5.2.1 Definition

DFT的三种形式

$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi k}{N}n}\qquad 0\le k\le N-1$$

$$X(e^{j\omega_k})=\sum_{n=0}^{N-1}x[n]e^{-j\omega_k n}\qquad \omega_k=\frac{2\pi k}{N}$$

$$0\le k\le N-1$$

$$X[k]=\sum_{n=0}^{N-1}x[n]W_N^{kn}\qquad 0\le k\le N-1$$

IDFT的三种形式

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\frac{2\pi k}{N}n}\qquad 0\le n\le N-1$$

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\omega_k n}\qquad 0\le n\le N-1$$

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-kn}\qquad 0\le n\le N-1$$

★★★ 复数乘法次数（**the number of complex multiplication**）： N^2

复数加法次数（**the number of complex addition**）： $N\cdot(N-1)$

5.2.1 Definition

- Example - Consider the length-N sequence

$$x[n]=\begin{cases}1, & n=0\\ 0, & 1\le n\le N-1\end{cases}$$

- Its N-point DFT is given by

$$X[k]=\sum_{n=0}^{N-1}x[n]W_N^{kn}=x[0]W_N^0=1$$

$$0\le k\le N-1$$

- Example - Consider the length-N sequence

$$y[n]=\begin{cases}1, & n=m\\ 0, & 0\le n\le m-1,\ m+1\le n\le N-1\end{cases}$$

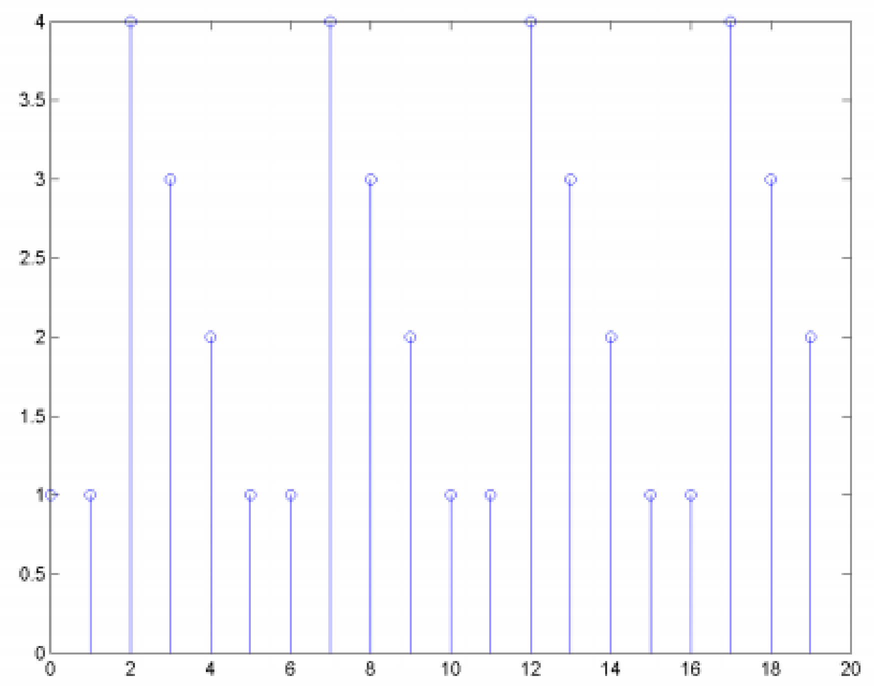
- Its N-point DFT is given by

$$Y[k]=\sum_{n=0}^{N-1}y[n]W_N^{kn}=y[m]W_N^{km}=W_N^{km}$$

$$0\le k\le N-1$$

5.9 **(a)** $Y_a[k] = \sum_{n=0}^{N-1} \alpha^n W_N^{kn} = \sum_{n=0}^{N-1} (\alpha W_N^k)^n = \dfrac{1-\alpha^N W_N^{kN}}{1-\alpha W_N^k} = \dfrac{1-\alpha^N}{1-\alpha W_N^k}$.

(b) Since $Y[k] = X[\langle k \rangle_5]$ for $0 \le k \le 20$, a sketch of $Y[k]$ will include a repetition of $X[k]$ 4 times as shown below:



5.2.3 Matrix Relations

★★★

- **Example: Compute 4 points DFT of sequences *x*[*n*]={5,0,-3,4}**

$$D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

说说为什么此处值是 –j？

$$\mathbf{X} = D_4 \mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8+j4 \\ -2 \\ 8-j4 \end{bmatrix}$$

Sampling the DTFT

频域采样引起时域序列的周期复制

we arrive at the desired relation

$$\boxed{y[n]=\sum_{m=-\infty}^{\infty}x[n+mN],\ \ 0\le n\le N-1}$$

$$x[n]\ \xrightarrow{\text{DTFT}}\ X(e^{j\omega})\ \xrightarrow{\text{采样}}\ Y[k]=X(e^{j\omega_k})\ \xrightarrow{\text{IDFT}}\ y[n]$$

- Thus $y[n]$ is obtained from $x[n]$ by adding an infinite number of shifted replicas of $x[n]$, with each replica shifted by an integer multiple of N sampling instants, and observing the sum only for the interval $0\le n\le N-1$

这一小节结论的数学公式表示就是公式（5.49）。公式中y[n]代表什么信号，x[n]代表什么信号，两者的关系是什么。这个公式说明了频域采样带来的时域信号怎样的变化。

★★★ Sampling the DTFT

- Example - Let $\{x[n]\} = \{\underset{\uparrow}{0} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\}$
- By sampling its DTFT $X(e^{j\omega})$ at $\omega_k = 2\pi k/4$, $0 \le k \le 3$ and then applying a 4-point IDFT to these samples, we arrive at the sequence $y[n]$ given by

$$y[n] = x[n] + x[n+4] + x[n-4] \text{ , } 0 \le n \le 3$$

- i.e.

$$\{y[n]\} = \{\underset{\uparrow}{4} \quad 6 \quad 2 \quad 3\}$$

$\Rightarrow$ $\{x[n]\}$ cannot be recovered from $\{y[n]\}$

5.68 From Eqn. (5.49) we have $y[n] = \sum_{m=-\infty}^{\infty} x[n+5m],\ 0 \le n \le 4.$

Therefore: $y[n] = x[n+5] + x[n] + x[n-5] = \{5, \quad 9, \quad 13, \quad 8, \quad 1\},\ 0 \le n \le 4.$

5.25 From Eq. (5.49) we have $x_i[n] = \sum_{m=-\infty} x[n+mN],\ 0 \le n \le N-1$. Let $x[n]$ be a length-M sequence defined for $0 \le n \le M-1$. If $M \le N$, then $x[n]$ can be recovered from $x_i[n]$ by extracting N samples from $x_i[n]$ in the range $0 \le n \le N-1$. If $M > N$, then $x[n]$ cannot be recovered from $x_i[n]$ because of time-domain aliasing.

(a) $x_1[n] = \sum_{m=-\infty}^{\infty} x[n+12m] = x[n+12] + x[n] + x[n-12],\ 0 \le n \le 11.$

Since $M = 12 > 9 = N$, $x[n]$ is recoverable from $x_1[n]$. In fact, $x[n]$ is given by the first 9 samples of $x_1[n]$ because of the above formula:

$$x_1[n] = \{1,\ -3,\ 4,\ -5,\ 7,\ -5,\ 4,\ -3,\ 1,\ 0,\ 0,\ 0\}, 0 \le n \le 11,$$

(b) $x_2[n] = \sum_{m=-\infty}^{\infty} x[n+8m] = x[n+8] + x[n] + x[n-8],\ 0 \le n \le 7.$

This time, since $M = 8 < 9 = N$, $x[n]$ is not recoverable from $x_2[n]$. In fact, the repeated copies overlap to form:

$$x_2[n] = [2,\ -3,\ 4,\ -5,\ 7,\ -5,\ 4,\ -3], 0 \le n \le 7$$

§ 5.4 Circular Convolution

长度为2N-1

Linear Convolution: $$y_L[n] = \sum_{k=0}^{N-1} x[k]h[n-k] = \sum_{k=0}^{N-1} h[k]x[n-k]$$

是周期序列，取一个周期长度为N

Circular Convolution: $$y_C[n] = \sum_{k=0}^{N-1} x[k]h[\langle n-k \rangle_N] = \sum_{k=0}^{N-1} h[k]x[\langle n-k \rangle_N]$$

- Since the operation defined involves two length-N sequences, it is often referred to as an N-point circular convolution, denoted as

$$y[n] = g[n] \circledN h[n]$$

- The circular convolution is commutative, i.e.

$$g[n] \circledN h[n] = h[n] \circledN g[n]$$

这个N在循环卷积定义中怎样体现？

$$y_C[n] = \sum_{k=0}^{N-1} x[k]h[\langle n-k \rangle_N]$$

$$= \sum_{k=0}^{N-1} h[k]x[\langle n-k \rangle_N]$$

★★★ 时域计算循环卷积计算四种方法

- Example - Determine the 4-point circular convolution of the two length-4 sequences:

$$\{g[n]\}=\{\underset{\uparrow}{1}\quad 2\quad 0\quad 1\},\ \{h[n]\}=\{\underset{\uparrow}{2}\quad 2\quad 1\quad 1\}$$

法2：

n:	0	1	2	3	$\langle 4\rangle_4$	$\langle 5\rangle_4$	$\langle 6\rangle_4$
$g[n]$:	1	2	0	1			
$h[n]$:	2	2	1	1			
	2	4	0	2			
	2	2	4	0			
	0	1	1	2			
	2	0	1	1			
$y_c[n]$:	6	7	6	5			

