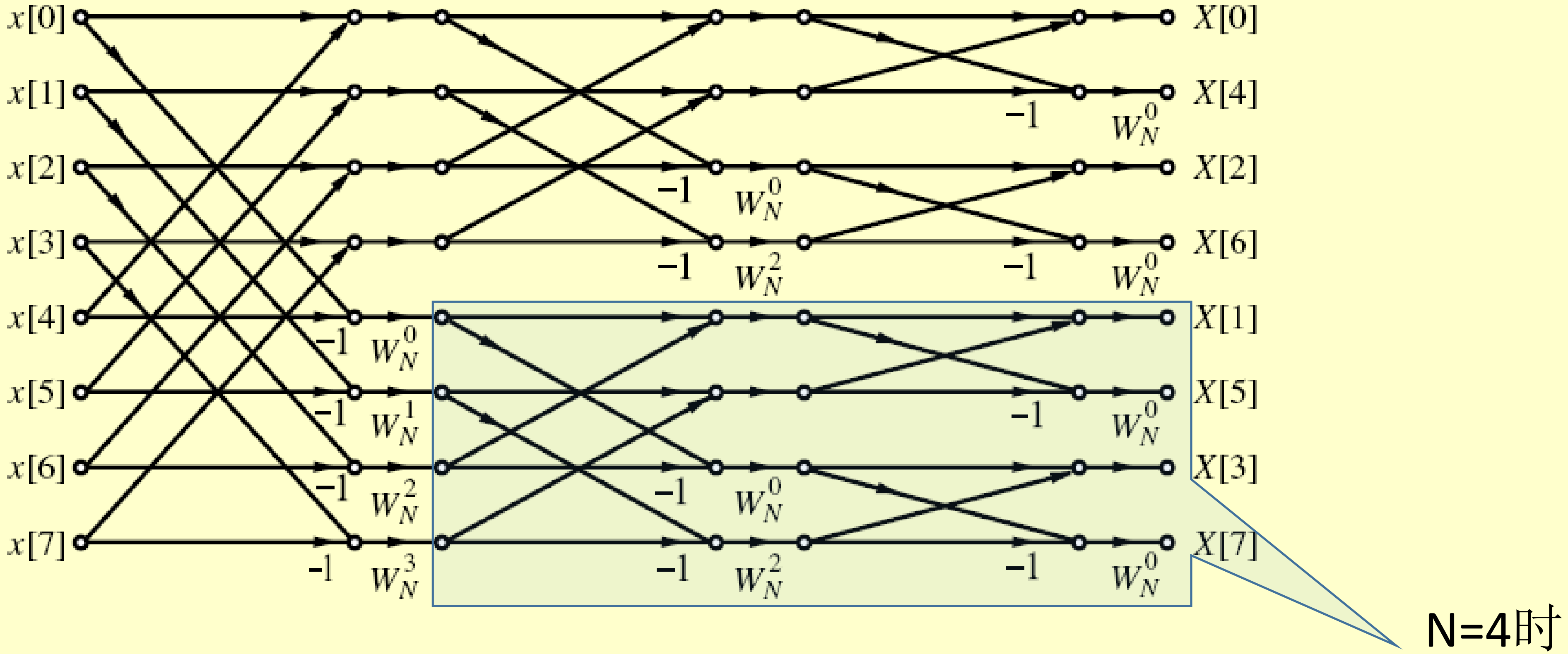
Decimation-in-Time FFT Algorithm

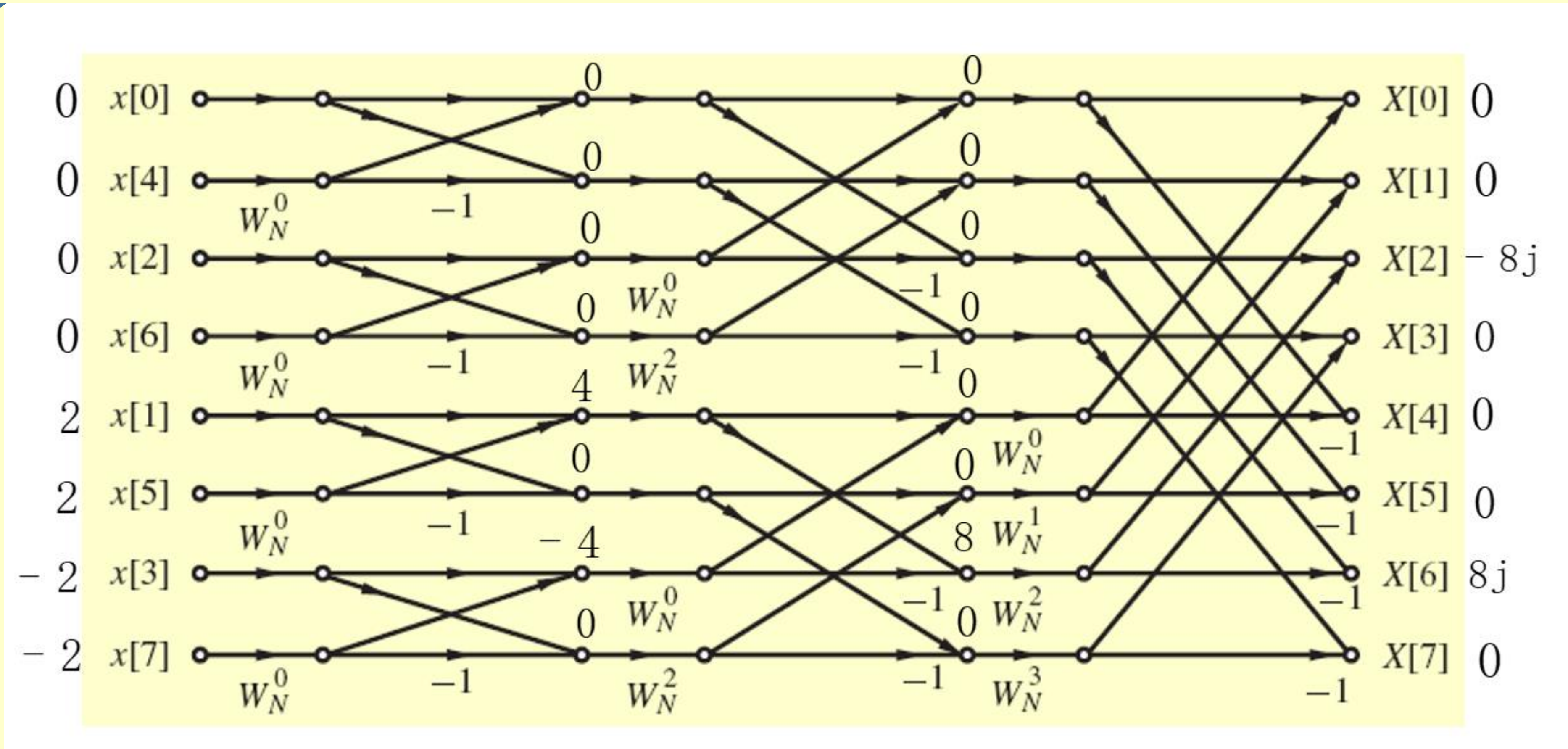


Decimation-in-Frequency FFT Algorithm

★★★ • Complete flow-graph of the decimation-in-frequency FFT computation scheme for $N = 8$



附：例题　$x[n]=[0\quad 2\quad 0\quad -2\quad 0\quad 2\quad 0\quad -2]$

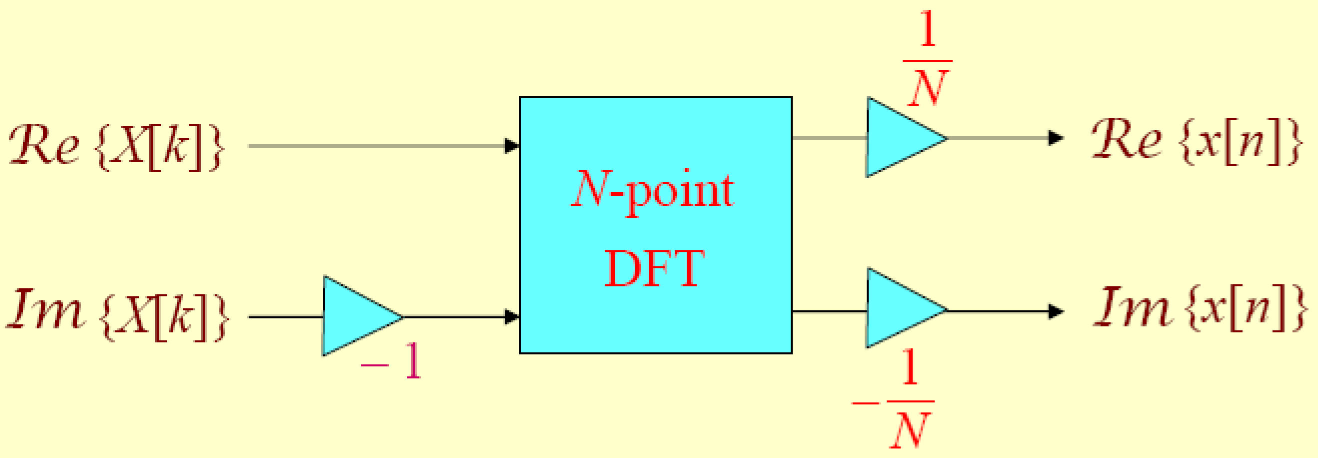


Inverse DFT Computation

- Desired IDFT $x[n]$ is then obtained as

$$x[n] = \frac{1}{N}\left\{\sum_{k=0}^{N-1} X^*[k] W_N^{nk}\right\}^*$$

- Inverse DFT computation is shown below:



$$x[n] = \frac{1}{N}\left\{FFT\{X^*[k]\}\right\}^*$$

附： Fast Convolution（快速卷积）

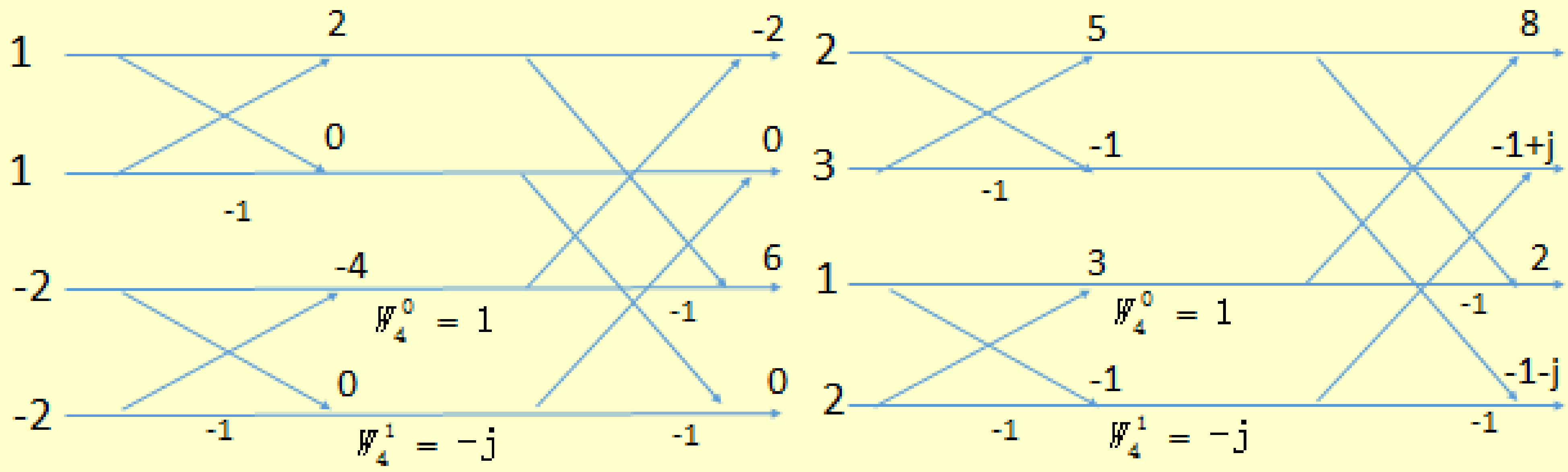
Example：

Let $x_1(n)=\{1,-2,1,-2\}$ and $x_2(n)=\{2,1,3,2\}$

1）Calculate the 4-point circular convolution using FFT

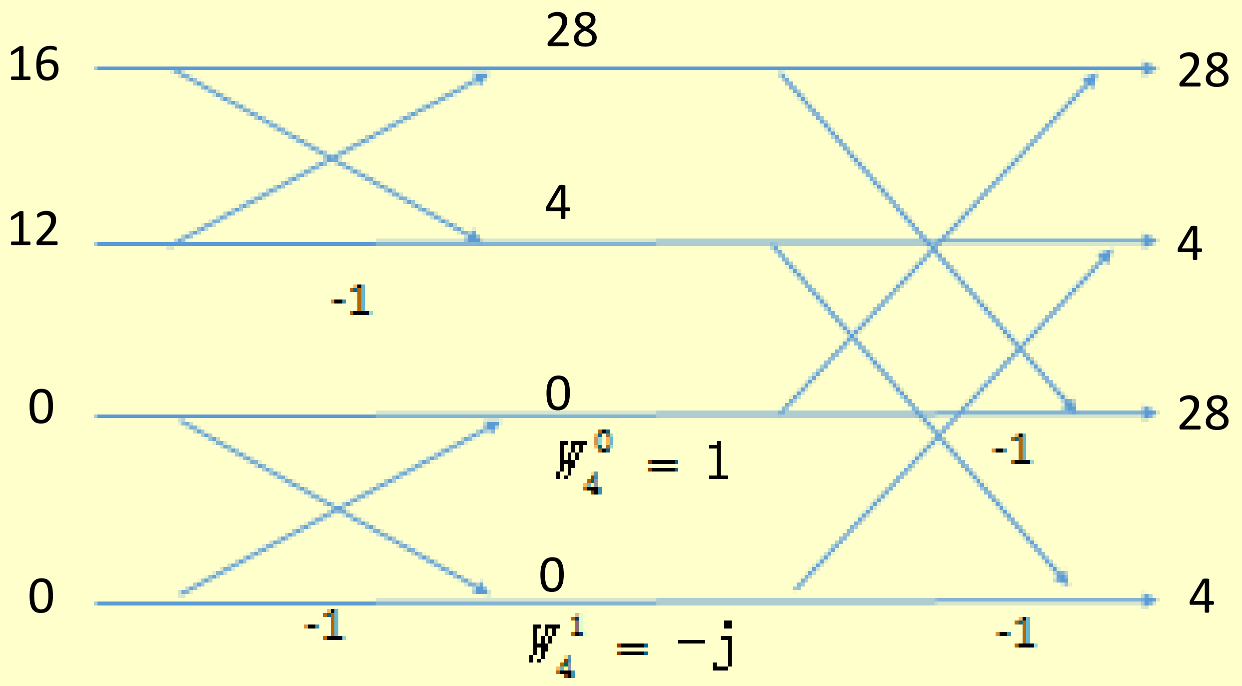
2）Calculate the linear convolution using FFT.

