SUBJECT: Bayesian Decision and Risk Analysis
SUBJECT CODE: ECS773P

Course Work 1: Tricky Magicians

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Part 1-Background

1. QUESTION-1.1

SOLUTION:

Binomial Distribution:

- The binomial distribution is the probability distribution function which is used in summarizing the likelihood.
- So that, the values would take one or more independent values under a given set of parameters

Formulae for binomial distribution is given as,

$$\frac{n!}{r! \times (n-r)!} \times p^r (1-p)^{n-r}$$

Where,

n → Number of trails
p → Probability of Success

Using Circumstances:

- Binomial distribution only has the two events which is a binary values 0 an 1.
- In binomial distribution 1 is denoted for success and 0 denotes failure for the given number of trails.
- It represents the probability of success in n number of trails, as probability of success for each trail is given.
- Binomial distribution concludes that the number of trails when each trail has same probability of achieving a single particular value.
- This helps in achieving the probability of success upon the given number of trails.

2. QUESTION-1.2

SOLUTION:

1.2) The Probability density function of beta distribution is computed by the formulae below,

$$f(x) = \begin{cases} \dfrac{(x-b)^{u-1} (c-x)^{v-1}}{B(u,v)\,(c-b)^{(u+v-1)}} & b \le x \le c,\ u,v>0 \\ 0 & \text{otherwise} \end{cases}$$

where,

$u, v \rightarrow$ shape Parameters

$b, c \rightarrow$ upper and the lower bounds

$B(u,v) \rightarrow$ Beta function.

formula for beta distribution is,

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1}\, dt$$

A continous random variable x is said to beta distribution function with given parameters u, v. the Probability density function is given by.

$$f(x) = \frac{x^{u-1}(1-x)^{v-1}}{B(u,v)} \qquad 0 \le x \le 1,\ u,v>0$$

Expected value.
we know,

$$E(x) = \int_0^1 x\, f(x)\, dx$$

$$= \int_0^1 x \left(\frac{x^{u-1}(1-x)^{v-1}}{B(u,v)} \right) dx$$

$$= \frac{1}{B(u,v)} \int_0^1 x^{(u+1)-1} (1-x)^{v-1}\, dx$$

$$= \frac{1}{B(u,v)} B(u+1, v) = \frac{u}{u+v} \left(\frac{\Gamma(u)\,\Gamma(v)\,\Gamma(u+v)}{\Gamma(u)\,\Gamma(v)\,\Gamma(u+v)} \right)$$

$$\therefore \boxed{E(x) = \frac{u}{u+v}}$$

3. QUESTION-1.3

SOLUTION:

Role of Parameters in determining Shape:

The shape parameter will effect in the general shape of the distribution. These parameters are mostly observed from the prior distribution data or the estimated data from the prior belief data.

Characteristics of Shape Parameter:

- When the graph lies in the horizontal axis of a cartesian plane, shape will not get affected or undergo any changes.
- The prior belief or the current data does not undergo any shrinking or squeezing. It defines the shape of the graph for the distribution.
- If the parameter of scaling on normal distribution is equal to one standard deviation, In this case squeezing occurs.

Prior belief in Coin Flipping:

- When the coin is flipped, there occurs the chances of two outcomes which can be either heads or tails, then the series of coin flipping are performed and the observations are noted.
- Fairness of the coin is determined by performing the above experiment, when sufficient trails of tossing coins are done, we can determine the probability of observing heads or tails. When the both probabilities are equal, it can be day that the coin is fair, In other case, if it is not equal and tends to fail, then it can be conclude that the coin is biased.

4. QUESTION-1.4

SOLUTION:

1.4) In Beta-Binomial formulation,

$$x_i = Bin(n_i, p_i) \rightarrow 1$$, where x_i is the binomial distribution

where we can say that,

$$P(x|n,p) = Bin(n,p) \rightarrow 2$$

Prior distribution $\pi(p) = Beta(\alpha, \beta)$

Likelihood of the distribution is

$$P(x/n,p) = \binom{n}{x} p^x (1-p)^{n-x} \rightarrow 3$$

p can be modelled as the beta function (α, β) are number of success (1) and failures (0) in finite number of trails.

The Prior model is,

$$P(p/\alpha, \beta) = Beta(\alpha, \beta) \rightarrow 4$$

The Beta distribution is the conditional posterior distribution which is a Beta distribution

By Bayes theorem,

$$P(p|x,\alpha,\beta) = p^{\sum_{i=1}^{n} x_i} (1-p)^{n-\sum_{i=1}^{n} x} \times \frac{1}{B(\alpha,\beta)} p^{\alpha-1} (1-p)^{\beta-1}$$

$$= \frac{\sqrt{\alpha+\beta}}{\sqrt{\alpha}\sqrt{\beta}} p^{\sum_{i=1}^{n} x+\alpha-1} \cdot (1-p)^{n-\sum_{i=1}^{n} x_i+\beta-1}$$

$$= Beta\left(\sum_{i=1}^{n} \alpha+x, n-\sum_{i=1}^{n} \beta+x\right) \rightarrow 5$$

Posterior distribution,

$$\boxed{P(p|x,\alpha,\beta) = Beta(\alpha+x, \beta+n-x)}$$

For $P(x|n,p)$,

$$P(x|n,p) = Beta(\alpha+x, \beta+n-x) \rightarrow 6$$

Part 2-Basic Analysis

1. QUESTION-2.1

SOLUTION:

We have modelled the chart in the following way,

- A node is assumed for the trails, in this we have initialized the uniform distribution and scenario is set 30, since total trails =30
- Secondly, We create a node for the prior distribution in which we have used the beta approximation, In order to find the beta distribution we need alpha and beta values.
- We have given a parameter 20:10, 10:20 which we use as the alpha beta value and the beta distribution is set.
- Since our aim is find the total number of heads, we create a node to find the heads, in this we have used the binomial distribution and the trails and prior beliefs are passed and the values are observed
- Two parameters are given so that two cases are taken.

CASE-1, For parameters 20:10

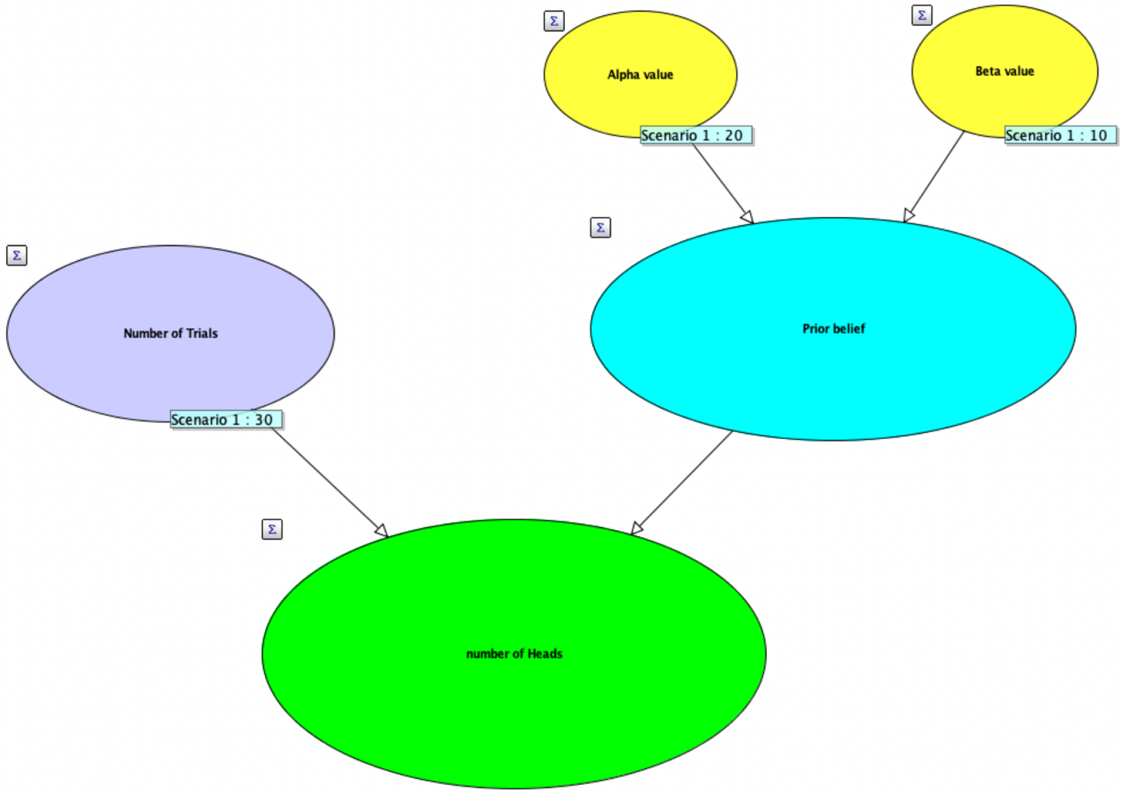
The hypothesised prior Beta distributions are modelled using parameter ratios 20:10.

Alpha= 20
Beta= 10
Number of Trails = 30

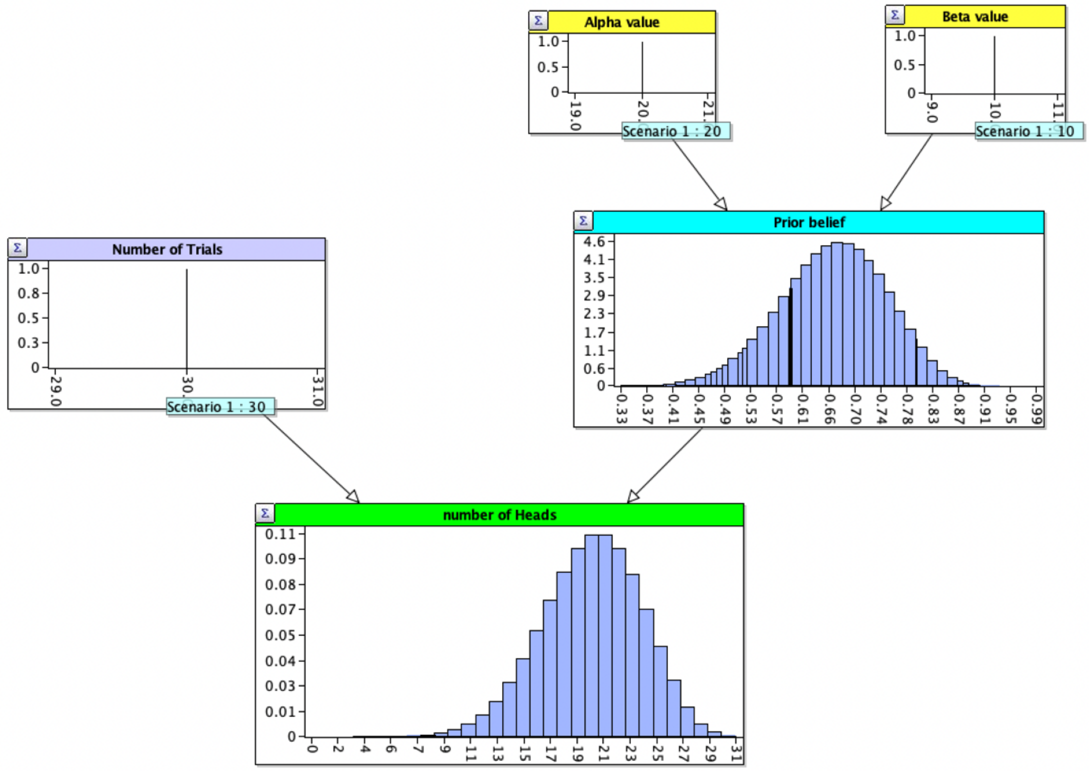
Theoretically,

The mean and mode are about 2/3rd, so the expected probability of a coin is to be around that region, especially, since the standard deviation is less than 0.16 which is quite small, the model suggests that it is biased towards heads.