法1
- The result is a length-4 sequence $y_C[n]$ given by

$$y_C[n]=g[n]④h[n]=\sum_{m=0}^{3}g[m]h[\langle n-m\rangle_4],\quad 0\le n\le 3$$

- From the above we observe

$$y_C[0]=\sum_{m=0}^{3}g[m]h[\langle -m\rangle_4]$$
$$=g[0]h[0]+g[1]h[3]+g[2]h[2]+g[3]h[1]$$
$$=(1\times 2)+(2\times 1)+(0\times 1)+(1\times 2)=6$$

- Likewise $y_C[1]=\sum_{m=0}^{3}g[m]h[\langle 1-m\rangle_4]$

$$=g[0]h[1]+g[1]h[0]+g[2]h[3]+g[3]h[2]$$

$$y_C[2]=\sum_{m=0}^{3}g[m]h[\langle 2-m\rangle_4]$$
$$=g[0]h[2]+g[1]h[1]+g[2]h[0]+g[3]h[3]$$

$$y_C[3]=\sum_{m=0}^{3}g[m]h[\langle 3-m\rangle_4]$$
$$=g[0]h[3]+g[1]h[2]+g[2]h[1]+g[3]h[0]$$

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方法3：用矩阵形式

$$\begin{bmatrix} y_c[0] \\ y_c[1] \\ y_c[2] \\ \vdots \\ y_c[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[1] \\ h[1] & h[0] & h[N-1] & \cdots & h[2] \\ h[2] & h[1] & h[0] & \cdots & h[3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h[0] \end{bmatrix} \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ \vdots \\ g[N-1] \end{bmatrix}$$

方法4：用循环卷积和线性卷积的关系

$$y_c(n) = \sum_{m=-\infty}^{+\infty} y_L(n+mN)$$

★★★ # 时域计算循环卷积计算四种方法

5.2 (a)

法1

$$\tilde{y}[0]=\sum_{r=0}^{5}\tilde{x}[r]\tilde{h}[-r]$$
$$=\tilde{x}[0]\tilde{h}[0]+\tilde{x}[1]\tilde{h}[5]+\tilde{x}[2]\tilde{h}[4]+\tilde{x}[3]\tilde{h}[3]+\tilde{x}[4]\tilde{h}[2]+\tilde{x}[5]\tilde{h}[1]$$
$$=(4\times(-1))+((-3)\times 2)+(2\times 0)+(0\times 1)+(1\times 0)+(1\times 2)=(-4)+(-6)+2=-8$$

$$\tilde{y}[1]=\sum_{r=0}^{5}\tilde{x}[r]\tilde{h}[1-r]$$
$$=\tilde{x}[0]\tilde{h}[1]+\tilde{x}[1]\tilde{h}[0]+\tilde{x}[2]\tilde{h}[5]+\tilde{x}[3]\tilde{h}[4]+\tilde{x}[4]\tilde{h}[3]+\tilde{x}[5]\tilde{h}[2]$$
$$=(4\times 2)+((-3)\times(-1))+(2\times 2)+(0\times 0)+(1\times 1)+(1\times 0)=8+3+4+1=16$$

$$\tilde{y}[2]=\sum_{r=0}^{5}\tilde{x}[r]\tilde{h}[2-r]$$
$$=\tilde{x}[0]\tilde{h}[2]+\tilde{x}[1]\tilde{h}[1]+\tilde{x}[2]\tilde{h}[0]+\tilde{x}[3]\tilde{h}[5]+\tilde{x}[4]\tilde{h}[4]+\tilde{x}[5]\tilde{h}[3]$$
$$=(4\times 0)+((-3)\times 2)+(2\times(-1))+(0\times 2)+(1\times 0)+(1\times 1)$$
$$=0-6-2+0+0+1=-7$$

★★★ # 时域计算循环卷积计算四种方法

5.2

法1

$$\tilde{y}[3] = \sum_{r=0}^{5} \tilde{x}[r]\tilde{h}[3-r]$$
$$= \tilde{x}[0]\tilde{h}[3] + \tilde{x}[1]\tilde{h}[2] + \tilde{x}[2]\tilde{h}[1] + \tilde{x}[3]\tilde{h}[0] + \tilde{x}[4]\tilde{h}[5] + \tilde{x}[5]\tilde{h}[4]$$
$$= (4\times 1) + ((-3)\times 0) + (2\times 2) + (0\times(-1)) + (1\times 2) + (1\times 0)$$
$$= 4 - 0 + 4 + 0 + 2 + 0 = 10$$

$$\tilde{y}[4] = \sum_{r=0}^{5} \tilde{x}[r]\tilde{h}[4-r]$$
$$= \tilde{x}[0]\tilde{h}[4] + \tilde{x}[1]\tilde{h}[3] + \tilde{x}[2]\tilde{h}[2] + \tilde{x}[3]\tilde{h}[1] + \tilde{x}[4]\tilde{h}[0] + \tilde{x}[5]\tilde{h}[5]$$
$$= (4\times 0) + ((-3)\times 1) + (2\times 0) + (0\times 2) + (1\times(-1)) + (1\times 2)$$
$$= 0 - 3 + 0 + 0 - 1 + 2 = -2$$

$$\tilde{y}[5] = \sum_{r=0}^{5} \tilde{x}[r]\tilde{h}[5-r]$$
$$= \tilde{x}[0]\tilde{h}[5] + \tilde{x}[1]\tilde{h}[4] + \tilde{x}[2]\tilde{h}[3] + \tilde{x}[3]\tilde{h}[2] + \tilde{x}[4]\tilde{h}[1] + \tilde{x}[5]\tilde{h}[0]$$
$$= (4\times 2) + ((-3)\times 0) + (2\times 1) + (0\times 0) + (1\times 2) + (1\times(-1)) = \mathbf{8+2+2-1=11}$$

Therefore, $\tilde{y}[n] = \{\ -8,\quad 16\quad -7\quad 10\quad -2\quad 11\ \},\ 0 \le n \le 5.$

★★★ 时域计算循环卷积计算四种方法

5.2

法3:
$$\begin{bmatrix} 4 & -3 & 2 & 0 & 1 & 1 \\ -3 & 2 & 0 & 1 & 1 & 4 \\ 2 & 0 & 1 & 1 & 4 & -3 \\ 0 & 1 & 1 & 4 & -3 & 2 \\ 1 & 1 & 4 & -3 & 2 & 0 \\ 1 & 4 & -3 & 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \\ -7 \\ 10 \\ -2 \\ 11 \end{bmatrix}$$

法4: Linear convolution:

$$y_L(n) = \{-4 \quad 11 \quad -8 \quad 8 \quad -4 \quad 11 \quad -4 \quad 5 \quad 1 \quad 2 \quad 2\}$$

$$y_c(n) = \sum_{m=-\infty}^{+\infty} y_L(n+mN)$$

$$y_c(n) = \{-8 \quad 16 \quad -7 \quad 10 \quad -2 \quad 11\}$$

★★★ 时域计算循环卷积计算四种方法

5.28 $$y_c(n)=\sum_{m=-\infty}^{+\infty}y_L(n+mN)$$

法4:

(b) First, we determine the linear convolution of $g[n]$ and $h[n]$:

$$y_L[n]=\{-6,\ \ 22,\ \ -3,\ \ -54,\ \ 77,\ \ 9,\ \ -28,\ \ 63,\ \ -6,\ \ 13,\ \ 12\}.$$

Applying the formula derived in Part (a) we arrive at

$$\begin{aligned}y_C[n]&=\{-6-28,\ \ 22+63,\ \ -3-6,\ \ -54+13,\ \ 77+12,\ \ 9\}\\&=\{-34,\ \ 85,\ \ -9,\ \ -41,\ \ 89,\ \ 9\}.\end{aligned}$$

5.32

法3:

Summing the four entries in the columns $n = 0, 1, 2,$ and 3, we finally arrive at the sequence $y[n]$ as indicated below:

$n:$	0	1	2	3
$x[n]:$	−3	2	−1	4
$h[n]:$	1	3	2	2
	−3	2	−1	4
	12	−9	6	−3
	−2	8	−6	4
	−4	2	−8	6
$y[n]:$	3	3	−9	11

★★★ 时域计算循环卷积计算四种方法

5.76 Given $g[n]=\{-3,\ 2,\ 5\},\ 0\le n\le 2$ and $h[n]=\{4,\ -3,\ 1,\ -4\},\ 0\le n\le 3$.

法4:

(a)

$$y_L[0]=g[0]h[0]=-12$$
$$y_L[1]=g[0]h[1]+g[1]h[0]=17$$
$$y_L[2]=g[0]h[2]+g[1]h[1]+g[2]h[0]=11$$
$$y_L[3]=g[0]h[3]+g[1]h[2]+g[2]h[1]=-1$$
$$y_L[4]=g[1]h[3]+g[2]h[2]=-3$$
$$y_L[5]=g[2]h[3]=-20$$

(b)

$$g_e[n]=\{-3,\ 2,\ 5,\ 0\}.$$

$$\begin{aligned}y_C[0]&=g_e[0]h[0]+g_e[1]h[3]+g_e[2]h[2]+g_e[3]h[1]\\&=g[0]h[0]+g[1]h[3]+g[2]h[2]=-15,\end{aligned}$$

$$\begin{aligned}y_C[1]&=g_e[0]h[1]+g_e[1]h[0]+g_e[2]h[3]+g_e[3]h[2]\\&=g[0]h[1]+g[1]h[0]+g[2]h[3]=3\end{aligned}$$

$$\begin{aligned}y_C[2]&=g_e[0]h[2]+g_e[1]h[1]+g_e[2]h[0]+g_e[3]h[3]\\&=g[0]h[2]+g[1]h[1]+g[2]h[0]=11\end{aligned}$$

$$\begin{aligned}y_C[3]&=g_e[0]h[3]+g_e[1]h[2]+g_e[2]h[1]+g_e[3]h[0]\\&=g[0]h[3]+g[1]h[2]+g[2]h[1]=-1\end{aligned}$$