3）Determine the number of complex multiplication in each case



$X_1[k]=[2 \quad 0 \quad 6 \quad 0]$ $\quad X_2[k]=[8 \quad -1+j \quad 2 \quad -1-j]$

$Y_C[k]=X_1[k]\cdot X_2[k]=[16 \quad 0 \quad 12 \quad 0]$ $\quad Y_c^*[k]=[16 \quad 0 \quad 12 \quad 0]$ $\quad y_C[n]=\dfrac{1}{N}\left\{\mathrm{FFT}\{Y_C^*[k]\}\right\}^*$



$y_C[n]=[7 \quad 1 \quad 7 \quad 1]$

b) $x_1(n)=\{1,-2,1,-2,0,0,0,0\}$ $\quad x_2(n)=\{2,1,3,2,0,0,0,0\}$

补零后两个序列分别都做8点FFT，重复a)中的步骤（略）

c) $N+3\dfrac{N}{2}\log_2^N=8+3\cdot(\dfrac{8}{2}\log_2^8)=44$ $\quad N+3\dfrac{N}{2}\log_2^N=4+3\cdot(\dfrac{4}{2}\log_2^4)=16$

★★

11.12 The k–th sample of an N–point DFT is given by $X[k] = \sum_{n=0}^{N-1} x[n]W_N^{nk}$. Thus, the computation of $X[k]$ requires N complex multiplications and $N-1$ complex additions. Now, each complex multiplication, in turn, requires 4 real multiplications and 2 real additions. Likewise, each complex addition requires 2 real additions. As a result, the N complex multiplications needed to compute $X[k]$ require a total of $4N$ real multiplications and a total of $2N-2$ real additions. Therefore, each sample of the N–point DFT involves $4N$ real multiplications and $4N-2$ real additions. Hence, the computation of all DFT samples thus requires $4N^2$ real multiplications and $(4N-2)N$ real additions.

Complex multiplication: $(a+bj)(c+dj)$

Complex addition: $(a+bj)+(c+dj)$

The k–th sample of an N–point DFT

The number of complex multiplication: N

The number of complex addition: N-1

The number of real multiplication: 4N

The number of real addition: 2N+2(N-1)=4N-2

11.21

（以复数乘法次数来衡量）

Direct computation of M DFT samples requires $M \cdot N$ complex multiplication

FFT computation:

$$\frac{N}{2}\log_2 N$$

$$\frac{N}{2}\log_2 N < M \cdot N \Rightarrow M > \frac{1}{2}\log_2 N$$

$$N = 32 \qquad M > \frac{\log_2 N}{2} = \frac{5}{2} \qquad \therefore \qquad M = 3$$

$$N = 64 \qquad M > \frac{\log_2 N}{2} = 3 \qquad \therefore \qquad M = 4$$

$$N = 128 \qquad M > \frac{\log_2 N}{2} = \frac{7}{2} \qquad \therefore \qquad M = 4$$

★★ **11.32** **(a)** zero-valued samples to be added is $256 - 197 = 59$.

(b) Direct computation of a 256-point DFT of a length-197 sequence requires

complex multiplications $256 \times 197 = 50432$

complex additions $256 \times (197 - 1) = 50176$

(c) A 256-point Cooley-Tukey type FFT algorithm requires

complex multiplications $128 \times \log_2(256) = 1024$

complex additions $256 \times \log_2(256) = 2048$

10.17 ★★★

(a) Given: $\omega_p = 0.42\pi,\ \omega_s = 0.58\pi,\ \delta_p = 0.002,\ \delta_s = 0.008.$

Thus: $\Delta\omega = 0.16\pi, \alpha_s = -20\log_{10}\delta_s = 41.93$ dB.

From Table 10.2, we see that for fixed-window functions,

we can achieve the minimum stopband attenuation by using Hann,

Hamming, or Blackman windows. Hann will have the lowest filter length:

Since $M = \dfrac{3.11\pi}{0.16\pi} = 19.43,$ $\Longrightarrow$ $M = 19.5$或20 $\Longrightarrow$ $N_{Haan} = \lceil 2M+1 \rceil = 40$或$41$

$$h[n] = \begin{cases} \dfrac{\sin[0.5\pi(n-19.5)]}{\pi(n-19.5)} \cdot w_{Hann}(n-19.5) & 0 \le n \le 39 \\ 0 & otherwise \end{cases}$$ 或 $$h[n] = \begin{cases} \dfrac{\sin[0.5\pi(n-20)]}{\pi(n-20)} \cdot w_{Hann}(n-20) & 0 \le n \le 40 \\ 0 & otherwise \end{cases}$$

The corresponding frequency response, generated with the code, is shown below:

补充例题：★★★

Design a lowpass digital FIR filter with cutoff frequency $\omega_c = 0.4\pi$ using a rectangular window of length L=21.

a) Determine the impulse response $h[n]$ of the designed filter

b) Sketch the magnitude frequency response $|H(e^{j\omega})|$ of the designed filter

c)Would the transition width and the ripple (波纹)size of the stopband be improved if a rectangular window of length L=41 is used ?

d) Would the transition width and the ripple (波纹)size of the stopband be improved if a hamming window of length L=21 is used ?

a. The impulse response of ideal lowpass filter is:

$$h_d[n] = \frac{\sin \omega_c n}{\pi n} \qquad -\infty < n < +\infty$$

$$h[n] = \frac{\sin 0.3\pi(n-10)}{\pi(n-10)} \qquad 0 \le n \le 20$$

b. p466. Figure10.2

c. If a rectangular window of length *L*=41 is used the transition width will get narrower and the stopband ripple （纹波）size will not get smaller but the width of stopband ripple will get smaller.

d. If a hamming window of length *L*=21 is used the transition width will get wider and the stopband ripple （纹波）size will get smaller.

★★★ Gibbs Phenomenon

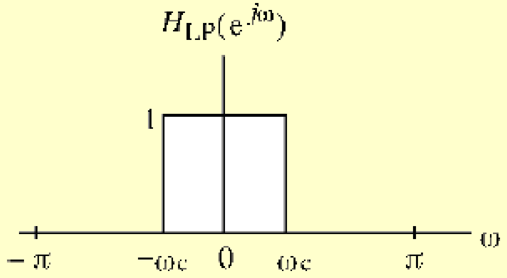
- As can be seen, as the length of the lowpass filter is increased, the number of ripples in both passband and stopband increases, with a corresponding decrease in the ripple widths
- Height of the largest ripples remain the same independent of length

（近似不变,随着窗长变长，而越来越趋近于恒值）

- The transition width decreases with increasing L. Note also that for any L, the windowed response $H(e^{j\omega})$ is always equal to 0.5 at the cutoff frequency .
- The largest ripples tend to cluster near the passband-to-stopband discontinuity (from both sides) and do not get smaller with L. Instead , the size remains approximately constant , about 8.9 percent , independent of L, Eventually , as $L \rightarrow \infty$, these ripples get squeezed onto the discontinuity at $\omega = \omega_c$ occupying a set of measure zero .

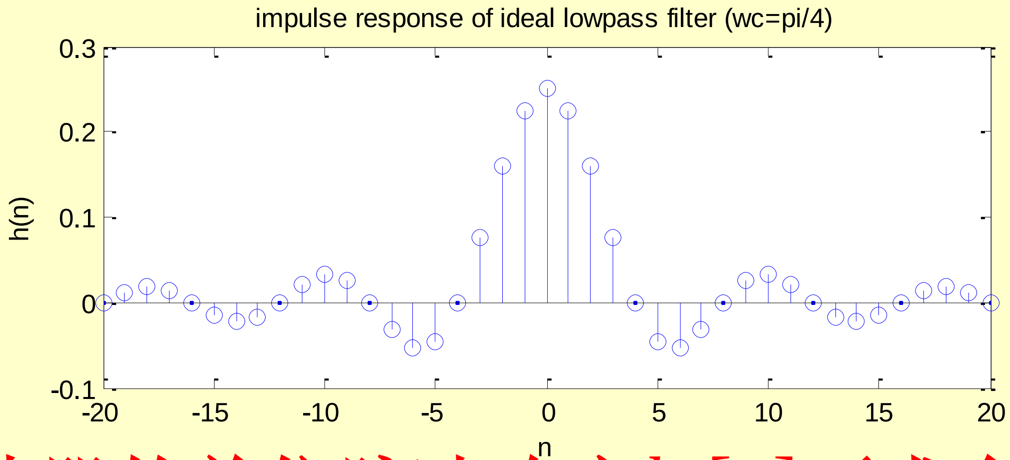
★★★ 附：求理想低通滤波器的单位脉冲响应

$$H_d(e^{j\omega})=\begin{cases}1 & |\omega|\le\omega_c\\ 0 & \omega_c<|\omega|\le\pi\end{cases}$$



$$h_d[n]=\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}H_d(e^{j\omega})e^{j\omega n}d\omega=\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}e^{j\omega n}d\omega$$

$$=\frac{1}{2\pi}\frac{e^{j\omega_c n}-e^{-j\omega_c n}}{jn}=\frac{\sin\omega_c n}{\pi n}\qquad -\infty<n<+\infty$$



理想滤波器的单位脉冲响应$h_d[n]$要求会用**IDTFT**求出来

★★

10.3 (a) From Eq. (10.17): $H_{HP}(e^{j\omega}) = \begin{cases} 0, & |\omega| < \omega_c, \\ 1, & \omega_c \le |\omega| \le \pi, \end{cases}$

$$h_{HP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{HP}\left(e^{j\omega}\right)e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_c}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_c n}}{jn} - \frac{e^{-j\pi n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi n}}{jn} - \frac{e^{j\omega_c n}}{jn}\right]$$

Using the properties of the sinc function, we arrive at: $h_{HP}[0] = 1 - \frac{\omega_c}{\pi}$.

Therefore: $h_{HP}[n] = \begin{cases} 1 - \frac{\omega_c}{\pi}, & n = 0, \\ -\frac{\sin(\omega_c n)}{\pi n}, & \text{otherwise.} \end{cases}$

(b) From Eq. (10.18): $H_{BP}(e^{j\omega}) = \begin{cases} 0, & |\omega| < \omega_{c1}, \\ 1, & \omega_{c1} \le |\omega| \le \omega_{c2}, \\ 0, & \omega_{c2} \le |\omega| \le \pi, \end{cases}$

$$h_{BP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{BP}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_{c2}}^{-\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_{c1}}^{\omega_{c2}}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c1}n}}{jn} - \frac{e^{-j\omega_{c2}n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c2}n}}{jn} - \frac{e^{j\omega_{c1}n}}{jn}\right]$$

$$= \frac{\sin(\omega_{c2}n)}{\pi n} - \frac{\sin(\omega_{c1}n)}{\pi n}.$$

(c) From Eq. (10.20): $H_{BS}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_{c1}, \\ 0, & \omega_{c1} \le |\omega| \le \omega_{c2}, \\ 1, & \omega_{c2} \le |\omega| \le \pi. \end{cases}$

$$h_{BS}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{BS}(e^{j\omega})e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}} e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_{c2}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_{c2}}^{\pi} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_{c1}}^{\omega_{c1}}$$

$$= \frac{\sin(\pi n)}{\pi n} - \frac{\sin(\omega_{c2} n)}{\pi n} + \frac{\sin(\omega_{c1} n)}{\pi n}.$$

Using the properties of the sinc function, we arrive at: $h_{BS}[0] = 1 - \frac{\omega_{c2} - \omega_{c1}}{\pi}$.

Therefore: $$h_{HP}[n] = \begin{cases} 1 - \dfrac{\omega_{c2} - \omega_{c1}}{\pi}, & n = 0, \\ \dfrac{\sin(\omega_{c1} n)}{\pi n} - \dfrac{\sin(\omega_{c2} n)}{\pi n}, & \text{otherwise.} \end{cases}$$

Supplementary Problem:

Using the bilinear transformation and a lowpass analog Butterworth prototype filter, design a lowpass digital filter operating at a rate of 40 kHz and having the following specifications: passband edge frequency being 10 kHz with the attenuation 3 dB, stopband edge frequency being 15 kHz with the attenuation 35 dB.

$$f_p = 10k\,\text{Hz},\ f_s = 15k\,\text{Hz}, F_T = 40k\,\text{Hz}$$

$$\alpha_p = 3dB, \alpha_s = 35dB$$

$$\omega_p = \frac{2\pi \cdot f_p}{F_T} = \frac{2\pi \cdot 10}{40} = \frac{\pi}{2}, \omega_s = \frac{2\pi \cdot f_s}{F_T} = \frac{2\pi \cdot 15}{40} = \frac{3\pi}{4}$$

$$\Omega_p = tg(\frac{\omega_p}{2}) = tg(\frac{\pi}{4}) = 1, \Omega_s = tg(\frac{\omega_s}{2}) = tg(\frac{3\pi}{8}) = 2.4142$$

$$\begin{cases} 10 1og[1+(\frac{\Omega_p}{\Omega_c})^{2N}] = \alpha_p \\ 10 1og[1+(\frac{\Omega_s}{\Omega_c})^{2N}] = \alpha_s \end{cases}$$

$$N_{exact}=\ln\left(\sqrt{\frac{10^{\alpha_s/10}-1}{10^{\alpha_p/10}-1}}\right)\Big/\ln\left(\frac{\Omega_s}{\Omega_p}\right)=4.5744$$

$$\therefore N=5$$

$$\Omega_c=\frac{\Omega_p}{(10^{\alpha_p/10}-1)^{\frac{1}{2N}}}=1.0005\approx 1$$

$$\text{或}\Omega_c=\frac{\Omega_s}{(10^{\alpha_s/10}-1)^{\frac{1}{2N}}}=1.0784\approx 1$$

$$H_a(s)=\frac{1}{(1+s)(1+0.618s+s^2)(1+1.618s+s^2)}$$

$$H(z)=H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$$

★★★ ***Supplementary Problem:***

Using the bilinear transformation and a first-order lowpass analog Butterworth filter $H_a(s) = \dfrac{\Omega_c}{s+\Omega_c}$, design a lowpass digital filter with 3-dB cutoff frequency 2.5kHz and operating at a sampling rate of 10kHz.