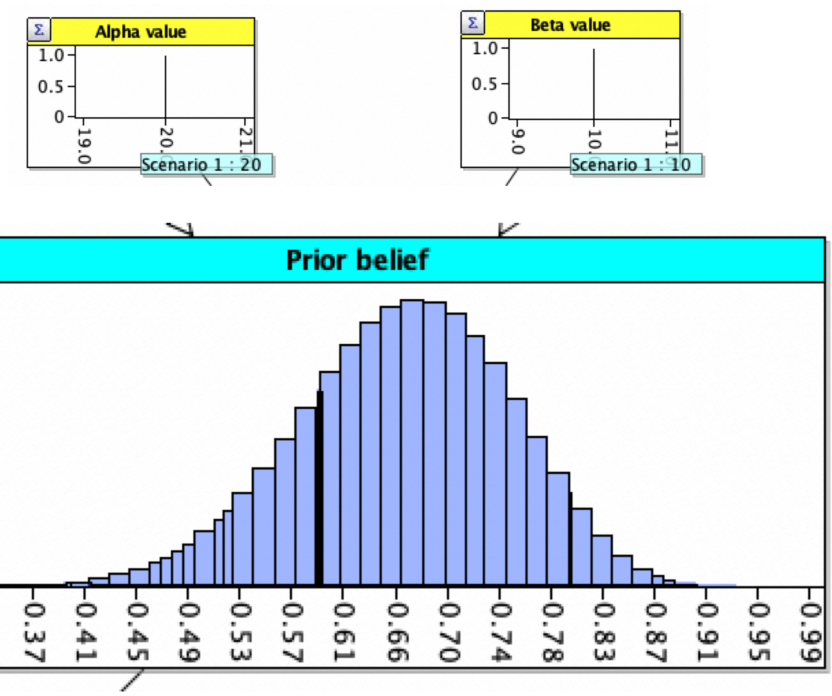
RISK MAP:



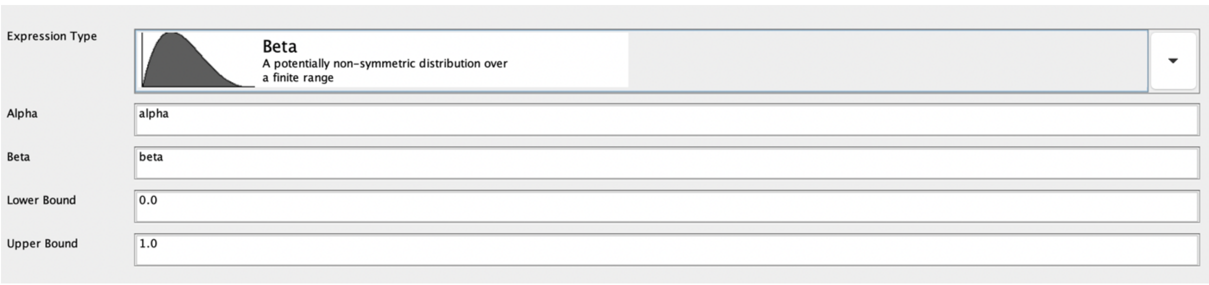
RISK MAP WITH GRAPHS:



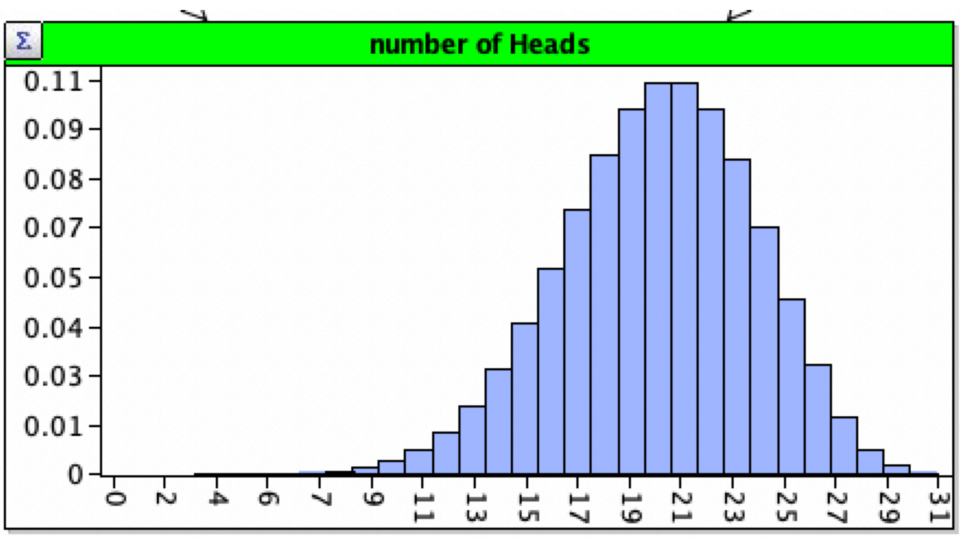
Prior distribution:



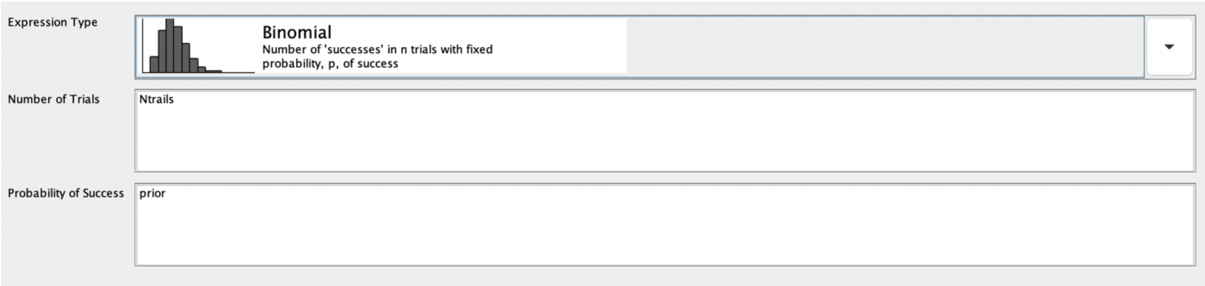
Prior belief assumptions:



Chances of heads:



Getting Number of heads:



CASE-2, For parameters 10:20

The hypothesised prior Beta distributions are modelled using parameter ratios 20:10.

