TECHNISCHE UNIVERSITÄT MÜNCHEN
Fakultät für Elektrotechnik und Informationstechnik
Lehrstuhl für Datenverarbeitung
Prof. Dr. Martin Kleinsteuber

Additional Exercises, 7.5.2018

Theoretical Exercises

Task 1. **Orthogonal Projections**. Assume $p > k$. Let $\mathbf{U} \in \mathbb{R}^{p\times k}$ have orthonormal columns, i.e. $\mathbf{U}^\top\mathbf{U} = \mathbf{I}_k$, with $\mathbf{I}_k$ being the identity matrix of dimension k. Let $\mathbf{T} = \mathbf{U}\mathbf{U}^\top$ denote the orthogonal projection matrix onto $\operatorname{span}\{\mathbf{u}^1, \dots, \mathbf{u}^k\}$.

a) Show that for any $n \in \mathbb{N}^+$, the equation

$$\mathbf{T}^n = \prod_{i=1}^{n} \mathbf{T} = \mathbf{T}$$

holds.

Solution: Proof by induction. For $n = 1$, this is trivial. For $n > 1$, we have

$$\mathbf{T}^{n+1} = \mathbf{T}^n\mathbf{T} = \mathbf{T}\mathbf{T} = \mathbf{U}(\mathbf{U}^\top\mathbf{U})\mathbf{U}^\top = \mathbf{U}\mathbf{U}^\top = \mathbf{T}.$$

b) Show that for any $\mathbf{x} \in \mathbb{R}^p$, choosing $\mathbf{y} = \mathbf{T}\mathbf{x}$, with $\mathbf{T}$ defined as in a), solves the minimization problem

$$\text{minimize } \|\mathbf{y} - \mathbf{x}\|^2 \text{ s.t. } \mathbf{y} \in \operatorname{span}\{\mathbf{u}^1, \dots, \mathbf{u}^k\}.$$

Solution: There is a $\mathbf{s} \in \mathbb{R}^k$ such that $\mathbf{y} = \mathbf{U}\mathbf{s}$. Furthermore, there is a matrix $\mathbf{U}_\perp \in \mathbb{R}^{p\times(p-k)}$, such that $\tilde{\mathbf{U}} = [\mathbf{U}\ \mathbf{U}_\perp]$ is orthogonal. Therefore, we can write

$$\begin{aligned}
\|\mathbf{y} - \mathbf{x}\|^2 &= \|\mathbf{U}\mathbf{s} - \mathbf{x}\|^2 \\
&= \|\tilde{\mathbf{U}}^\top(\mathbf{U}\mathbf{s} - \mathbf{x})\|^2 \\
&= \left\| \begin{bmatrix} \mathbf{s} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{U}^\top\mathbf{x} \\ \mathbf{U}_\perp^\top\mathbf{x} \end{bmatrix} \right\|^2 \\
&= \|\mathbf{s} - \mathbf{U}^\top\mathbf{x}\|^2 + \|\mathbf{U}_\perp^\top\mathbf{x}\|^2
\end{aligned}$$

The second term does not depend on $\mathbf{s}$ and the first term vanishes when we choose $\mathbf{s} = \mathbf{U}^\top\mathbf{x}$, making

$$\mathbf{y} = \mathbf{U}\mathbf{s} = \mathbf{U}\mathbf{U}^\top\mathbf{x} \tag{1}$$

a solution of the minimization problem.

c) Derive the following equality.

$$\operatorname{tr}(\mathbf{T}) = k$$

Solution:

$$\operatorname{tr}(\mathbf{T}) = \operatorname{tr}(\mathbf{U}\mathbf{U}^\top) = \operatorname{tr}(\mathbf{U}^\top\mathbf{U}) = \operatorname{tr}(\mathbf{I}_k) = k \tag{2}$$

d) Derive the following inequalities.