Solution:

$$\omega_c = \frac{2\pi f_c}{F_T} = \frac{2\pi \times 2.5}{10} = 0.5\pi$$

3-dB frequency of the analog filter:

$$\Omega_c = \tan\left(\frac{\omega_c}{2}\right) = \tan(0.25\pi) = 1$$

$$H(z) = H_a(s)\Big|_{\frac{1-z^{-1}}{1+z^{-1}}} = \frac{1}{\dfrac{1-z^{-1}}{1+z^{-1}}+1} = \frac{1+z^{-1}}{2}$$

9.3 Design of Lowpass IIR Filters

IIR Lowpass Digital Filter Design Using Bilinear Transformation

★★★ • Example - Design a lowpass Butterworth digital filter with $\omega_p = 0.25\pi$, $\omega_s = 0.55\pi$, $\alpha_p \le 0.5\,\text{dB}$, and $\alpha_s \ge 15\ \text{dB}$

第一步:• Prewarping we get

$$\Omega_p = \tan(\omega_p/2) = \tan(0.25\pi/2) = 0.4142136$$

$$\Omega_s = \tan(\omega_s/2) = \tan(0.55\pi/2) = 1.1708496$$

$$\left(20\log_{10}(\sqrt{1+\varepsilon^2}) = 0.5 \qquad 20\log_{10} A = 15\right)$$

IIR Lowpass Digital Filter Design Using Bilinear Transformation

第二步：设计模拟滤波器

$$\begin{cases} 10\log[1+(\frac{\Omega_p}{\Omega_c})^{2N}]=\alpha_p=0.5 \\ 10\log[1+(\frac{\Omega_s}{\Omega_c})^{2N}]=\alpha_s=15 \end{cases} \Rightarrow \begin{cases} (\frac{\Omega_p}{\Omega_c})^{2N}=10^{\frac{\alpha_p}{10}}-1=10^{\frac{0.5}{10}}-1 \\ (\frac{\Omega_s}{\Omega_c})^{2N}==10^{\frac{\alpha_s}{10}}-1=10^{\frac{15}{10}}-1 \end{cases}$$

$$\Rightarrow (\frac{\Omega_p}{\Omega_s})^{2N}=\frac{10^{\frac{\alpha_p}{10}}-1}{10^{\frac{\alpha_s}{10}}-1}=\frac{10^{\frac{0.5}{10}}-1}{10^{\frac{15}{10}}-1}$$

$$\Rightarrow N_{exact}=\ln(\sqrt{\frac{10^{\alpha_s/10}-1}{10^{\alpha_p/10}-1}})\Big/\ln(\frac{\Omega_s}{\Omega_p})=2.6587$$

$$\therefore N=3$$

IIR Lowpass Digital Filter Design Using Bilinear Transformation

第二步：设计模拟滤波器

- To determine Ω_c we use

$$\left|H_a(j\Omega_p)\right|^2 = \frac{1}{1+(\Omega_p/\Omega_c)^{2N}} = \frac{1}{1+\varepsilon^2}$$

- We then get

$$\Omega_c = 1.419915(\Omega_p) = 0.588148$$

- 3rd-order lowpass Butterworth transfer function for $\Omega_c = 1$ is

$$H_{an}(s) = \frac{1}{(s+1)(s^2+s+1)}$$

- Denormalizing to get $\Omega_c = 0.588148$ we arrive at

$$H_a(s) = H_{an}\left(\frac{s}{0.588148}\right)$$

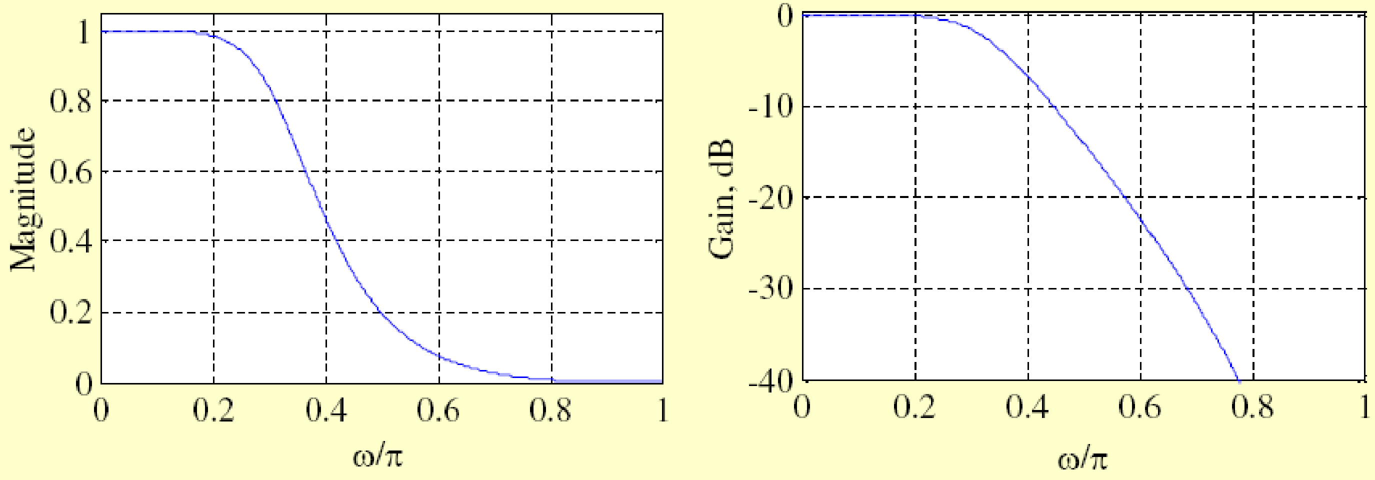
IIR Lowpass Digital Filter Design Using Bilinear Transformation

第三步：得到数字滤波器传递函数

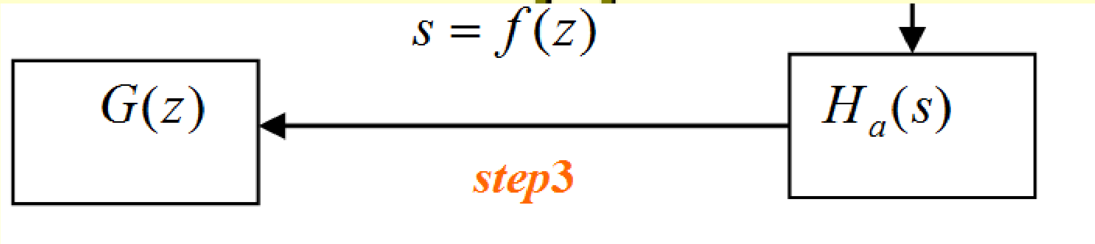
- Applying bilinear transformation to $H_a(s)$ we get the desired digital transfer function

$$G(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$$

- Magnitude and gain responses of $G(z)$ shown below:



Basic Approaches



The basic idea behind the conversation of an analog transfer function $H_a(s)$ into a digital IIR transfer function $H(z)$ is to apply a mapping from the s-domain to the z-domain so that the essential properties of the analog frequency response are preserved.

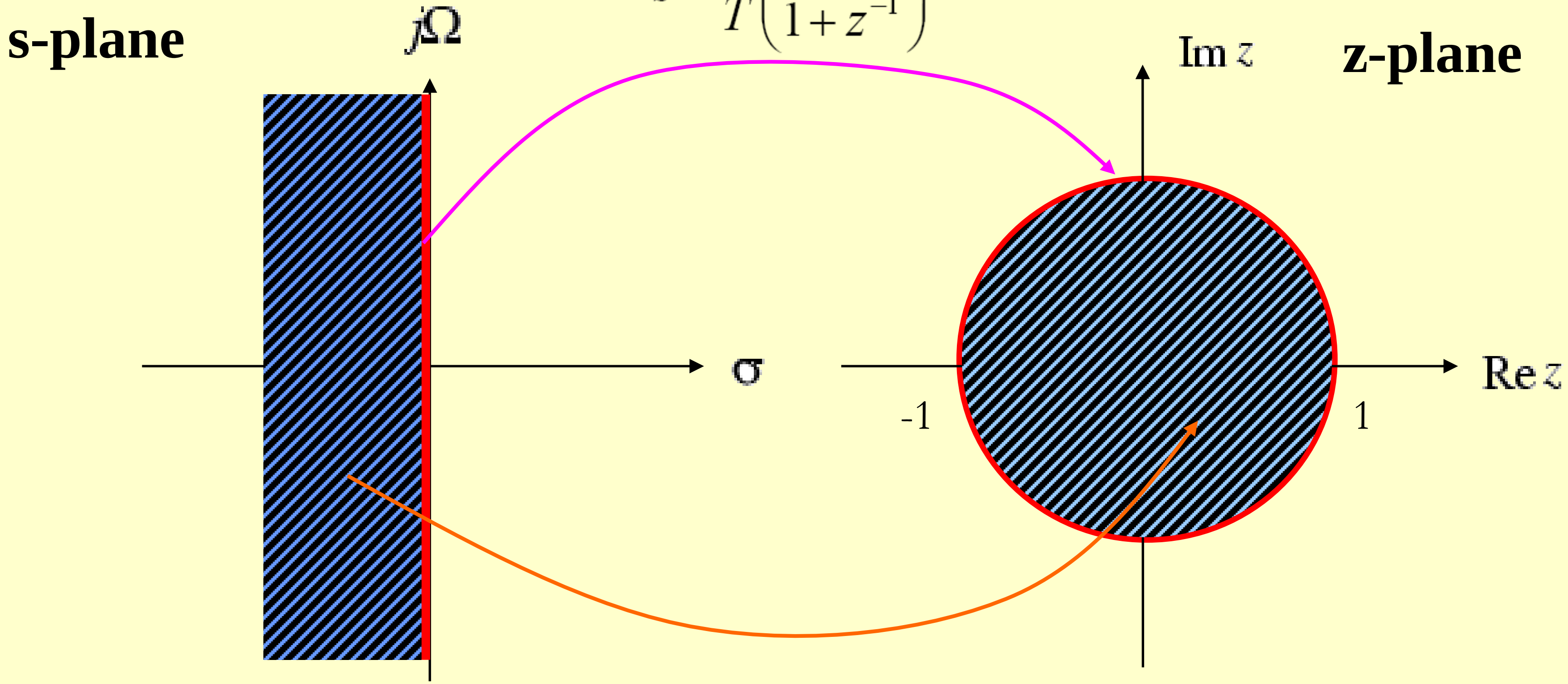
(a) **Mapping the imaginary axis on s-plane to unit circle on z-plane, that is $e^{j\omega} \to j\Omega$**

(b) **Mapping the left-hand of s-plane into the interior of unit circle on the z-plane which guarantees $H(z)$ will be stable and causal**

Bilinear Transformation

- Mapping of s-plane into the z-plane

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$



(a) **Mapping the imaginary axis on s-plane to unit circle on z-plane, that is** $e^{j\omega} \to j\Omega$

(b) **Mapping the left-hand of s-plane into the interior of unit circle on the z-plane which guarantees** $H(z)$ **will be stable and causal**

★★ **9.1** (a) $\delta_p = 1 - 10^{-\alpha_p/20} = 1 - 10^{-0.24/20} = 0.0273,$

$$\delta_s = 10^{-\alpha_s/20} = 10^{-49/20} = 0.0035.$$

$$-20\log_{10}(1-\delta_p) = \alpha_p \Rightarrow \delta_p = 1-10^{(-\frac{\alpha_p}{20})}$$

$$-20\log_{10}(\delta_s) = \alpha_s \Rightarrow \delta_s = 10^{(-\frac{\alpha_s}{20})}$$