5.43 Since $x[n]$ is a length-11 real sequence, its DFT satisfies $X[k] = X^*[\langle -k \rangle_{11}]$. Thus:

$$X[1] = X^*[\langle -1 \rangle_{11}] = X^*[10] = 1.5 + j5.31,$$
$$X[3] = X^*[\langle -3 \rangle_{11}] = X^*[8] = -3.34 - j3.69,$$
$$X[5] = X^*[\langle -5 \rangle_{11}] = X^*[6] = -7.55 - j13.69,$$
$$X[7] = X^*[\langle -7 \rangle_{11}] = X^*[4] = -12.44 - j12.7,$$
$$X[9] = X^*[\langle -9 \rangle_{11}] = X^*[2] = 2.49 + j19.12.$$

$$X[11] = X^*[\langle 11 \rangle_{11}] = X^*[0] = -12.61$$

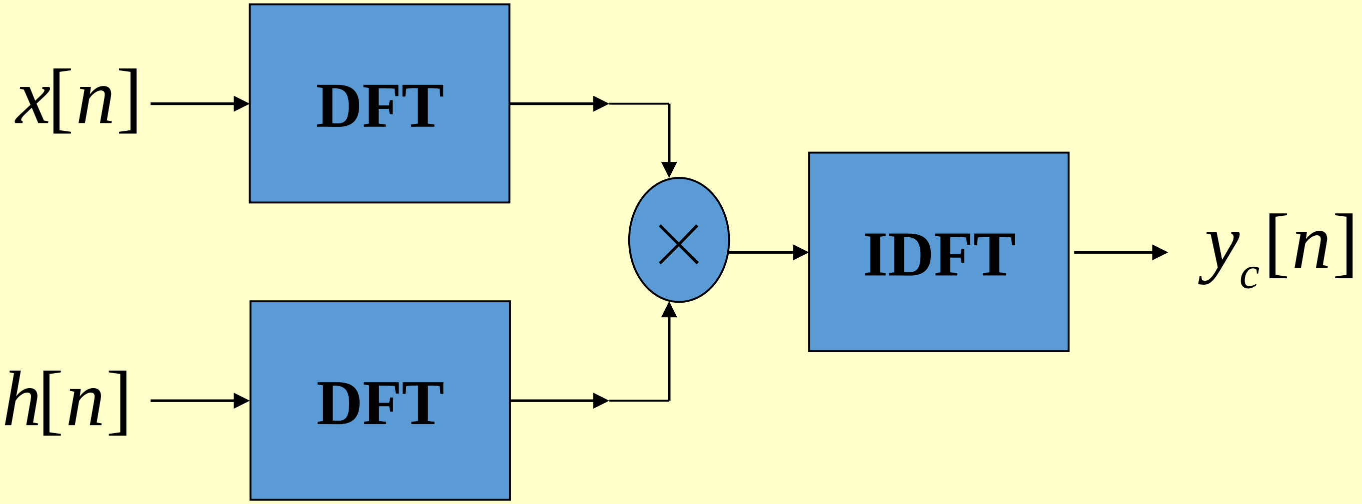
5.45 Since the DFT $X[k]$ is real-valued, $x[n]$ is circularly even:

$x[n] = x[\langle -n \rangle_{10}]$. Therefore:

$$x[2] = x[\langle -2 \rangle_{10}] = x[8] = 6.28,$$
$$x[6] = x[\langle -6 \rangle_{10}] = x[4] = -3.1,$$
$$x[7] = x[\langle -7 \rangle_{10}] = x[3] = 8.58,$$
$$x[9] = x[\langle -9 \rangle_{10}] = x[1] = 6.2.$$

The circular convolution can also be computed using a DFT-based approach（在频域计算循环卷积必须掌握）

★★★



循环卷积定理：

N-point circular convolution	$\sum_{m=0}^{N-1} g[m]h[\langle n-m\rangle_N]$	$G[k]H[k]$

The circular convolution can also be computed using a DFT-based approach

5.76 (c)

(c) To calculate the circular convolution, we first compute the DFTs and form their products samplewise:

$$\begin{bmatrix} G_e[0] \\ G_e[1] \\ G_e[2] \\ G_e[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -8-2j \\ 0 \\ -8+2j \end{bmatrix},$$

$$\begin{bmatrix} H[0] \\ H[1] \\ H[2] \\ H[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -2 \\ 3-j \\ 12 \\ 3+j \end{bmatrix}.$$

$$\begin{bmatrix} Y_C[0] \\ Y_C[1] \\ Y_C[2] \\ Y_C[3] \end{bmatrix} = \begin{bmatrix} G_e[0]H[0] \\ G_e[1]H[1] \\ G_e[2]H[2] \\ G_e[3]H[3] \end{bmatrix} = \begin{bmatrix} -8 \\ -26+2j \\ 0 \\ -26-2j \end{bmatrix}.$$

The circular convolution can also be computed using a DFT-based approach

5.76 (c) ★★★

We then determine the IDFT of $Y_C[k]$:

$$\begin{bmatrix} y_C[0] \\ y_C[1] \\ y_C[2] \\ y_C[3] \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} -8 \\ -26+2j \\ 0 \\ -26-2j \end{bmatrix} = \begin{bmatrix} -15 \\ -3 \\ 11 \\ -1 \end{bmatrix}.$$

The circular convolution can also be computed using a DFT-based approach（在频域计算循环卷积必须掌握）

★★★

- **Example 5.11 -** Consider the two length-4 sequences repeated below for convenience:

$$\begin{bmatrix} G[0] \\ G[1] \\ G[2] \\ G[3] \end{bmatrix} = D_4 \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ g[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

$$\begin{bmatrix} H[0] \\ H[1] \\ H[2] \\ H[3] \end{bmatrix} = D_4 \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

$$\begin{bmatrix} Y_c[0] \\ Y_c[1] \\ Y_c[2] \\ Y_c[3] \end{bmatrix} = \begin{bmatrix} G[0]H[0] \\ G[1]H[1] \\ G[2]H[2] \\ G[3]H[3] \end{bmatrix} = \begin{bmatrix} 24 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

- A 4-point IDFT of $Y_C[k]$ yields

$$\begin{bmatrix} y_C[0] \\ y_C[1] \\ y_C[2] \\ y_C[3] \end{bmatrix} = \frac{1}{4} D_4^* \begin{bmatrix} Y_C[0] \\ Y_C[1] \\ Y_C[2] \\ Y_C[3] \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 24 \\ -j2 \\ 0 \\ j2 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 6 \\ 5 \end{bmatrix}$$

The circular convolution can also be computed using a DFT-based approach

5.55 Using the concept similar to Example 5.12, we first calculate the 4-point DFT $X[k]$ of $x[n]$:

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1+j5 \\ 2 \\ 1-j5 \end{bmatrix}.$$

Next, we use the relationship:

$$Y[k] = \left\{ \frac{W[k]}{X[k]} \right\} = \frac{\{20, \quad 7+j9, \quad -10, \quad 7-j9\}}{\{4, \quad 1+j5, \quad 2, \quad 1-j5\}} = \{5, \quad 2-j, \quad -5, \quad 2+j\}.$$

Finally, we compute the inverse DFT $y[n]$ of $Y[k]$:

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 5 \\ 2-j \\ -5 \\ 2+j \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}.$$

Relationship between linear convolution and circular convolution

$$\boxed{y_c(n) = \sum_{m=-\infty}^{+\infty} y_L(n+mN)}$$

The condition for that linear convolution is equal to circular convolution is:

$$\boxed{N \geq N_g + N_h - 1}$$

N : The point of circular convolution

N_g : *the point of* $g[n]$

N_h : *the point of* $h[n]$

★★★ 在时域用循环卷积方法计算线性卷积

Example 5.13

在时域算满足 $N \geq N_g + N_h - 1$ 点的循环卷积，使其等于线性卷积

根据线性卷积和循环卷积相等的条件：$N \geq N_g + N_h - 1$

$$g_e[n] = \begin{cases} g[n] & 0 \leq n \leq 3 \\ 0 & 4 \leq n \leq 6 \end{cases} \qquad h_e[n] = \begin{cases} h[n] & 0 \leq n \leq 3 \\ 0 & 4 \leq n \leq 6 \end{cases}$$

$$y_c[n] = \sum_{m=0}^{6} g_e[m] h_e[< n - m >_7]$$

$$y_c[0] = \sum_{m=0}^{6} g_e[m] h_e[< 0 - m >_7] =$$

$$= g_e[0]h_e[0] + g_e[1]h_e[6] + g_e[2]h_e[5] + g_e[3]h_e[4] + g_e[4]h_e[3] + g_e[5]h_e[2] + g_e[6]h_e[1]$$

$$= g_e[0]h_e[0] = g[0]h[0] = 1 \times 2 = 2$$

$$\vdots$$

★★★ 在时域用循环卷积方法计算线性卷积

5.76

(d) The extended sequences are as follows:

$$g_e[n]=[-3,\ \ 2,\ \ 5,\ \ 0,\ \ 0,\ \ 0],\quad h_e[n]=[4,\ \ -3,\ \ 1,\ \ -4,\ \ 0,\ \ 0].$$

$$y_C[0]=g_e[0]h_e[0]+g_e[1]h_e[5]+g_e[2]h_e[4]+g_e[3]h_e[3]+g_e[4]h_e[2]+g_e[5]h_e[1]$$
$$=g[0]h[0]=-12=y_L[0],$$

$$y_C[1]=g_e[0]h_e[1]+g_e[1]h_e[0]+g_e[2]h_e[5]+g_e[3]h_e[6]+g_e[4]h_e[3]+g_e[5]h_e[2]$$
$$=g[0]h[1]+g[1]h[0]=17=y_L[1],$$

$$y_C[2]=g_e[0]h_e[2]+g_e[1]h_e[1]+g_e[2]h_e[0]+g_e[3]h_e[5]+g_e[4]h_e[4]+g_e[5]h_e[3]$$
$$=g[0]h[2]+g[1]h[1]+g[2]h[0]=11=y_L[2],$$

$$y_C[3]=g_e[0]h_e[3]+g_e[1]h_e[2]+g_e[2]h_e[1]+g_e[3]h_e[0]+g_e[4]h_e[5]+g_e[5]h_e[4]$$
$$=g[0]h[3]+g[1]h[2]+g[2]h[1]=-1=y_L[3],$$

$$y_C[4]=g_e[0]h_e[4]+g_e[1]h_e[3]+g_e[2]h_e[2]+g_e[3]h_e[1]+g_e[4]h_e[0]+g_e[5]h_e[5]$$
$$=g[1]h[3]+g[2]h[2]=-3=y_L[4],$$

$$y_C[5]=g_e[0]h_e[5]+g_e[1]h_e[4]+g_e[2]h_e[3]+g_e[3]h_e[2]+g_e[4]h_e[1]+g_e[5]h_e[0]$$
$$=g[2]h[3]=-20=y_L[5].$$