Alpha= 10

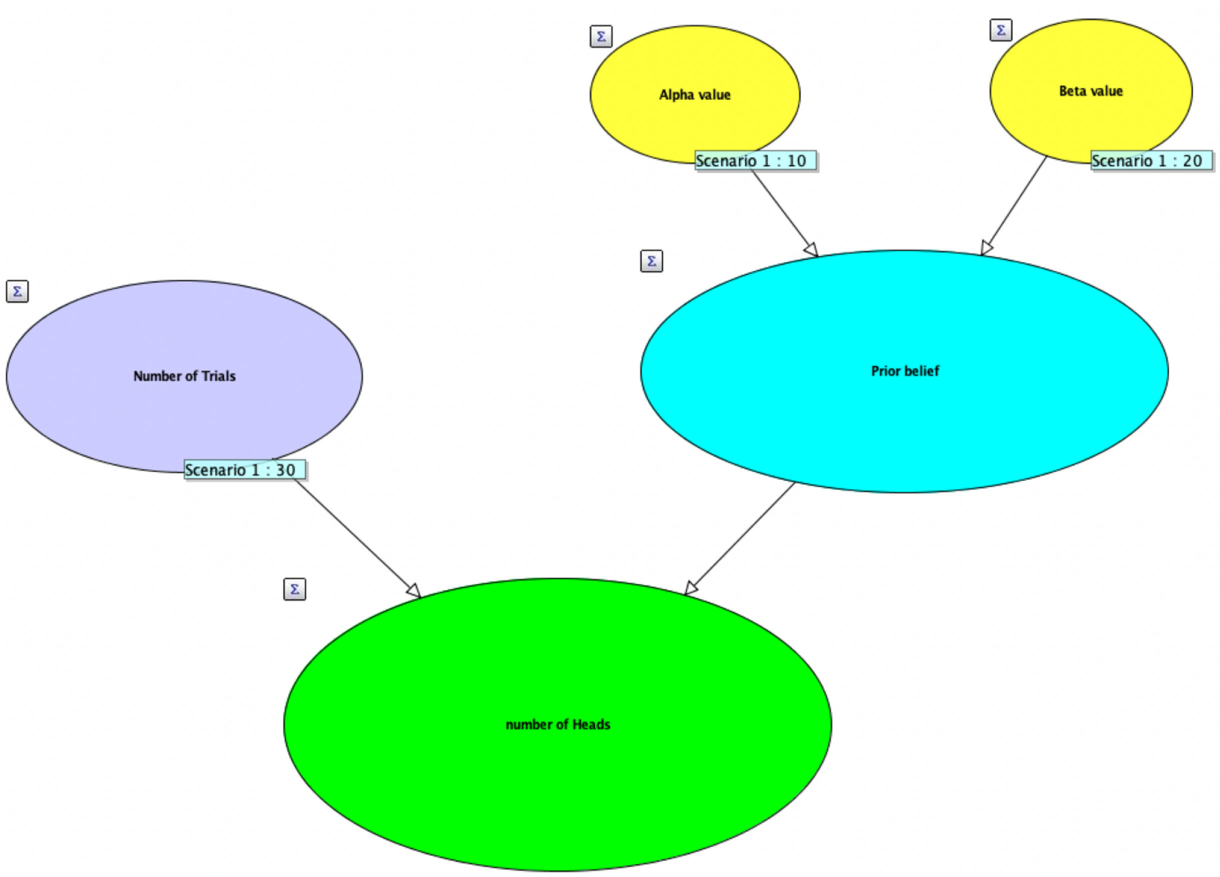
Beta= 20

Number of Trails = 30

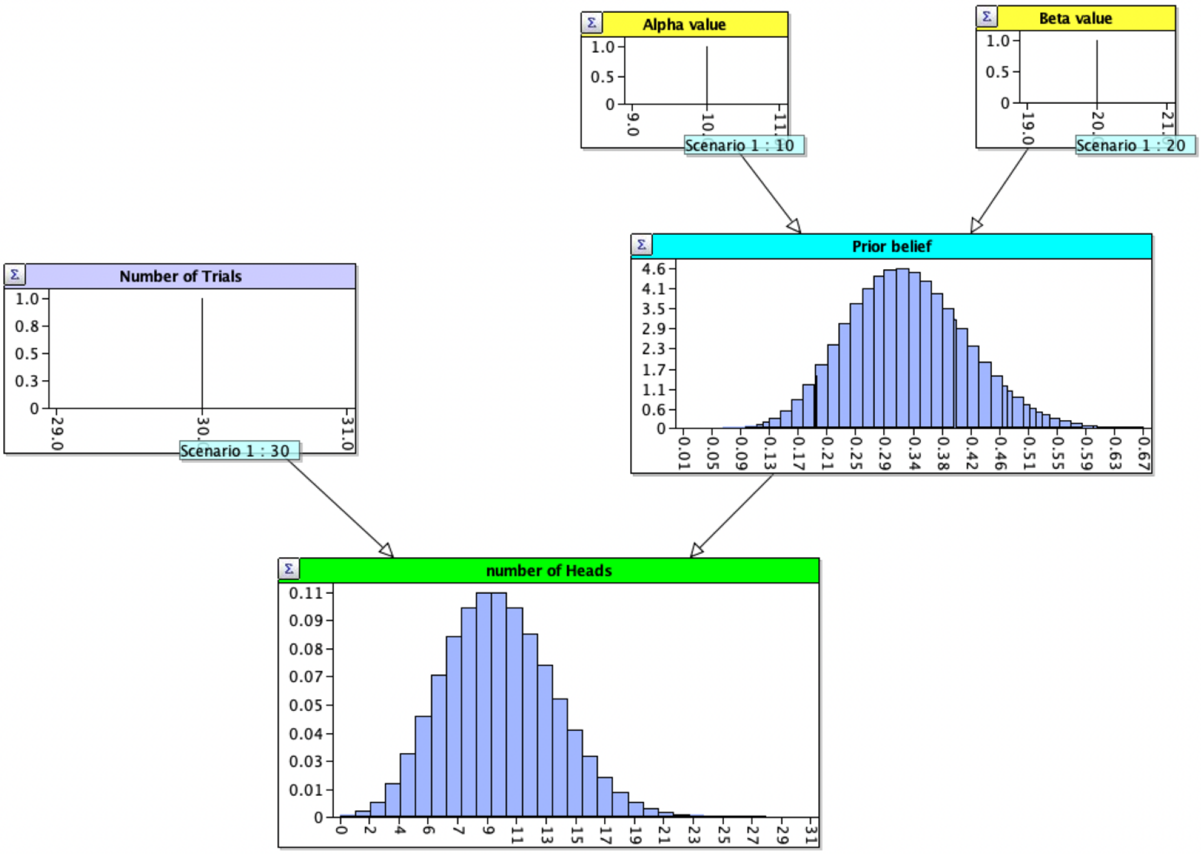
Theoretically,

It is similar to case 1, but the only difference is it is biased towards tails.

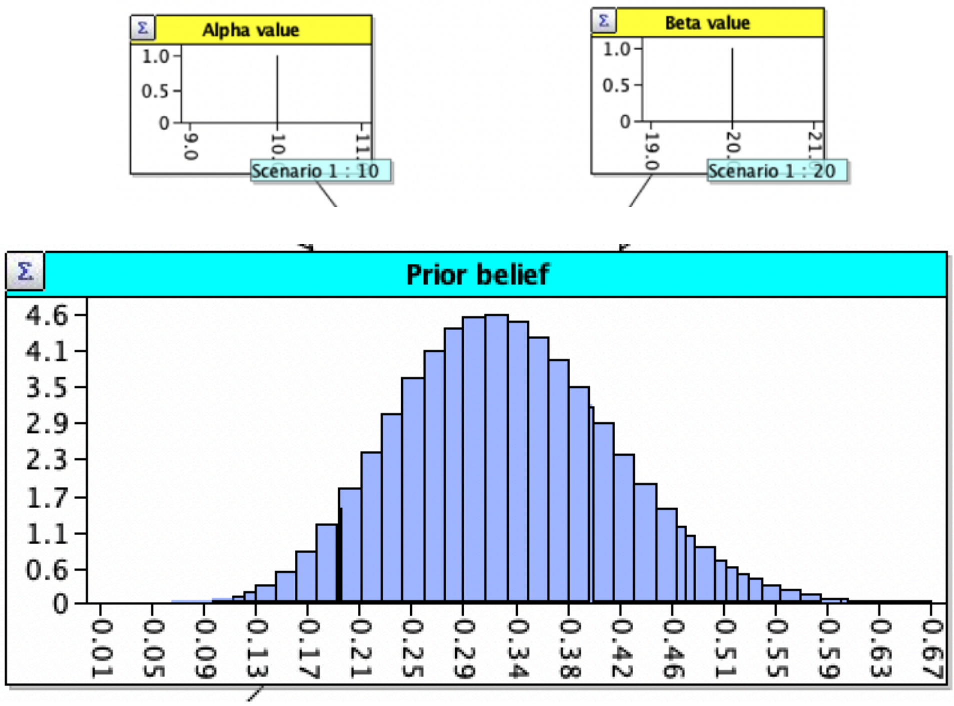
RISK MAP:



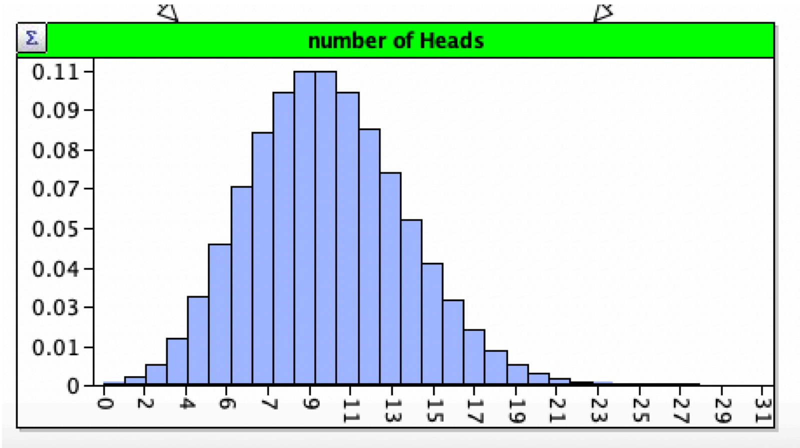
RISK MAP WITH GRAPHS:



Prior belief:



Chances of heads:



Theoretical solution:

2.1) P = Probability of heads.

$H_0 = 1/2$

Hypothesis.

a) $H_1 \rightarrow P < 1/2$

b) $H_1 \rightarrow P > 1/2$

Case.1

$\alpha = 20$, $\beta = 10$

$$\text{Mode} = \frac{\alpha - 1}{\alpha + \beta - 2} = \frac{20 - 1}{28}$$

$$\text{Mode} = 19/28$$

$$\text{Mean} = \frac{\alpha}{\alpha + \beta} = \frac{20}{20+10} = \frac{20}{30}$$

$$\text{Mean} = 2/3$$

$$\text{Variance} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{20 \times 10}{(20+10)^2[20+10+1]}$$

$$= \frac{200}{(30)^2(31)} < 0.025$$

$\therefore$ the standard deviation is < 0.16

The mean and mode are about $2/3^{rd}$, so the expect probability (P) of a coin to be around that region Especially since the standard deviation is < 0.16 which is Quite small.

Hence the model suggests, biased towards heads.

Case2

$\alpha = 10$, $\beta = 20$

It is similar to case 1, but it is biased towards tails.

2. QUESTION-2.2

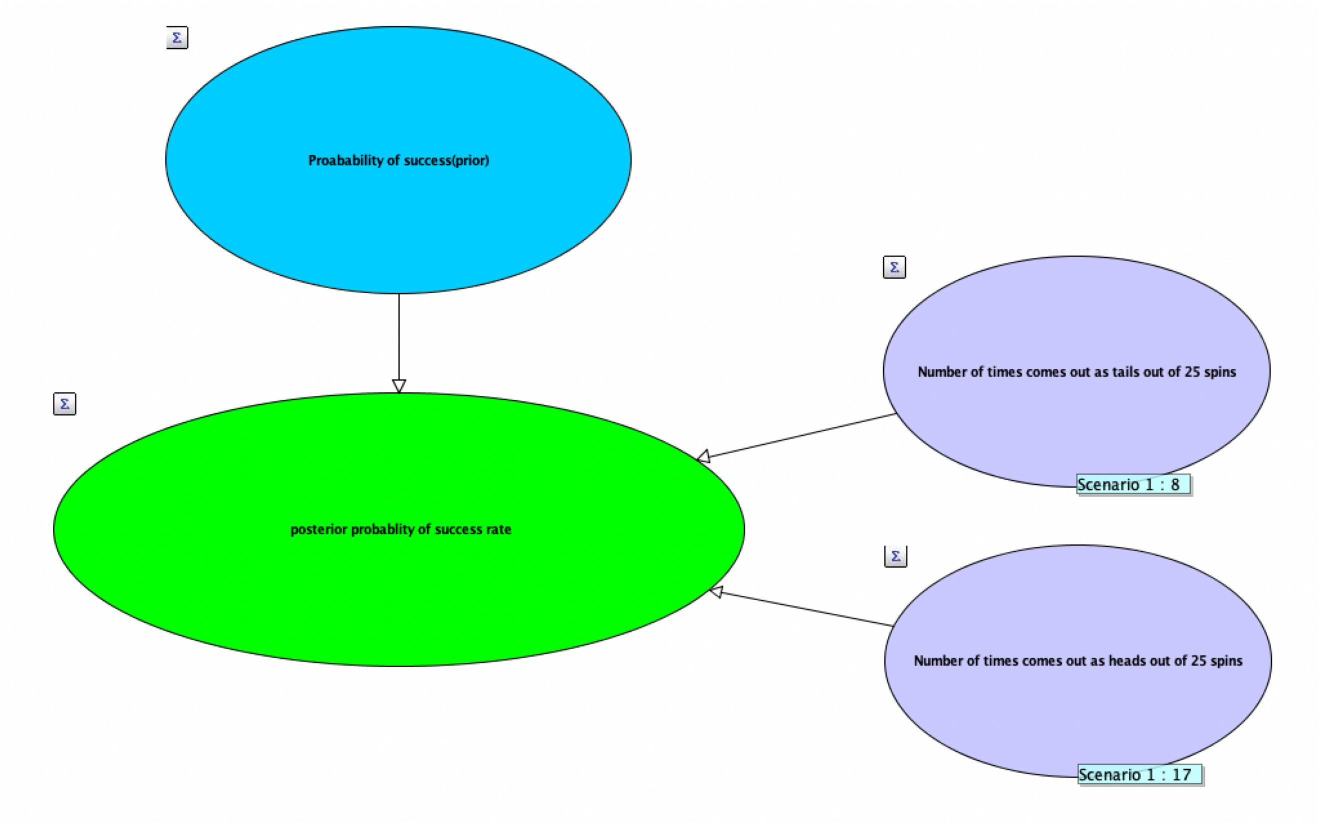
SOLUTION:

We have designed the model in the following way,

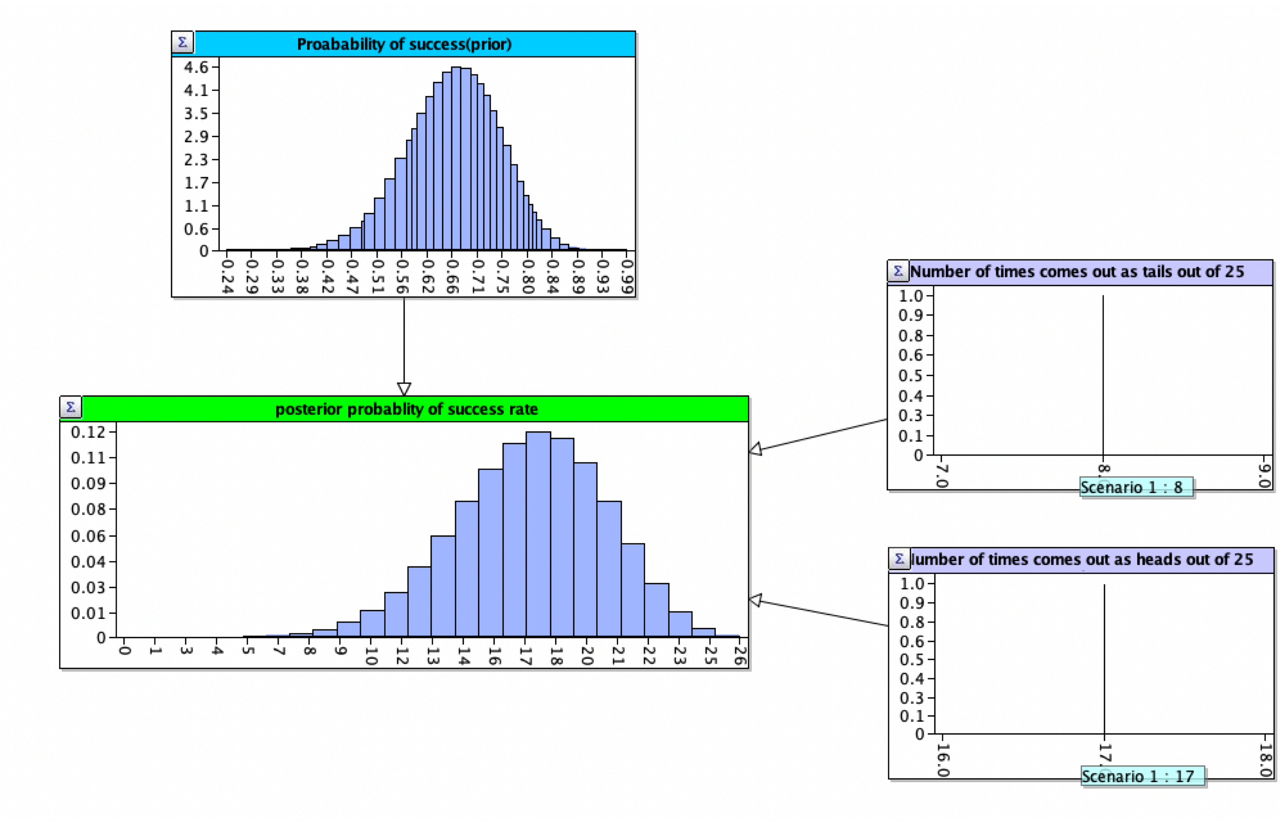
- As per the question there occurs 25 spins, which is the mixture of heads and tails
- There are 17 heads and 8 tails.
- Two nodes are created, one for number of times comes out as heads and other for tails.
- Then, the prior distribution is modelled using beta distribution of parameter 20:10, 10:20
- Finally, posterior probability is calculated, for this a node is created using the binomial distribution with sum of two trails and the prior belief.

CASE-1, For parameters 20:10 (alpha=20, beta=10)

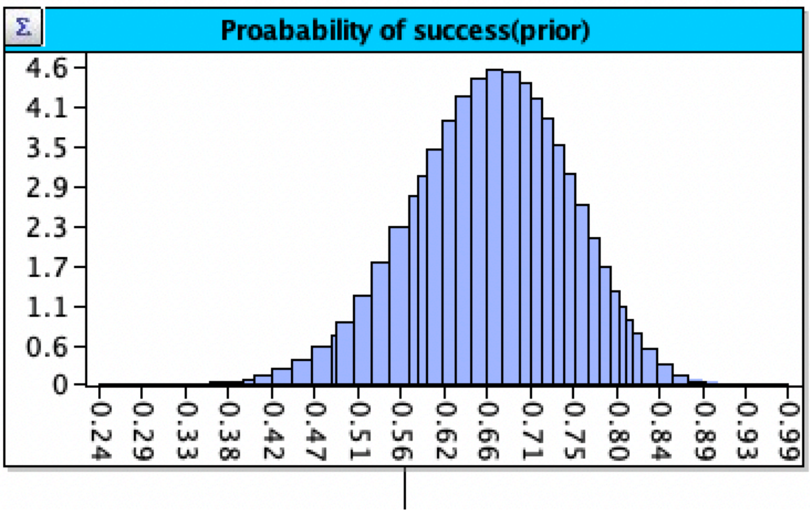
RISK MAP:

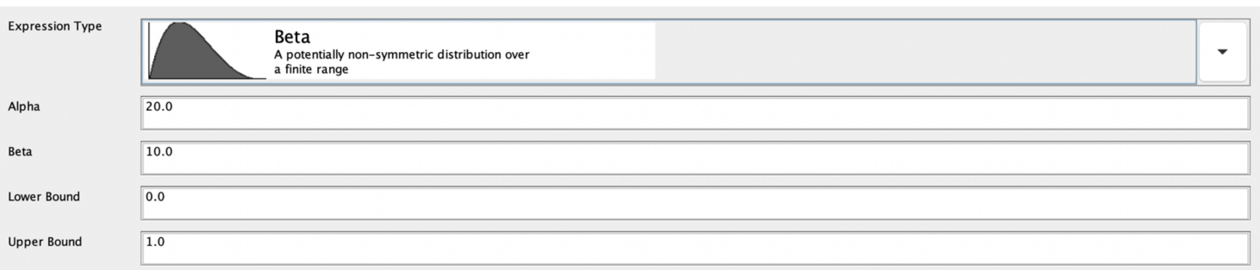


RISK MAP WITH GRAPHS:

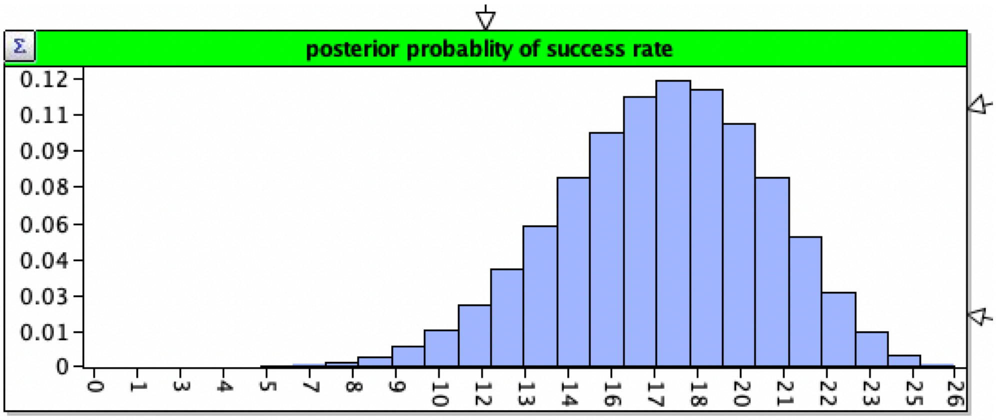


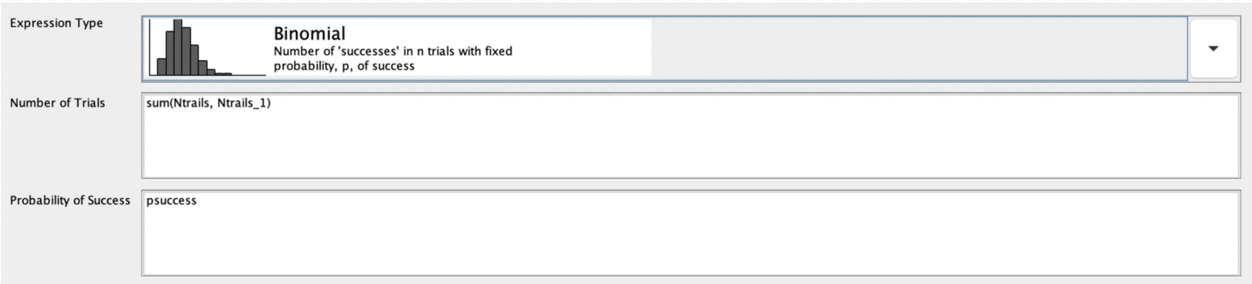
Prior belief:



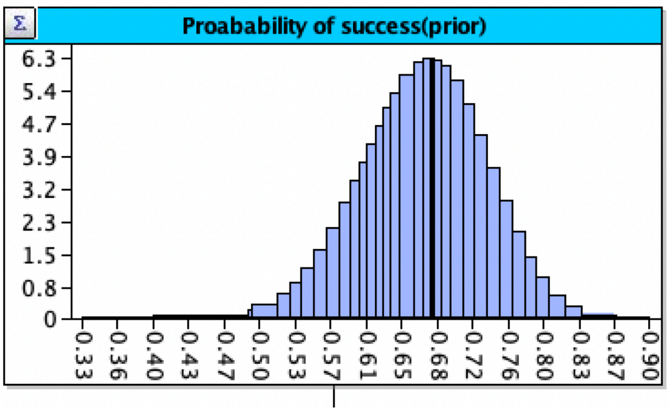


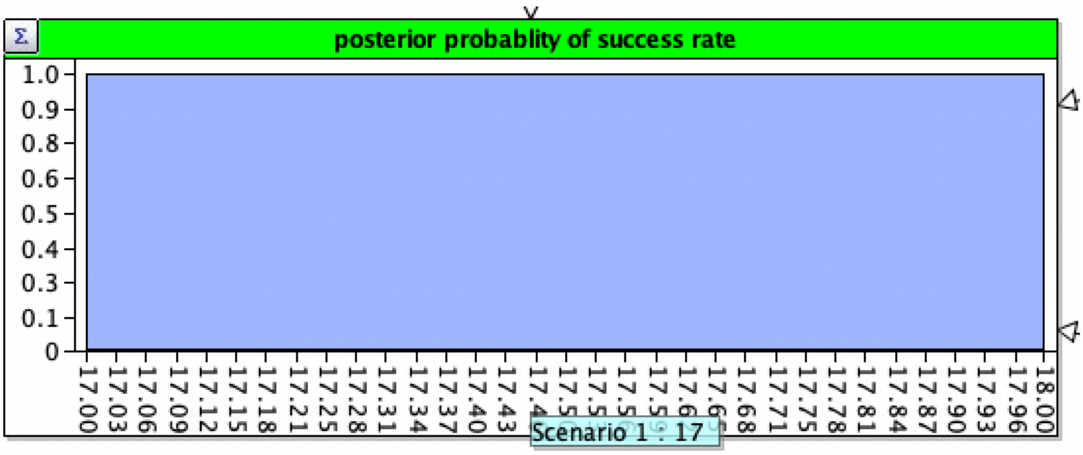
Posterior probability:



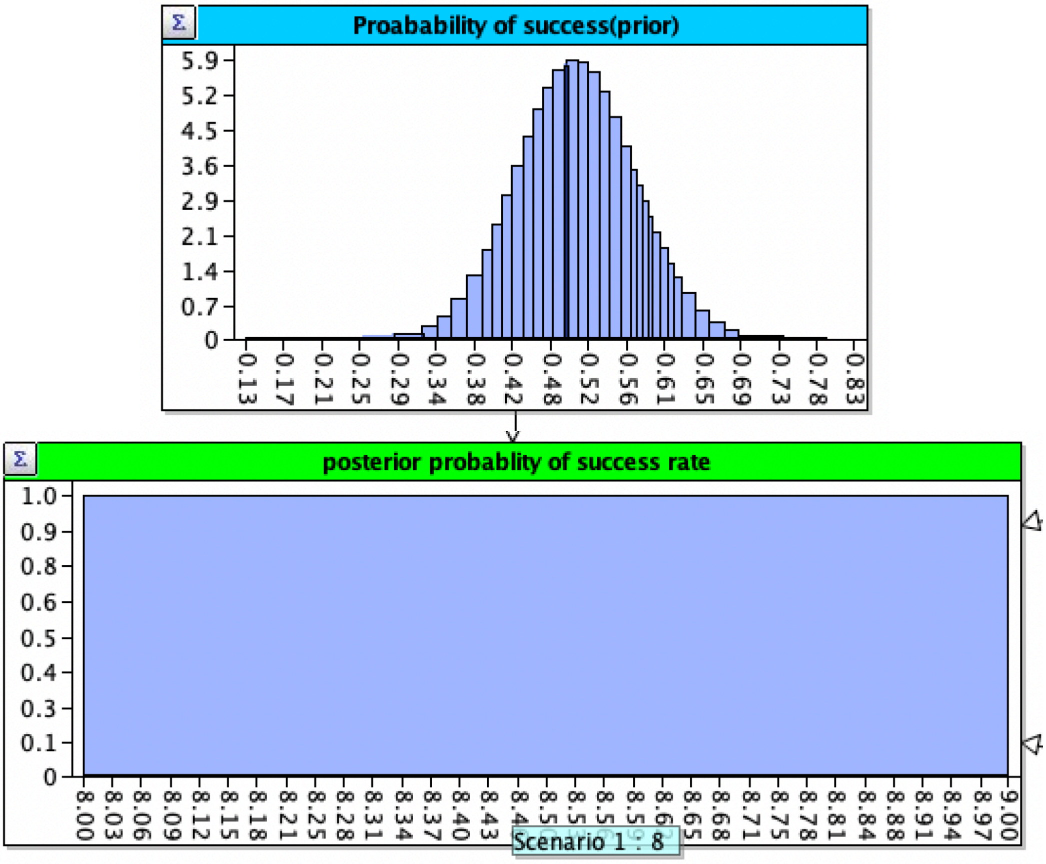


Posterior probability with observation=17



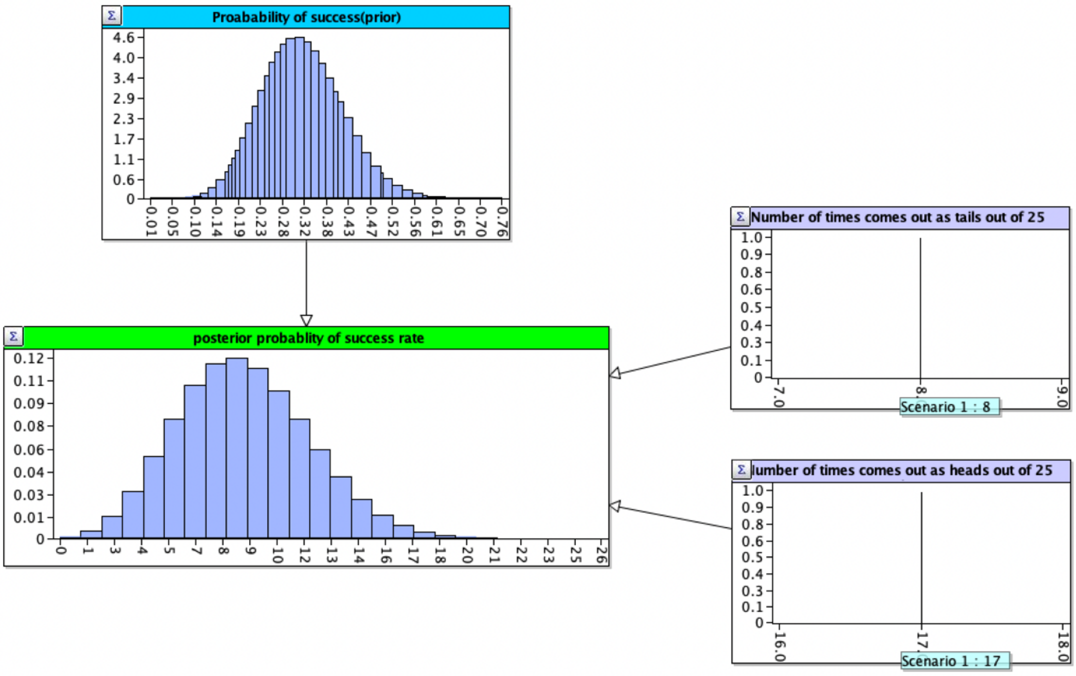


Posterior probability with observation=8

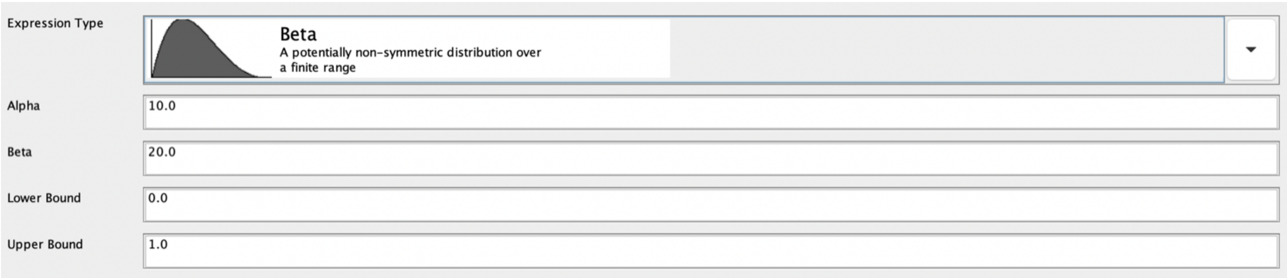


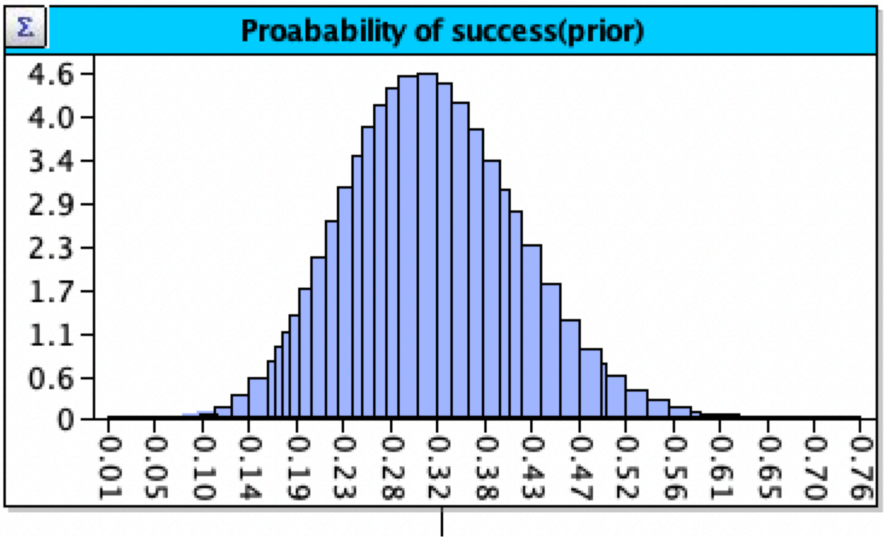
CASE-2, For parameters 10:20 (alpha=10, beta=20)

RISK MAP:

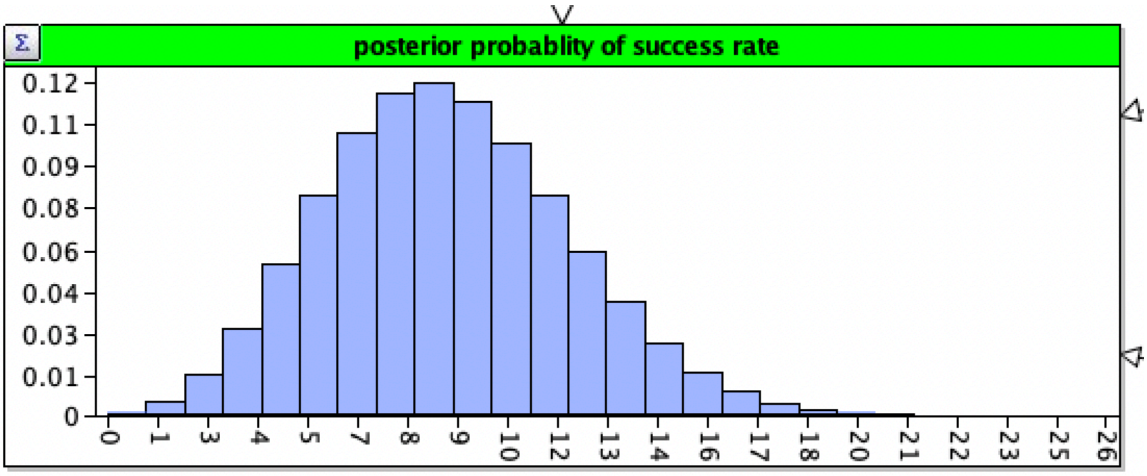


Prior distribution:

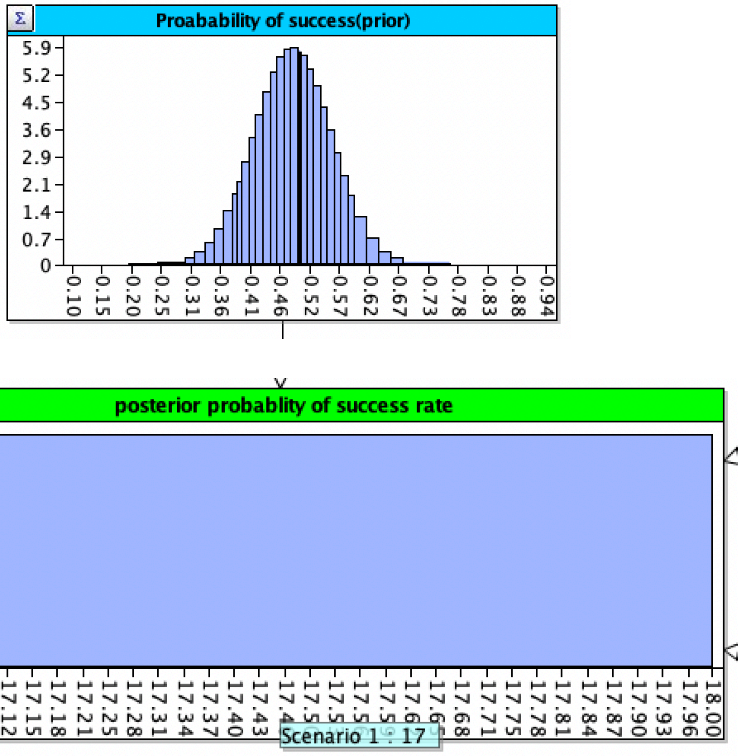




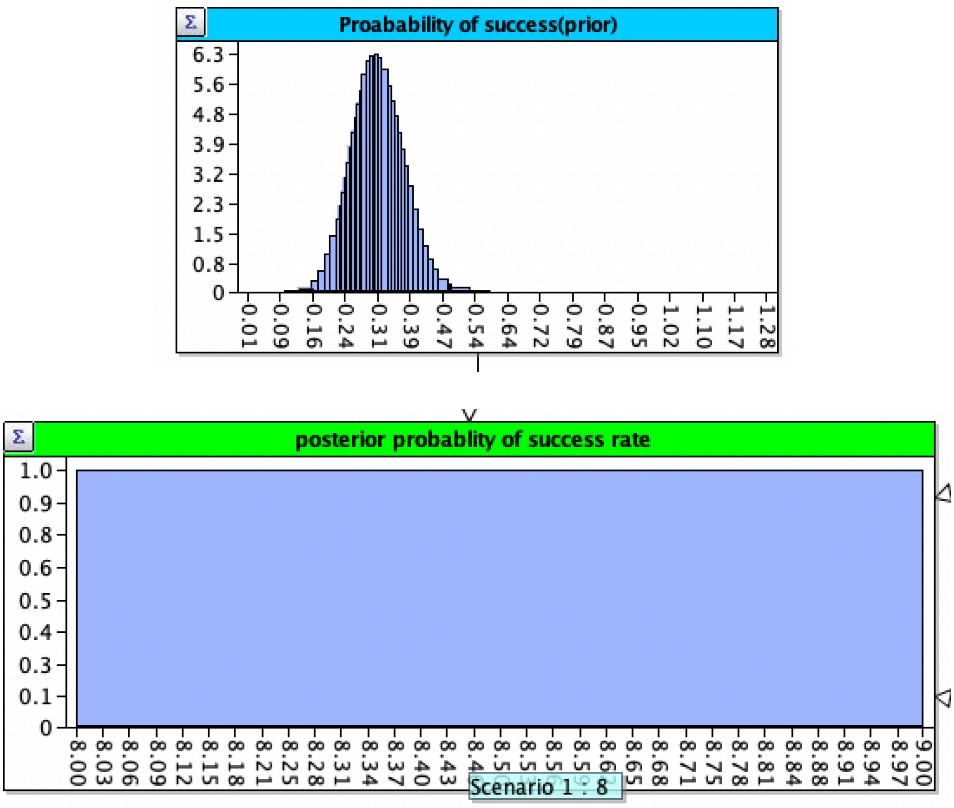
Posterior probability:



Posterior probability with observation=17



Posterior probability with observation=8



Theoretical solution:

2.2) The Question shows there occurs 25 spins and the results are given:

From that I observed, Total = 25

Tails = 8

Heads = 17

From Bayes theorem,

$$\pi(p/x) = \frac{f_X(x|P)\,\pi(p)}{f_X(x)}$$

where, x is the vector of coin spin results.

$$f_X(x|P) = p^{17}(1-p)^8$$

$$\pi(p) \propto p^{\alpha-1}(1-p)^{\beta-1}$$

As, $p \sim Beta(\alpha\beta)$ is prior

$$\pi(p|x) \propto f_X(x|p) \times \pi(p)$$

$$\propto p^{17}(1-p)^8 \times p^{\alpha-1}(1-p)^{\beta-1}$$

$\therefore$ Posterior distribution,

$$p \sim Beta(\alpha+17,\ \beta+8)$$

Part 3- Advanced Analysis

1. QUESTION-3.1

ANSWER:

- In part2 the chances of shaven on head or tail is not specify, but in this part its is specified that, there is 2:1 chance that the coin is shaved on the head side.
- 2:1 chance denotes that is **2/3 possibility of success** which is around **0.67**. From this we can tell that head side is shaved 0.67 which is 2/3.
- Likelihood is estimated with the total trails and probability of success and then the prior distribution is calculated.

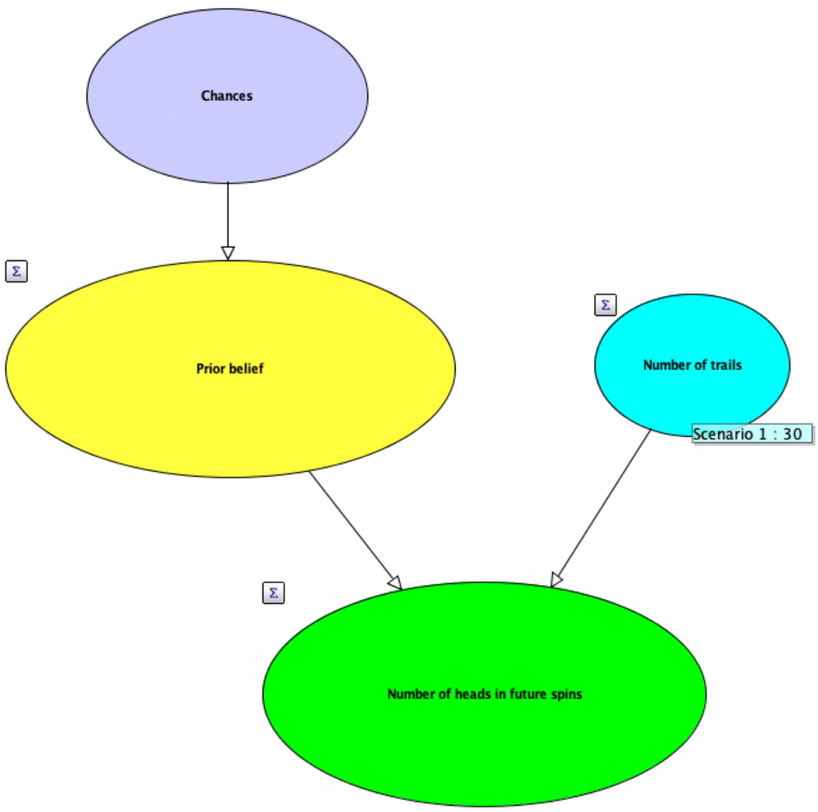
2. QUESTION-3.2

SOLUTION:

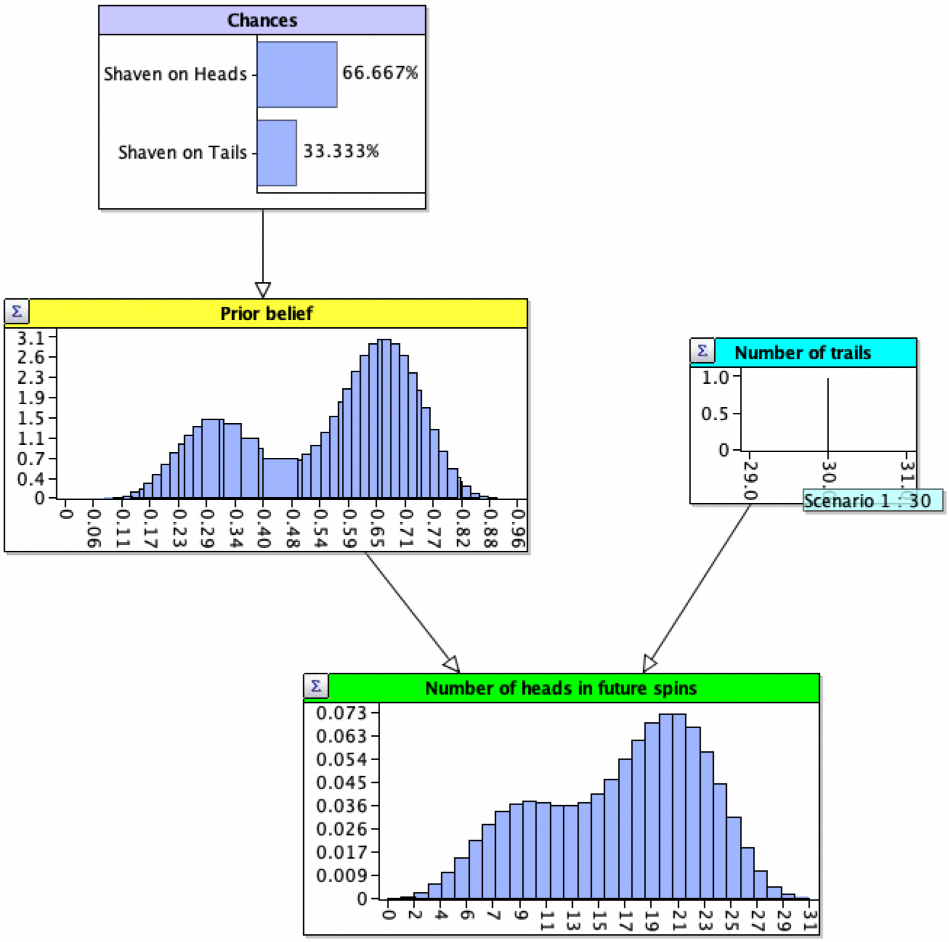
We modelled using the number of trails given in part -2 which is of 30 spins, which has the 2:1 probability of shaven on heads which gives 2/3 probability. Using this prior is calculated with partitioned parameter and the model is formed.