★★ **9.11**

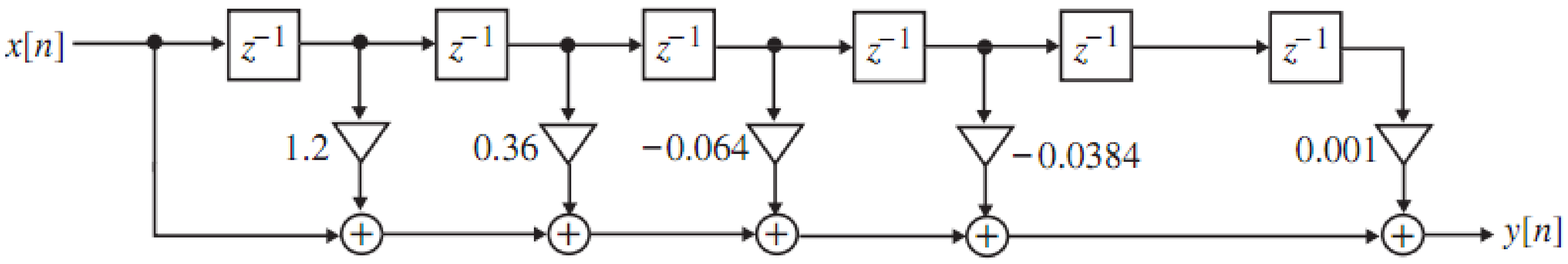
$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \Rightarrow z = \frac{1+\frac{T}{2}s}{1-\frac{T}{2}s}$$

$$H_a(s) = G_a(z)\Big|_{z=\frac{1+\frac{04}{2}s}{1-\frac{0.4}{2}s}}$$

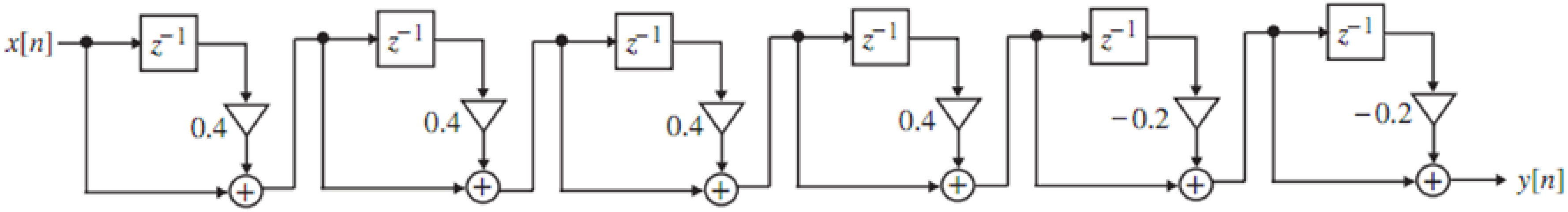
★★★ Canonical（direct II form ）, cascade

★★★ **8.13** **(a)** $H(z) = (1+0.4z^{-1})^4(1-0.2z^{-1})^2$
$$= 1+1.2z^{-1}+0.36z^{-2}-0.64z^{-3}-0.0384z^{-4}+0.001z^{-6}.$$

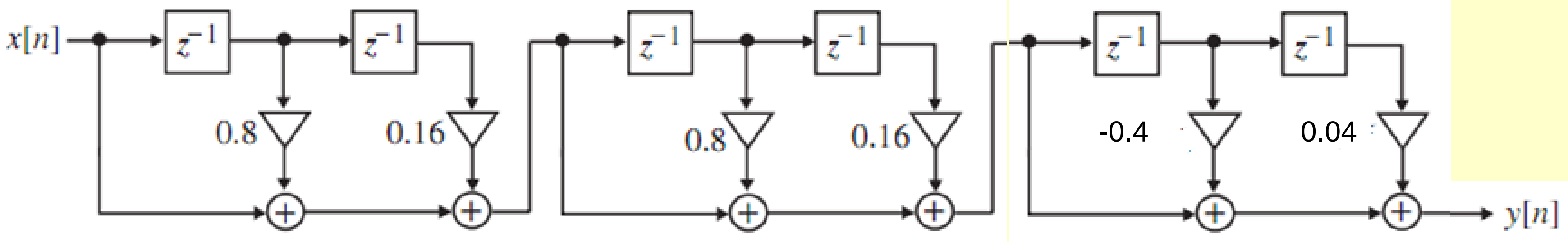
A direct form realization of $H(z)$ is shown below:



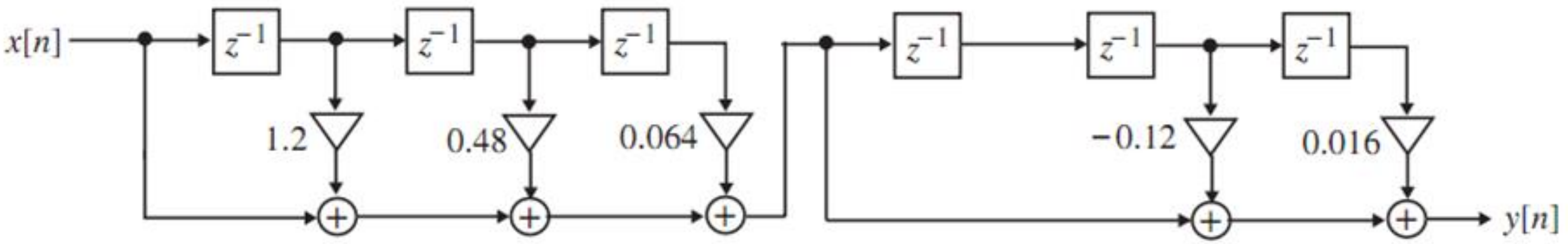
(b) A realization in the form of cascade of six first-order sections is shown below:



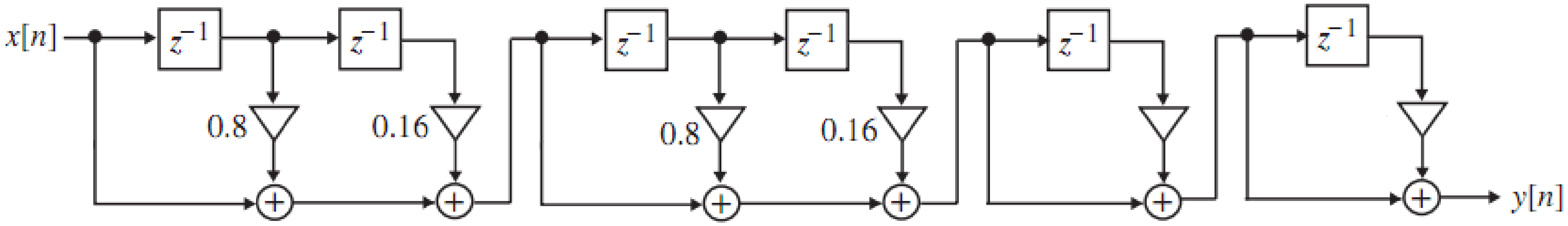
(c) A realization in the form of cascade of three second-order sections is shown below:



(d) A realization in the form of cascade of two third-order sections is shown below:



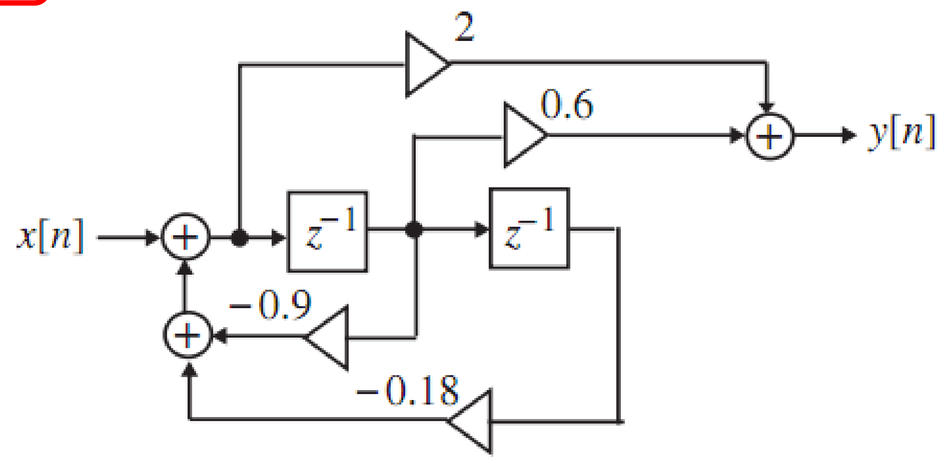
(e) A realization in the form of cascade of two first-order sections and two second-order sections is shown below:



★★★

8.24 **(a)** A $\boxed{\text{direct form II}}$ realization of $H_1(z)$ is shown below:

canonical



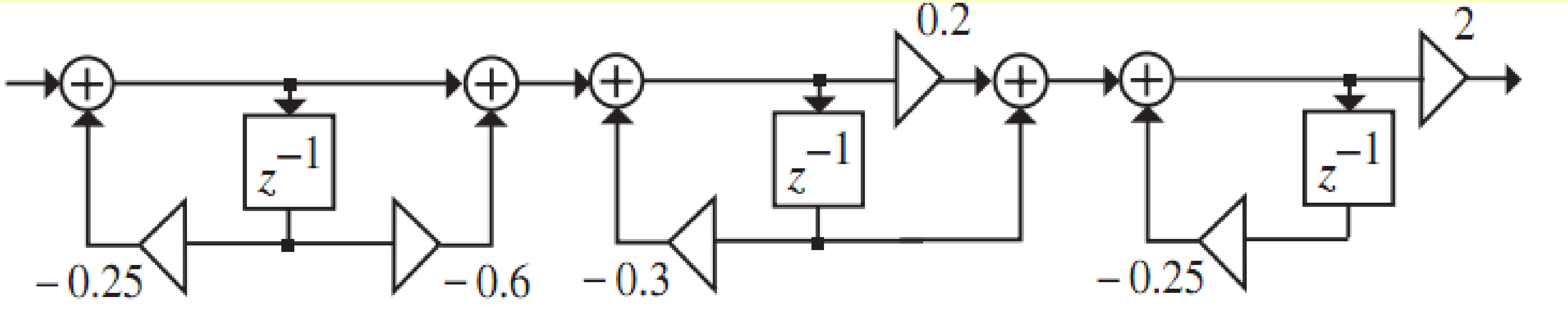
-0.2

★★★

8.28 **(a)** $H(z) = \dfrac{1-0.6z^{-1}}{1+0.25z^{-1}} \cdot \dfrac{0.2+z^{-1}}{1+0.3z^{-1}} \cdot \dfrac{2}{1+0.25z^{-1}} = \dfrac{0.4+1.76z^{-1}-1.2z^{-2}}{1+0.8z^{-1}+0.2125z^{-2}+0.0187z^{-3}}.$

(b)
$y[n] = 0.4x[n] + 1.76x[n-1] - 1.2x[n-2] - 0.8y[n-1] - 0.2125y[n-2] - 0.0187y[n-3].$

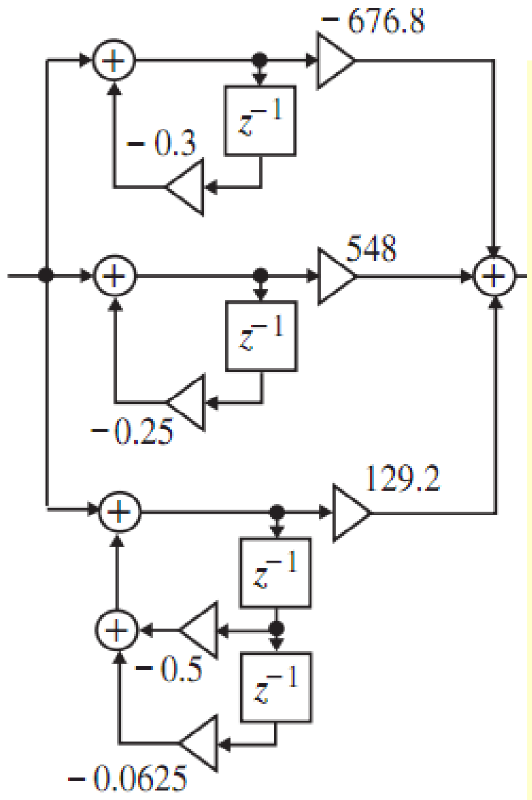
(c) A cascade realization of $H(z)$ is shown below:



★★★

(d) A partial-fraction expansion of $H(z)$ in z^{-1} obtained using `residuez` is given by

$$H(z) = \frac{548}{1+0.25z^{-1}} + \frac{129.2}{(1+0.25z^{-1})^2} + \frac{-676.8}{1+0.3z^{-1}}.$$ The Parallel Form I realization based on this expansion is shown



(e) The inverse z–transform of the partial-fraction of $H(z)$ given in Part (d) yields

$$h[n] = 548(-0.25)^n \mu[n] + 129.2(n+1)(-0.25)^n \mu[n] - 676.8(-0.3)^n \mu[n].$$

★★★ 补充：课堂测验题

★★★ 补充：

* 量化误差的三个来源：

a) the quantization of the input and output signals

b) roundoff errors in the internal computations of the filter (滤波器中间计算舍入误差)

c) coefficient quantization(系数量化误差）

Quantization Effects (量化效应) : 因量化误差对系统产生的影响。

7.39 (a)

$$z_1 = 1 \Rightarrow z_1^* = 1, \frac{1}{z_1} = \frac{1}{z_1^*} = 1$$

$$z_2 = 0.6 \Rightarrow z_2^* = 0.6, \frac{1}{z_2} = \frac{1}{z_2^*} = -\frac{5}{3}$$

$$z_3 = -1+j \Rightarrow z_3^* = -1-j, \frac{1}{z_3} = \frac{-1-j}{2}, \frac{1}{z_3^*} = \frac{-1+j}{2}$$

$$H(z) = (1-z^{-1})^2(1+0.6z^{-1})(1+\frac{5}{3}z^{-1})(1-(-1+j)z^{-1})$$

$$(1-(-1-j)z^{-1})(1-(\frac{-1-j}{2})z^{-1})(1-(\frac{-1+j}{2})z^{-1})$$

Since a Type 1 FIR transfer function can have

no zeros or an even number zeros at $z = 1$

$H_1(z)$ must have another zero at $z = 1$.

7.3.1 Zero Locations of Linear-Phase Transfer Functions

★★★ **EXAMPLE 7.8**

Determination of a Linear-Phase FIR Transfer Function from Its Zero Locations

A length-9 Type 1 real coefficient FIR filters has the following zeros:

$$z_1 = -0.5, \quad z_2 = 0.3 + j0.5, \quad z_3 = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

Determine the locations of the remaining zeros and the expression for the transfer function .