★★★ 在频域用循环卷积方法计算线性卷积

- Let $g[n]$ and $h[n]$ be two finite-length sequences of length N and M, respectively
- Denote $L = N + M - 1$
- Define two length-L sequences

$$g_e[n] = \begin{cases} g[n], & 0 \le n \le N-1 \\ 0, & N \le n \le L-1 \end{cases}$$

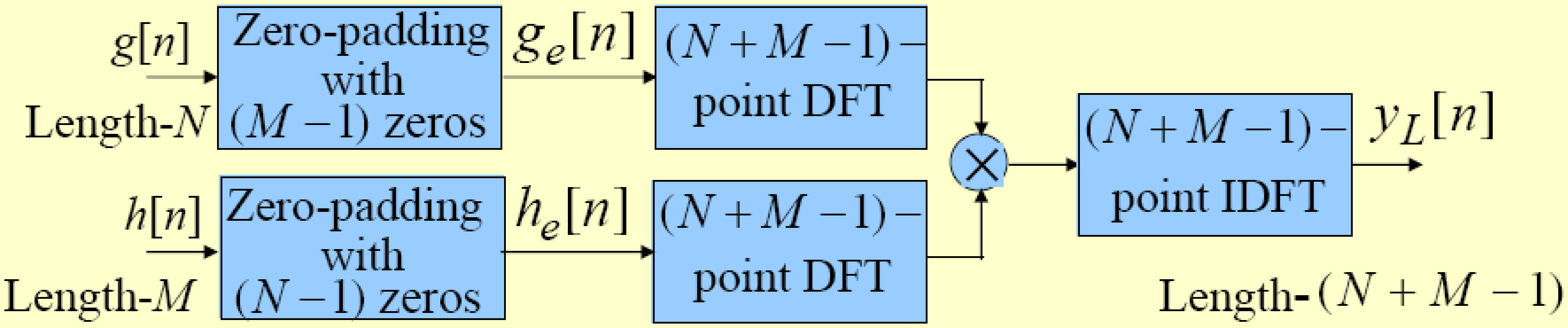
$$h_e[n] = \begin{cases} h[n], & 0 \le n \le M-1 \\ 0, & M \le n \le L-1 \end{cases}$$

★★★ # 在频域用循环卷积方法计算线性卷积

- Then

$$y_L[n] = g[n] \circledast h[n] = y_C[n] = g_e[n] \textcircled{\scriptsize L} h_e[n]$$

- The corresponding implementation scheme is illustrated below



在频域算满足 $N \geq N_g + N_h - 1$ 点的循环卷积，使其等于线性卷积

问题：请例举出能计算两个序列线性卷积的方法？

1. $$y[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]=\sum_{k=-\infty}^{+\infty}h[k]x[n-k]$$

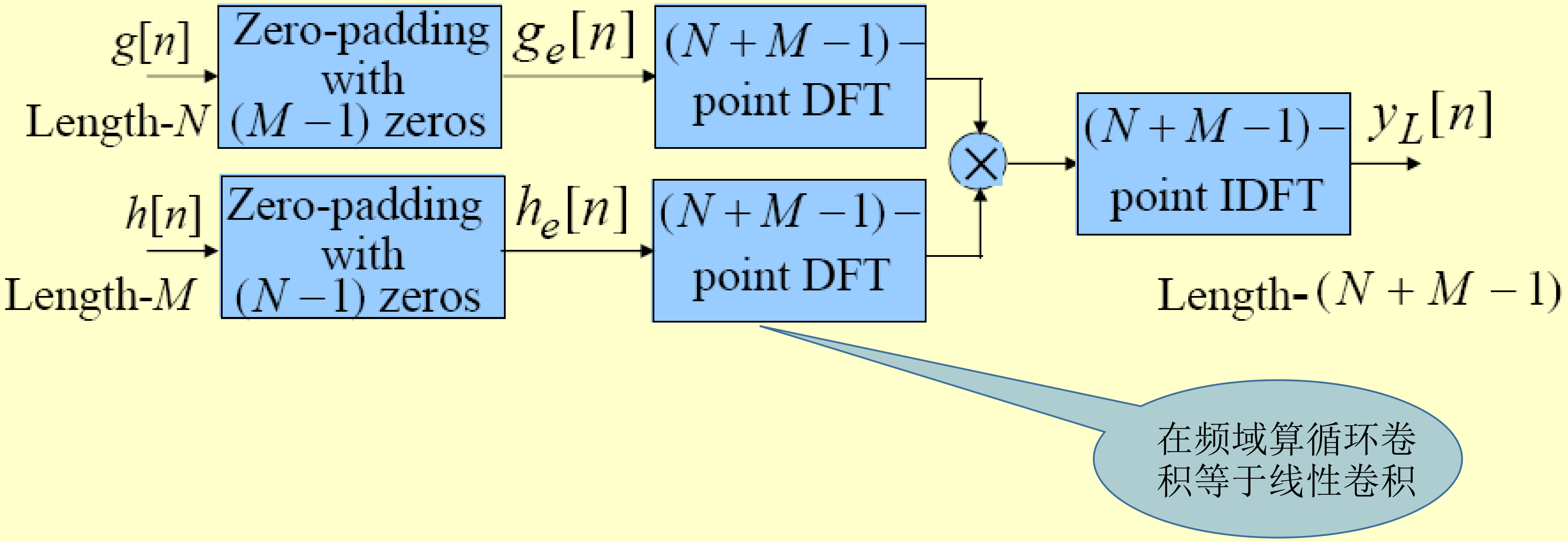
2. $$Y(z)=H(z)X(z)\xRightarrow{\text{逆z变换}}\mathrm{y[n]}$$

3. $$y[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}H(e^{j\omega})X(e^{j\omega})e^{j\omega n}d\omega$$

4. $$y_L[n]=g[n]\circledast h[n]=y_C[n]=g_e[n]\,Ⓛ\,h_e[n]$$

在时域算循环卷积等于线性卷积

5.



思考题：为什么要定义和研究循环卷积的概念？用循环卷积计算线性卷积有优势吗？

4.3 ★★★

(b) Given $y[n] = x[2-n] + \alpha$ with α nonzero constant.

For an input $x_i[n]$, $i = 1,2$, the output is $y_i[n] = x_i[2-n] + \alpha, i = 1,2.$

Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is:

$y[n] = x[2-n] + \alpha = Ax_1[2-n] + Bx_2[2-n] + \alpha.$

On the other hand

$Ay_1[n] + By_2[n] = Ax_1[2-n] + A\alpha + Bx_1[2-n] + B\alpha \neq y[n].$

Hence the system is nonlinear.

Time-invariance:

$$x[n-D] \xrightarrow{H} y_D[n] = x[2-n-D] + \alpha$$

$$y[n-D] = x[2-(n-D)] + \alpha = x[2-n+D] + \alpha$$

$$\therefore \quad y_D[n] \neq y[n-D]$$

Hence, the system is not time-invariant.

Causality:

★★★

当n=0时 $y[0]=x[2]+\alpha$ 当n=−1时 $y[-1]=x[3]+\alpha$

To compute the output, the input at future time is used. Thus the system is non-causal.

Note: 不能用h[n]去判断，因为系统不是线性时不变系统。

Stability:

For a bounded input $|x[n]| \le B < \infty$, the magnitude of the output samples are:

$$|y(n)| = |x(2-n)+\alpha| \le |x(2-n)| + |\alpha| \le B + |\alpha| \le \infty$$

As the output is also a bounded sequence, the system is BIBO stable.