CASE-1, For parameters, shaven on heads = 20:10 and tails= 10:20

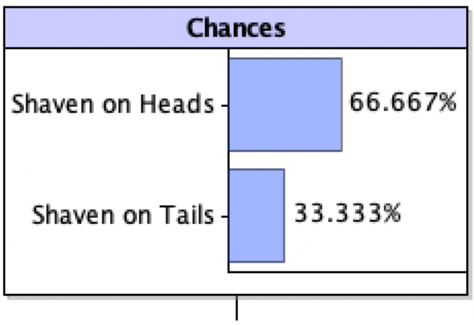
RISK MAP:



RISK MAP WITH GRAPHS:

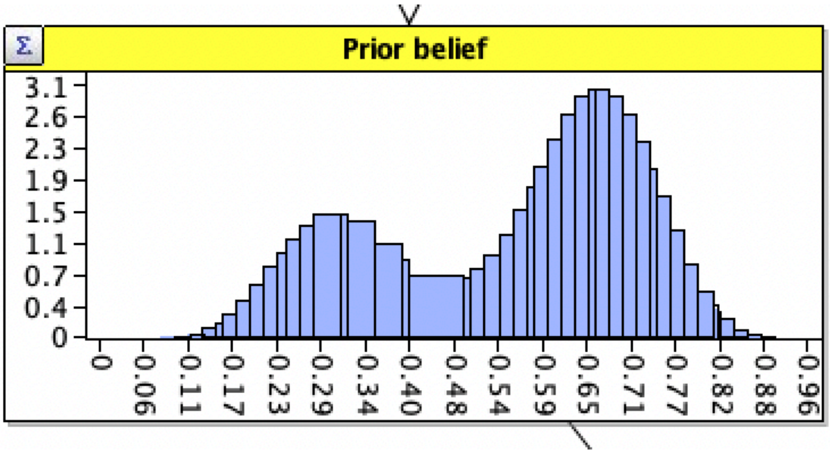


Chances of shaven heads and tails:



Shaven on ...	0.6666667
Shaven on ...	0.3333333

Prior distribution:



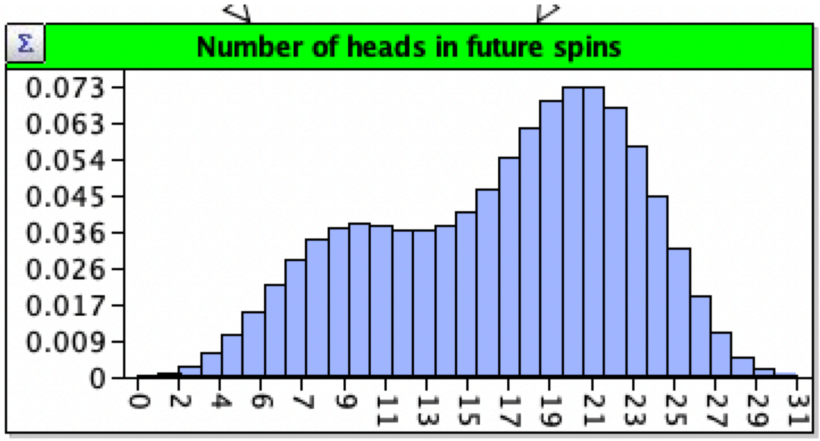
Chances	Shaven on Heads	Shaven on Tails
Expressions	Beta(20.0,10....	Beta(10.0,20....

Expression Type	Beta A potentially non-symmetric distribution over a finite range
Alpha	20.0
Beta	10.0
Lower Bound	0.0
Upper Bound	1.0

Expression Type	Beta A potentially non-symmetric distribution over a finite range
Alpha	10.0
Beta	20.0
Lower Bound	0.0
Upper Bound	1.0

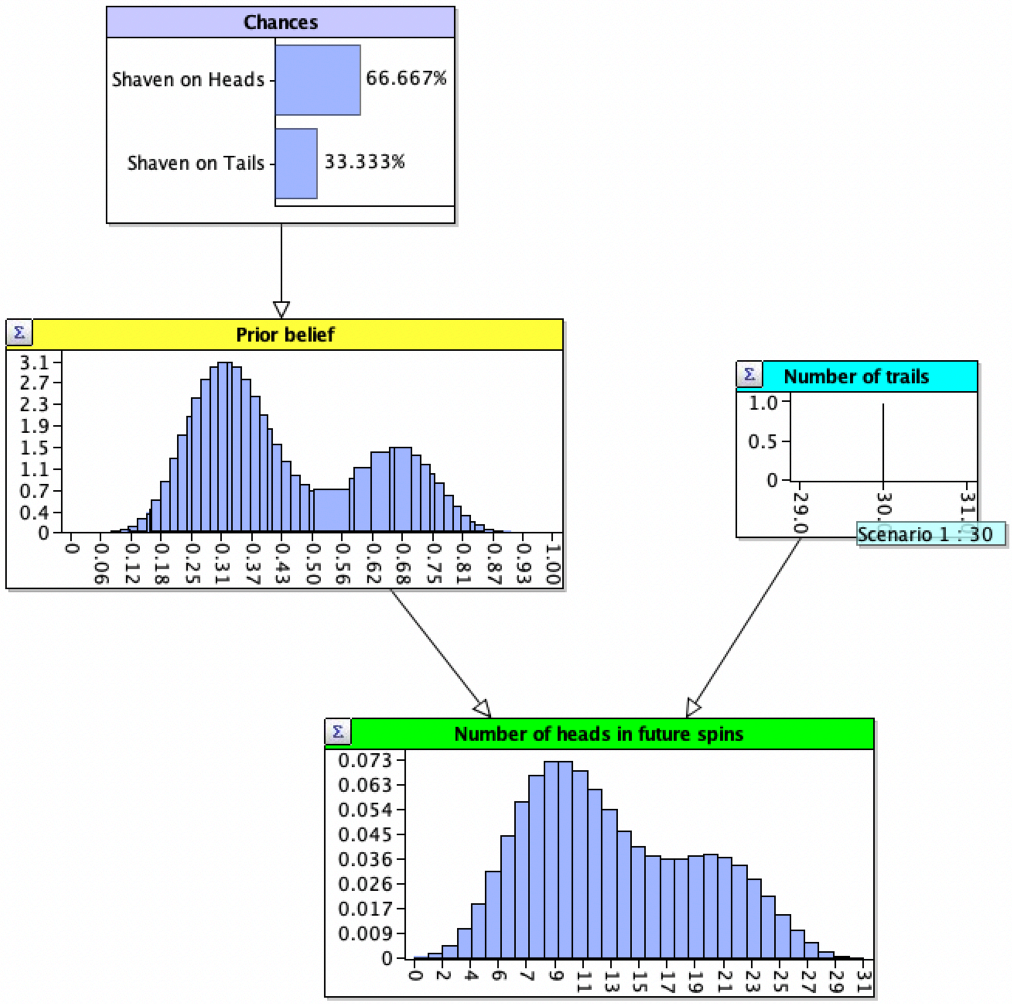
Chances of heads:

Expression Type	Binomial Number of 'successes' in n trials with fixed probability, p, of success
Number of Trials	Ntrails
Probability of Success	dist

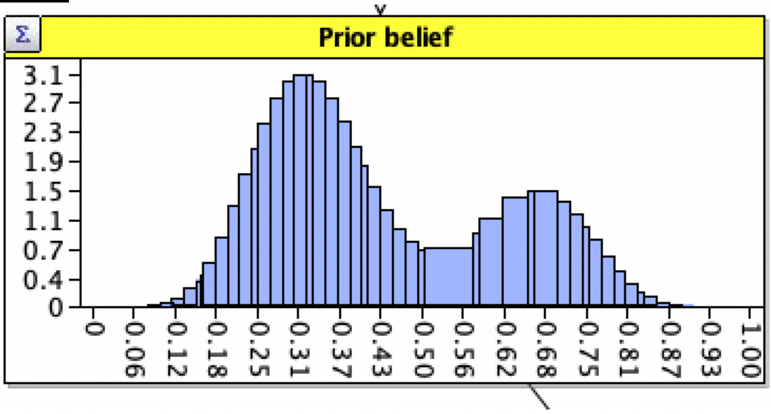


CASE-2, For parameters, shaven on heads = 20:10 and tails= 10:20

RISK MAP:

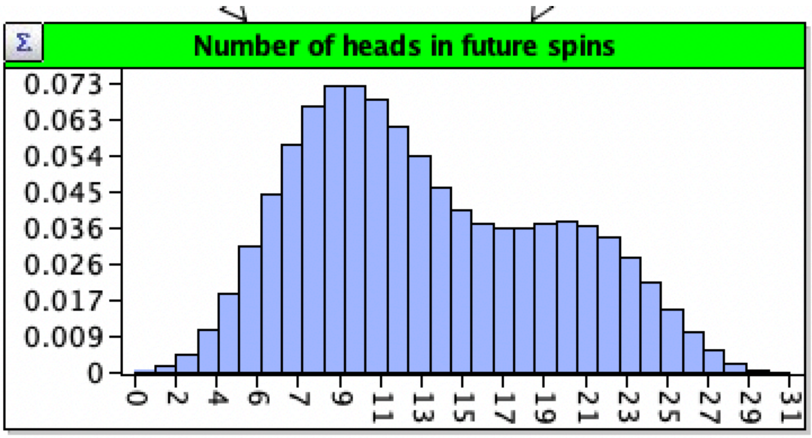


Prior distribution:



Chances	Shaven on Heads	Shaven on Tails
Expressions	Beta(10.0,20....	Beta(20.0,10....