7.3.1 Zero Locations of Linear-Phase Transfer Functions

$$z_4 = \frac{1}{z_1} = -2, \quad z_5 = \quad z_2^* = 0.3 - j0.5$$

$$z_6 = \frac{1}{z_2} = 0.12 - j0.1993$$

$$z_7 = z_6^* = 0.12 + j0.1993$$

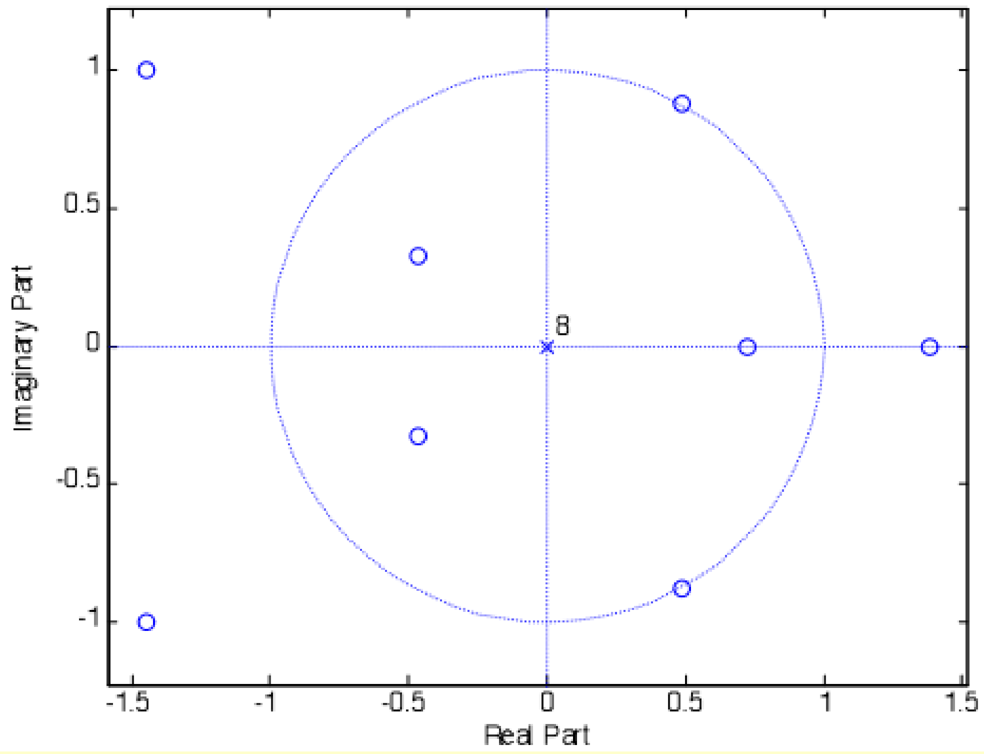
$$z_8 = z_3^* = -\frac{1}{2} - j\frac{\sqrt{3}}{2} (z_3 \quad is \quad on \quad unit \quad circle)$$

$$H(z) = \prod_{i=1}^{8} (1 - z_i z^{-1})$$

7.45 **(a)** Type 1: $\{h[n]\} = \{2,\ 1.5,\ -3.2,\ -5.2,\ 6.4,\ -5.2,\ -3.2,\ 1.5,\ 2\}$. Hence,

$$H(z) = 2 + 1.5z^{-1} - 3.2z^{-2} - 5.2^{-3} + 6.4z^{-4} - 5.2z^{-5} - 3.2z^{-6} + 1.5z^{-7} + 2z^{-8}.$$

The zero plot obtained using the M-file `zplane` is shown below:



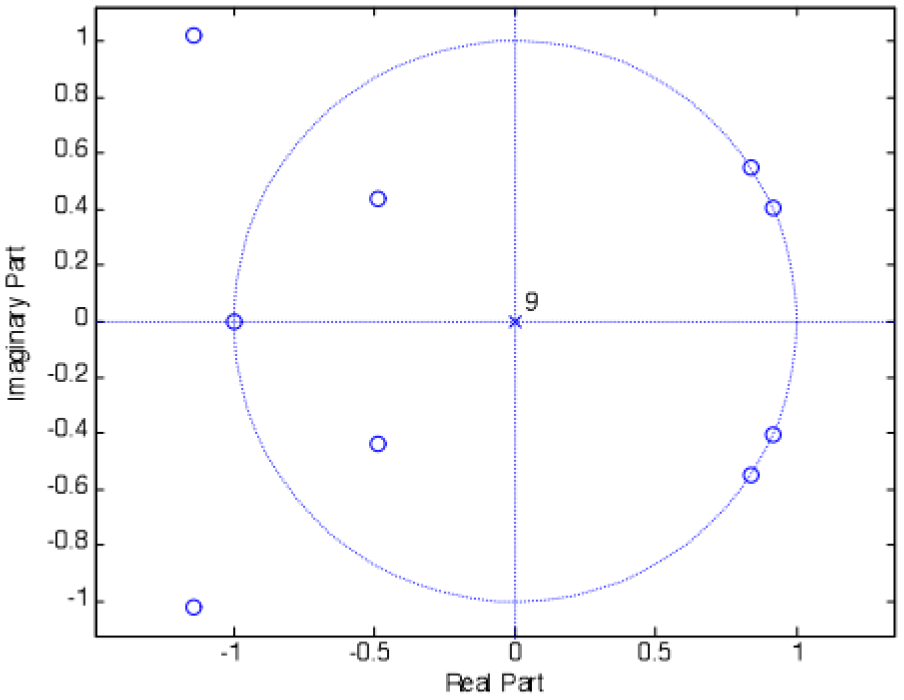
It can be seen from the above that complex zeros on the unit circle appear in complex-conjugate pairs, and zeros not on the unit circle appear in mirror-image symmetry. There are no zeros at $z = 1$ or $z = -1$.

(b) Type 2: $\{h[n]\} = \{2,\ \ 1.5,\ \ -3.2,\ \ -5.2,\ \ 6.4,\ \ 6.4,\ \ -5.2,\ \ -3.2,\ \ 1.5,\ \ 2\}$.

Hence,

$$H(z) = 2 + 1.5z^{-1} - 3.2z^{-2} - 5.2^{-3} + 6.4z^{-4} + 6.4z^{-5} - 5.2z^{-6} - 3.2z^{-7} + 1.5z^{-8} + 2z^{-9}.$$

The zero plot obtained using the M-file `zplane` is shown below:



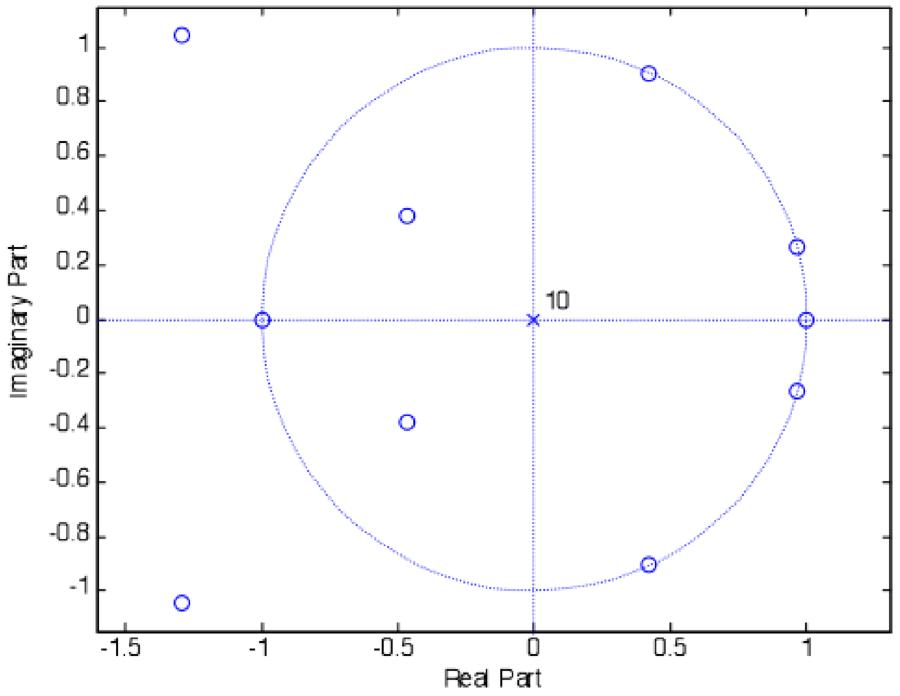
It can be seen from the above that complex zeros on the unit circle appear in complex-conjugate pairs, and zeros not on the unit circle appear in mirror-image symmetry. There are no zeros at $z = 1$ and one zero at $z = -1$.

(c) Type 3:

$\{h[n]\} = \{2,\ \ 1.5,\ \ -3.2,\ \ -5.2,\ \ 6.4,\ \ 0,\ \ -6.4,\ \ 5.2,\ \ 3.2,\ \ -1.5,\ \ -2\}$. Hence,

$H(z) = 2 + 1.5z^{-1} - 3.2z^{-2} - 5.2^{-3} + 6.4z^{-4} - 6.4z^{-6} + 5.2z^{-7} + 3.2z^{-8} - 1.5z^{-9} - 2z^{-10}$.

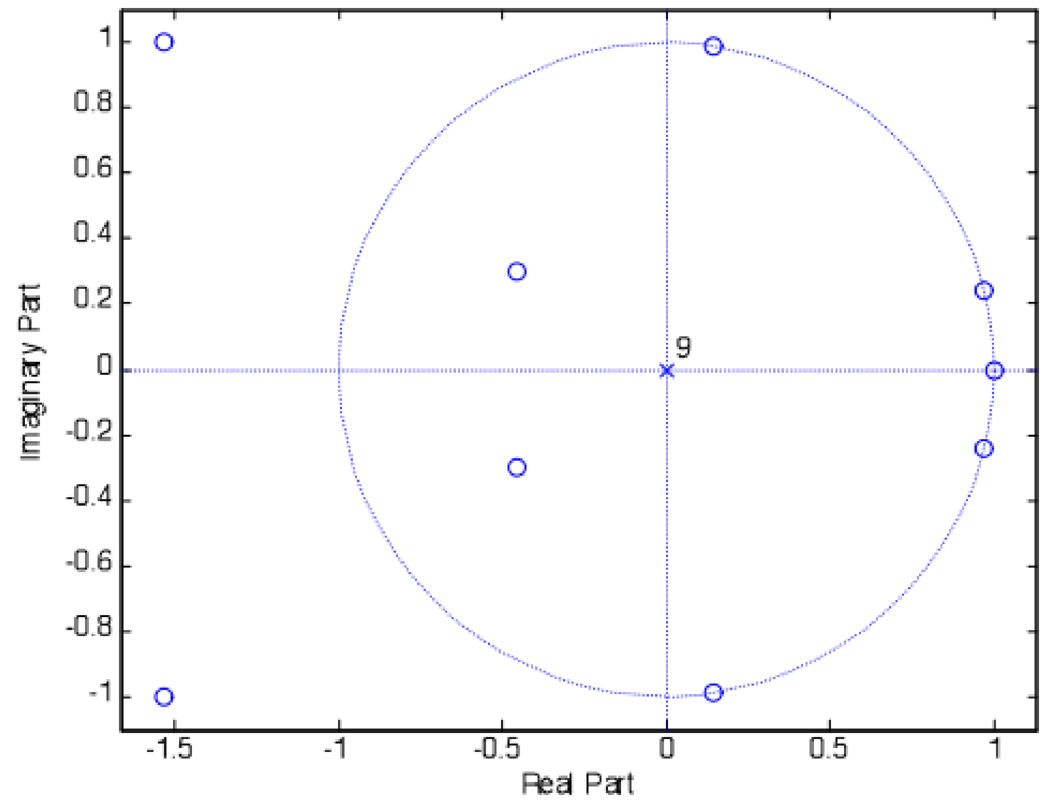
The zero plot obtained using the M-file `zplane` is shown below:



It can be seen from the above that complex zeros on the unit circle appear in complex-conjugate pairs, and zeros not on the unit circle appear in mirror-image symmetry. There is one zero at $z = 1$ and one zero at $z = -1$.

(d) Type 4: $\{h[n]\} = \{2, \ \ 1.5, \ \ -3.2, \ \ -5.2, \ \ 6.4, \ \ -6.4, \ \ 5.2, \ \ 3.2, \ \ -1.5, \ \ -2\}$.
Hence,
$$H(z) = 2 + 1.5z^{-1} - 3.2z^{-2} - 5.2^{-3} + 6.4z^{-4} - 6.4z^{-5} + 5.2z^{-6} + 3.2z^{-7} - 1.5z^{-8} - 2z^{-9}.$$

The zero plot obtained using the M-file `zplane` is shown below:



It can be seen from the above that complex zeros on the unit circle appear in complex-conjugate pairs, and zeros not on the unit circle appear in mirror-image symmetry. There is one zero at $z = 1$ and no zeros at $z = -1$.

★★★ **7.55** $H_0(z) = \frac{1}{2}(1+z^{-1}).$

$$H(e^{jw}) = \frac{1}{2}(1+e^{-jw}) = \frac{1}{2}(e^{j\frac{w}{2}}e^{-j\frac{w}{2}} + e^{-j\frac{w}{2}}e^{-j\frac{w}{2}})$$

$$= \frac{1}{2}e^{-j\frac{w}{2}}(e^{j\frac{w}{2}} + e^{-j\frac{w}{2}}) = e^{-j\frac{w}{2}}\cos(\frac{w}{2})$$

$$\left|H_0(e^{j\omega})\right| = \cos(\omega/2)$$

$$G(z) = \left(H_0(z)\right)^M \Rightarrow \left|G(e^{j\omega})\right|^2 = \left|H_0(e^{j\omega})\right|^{2M} = \left(\cos(\omega/2)\right)^{2M}$$

The 3–dB cutoff frequency ω_c of

$G(z)$ is thus given by $\left(\cos(\omega_c/2)\right)^{2M} = \frac{1}{2}$

Hence, $\omega_c = 2\cos^{-1}(2^{-1/2M})$

★★★ **7.57** To suppress a sinusoidal component with frequency ω_o, the transfer function $H(z)$ must have zeros at $z = e^{\pm j\omega_o}$.

