TECHNISCHE UNIVERSITÄT MÜNCHEN
Fakultät für Elektrotechnik und Informationstechnik
Lehrstuhl für Datenverarbeitung
PD Martin Kleinsteuber

`Information Retrieval in High Dimensional Data`

`Lab #8, Theoretical Exercises`, 5.7.2018

Convex Optimization

Task 1. Consider the optimization problem

$$\begin{aligned} &\text{minimize } x^2 + 1 \\ &\text{subject to } (x-2)(x-4) \leq 0. \end{aligned}$$

a) Provide the feasible set, the optimal value, and the optimal solution.

Solution: Feasible set: $\mathcal{S} = [2, 4]$, solution: $x^* = 2$, because due to monotonicity, the solution has to be the left limit of the feasible set, optimal value: $p^* = 5$.

b) Rewrite the problem as a set of KKT conditions.

Solution: The Lagrange function is given by

$$L(x, \lambda) = x^2 + 1 - \lambda(-x^2 + 6x - 8).$$

The KKT conditions are

$$\begin{aligned} \frac{\partial L}{\partial x}(x, \lambda) = 2x + 2\lambda x - 6\lambda = 0, \\ -(x-2)(x-4) \geq 0, \\ \lambda \geq 0, \\ \lambda(x-2)(x-4) = 0. \end{aligned}$$

c) State the dual problem. Find the dual optimal value and dual optimal solution. Does strong duality hold?

Solution: The Lagrange function is a squared function and thus its infimum can be easily determined by setting its derivative to 0:

$$\frac{\mathrm{d}}{\mathrm{d}x} L = 2(1 + \lambda)x - 6\lambda = 0 \;\Leftrightarrow\; x = \frac{3\lambda}{1 + \lambda}.$$

The *Lagrange dual function* is given by

$$
\begin{aligned}
L_D(\lambda) = \inf_x L(x, \lambda) &= L(\frac{3\lambda}{1+\lambda}, \lambda) \\
&= \frac{9\lambda^2(1+\lambda)}{(1+\lambda)^2} - \frac{18\lambda^2}{1+\lambda} + 8\lambda + 1 \\
&= \frac{-\lambda^2 + 9\lambda + 1}{1+\lambda} \\
&= -\lambda + 1 + \frac{9\lambda}{\lambda+1}.
\end{aligned}
$$

The dual problem is thus given by

$$
\begin{aligned}
&\text{maximize} \ -\lambda + 1 + \frac{9\lambda}{\lambda+1} \\
&\text{subject to } \lambda \geq 0.
\end{aligned}
$$

Differentiating with respect to λ yields

$$
\frac{\mathrm{d}}{\mathrm{d}\lambda} L_D = -\frac{\lambda^2 + 2\lambda - 8}{(\lambda+1)^2},
$$

which, together with the constraint $\lambda \geq 0$, yields the two candidates $\lambda_1 = 0, \lambda_2 = 2$. Furthermore, $L_D(2) \geq L_D(0)$, so we conclude $\lambda^* = 2$ and $L_D(\lambda^*) = 5 = p^*$. Thus, strong duality holds.

d) Determine the solution of the original problem by substituting the dual solution into the KKT conditions.

Solution: We get

$$
\begin{aligned}
\frac{\partial L}{\partial x}(x, \lambda^*) = 6x - 12 &= 0, \\
-(x-2)(x-4) &\geq 0, \\
2 &\geq 0, \\
2(x-2)(x-4) &= 0.
\end{aligned}
$$

These conditions are satisfied by $x^* = 2$. Note: This works, because strong duality holds. Otherwise, the dual solution would not lead to a unique point that fulfills the KKT conditions.