§ 4.2 Classification of Discrete-Time Systems

★★★ Shift-Invariant System

- Example - Consider the up-sampler

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_u[n]$$

with an input-output relation given by

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \ldots.. \\ 0, & \text{otherwise} \end{cases}$$

- For an input $x_1[n] = x[n-n_o]$ the output $x_{1,u}[n]$ is given by

$$x_{1,u}[n] = \begin{cases} x_1[n/L], & n = 0, \pm L, \pm 2L, \ldots.. \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} x[(n-Ln_o)/L], & n = 0, \pm L, \pm 2L, \ldots.. \\ 0, & \text{otherwise} \end{cases}$$

- However from the definition of the up-sampler

$$x_u[n-n_o]$$

$$= \begin{cases} x[(n-n_o)/L], & n = n_o, n_o \pm L, n_o \pm 2L, \ldots.. \\ 0, & \text{otherwise} \end{cases}$$

$$\neq x_{1,u}[n]$$

- Hence, the up-sampler is a time-varying system

§ 4.2 Classification of Discrete-Time Systems

Shift-Invariant System

★★★ $y[n]=x[2n]$

（b）对输入信号延时 D 个时间单位 $x_D[n]=x[n-D]$

相应的系统输出为 $y_D[n]=H\{x[n-D]\}=\underline{x[2n-D]}$

$y[n]$ 延时 D 个时间单位，得：

$y[n-D]=x[2(n-D)]=\underline{x[2n-2D]}$

显然，$y[n-D]\neq y_D[n]$。

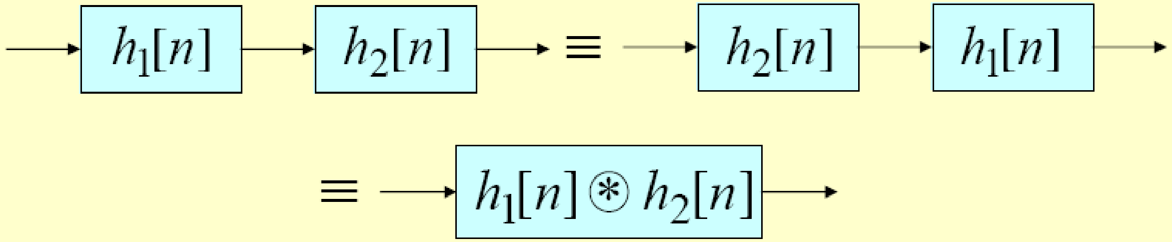
因此，下抽样器也是时变的。

An Example:

$$\{x_0,x_1,x_2,x_3,x_4,x_5,x_6,\ldots\}\xrightarrow{H}\{x_0,x_2,x_4,x_6,\ldots\}$$

$$\{0,x_0,x_1,x_2,x_3,x_4,x_5,x_6,\ldots\}\xrightarrow{H}\{0,x_1,x_3,x_5,\ldots\}$$

Cascade Connection

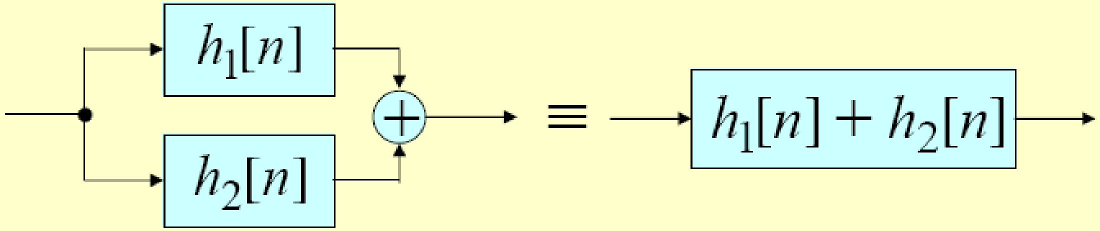


- Impulse response $h[n]$ of the cascade of two LTI discrete-time systems with impulse responses $h_1[n]$ and $h_2[n]$ is given by

$$h[n] = h_1[n] \circledast h_2[n]$$

$$H(z) = H_1(z) \cdot H_2(z)$$

Parallel Connection

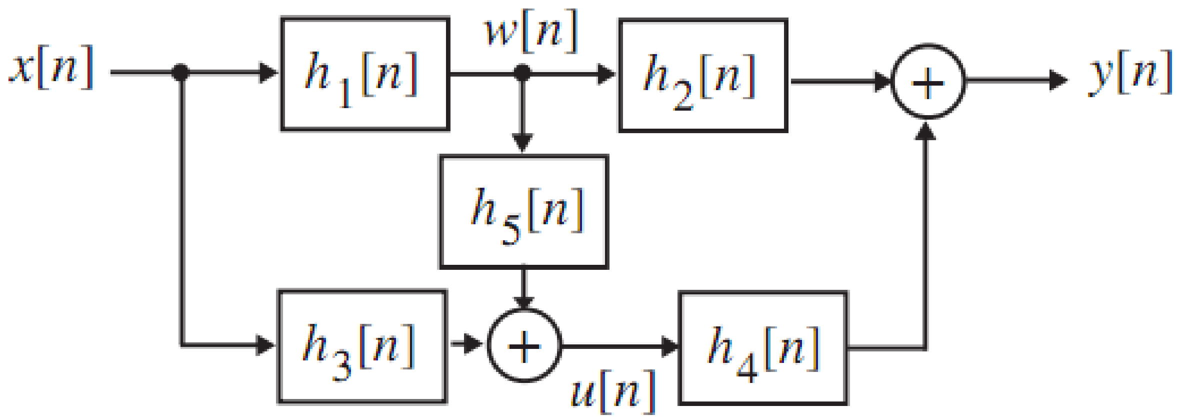


- Impulse response $h[n]$ of the parallel connection of two LTI discrete-time systems with impulse responses $h_1[n]$ and $h_2[n]$ is given by

$$h[n] = h_1[n] + h_2[n]$$

$$H(z) = H_1(z) + H_2(z)$$

4.30 **(a)** The structure of Fig. P4.1(a) with all signal variables is shown below:



Analysis of the above structure yields: $w[n] = h_1[n] \circledast x[n]$,

$u[n] = (h_3[n] + h_1[n] \circledast h_5[n]) \circledast x[n]$, and $y[n] = h_2[n] \circledast w[n] + h_4[n] \circledast u[n]$.

Substituting the first two equations into the last one yields

$$y[n] = h_1[n] \circledast h_2[n] \circledast x[n] + h_4[n] \circledast (h_3[n] + h_1[n] \circledast h_5[n]) \circledast x[n]$$

$$= \left[h_1[n] \circledast h_2[n] + h_4[n] \circledast (h_3[n] + h_1[n] \circledast h_5[n]) \right] \circledast x[n].$$

Hence, the impulse response of this structure is given by

$$h[n] = h_1[n] \circledast h_2[n] + h_4[n] \circledast (h_3[n] + h_1[n] \circledast h_5[n])$$

$$= h_1[n] \circledast h_2[n] + h_4[n] \circledast h_3[n] + h_4[n] \circledast h_1[n] \circledast h_5[n].$$

★★★ **4.20** **(a)** $\alpha^n \mu[n] \circledast \mu[n] = \sum_{k=-\infty}^{\infty} \alpha^k \mu[k]\mu[n-k] = \sum_{k=0}^{\infty} \alpha^k \mu[n-k]$

$$= \begin{cases} \sum_{k=0}^{n} \alpha^k, & n \geq 0, \\ 0, & n < 0, \end{cases} = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right)\mu[n].$$

★★ **4.23** The convolution of a sequence of length N and a sequence of length M will produce a sequence of length $L = N + M - 1$. Thus, the length of $x[n]$ can be computed by rearranging the equation and evaluating for $N = L - M + 1$. Rearranging the terms of the convolution formula, we can recursively compute $x[n]$ because successive samples of $y[n]$ are based purely on successive coefficients of $x[n]$. For example, since $y[0] = x[0]h[0]$, we can find $x[0] = y[0]/h[0]$. From here, we can use the following formula to compute all other terms within $x[n]$:

$$x[n] = \frac{1}{h[0]}\left[y[n] - \sum_{k=1}^{n} h[k]x[n-k]\right].$$

(a) The length of $x[n]$ is $8 - 4 + 1 = 5$. Using the above formula, we arrive at:

$$\{x[n]\} = \{2, \ -5, \ 3, \ 1, \ 7\}.$$

4.67 Given the difference equation:

$$y[n]+a_1y[n-1]+a_2y[n-2]=b_0x[n]+b_1x[n-1]+b_2x[n-2].$$

The frequency response is found as follows:

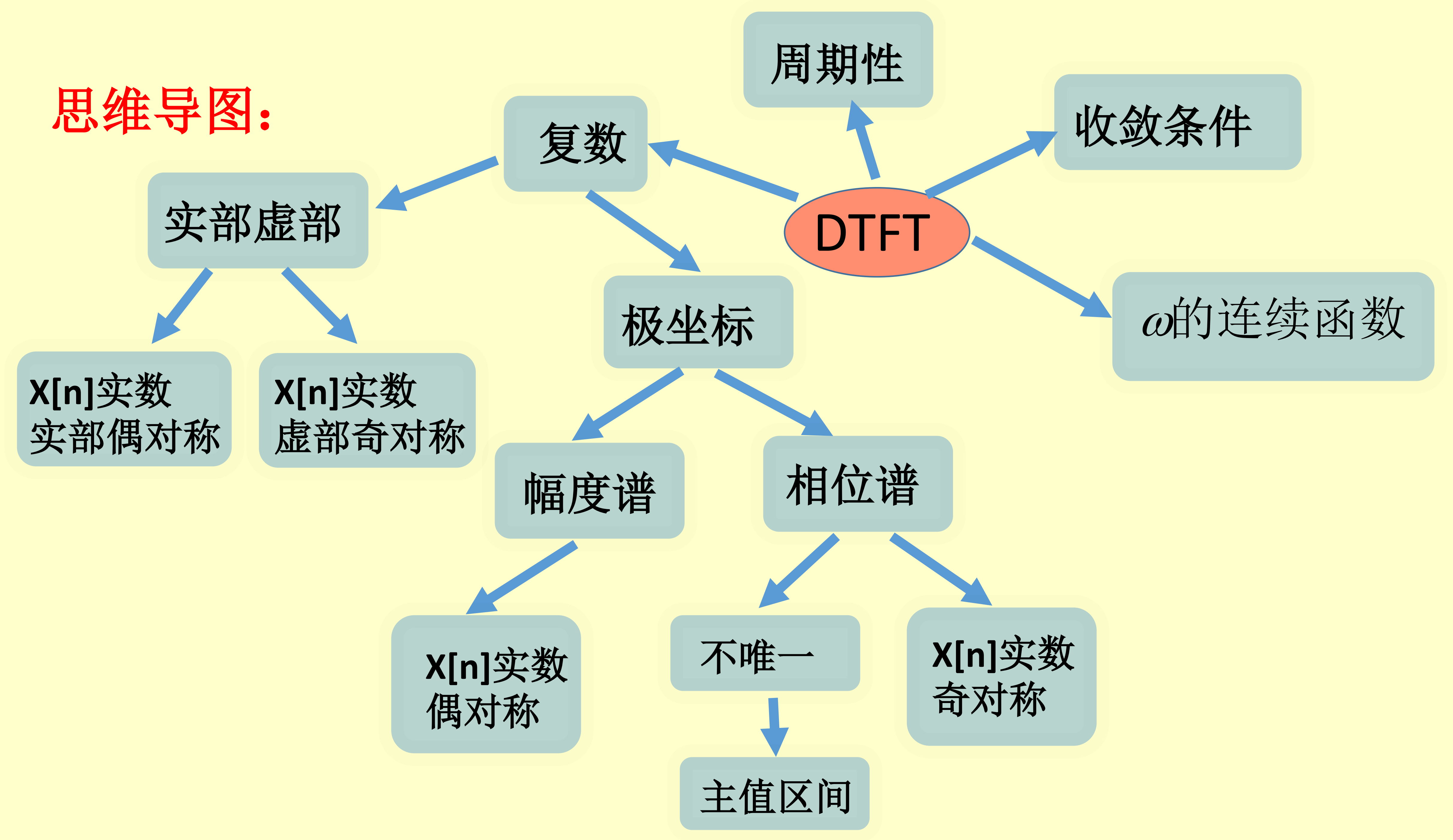
$$Y(e^{j\omega})+a_1e^{-j\omega}Y(e^{j\omega})+a_2e^{-2j\omega}Y(e^{j\omega})=b_0X\left(e^{j\omega}\right)+b_1e^{-j\omega}X(e^{j\omega})+b_2e^{-2j\omega}X(e^{j\omega})$$

which can be rewritten as

$$Y\left(e^{j\omega}\right)\left(1+a_1e^{-j\omega}+a_2e^{-2j\omega}\right)=X(e^{j\omega})\left(b_0+b_1e^{-j\omega}+b_2e^{-2j\omega}\right)$$

resulting in $\dfrac{Y(e^{j\omega})}{X(e^{j\omega})}=H(e^{j\omega})=\dfrac{b_0+b_1e^{-j\omega}+b_2e^{-j2\omega}}{1+a_1e^{-j\omega}+a_2e^{-j2\omega}}.$

思维导图：


DTFT
周期性
收敛条件
ω的连续函数
复数
实部虚部
X[n]实数
实部偶对称
X[n]实数
虚部奇对称
极坐标
幅度谱
X[n]实数
偶对称
相位谱
不唯一
主值区间
X[n]实数
奇对称

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

★★★

- Example - The DTFT of the unit sample sequence $\delta[n]$ is given by

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = \delta[0] = 1$$

★★★

- Example - Consider the causal sequence

$$x[n] = \alpha^n \mu[n], \quad |\alpha| < 1$$

- Its DTFT is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1-\alpha e^{-j\omega}}$$

as $\left|\alpha e^{-j\omega}\right| = |\alpha| < 1$

★★★

3.23 **(a)** $H_1(e^{j\omega}) = -4 + 3\cos(\omega) + 4\cos(2\omega) = -4 + 3\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right) + 4\left(\frac{e^{j2\omega} + e^{-j2\omega}}{2}\right)$

$= -4 + 1.5e^{j\omega} + 1.5e^{-j\omega} + 2e^{j2\omega} + 2e^{-j2\omega}$. Therefore,
$\{h_1[n]\} = \{2, \quad 1.5, \quad -4, \quad 1.5, \quad 2\}, \quad -2 \le n \le 2.$

★★

§ 3.2.3 Symmetry Relations

Table 3.1: Symmetry Relations of DTFT of a **real sequence**

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$
$x_{ev}[n]$	$X_{re}(e^{j\omega})$
$x_{od}[n]$	$jX_{im}(e^{j\omega})$
★Symmetry relations	$X(e^{j\omega}) = X^*(e^{-j\omega})$
	$X_{re}(e^{j\omega}) = X_{re}(e^{-j\omega})$
	$X_{im}(e^{j\omega}) = -X_{im}(e^{-j\omega})$
	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$
	$\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Note: $x_{ev}[n]$ and $x_{od}[n]$ denote the even and odd parts of $x[n]$, respectively.

实信号才满足

例题：

The DTFT $X(e^{j\omega})$ of a complex signal $x[n]$ satisfies

A $X(e^{j\omega}) = X(e^{-j\omega})$

B $X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$

C $\left|X(e^{j\omega})\right| = \left|X(e^{-j\omega})\right|$

D $X(e^{j\omega}) = X^*(e^{-j\omega})$

3.16

(b) $x_2[n] = n\alpha^n \mu[n]$ with $|\alpha| < 1$. Note $x_2[n] = n\,x[n]$.

Therefore, using the differentiation-in-frequency property in Table 3.4 we get

$$X_2(e^{j\omega}) = j\frac{dX(e^{j\omega})}{d\omega} = j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$= \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

★★★ 3.16

(c) $x_3[n] = \alpha^n \mu[n+1]$ with $|\alpha| < 1$. Its DTFT is given by

$$X_3(e^{j\omega}) = \sum_{n=-1}^{\infty} \alpha^n e^{-j\omega n} = \alpha^{-1} e^{j\omega} + \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \alpha^{-1} e^{j\omega}$$

$$+ \frac{1}{1-\alpha e^{-j\omega}} = \frac{1}{\alpha}\left(\frac{e^{j\omega}}{1-\alpha e^{-j\omega}}\right).$$

<法二> $x_3(n) = \dfrac{\alpha^{n+1}}{\alpha} u(n+1)$

$$X_3(e^{j\omega}) = \frac{e^{j\omega}}{\alpha} \cdot \frac{1}{1-\alpha e^{-j\omega}} = \frac{e^{j\omega}}{\alpha - \alpha^2 e^{-j\omega}}$$

$$\{\alpha^n u(n) \xrightarrow{DTFT} \frac{1}{1-\alpha e^{-j\omega}}\}$$

★★

3.21 **(a)** $X_a(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k)$. Hence, $x_a[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\omega) e^{j\omega n} d\omega$

$$= \frac{1}{2\pi}$$

(b) $X_b(e^{j\omega}) = \frac{e^{j\omega}\left(1 - e^{j\omega N}\right)}{1 - e^{j\omega}} = e^{j\omega}\sum_{n=0}^{N-1} e^{j\omega n}.$

Let $m = -n$. $X_b(e^{j\omega}) = e^{j\omega}\sum_{m=0}^{-N+1} e^{-j\omega m}.$

Consider the DTFT $X(e^{j\omega}) = \sum_{m=0}^{-N+1} e^{-j\omega m}$. Its inverse is given by

$x[n] = \begin{cases} 1, & -(N-1) \le n \le 0, \\ 0, & \text{otherwise.} \end{cases}$ Therefore, by the time-shifting property of the DTFT, the inverse DTFT of $X_b(e^{j\omega}) = e^{j\omega} X(e^{j\omega})$ is given by $x_b[n] = x[n+1] = \begin{cases} 1, & -N \le n \le -1, \\ 0, & \text{otherwise.} \end{cases}$

(d) $X_d(e^{j\omega}) = \dfrac{-\alpha e^{-j\omega}}{(1-\alpha e - j^{j\omega})^2}$ with $|\alpha| < 1.$

$$x_o[n] = \alpha^n \mu[n] \xrightarrow{DTFT} X_o(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}.$$

$$\frac{dX_o(e^{j\omega})}{d\omega} = \frac{(-1)\cdot(-\alpha)\cdot(-j)\cdot e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} = \frac{-j\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

频域微分性质：

$$nx_0(n) \xrightarrow{DTFT} j\frac{dX_0(e^{j\omega})}{d\omega} = \frac{-j^2\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

$$\therefore -n\alpha^n u(n) \xrightarrow{DTFT} \frac{-\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

(d)<法二>

$$x(n)=(n+1)\alpha^{n}u(n)\xrightarrow{DTFT}X(e^{j\omega})=\frac{1}{(1-\alpha e^{-j\omega})^{2}}$$

$$\Theta\ x(n-1)=(n+1-1)\cdot\alpha^{n-1}u(n-1)=n\alpha^{n-1}u(n-1)$$

$$\therefore x(n-1)\xrightarrow{DTFT}X(e^{j\omega})=\frac{e^{-j\omega}}{(1-\alpha e^{-j\omega})^{2}}$$

$$\therefore X_{d}(e^{j\omega})=\frac{-\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^{2}}\xrightarrow{IDTFT}-\alpha x(n-1)=-\alpha\cdot n\alpha^{n-1}u(n-1)$$

$$=-n\alpha^{n}u(n-1)=-n\alpha^{n}u(n)$$

★★

3.29 (a) $X_a(e^{j\omega}) = \sum_{m=0}^{M-1} x_a[n]e^{-j\omega m} = \sum_{m=0}^{N-1} x_a[n]e^{-j\omega m} + \sum_{m=N}^{M-1} x_a[n]e^{-j\omega m}$

$$= \sum_{m=0}^{N-1} x[n]e^{-j\omega m} + \sum_{m=N}^{M-1} 0 e^{-j\omega m} = X(e^{j\omega}).$$