Thus we have

$$H(z) = (1 - e^{j\omega_o} z^{-1})(1 - e^{-j\omega_o} z^{-1}) = 1 - 2\cos\omega_o z^{-1} + z^{-2}.$$

The notch frequencies are

(i) Here $2\cos\omega_o = 1$ and hence, $\omega_o = \pi/3$,

(ii) $x(n) = \sin(w_0 n)\Big|_{w_0 = \frac{\pi}{3}}$

(iii) $y[n] = \left|H(e^{jw_0})\right| \sin(w_0 n + \arg H(e^{jw_0}))$

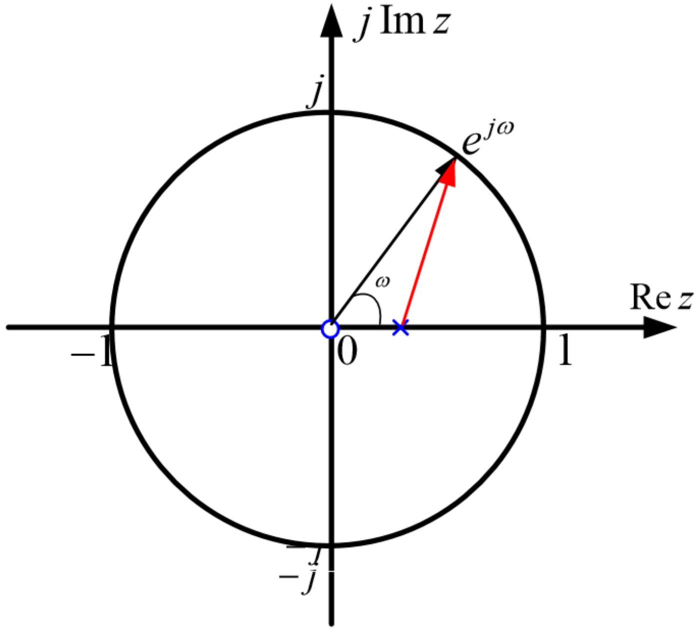
$$H(e^{jw}) = H(z)\Big|_{z=e^{jw}} = (1 - e^{j\frac{\pi}{3}} e^{-jw})(1 - e^{-j\frac{\pi}{3}} e^{-jw})$$

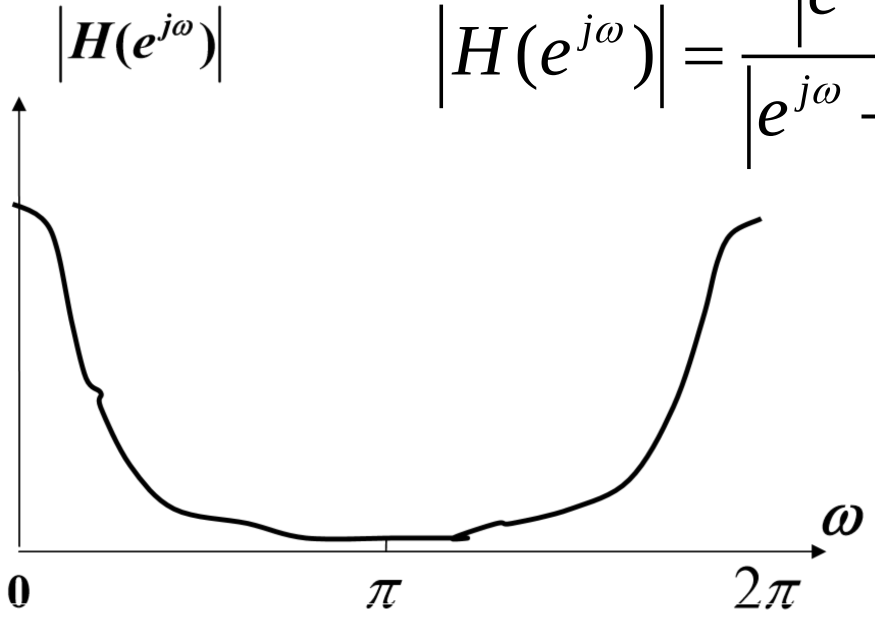
$$\left|H(e^{j\frac{\pi}{3}})\right| = \left|(1 - e^{j\frac{\pi}{3}} e^{-j\frac{\pi}{3}})(1 - e^{-j\frac{\pi}{3}} e^{-j\frac{\pi}{3}})\right| = 0 \quad \Longrightarrow \quad y[n] = 0$$

Geometric Interpretation of Frequency Response Computation

例: 一阶系统 $H(z)=\dfrac{z}{z-a}$, $|z|>|a|$

$$h[n]=a^n u[n]$$





$$|H(e^{j\omega})|=\frac{|e^{j\omega}|}{|e^{j\omega}-a|}=\frac{1}{|e^{j\omega}-a|}$$

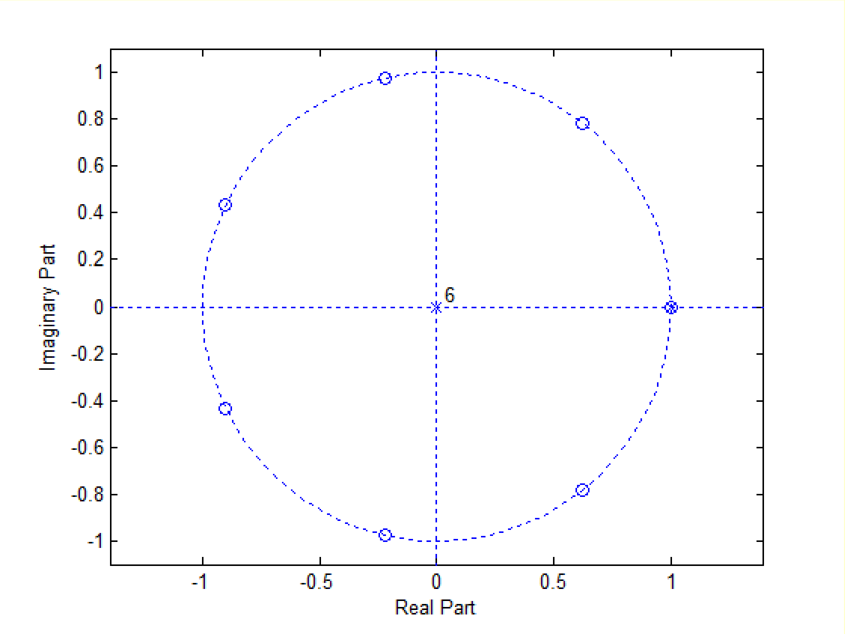
Geometric Interpretation of Frequency Response Computation

$$h[n]=\mu[n]-\mu[n-8]\Rightarrow H(z)=\sum_{n=0}^{7} z^{-n}=\frac{1-z^{-8}}{1-z^{-1}}$$

$$H(z)=\frac{1-z^{-8}}{1-z^{-1}}=\frac{z^{8}-1}{z^{7}(z-1)}$$

$$zeros: \qquad z^{8}=1=e^{j2k\pi}\Rightarrow z_{0k}=e^{j\frac{2k\pi}{8}}=e^{j\frac{k\pi}{4}} \qquad k=0\cdots 7$$

$$poles: \qquad z=0, z=1$$



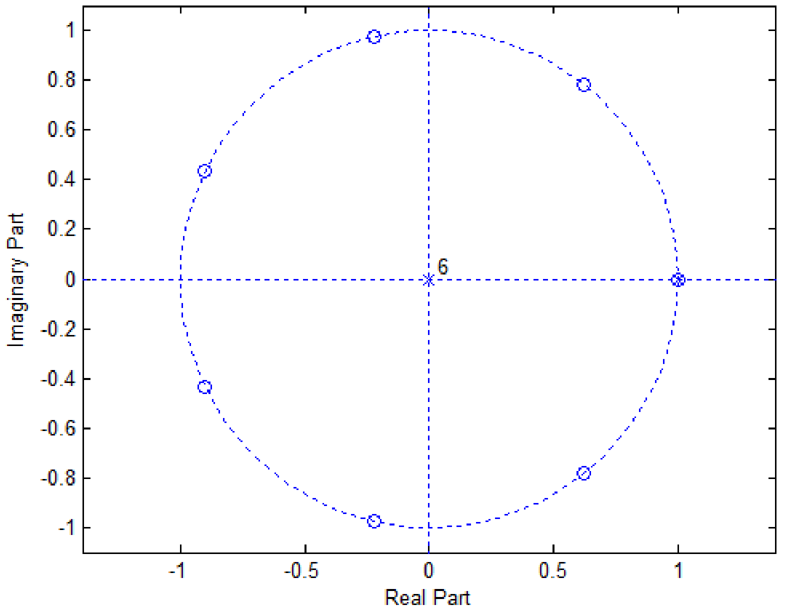
Geometric Interpretation of Frequency Response Computation

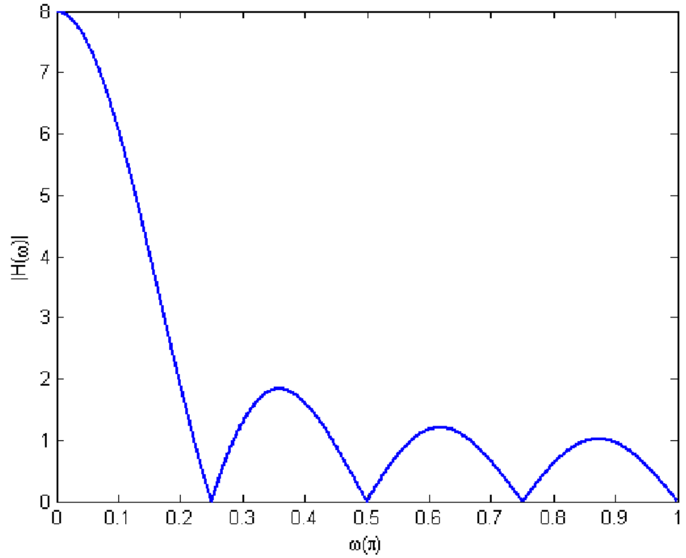
$$H(z)=\frac{z^8-1}{z^7(z-1)}\Rightarrow H(z)=\frac{\prod_{k=0}^{7}(z-z_{0k})}{z^7(z-1)}$$

$$H(e^{j\omega})=\frac{\prod_{k=0}^{7}(e^{j\omega}-z_{0k})}{e^{j\omega7}(e^{j\omega}-1)}\Rightarrow\left|H(e^{j\omega})\right|=\frac{\prod_{k=0}^{7}\left|e^{j\omega}-z_{0k}\right|}{\left|e^{j\omega}-1\right|}=\prod_{k=1}^{7}\left|e^{j\omega}-z_{0k}\right|$$

LTI系统：$h[n]=\mu[n]-\mu[n-8]$

其幅频响应如图：





★★★会求下列滤波器的零极点（zeros，poles），画零级图（pole/zero patterns），判断是什么选频特性的滤波器？画出频响草图（sketch magnitude response）.

$$H_0(z)=\frac{1}{2}(1+z^{-1})=\frac{z+1}{2z}$$

$$H_1(z)=\frac{1}{2}(1-z^{-1})$$

$$H(z)=\frac{K}{1-\alpha z^{-1}},\quad 0<\alpha<1$$

$$H(z)=\frac{K(1+z^{-1})}{1-\alpha z^{-1}},\quad 0<\alpha<1$$

$$H(z)=\frac{K}{1-\alpha z^{-1}},\quad -1<\alpha<0$$

$$H(z)=\frac{K(1+z^{-1})}{1-\alpha z^{-1}},\quad -1<\alpha<0$$

$$H_{LP}(z)=\frac{K(1+z^{-1})}{1-\alpha z^{-1}},\quad 0<|\alpha|<1$$

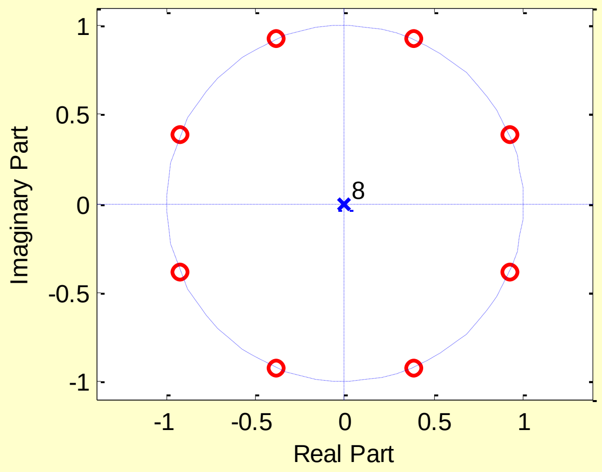
$$H(z)=\frac{(1-z^{-1})(1+z^{-1})}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})}$$

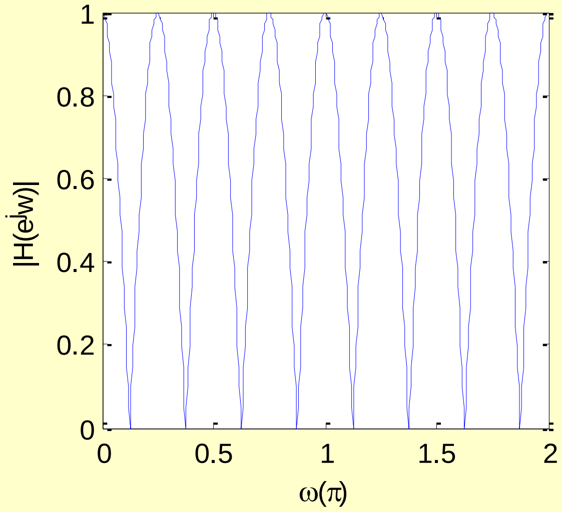
$$H(z)=\frac{(1-e^{j\omega_0}z^{-1})(1+e^{-j\omega_0}z^{-1})}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})}$$

★★★ 会求下列滤波器的零极点（zeros，poles），画零级图（pole/zero patterns），判断是什么选频特性的滤波器？画出频响草图（sketch magnitude response）.