$$t_{i,i} \in [0,1] \ \forall i \in \{1, \ldots, p\}.$$

Solution: From the definition of $\mathbf{T}$, we can see that the entries on its diagonals are the squared euclidean norms of the rows $\mathbf{U}_i$ of $\mathbf{U}$:

$$t_{i,i} = \|\mathbf{U}_i^\top\|^2$$

Thus, we can write

$$1 = \left(\tilde{\mathbf{U}}\tilde{\mathbf{U}}^\top\right)_{i,i} = \|\tilde{\mathbf{U}}_i^\top\|^2 = \|\mathbf{U}_i^\top\|^2 + \|\mathbf{U}_{\perp i}^\top\|^2 \geq \|\mathbf{U}_i^\top\|^2 \geq 0.$$

Task 2. **EVD and SVD**. Consider the matrix $\mathbf{X}$, defined as

$$\mathbf{X} = \begin{bmatrix} 4 & 5 & 3 \\ -2 & 0 & -4 \end{bmatrix}.$$

a) Compute an eigenvalue decomposition (EVD) $\mathbf{U}\mathbf{\Lambda}\mathbf{U}^{-1} = \mathbf{X}\mathbf{X}^\top$ of $\mathbf{X}\mathbf{X}^\top$. What can you say straightaway about the structure of the EVD?

Solution: Since

$$\mathbf{X}\mathbf{X}^\top = \begin{bmatrix} 50 & -20 \\ -20 & 20 \end{bmatrix}$$

is symmetric, we can find as many linear independent eigenvectors as eigenvalues (counted by multiplicity), the eigenvalues are real-valued and the eigenvectors can be chosen to be orthonormal. For any $\mathbf{y} \in \mathbb{R}^2$, we have

$$\mathbf{y}^\top\mathbf{X}\mathbf{X}^\top\mathbf{y} = \|\mathbf{X}^\top\mathbf{y}\|^2 \geq 0,$$

which means that the matrix is positive-semidefinite and thus its eigenvalues are non-negative. For the eigenvalues, we need to solve

$$\begin{aligned} \det(\mathbf{X}\mathbf{X}^\top - \lambda\mathbf{I}) &= 0 \\ \Leftrightarrow \begin{vmatrix} 50-\lambda & -20 \\ -20 & 20-\lambda \end{vmatrix} &= 0 \\ \Leftrightarrow (50-\lambda)(20-\lambda) - 400 &= 0 \\ \Leftrightarrow \lambda^2 - 70\lambda + 600 &= 0 \\ \Leftrightarrow (\lambda-35)^2 - 625 &= 0, \end{aligned}$$

which yields the eigenvalues $\lambda_1 = 60$ and $\lambda_2 = 10$. We first compute the second eigenvector by solving

$$\left(\mathbf{X}\mathbf{X}^\top - \lambda_2 \mathbf{I}\right) \boldsymbol{u}_2 = \begin{bmatrix} 40 & -20 \\ -20 & 10 \end{bmatrix} \boldsymbol{u}_2 = \mathbf{0} \Rightarrow \boldsymbol{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix},$$

and infer $\boldsymbol{u}_1$ from orthogonality: $\boldsymbol{u}_1 = \left[2/\sqrt{5},\ -1/\sqrt{5}\right]^\top$.

b) Compute an economy-size singular value decomposition (SVD) $\mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top = \mathbf{X}$ of $\mathbf{X}$, where "economy-size" implies that $\mathbf{\Sigma}$ is a square matrix.

Solution: We already computed the left singular vectors $\mathbf{U}$ and $\mathbf{\Sigma} = \operatorname{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2})$. For the right singular vectors, we compute

$$\begin{aligned}
\mathbf{V} &= \mathbf{V}\mathbf{\Sigma}\mathbf{U}^\top\mathbf{U}\mathbf{\Sigma}^{\mathbf{-1}} = \mathbf{X}^\top\mathbf{U}\mathbf{\Sigma}^{-1} \\
&= \begin{bmatrix} 4 & -2 \\ 5 & 0 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{60}} & \\ & \frac{1}{\sqrt{10}} \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 \\ 2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{12}} & \\ & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix},
\end{aligned}$$

which completes the SVD.