(b) $X_b(e^{j\omega}) = \sum_{m=0}^{M-1} x_b[n]e^{-j\omega m} = \sum_{m=0}^{M-N-1} x_b[n]e^{-j\omega m} + \sum_{m=M-N}^{M-1} x_b[n]e^{-j\omega m}$

$$= \sum_{n=0}^{M-N-1} 0 \cdot e^{-j\omega n} + \sum_{n=M-N}^{M-1} x_b(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=M-N}^{M-1} x(n-M+N) \cdot e^{-j\omega n} \overset{k=n-(M-N)}{=} \sum_{k=0}^{N-1} x(k) \cdot e^{-j\omega(k+M-N)}$$

$$= e^{-j\omega(M-N)} \sum_{k=0}^{N-1} x(k) \cdot e^{-j\omega k} = e^{-j\omega(M-N)} X(e^{j\omega})$$

★★

3.29的结论记住：

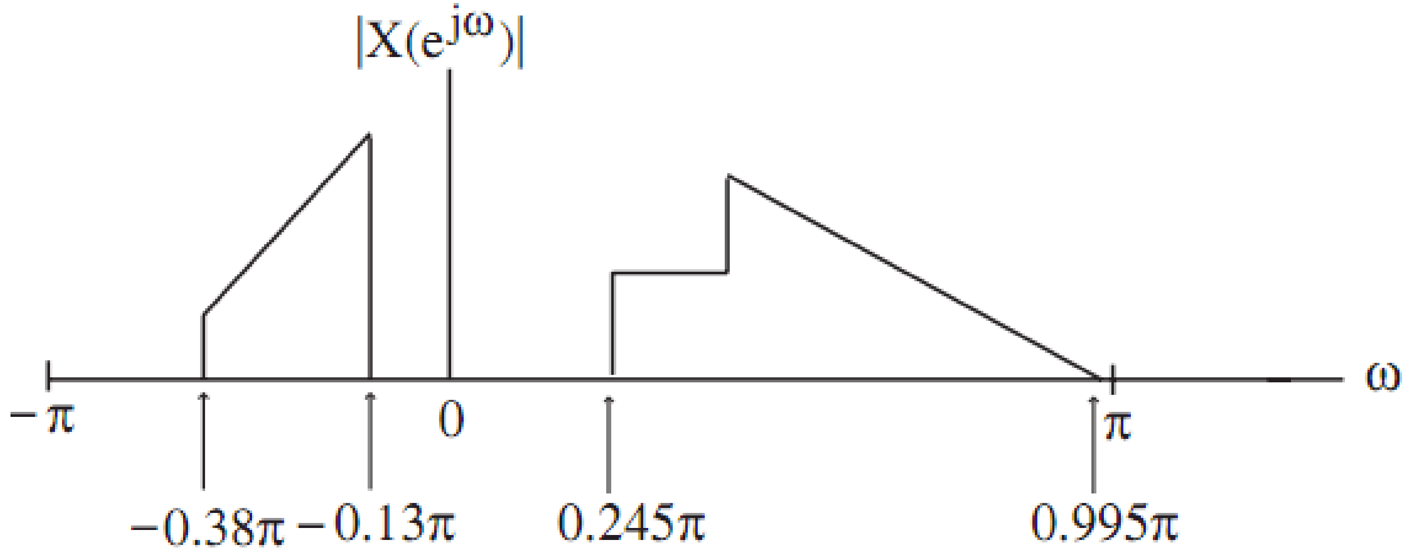
3.29应用：

$$x[n]=\{x_0,x_1\cdots\cdots\}\xrightarrow{DTFT}X(e^{j\omega})$$

$$x_1[n]=\{0,0,x_0,x_1\cdots\cdots\}\xrightarrow{DTFT}X_1(e^{j\omega})=e^{-j2\omega}X(e^{j\omega})$$

$$x_2[n]=\{x_0,x_1\cdots\cdots0,0,\}\xrightarrow{DTFT}X_2(e^{j\omega})=X(e^{j\omega})$$

3.30



The magnitude function is non-symmetric, so $x[n]$ is a complex sequence.

$$3.62\pi - 4\pi = -0.38\pi \qquad 3.87\pi - 4\pi = -0.13\pi$$

$$4.245\pi - 4\pi = 0.245\pi \qquad 4.995\pi - 4\pi = 0.995\pi$$

3.31

$$X(e^{j\omega})=\sum x(n)e^{-j\omega n}$$

$$X^*(e^{j\omega})=\sum x^*(n)e^{j\omega n}\overset{\boxed{x(n)\text{是实数}}}{=}\sum x(n)e^{j\omega n}\overset{n=-m}{=}\sum x(-m)e^{-j\omega m}$$

$$\begin{cases}\overset{\boxed{x(m)\text{是偶函数}}}{=}\sum x(m)e^{-j\omega m}=X(e^{j\omega})\Rightarrow \boxed{X(e^{j\omega})\text{是实数}}\\ \overset{\boxed{x(m)\text{是奇函数}}}{=}\sum -x(m)e^{-j\omega m}=-X(e^{j\omega})\Rightarrow \boxed{X(e^{j\omega})\text{是虚数}}\end{cases}$$

From Table 3.2 we observe that an even real-valued sequence has a real-valued DTFT and an odd real-valued sequence has an imaginary-valued DTFT.

(a) Since $x_1[n]$ is an odd sequence, it has an imaginary-valued DTFT.

(b) Since $x_2[n]$ is an even sequence, it has a real-valued DTFT.

(c) $x_3[-n]=\dfrac{\sin(-\omega_c n)}{-\pi n}=\dfrac{-\sin(\omega_c n)}{-\pi n}=\dfrac{\sin(\omega_c n)}{\pi n}=x_3[n]$. Since, $x_3[n]$ is an even sequence, it has a real-valued DTFT.

(d) Since $x_4[n]$ is an odd sequence, it has an imaginary-valued DTFT.

(e) Since $x_5[n]$ is an odd sequence, it has an imaginary-valued DTFT.

★★★

3.46 **(a)** $x_1[n] = g[n-4]$,

(b) $x_2[n] = g[n]e^{-j0.5\pi n}$,

(c) $x_3[n] = 3g[n] + 4g[-n]$,

(d) $x_4[n] = \frac{1}{j} n g[n]$,

(e) $x_5[n] = g_{odd}[n] = \frac{1}{2}\left[g[n] - g[-n]\right]$.

★★★

3.48

(a) $X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] = \sum_{n=-5}^{3} x[n] = -3$,

(b) $X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\pi n} = \sum_{n=-5}^{3} x[n]e^{-j\pi n} = -21$,

(c) $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 2\pi x[0] = -10\pi$,

(d) $\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 2\pi \sum_{n=-5}^{3} |x[n]|^2 = 526\pi$,

3.60 Sampling period $T = \frac{4}{8500}$ sec. Hence, the sampling frequency is $F_T = \frac{1}{T} = 2125$ Hz. Therefore, the highest frequency component that could be present in the continuous-time signal has a frequency $\frac{2125}{2} = 1062.5$ Hz.

3.61

Since the continuous-time signal $x_a(t)$ is being sampled at 3.0 kHz rate, the sampled version of its i-th sinusoidal component with a frequency F_i will generate discrete-time sinusoidal signals with frequencies $F_i \pm 3000n, -\infty < n < \infty$. Hence, the frequencies F_{im} generated in the sampled version associated with the sinusoidal components present in are as follows:

$$F_1 = 300 \text{ Hz} \Rightarrow F_{1m} = 300, 2700, 3300, \ldots \text{ Hz}$$
$$F_2 = 500 \text{ Hz} \Rightarrow F_{2m} = 500, 2500, 3500, \ldots \text{ Hz}$$
$$F_3 = 1200 \text{ Hz} \Rightarrow F_{3m} = 1200, 1800, 4200, \ldots \text{ Hz}$$
$$F_4 = 2150 \text{ Hz} \Rightarrow F_{4m} = 850, 2150, 5150, \ldots \text{ Hz}$$
$$F_5 = 3500 \text{ Hz} \Rightarrow F_{5m} = 500, 3500, 6500, \ldots \text{ Hz}$$

After filtering by a lowpass filter with a cutoff at 900 Hz, the frequencies of the sinusoidal components in $y_a(t)$ are 300, 500, 850 Hz.

★★ § 2.5 The Sampling Process

- The three continuous-time signals

$$g_1(t) = \cos(6\pi t)$$
$$g_2(t) = \cos(14\pi t)$$
$$g_3(t) = \cos(26\pi t)$$

of frequencies 3 Hz, 7 Hz, and 13 Hz, are sampled at a sampling rate of 10 Hz, i.e. with $T = 0.1$ sec. generating the three sequences

已知连续时间正弦信号会求出采样后序列表达式

$$g_1(n) = g_1(nT) = \cos(6\pi nT) \overset{T=0.1}{=} \cos(0.6\pi n)$$

$$g_2[n] = \cos(1.4\pi n) = \cos\big((2\pi - 0.6\pi)n\big) = \cos(0.6\pi n)$$

$$g_3[n] = \cos(2.6\pi n) = \cos\big((2\pi + 0.6\pi)n\big) = \cos(0.6\pi n)$$

- As a result, all three sequences are identical and it is difficult to associate a unique continuous-time function with each of these sequences

★★★ 2.3

(c) $e[n] = x[-n] = \{0 \quad 2 \quad -3 \quad 6 \quad -1 \quad 0 \quad 2\}, \quad -3 \le n \le 3.$

(d) $$u[n] = \{8 \quad 2 \quad -7 \quad -3 \quad 0 \quad 1 \quad 1 \quad 0 \quad \underset{\uparrow}{\underline{2}} \quad 0 \quad -1 \quad 6 \quad -3 \quad 2\} \qquad -8 \le n \le 6$$