$$G_0(z)=H_0(z^L)=\tfrac{1}{2}(1+z^{-L})$$

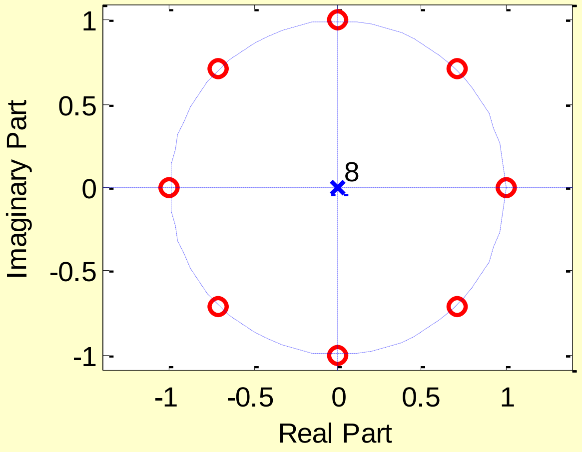
$$1+z^{-L}=0\Rightarrow z^L=-1=e^{j\pi}e^{j2k\pi}\Rightarrow z_{0k}=e^{j\frac{(2k+1)\pi}{L}}$$

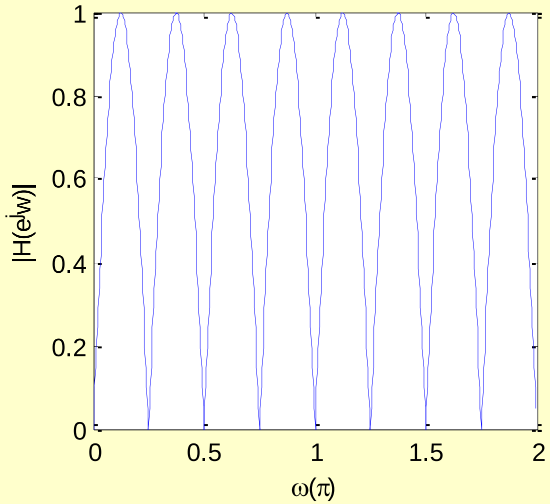




$$G_1(z)=H_1(z^L)=\tfrac{1}{2}(1-z^{-L})$$

$$1-z^{-L}=0\Rightarrow z^L=1=e^{j2k\pi}\Rightarrow z_{0k}=e^{j\frac{2k\pi}{L}}$$





★★

四种类型实系数线性相位FIR系统都适合做什么类型的滤波器，为什么？

Type 1	Type 2	Type 3	Type 4
No restriction Can design any type	Cannot design highpass and bandstop Zero at $\omega = \pi$	Cannot design lowpass, highpass, and bandstop Zero at $\omega = 0$ and $\omega = \pi$	Cannot design lowpass, and bandstop Zero at $\omega = 0$

zeros:		$z = -1$	$z = 1$ $\quad z = -1$	$z = 1$

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} \qquad H(e^{j0}) = H(z)\big|_{z=e^{j0}=1} \qquad H(e^{j\pi}) = H(z)\big|_{z=e^{j\pi}=-1}$$

★★ 线性相位FIR系统的单位脉冲响 $h[n]$ 应该满足什么条件?

$$h[n]=h[N-n],\qquad 0\le n\le N$$
$$(c=-N/2)$$

$$h[n]=-h[N-n],\qquad 0\le n\le N$$
$$(c=-N/2)$$

四种线性相位系统的相频响应是什么具体函数形式?

$$\arg\{H(e^{j\omega})\}=\theta(\omega)=-\frac{N}{2}\omega+\beta \qquad (7.62)$$

β Is either 0 or π for either Type 1 or Type 2 filters

β Is either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ for either Type 3 or Type 4 filters

线性相位FIR系统传递函数 $H(z)$满足什么条件?

$$H(z)=z^{-N}H(z^{-1})\quad H(z)=-z^{-N}H(z^{-1})$$

★★ 7.5 Inverse Systems

$$h_1[n] \circledast h_2[n] = \delta[n] \qquad (7.111)$$

$$H_1(z)H_2(z) = 1 \qquad (7.112)$$

$H_1(z)$ *is the inverse filter of* $H_2(z)$, *and vice versa*

$$H_2(z) = \frac{1}{H_1(z)} \qquad (7.113)$$

$$H_1(z) = \frac{P(z)}{D(z)} \qquad H_2(z) = \frac{D(z)}{P(z)} \qquad (7.114)\ (7.115)$$

The poles(zeros) of the inverse system $H_2(z)$ are the zeros(poles) of the system $H_1(z)$

Example 7.14

序列	z变换	收敛域
$\delta[n]$	1	整个z平面
$\alpha^n \mu[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$-\alpha^n \mu[-n-1]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
$\mu[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$r^n(\cos\omega_o n)\mu[n]$	$\dfrac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
$r^n(\sin\omega_o n)\mu[n]$	$\dfrac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

必须记住

6.3 Region of Convergence of a Rational z-Transform

Determine the ROC of the following sequence

(1) $x[n]=\delta[n]$ → $X(z)=\sum_{n}\delta[n]z^{-n}=1$

ROC：整个z平面

(2) $x[n]=\delta[n+2]+\delta[n-2]$ → $X(z)=z^{2}+z^{-2}$

ROC：$z\neq 0, |z|\neq\infty$

(3) $x[n]=0.2^{n}(\mu[n]-\mu[n-5])$ → $X(z)=\sum_{n=0}^{4}0.2^{n}z^{-n}$

ROC：$z\neq 0$

单选题 1分

设置

The ROC of sequence $x[n]=0.2^n\mu[n+5]$ **is**

A $|z|>0.2$

B $|z|>0.2 \quad |z|\neq\infty$

C $|z|<0.2$

D $|z|>0.2 \quad z\neq 0$

提交

6.3 Region of Convergence of a Rational z-Transform

★ ★ ★ **Example:** Determine the z-transform and corresponding ROCs.

1) $x[n] = (0.8)^n \mu[n] + (1.25)^n \mu[n]$ $\quad |z| > 1.25$

2) $x[n] = (0.8)^n \mu[n] - (1.25)^n \mu[-n-1]$ $\quad 0.8 < |z| < 1.25$

3) $x[n] = -(0.8)^n \mu[-n-1] - (1.25)^n \mu[-n-1]$ $\quad |z| < 0.8$

4) $x[n] = -(0.8)^n \mu[-n-1] + (1.25)^n \mu[n]$ $\quad$ **ROC does not exist**

$$X(z) = \frac{1}{1-0.8z^{-1}} + \frac{1}{1-1.25z^{-1}}$$

6.2 (a)

$$X_a(z) = \frac{\alpha}{1 - z^{-1}}, \text{ROC: } |z| < 1$$

(b)

$$X_b(z) = \frac{1}{1 - \alpha z^{-1}} \qquad |z| < |\alpha|$$

6.5 **(a)** $Z\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = \delta[0] = 1$, which converges everywhere in the z-plane.

(b) $x[n] = \alpha^n \mu[n]$. From Table 6.1, $Z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \dfrac{1}{1-\alpha z^{-1}}, |z| > |\alpha|$.

Let $g[n] = nx[n]$. Then, $Z\{g[n]\} = G(z) = \sum_{n=-\infty}^{\infty} nx[n] z^{-n}$. Now,

$\dfrac{dX(z)}{dz} = -\sum_{n=-\infty}^{\infty} ng[n] z^{-n-1}$. Hence, $z\dfrac{dX(z)}{dz} = -\sum_{n=-\infty}^{\infty} nx[n] z^{-n} = -G(z)$, or,

$$G(z) = -z\frac{dX(z)}{dz} = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}, |z| > |\alpha|.$$

(c) $x[n] = r^n \sin(\omega_o n)\mu[n] = \dfrac{r^n}{2j}(e^{j\omega_o n} - e^{-j\omega_o n})\mu[n]$. Using the results of Example 6.1 and the linearity property of the z-transform we get

$$Z\{r^n \sin(\omega_o n)\mu[n]\} = \frac{1}{2j}\left(\frac{1}{1-re^{j\omega_o}z^{-1}}\right) - \frac{1}{2j}\left(\frac{1}{1-re^{-j\omega_o}z^{-1}}\right)$$

$$= \frac{\frac{r}{2j}(e^{j\omega_o} - e^{-j\omega_o})z^{-1}}{1-r(e^{j\omega_o}+e^{-j\omega_o})z^{-1}+r^2z^{-2}} = \frac{r\sin(\omega_o)z^{-1}}{1-2r\cos(\omega_o)z^{-1}+r^2z^{-2}}, \quad \text{ROC: } |z| > |r|.$$

★★★ **6.7** $\mathcal{Z}\{(0.6)^n \mu[n]\} = \dfrac{1}{1-0.6z^{-1}}, |z| > 0.6;\ \mathcal{Z}\{(-0.8)^n \mu[n]\} = \dfrac{1}{1+0.8z^{-1}}, |z| > 0.8;$

$$\mathcal{Z}\{-(0.6)^n \mu[-n-1]\} = \frac{1}{1-0.6z^{-1}}, |z| < 0.6;$$

$$\mathcal{Z}\left\{-(-0.8)^n u(-n-1)\right\} = \frac{1}{1+0.8z^{-1}}, |z| < 0.8$$

(a) $\mathcal{Z}\{x_1[n]\} = \dfrac{1}{1-0.6z^{-1}} + \dfrac{1}{1+0.8z^{-1}}$ $\quad |z| > 0.8.$

(b) $\mathcal{Z}\{x_2[n]\} = \dfrac{1}{1-0.6z^{-1}} + \dfrac{1}{1+0.8z^{-1}}$ $\quad 0.6 < |z| < 0.8.$

(c) $\mathcal{Z}\{x_3[n]\} = \dfrac{1}{1-0.6z^{-1}} + \dfrac{1}{1+0.8z^{-1}}$ $\quad |z| < 0.6$

(d) Since the ROC of the first term is $|z| < 0.6$ and that of the second term is $|z| > 0.8$, the z–transform of $x_4[n]$ does not converge. Hence, none of the sequences have the same z-transform.

★★★ **6.8** （i）

<法一> $x_1(n)=0.2^n u(n+1)$

$$X_1(z)=\sum_{n=-1}^{+\infty}0.2^n z^{-n}=\frac{(0.2)^{-1}z[1-(0.2z^{-1})^{\infty}]}{1-0.2z^{-1}}=\frac{5z}{1-0.2z^{-1}}$$

$$ROC:|z|>0.2,|z|\neq\infty$$

<法二>

利用延时性质：

$$x_1(n)=\frac{0.2^{n+1}}{0.2}u(n+1)\xrightarrow{Z}\mathrm{X}_1(z)=\frac{1}{0.2}\frac{z}{1-0.2z^{-1}}=\frac{5z}{1-0.2z^{-1}}$$

★★★ **6.8 （iv）**

<法一> $x_4(n)=(-0.5)^n u(-n-3)$

$$X_4(z)=\sum_{n=-\infty}^{-3}(-0.5)^n z^{-n}=\sum_{n=3}^{+\infty}(-0.5)^{-n}z^n=\sum_{n=3}^{+\infty}(-2)^n z^n$$

$$=\frac{(-2)^3z^3[1-(-2z)^\infty]}{1+2z}=\frac{-4z^2}{1+0.5z^{-1}}$$

$$ROC:|z|<\frac{1}{2}$$

<法二>

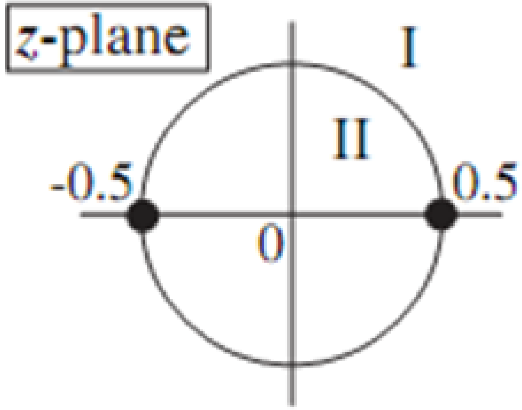
利用延时性质：

$$x_4(n)=-\frac{(-0.5)^{n+2}}{(-0.5)^2}u[-(n+2)-1]\xrightarrow{Z}X_4(z)=\frac{1}{(-0.5)^2}\frac{z^2}{1+0.5z^{-1}}=\frac{-4z^2}{1+0.5z^{-1}}$$

§ 6.4 The Inverse z-Transform

Example 5.5.4: Compute all possible inverse z-transforms of

$$X(z) = \frac{6 + z^{-1}}{1 - 0.25z^{-2}}$$



Solution: Because the numerator has degree one in z^{-1}, we have the PF expansion:

$$X(z) = \frac{6 + z^{-1}}{1 - 0.25z^{-2}} = \frac{6 + z^{-1}}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})} = \frac{A_1}{1 - 0.5z^{-1}} + \frac{A_2}{1 + 0.5z^{-1}}$$

where

$$A_1 = \left[\frac{6 + z^{-1}}{1 + 0.5z^{-1}}\right]_{z=0.5} = 4, \qquad A_2 = \left[\frac{6 + z^{-1}}{1 - 0.5z^{-1}}\right]_{z=-0.5} = 2$$