(e) $$v(n) = \{-8 \quad 4 \quad -42 \quad -18\} \quad -2 \le n \le 1$$

★★ 2.4

$$y(n) = p_0 x(n) + p_1 x(n-1) + p_2 x(n-2) - d_1 y(n-1) - d_2 y(n-2)$$

★★★ 2.21

(a) $\textit{conjugate} \quad \textit{symmetric} \quad \textit{part}: \quad \frac{1}{2}(x[n]+x^*[-n])$

$$x_{cs}[n]=\frac{1}{2}\{3+4j \quad 5-12j \quad 8 \quad 5+12j \quad -3-4j\}$$

$\textit{conjugate} \quad \textit{antisymmetric} \quad \textit{part}: \quad \frac{1}{2}(x[n]-x^*[-n])$

$$x_{ca}[n]=\frac{1}{2}\{1+2j \quad -1-2j \quad -10j \quad 1-2j \quad -1+2j\}$$

(c) From the properties and formulas for sine and cosine:

$$x_{3,cs}[n]=\frac{1}{2}\left[\left(j\cos\left(\frac{2\pi n}{7}\right)-\sin\left(\frac{2\pi n}{4}\right)\right)+\left(-j\cos\left(\frac{2\pi n}{7}\right)+\sin\left(\frac{2\pi n}{4}\right)\right)\right]=0,$$

$$x_{3,ca}[n]=\frac{1}{2}\left[\left(j\cos\left(\frac{2\pi n}{7}\right)-\sin\left(\frac{2\pi n}{4}\right)\right)-\left(-j\cos\left(\frac{2\pi n}{7}\right)+\sin\left(\frac{2\pi n}{4}\right)\right)\right]$$

$$=\frac{1}{2}\left[j\cos\left(\frac{2\pi n}{7}\right)-\sin\left(\frac{2\pi n}{4}\right)+j\cos\left(\frac{2\pi n}{7}\right)-\sin\left(\frac{2\pi n}{4}\right)\right]=j\cos\left(\frac{2\pi n}{7}\right)-\sin\left(\frac{2\pi n}{4}\right)$$

2.22

Since $x[n]$ is conjugate symmetric it satisfies the condition $x[n] = x^*[-n]$ and since $y[n]$ is conjugate antisymmetric it satisfies the condition $y[n] = -y^*[-n]$.

(a) $g^*[-n] = x^*[-n]x^*[-n] = x[n]x[n] = g[n]$.

Thus, $g[n]$ is conjugate symmetric.

(b) $u^*[-n] = x^*[-n]y^*[-n] = x[n](-y[n]) = -u[n]$.

Thus, $u[n]$ is conjugate antisymmetric.

(c) $v^*[-n] = y^*[-n]y^*[-n] = (-y[n])(-y[n]) = v[n]$.

Thus, $u[n]$ is conjugate symmetric

★★★ 给出两个序列会计算卷积：

1. 作业 **2.13** $\{x[n]\} = [2, \quad -3, \quad 4, \quad 1], -1 \le n \le 2$ and

$\{h[n]\} = [-3, \quad 5, \quad -6, \quad 4], -2 \le n \le 1.$

Thus, $y[-1] = x[-1]h[0] + x[0]h[-1] + x[1]h[-2] = 2 \times (-6) + (-3) \times 5 + 4 \times (-3) = -39.$

2.

$$y[n] = a^n \mu[n] * \mu[n]$$

$$y[n] = \sum_m a^m \mu[m]\mu[n-m]$$

$$\begin{cases} m \ge 0 \\ n-m \ge 0 \end{cases} \Rightarrow 0 \le m \le n$$

$$\therefore y[n] = \sum_{m=0}^{n} a^m = \frac{1-a^{n+1}}{1-a} \cdot \mu[n]$$

★★★ 给出两个序列会计算卷积：

3.

n:	0	1	2	3	4	5
$g[n]$:	1	2	3			
$h[n]$:	3	2	1			
	3	6	9			
		2	4	6		
			1	2	3	
	3	8	14	8	3	

	1	2	3
3	3	6	9
2	2	4	6
1	1	2	3

4. $\delta[n-m]\circledast\delta[n-r]=\delta[n-m-r]$.

作业 2.47

(a) $y_1[n]=x_1[n]\circledast h_1[n]$

$$=\left(2\delta[n-1]-2\delta[n+1]\right)\circledast\left(-\delta[n-2]-1.5\delta[n]+\delta[n+3]\right)$$

$$=-2\delta[n-1]\circledast\delta[n-2]-3\delta[n-1]\circledast\delta[n]+2\delta[n-1]\circledast\delta[n+3]$$

$$+2\delta[n+1]\circledast\delta[n-2]+3\delta[n+1]\circledast\delta[n]-2\delta[n+1]\circledast\delta[n+3]$$

$$=-2\delta[n-3]-3\delta[n-1]+2\delta[n+2]+2\delta[n-1]+3\delta[n+1]-2\delta[n+4]$$

$$=-2\delta[n-3]-\delta[n-1]+3\delta[n+1]+2\delta[n+2]-2\delta[n+4].$$

★★

2.39 The fundamental period N of a periodic sequence with an angular frequency ω_0 satisfies Equation (2.53a) with the smallest value of N and r.

(d) We first determine the fundamental period N_1 of $\sin(0.15\pi n)$ In this case, the equation reduces to $0.15\pi N_1 = 2\pi r_1$, which is satisfied with $N_1 = 40$ $N_1 = 40$ and $r_1 = 3$.

Next, we determine the fundamental period N_2 of $\cos(0.12\pi n - 0.1\pi)$. In this case, the equation reduces to $0.12\pi N_2 = 2\pi r_2$, which is satisfied with $N_2 = 50$ and $r_2 = 3$. Hence the fundamental period is given by

$\mathrm{LCM}(N_1, N_2) = \mathrm{LCM}(50,40) = 200.$

★★ 2.43

a) $x[n] = 2\delta[n+3] - \delta[n+1] + 6\delta[n] - 3\delta[n-1] + 2\delta[n-2]$

b) $x[n] = 2\mu[n+3] - 2\mu[n+2] - \mu[n+1] + 7\mu[n] - 9\mu[n-1]$

$+ 5\mu[n-2] - 2\mu[n-3]$

补充：FIR filter 和 IIR filter 的判定：

1）以脉冲响应（impulse response）h[n]判定

FIR：h[n] 有限长
IIR：h[n] 无限长

2）差分方程（difference equation）判定

FIR：没有y[n-k]
IIR：有y[n-k]

3）传递函数（transfer function）判定

FIR：没有分母多项式
IIR：有分母多项式（特例：FIR滤波器的递归实现）

4）框图表示（block diagram representation）判定

FIR：没有输出反馈
IIR：有输出反馈

补充：FIR filter 和 IIR filter 的判定：

一种特殊情况：FIR滤波器的递归实现

$$h[n]=u[n]-u[n-N]$$

$$H(z)=\sum_{n=0}^{N-1} z^{-n}=\frac{1-z^{-N}}{1-z^{-1}}$$

本质上是FIR滤波器，但看起来H（z）有分母多项式，有极点，但实际上可以零极点相消，极点不是有效极点

确定下列系统中哪些是IIR滤波器

A $H(z)=\dfrac{1}{1-\alpha z^{-1}}$

B $y[n]=\dfrac{1}{2}y[n-1]+x[n]$

C $y[n]=x[n]+x[n-1]$

D $h[n]=\mu[n]-\mu[n-4]$

提交

补充： the order of the system

1）差分方程（ difference equation）

$$y[n]=-\sum_{k=1}^{N}d_k y[n-k]+\sum_{k=0}^{M}p_k x[n-k]$$

order: max(M,N)

2）卷积等式（convolutional equation）

$$y[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]$$

FIR： $$y[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]=\sum_{k=0}^{M}p_k x[n-k]$$

$$\boldsymbol{h[k]=p_k} \qquad \boldsymbol{d_k=0}$$

IIR：*h*[n]无限长，卷积等式不可计算，研究满足差分方程的 *h*[n],即*x*[n],*y*[n]也满足同样的差分方程。

order: max(M,N)

补充： the order of the system

3）冲激响应h[n]

order:　FIR：length{h[n]}- 1=M

IIR：　不能用h[n]的长度定义

4）传递函数：

$$H(z)=\frac{\sum_{k=0}^{M}p_k z^{-k}}{1+\sum_{k=1}^{N}d_k z^{-k}}$$

order: max(M,N)

补充： Equivalent Descriptions of FIR DF 、IIR DF

1）差分方程（ difference equation）

2）冲激响应（impulse response）h(n)

3）卷积等式（convolutional equation）

时域

4）传递函数 （transfer function ） H(z)

5）频率响应（frequency response ）

6）零/极 图（pole/zero pattern）

变换域

7） 框图实现（Block diagram realization and sample processing algorithm）

注：处于中心地位的是 transfer function H(z) ,每一种描述形式都可以帮助我们从特定的某 一角度深入理解滤波器的本质。

9.1.2 Selection of the Filter Type

Selection of Filter Type

- Advantages in using an FIR filter -

  (1) Can be designed with exact linear phase,

  (2) Filter structure always stable with quantized coefficients

- Disadvantages in using an FIR filter -

  Order of an FIR filter, in most cases, is considerably higher than the order of an equivalent IIR filter meeting the same specifications, and FIR filter has thus higher computational complexity

9.1.2 Selection of the Filter Type

Selection of Filter Type

Filter Type Choice: FIR vs. IIR

	FIR	IIR
	■ No feedback (just zeros) ■ Always stable ■ Can be linear phase	■ Feedback (poles & zeros) ■ May be unstable ■ Difficult to control phase
BUT	■ High order (20-2000) ■ Unrelated to continuous-time filtering	■ Typ. < 1/10th order of FIR (4-20) ■ Derive from *analog prototype*