§ 6.4 The Inverse z-Transform

The two poles at ± 0.5 have the same magnitude and therefore divide the z-plane into two ROC regions I and II: $|z| > 0.5$ and $|z| < 0.5$. For the first ROC, both terms in the PF expansion are inverted causally giving:

$$x(n) = A_1 (0.5)^n u(n) + A_2 (-0.5)^n u(n)$$

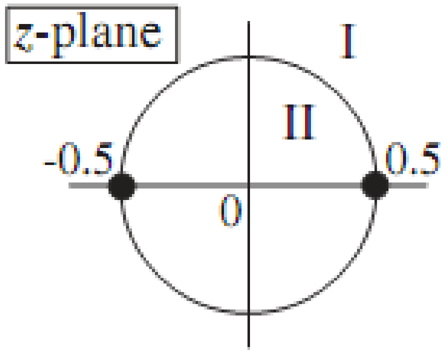
Because this ROC also contains the unit circle the signal $x(n)$ will be stable. For the second ROC, both PF expansion terms are inverted anticausally giving:

$$x(n) = -A_1 (0.5)^n u(-n-1) - A_2 (-0.5)^n u(-n-1)$$

§ 6.4 The Inverse z-Transform

Example 5.5.5: Determine all inverse z-transforms of

$$X(z) = \frac{10 + z^{-1} - z^{-2}}{1 - 0.25z^{-2}}$$



$$X(z) = \frac{10 + z^{-1} - z^{-2}}{1 - 0.25z^{-2}} = \frac{10 + z^{-1} - z^{-2}}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})}$$

$$= A_0 + \frac{A_1}{1 - 0.5z^{-1}} + \frac{A_2}{1 + 0.5z^{-1}}$$

$$A_0 = \left[\frac{10 + z^{-1} - z^{-2}}{1 - 0.25z^{-2}}\right]_{z=0} = \left[\frac{10z^2 + z - 1}{z^2 - 0.25}\right]_{z=0} = \frac{-1}{-0.25} = 4$$

$$A_1 = \left[\frac{10 + z^{-1} - z^{-2}}{1 + 0.5z^{-1}}\right]_{z=0.5} = 4, \qquad A_2 = \left[\frac{10 + z^{-1} - z^{-2}}{1 - 0.5z^{-1}}\right]_{z=-0.5} = 2$$

Again, there are only two ROCs I and II: $|z| > 0.5$ and $|z| < 0.5$. For the first ROC, the A_1 and A_2 terms are inverted causally, and the A_0 term inverts into a simple $\delta(n)$:

$$x(n) = A_0\delta(n) + A_1(0.5)^n u(n) + A_2(-0.5)^n u(n)$$

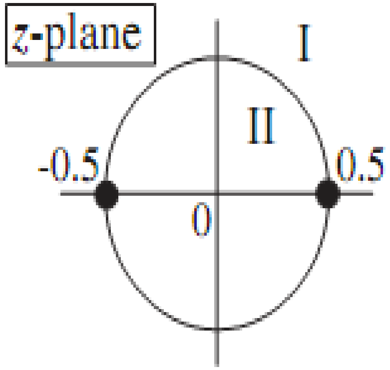
For the second ROC, we have:

$$x(n) = A_0\delta(n) - A_1(0.5)^n u(-n-1) - A_2(-0.5)^n u(-n-1)$$

★★★ § 6.4 The Inverse z-Transform

Example 5.5.6: Determine the causal inverse z-transform of

$$X(z) = \frac{6 + z^{-5}}{1 - 0.25z^{-2}}$$



The second technique is the "remove/restore" method. Ignoring the numerator we have

$$W(z) = \frac{1}{1 - 0.25z^{-2}} = \frac{0.5}{1 - 0.5z^{-1}} + \frac{0.5}{1 + 0.5z^{-1}}$$

which has the causal inverse

$$w(n) = 0.5(0.5)^n u(n) + 0.5(-0.5)^n u(n)$$

Once $w(n)$ is known, one can obtain $x(n)$ by restoring the numerator:

$$X(z) = (6 + z^{-5})W(z) = 6W(z) + z^{-5}W(z)$$

Taking inverse z-transforms of both sides and using the delay property, we find

$$x(n) = 6w(n) + w(n-5) = 3(0.5)^n u(n) + 3(-0.5)^n u(n) + 0.5(0.5)^{n-5} u(n-5) + 0.5(-0.5)^{n-5} u(n-5)$$

The two expressions for $x(n)$ from the two techniques are equivalent.

$$X(z) = (6 + z^{-5})W(z) = 6W(z) + z^{-5}W(z)$$

Taking inverse z-transforms of both sides and using the delay property, we find

$$x(n) = 6w(n) + w(n-5) = 3(0.5)^n u(n) + 3(-0.5)^n u(n) + 0.5(0.5)^{n-5} u(n-5) + 0.5(-0.5)^{n-5} u(n-5)$$

The two expressions for $x(n)$ from the two techniques are equivalent.

★★★

6.13 **(a)** $X_a(z) = \dfrac{7+3.6z^{-1}}{1+0.9z^{-1}+0.18z^{-2}} = \dfrac{2}{1+0.6z^{-1}} + \dfrac{5}{1+0.3z^{-1}}$.

left-sided: $|z| < 0.3$,

$$x_a[n] = -5(-0.3)^n u[-n-1] - 2(-0.6)^n u[-n-1]$$

two-sided: $0.3 < |z| < 0.6$,

$$x_b[n] = 5(-0.3)^n u[n] - 2(-0.6)^n u[-n-1]$$

right-sided: $|z| > 0.6$

$$x_c[n] = 2(-0.6)^n \mu[n] + 5(-0.3)^n \mu[n]$$

★★★

6.44 (a) A partial-fraction expansion of $H(z)$ in z^{-1}

$$A_0 = \left.\frac{-1.5z^{-1}+0.3z^{-2}}{(1+0.4z^{-1})(1-0.15z^{-1})}\right|_{z=0} = -5$$

$$A_1 = (1+0.4z^{-1})\left.\frac{-1.5z^{-1}+0.3z^{-2}}{(1+0.4z^{-1})(1-0.15z^{-1})}\right|_{z=-0.4} = 4.0909$$

$$A_1 = (1-0.15z^{-1})\left.\frac{-1.5z^{-1}+0.3z^{-2}}{(1+0.4z^{-1})(1-0.15z^{-1})}\right|_{z=0.15} = 0.9091$$

$H(z) = -5 + \dfrac{4.0909}{1+0.4z^{-1}} + \dfrac{0.9091}{1-0.15z^{-1}}$. Hence, from Table 6.1 we have

$h[n] = -5\delta[n] + 4.0909(-0.4)^n \mu[n] + 0.9091(0.15)^n \mu[n]$.

6.47 **(a)** Taking the *z*–transform of both sides of the difference equation we get

$$Y(z) = 0.4z^{-1}Y(z) + 0.05z^{-2}Y(z) + 3X(z).$$

Hence, $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{3}{1 - 0.4z^{-1} - 0.05z^{-2}}$.

(b) A partial-fraction expansion of using the M-file `residuez` yields

$H(z) = \dfrac{2.5}{1 - 0.5z^{-1}} + \dfrac{0.5}{1 + 0.1z^{-1}}$. Hence, from Table 6.1,

$h[n] = 2.5(0.5)^n \mu[n] + 0.5(-0.1)^n \mu[n].$

6.81

(a) The frequency response exists if the ROC contains the unit circle. Since $H(z)$ has poles at −0.3, 0.6, and −5, a two-sided sequence corresponding to an ROC of $0.6 < |z| < 5$ would allow the existence of the frequency response.

(b) The system can be stable if the ROC is $0.6 < |z| < 5$. However, it cannot be both stable and causal because this ROC corresponds to a two-sided sequence.

(c) $A(-0.3)^n u(n) + B(0.6)^n u(n) - C(-5)^n u(-n-1)$

第七周视频1

Finite-Dimensional Linear Time-Invariant IIR Discrete-Time System

★★

- Example - Consider the M-point moving-average FIR filter with an impulse response

$$h[n] = \begin{cases} 1/M, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

- Its transfer function is then given by

$$H(z) = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n} = \boxed{\frac{1-z^{-M}}{M(1-z^{-1})}} = \frac{z^M - 1}{M[z^M(z-1)]}$$

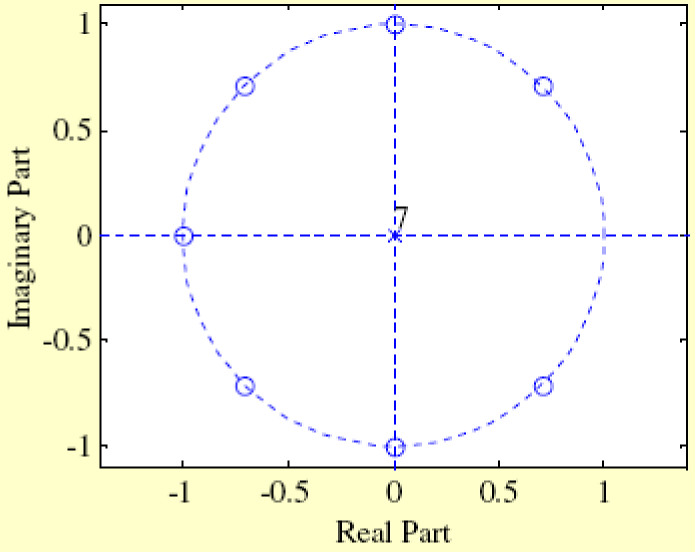
注意：从传递函数形式看好像是IIR滤波器，但本质上是FIR滤波器，因为分母可以消去，这样叫做FIR系统的递归实现

第七周视频1

Finite-Dimensional Linear Time-Invariant IIR Discrete-Time System

- The transfer function has M zeros on the unit circle at $z = e^{j2\pi k/M}$, $0 \le k \le M-1$
- There are $M-1$ poles at $z = 0$ and a single pole at $z = 1$
- The pole at $z = 1$ exactly cancels the zero at $z = 1$
- The ROC is the entire z-plane except $z = 0$

$M = 8$



Determine the zeros and poles of the Z-transform

$$X(z) = \frac{1}{1 - z^{-4}}$$

$$X(z) = \frac{1}{1 - z^{-4}} = \frac{z^4}{z^4 - 1}$$

$$zeros: z^4 = 0 \Rightarrow z_0 = 0$$

$$poles: z^4 - 1 = 0 \Rightarrow z^4 = 1 = e^{j2k\pi}$$

$$z_{p,k} = e^{j\frac{2k\pi}{4}} \qquad k = 0, 1, 2, 3$$

第七周视频1

Finite-Dimensional Linear Time-Invariant IIR Discrete-Time System

★★★

- Example - A causal LTI IIR digital filter is described by a constant coefficient difference equation given by

$$y[n]=x[n-1]-1.2x[n-2]+x[n-3]+1.3y[n-1]$$
$$-1.04y[n-2]+0.222y[n-3]$$

- Its transfer function is therefore given by

$$H(z)=\frac{z^{-1}-1.2z^{-2}+z^{-3}}{1-1.3z^{-1}+1.04z^{-2}-0.222z^{-3}}$$

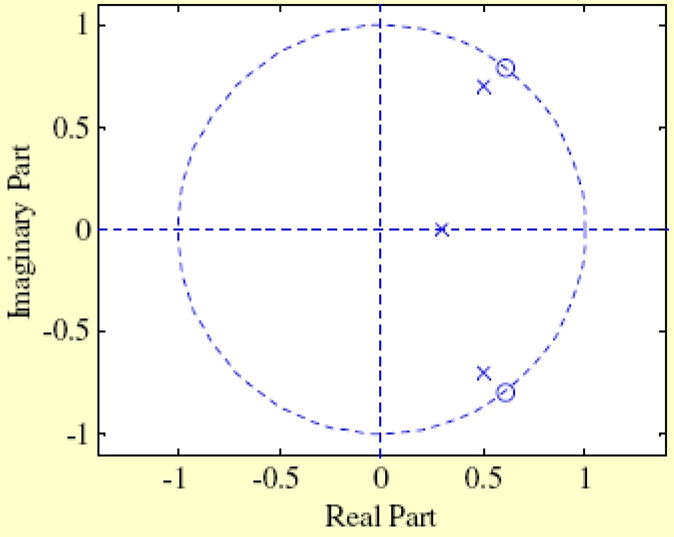
第七周视频1

Finite-Dimensional Linear Time-Invariant IIR Discrete-Time System

- Alternate forms:

$$H(z)=\frac{z^2-1.2z+1}{z^3-1.3z^2+1.04z-0.222}$$

$$=\frac{(z-0.6+j0.8)(z-0.6-j0.8)}{(z-0.3)(z-0.5+j0.7)(z-0.5-j0.7)}$$



- Note: Poles farthest from $z=0$ have a magnitude $\sqrt{0.74}$
- ROC: $|z|>\sqrt{0.74}$

	因果性	稳定性
时域 适用所有系统	计算当前时刻输出没有用到未来时刻输入	输入有界输出有界
时域 适用LTI系统	$h[n]\equiv 0, n<0$	$\sum_{n=-\infty}^{\infty} \lvert h[n] \rvert < \infty$
Z域 适用LTI系统	$ROC: R_{h-} < \lvert z \rvert \le \infty$	ROC: 包含单位圆
Z域 适用LTI系统	所有极点全在单位圆内部	

主观题 10分

设置

★★★

Consider a LTI ***stable*** system

$$H(z) = \frac{1}{1-0.5z^{-1}} + \frac{1}{1+2z^{-1}}$$

a) Determine the difference equation.
b) Determine the impulse response.
c) Determine the causality.

正常使用主观题需2.0以上版本雨课堂

作答

答案：

a） $H(z)=\dfrac{Y(z)}{X(z)}=\dfrac{1+2z^{-1}+1-0.5z^{-1}}{(1-0.5z^{-1})(1+2z^{-1})}=\dfrac{2+1.5z^{-1}}{1+1.5z^{-1}-z^{-2}}$

$(1+1.5z^{-1}-z^{-2})Y(z)=(2+1.5z^{-1})X(z)$

$y[n]+1.5y[n-1]-y[n-2]=2x[n]+1.5x[n-1]$

$y[n]=y[n-2]-1.5y[n-1]+2x[n]+1.5x[n-1]$

b） 因为极点为 $z_{p_1}=0.5, z_{p_2}=-2$

所以可能的ROC为 $|z|<0.5, 0.5<|z|<2, |z|>2$

又因为系统稳定，则ROC应该包含单位圆，所以ROC为 $0.5<|z|<2$

$\therefore h[n]=(0.5)^n\mu[n]-(-2)^n\mu[-n-1]$

c） $\because n<0$时，$h[n]\neq 0$，所以系统是非因果的；

或者说

$\because$ ROC为$0.5<|z|<2$,所以系统是非因果的。

6.42 $H(z) = H_1(z)H_3(z) + \left(1 + H_1(z)\right)H_2(z)$

$= 11.06 + 8.51z^{-1} + 5.28z^{-2} + 5.12z^{-3} + 1.19z^{-4}$.