Chances of heads:



3. QUESTION-3.3

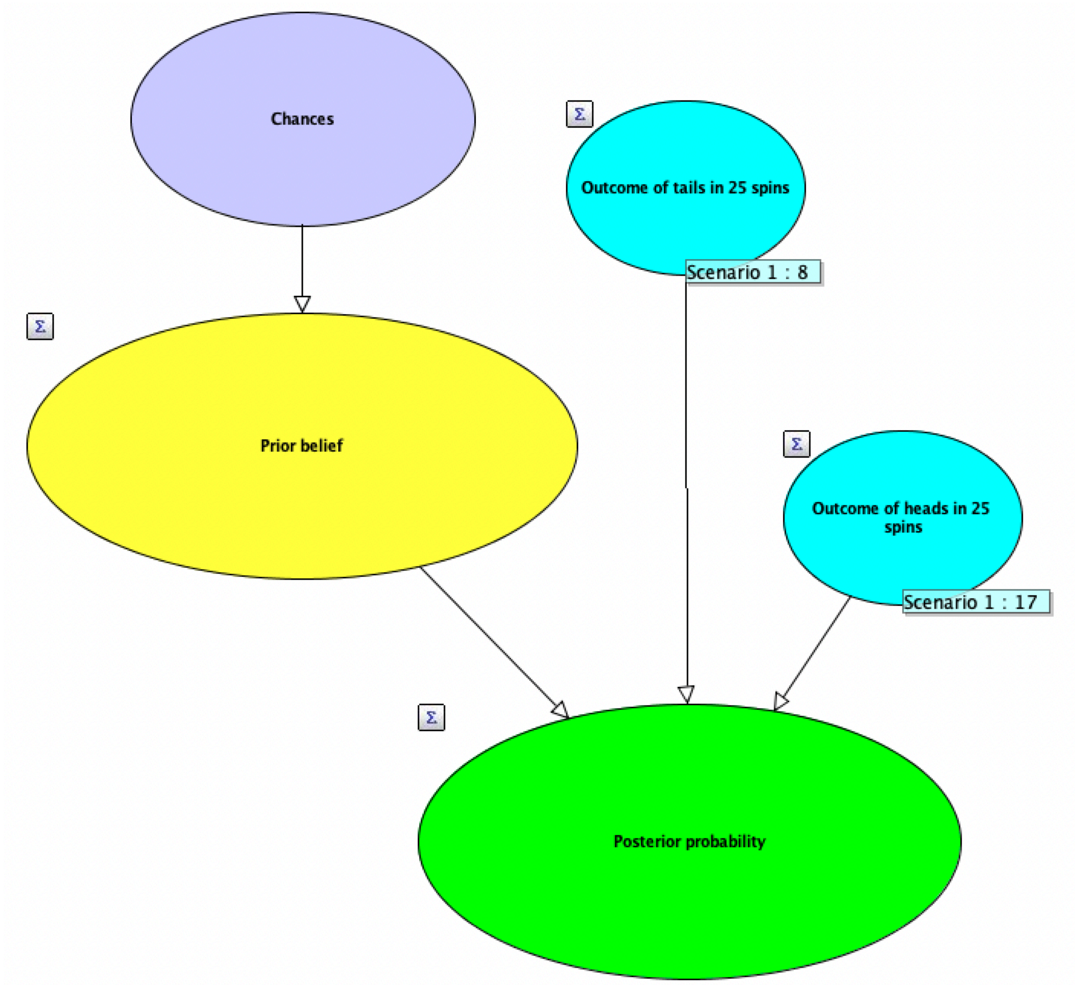
SOLUTION:

We have designed the model in the following way,

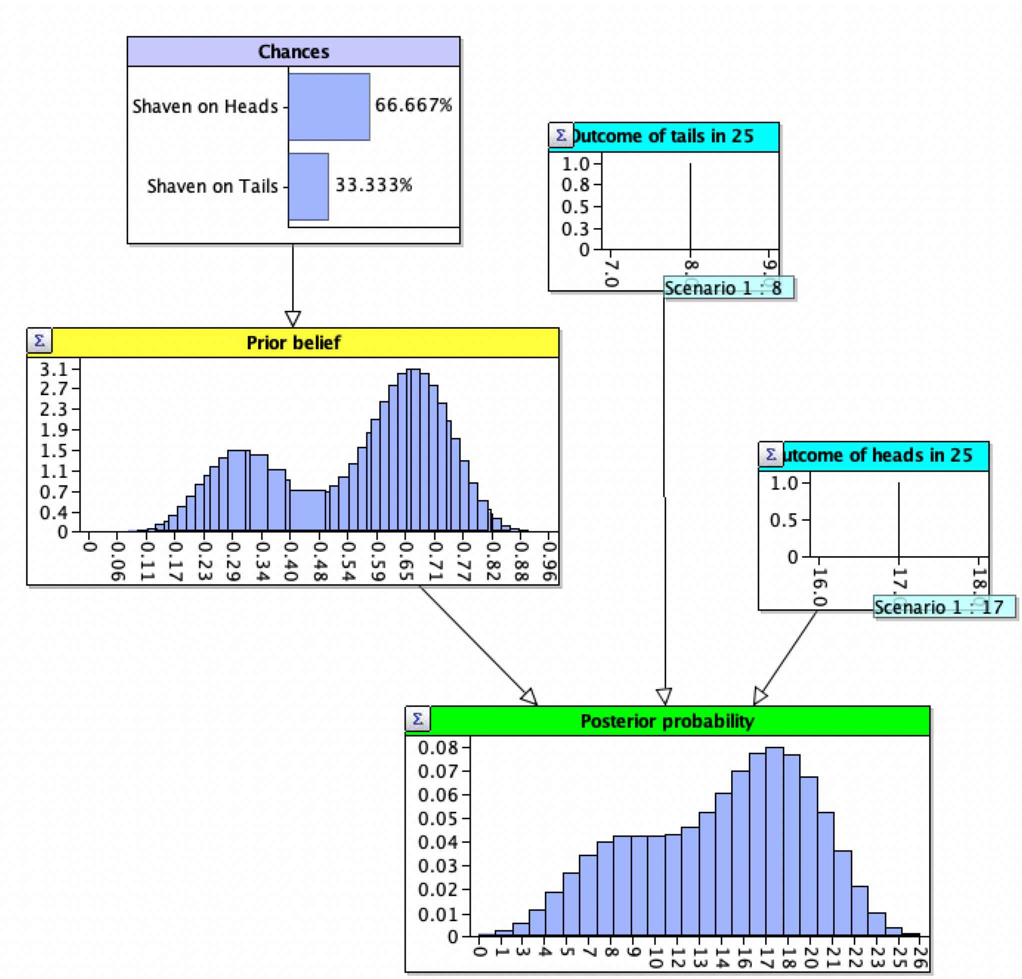
- As per the question there occurs 25 spins, which is the mixture of heads and tails
- There are 17 heads and 8 tails.
- Two nodes are created, one for number of times comes out as heads and other for tails.
- Then, the prior is calculated for 2:1 chances to be shaven on head which is 0.67 and the prior belief is shown using the beta distribution.
- Finally, posterior probability is calculated, for this a node is created using the binomial distribution with sum of two trails and the prior belief.

CASE-1, For parameters, shaven on heads = 20:10 and tails= 10:20

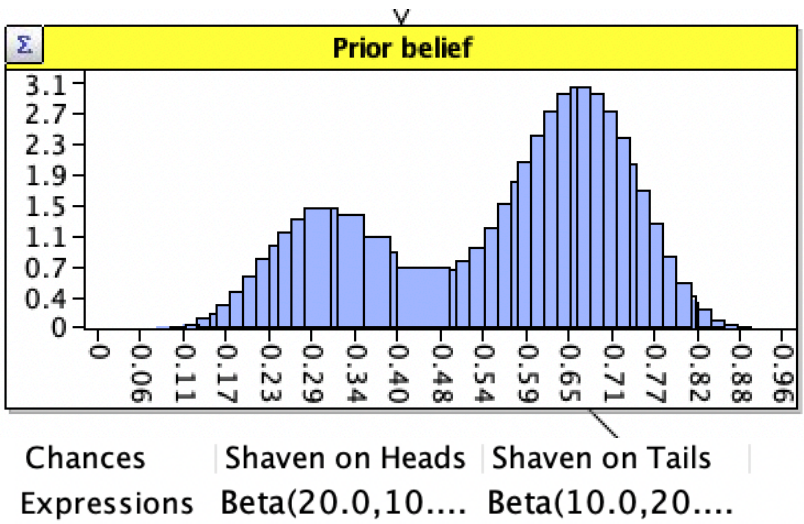
RISK MAP:

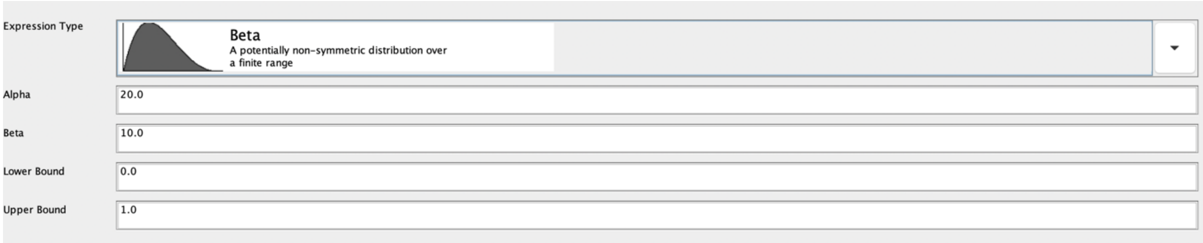


RISK MAP WITH GRAPHS:

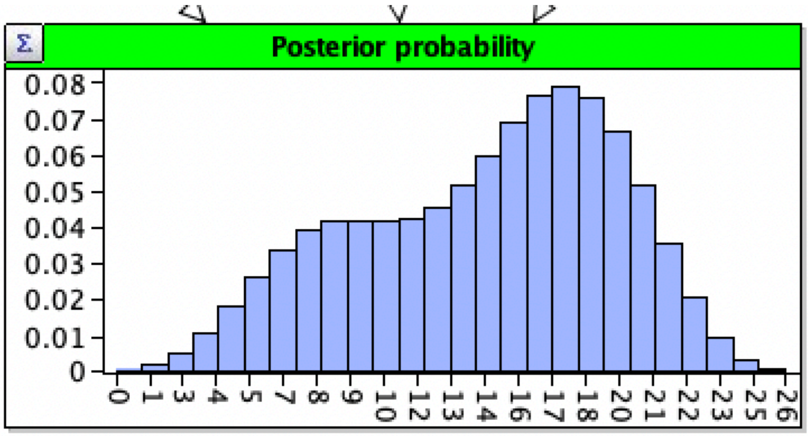


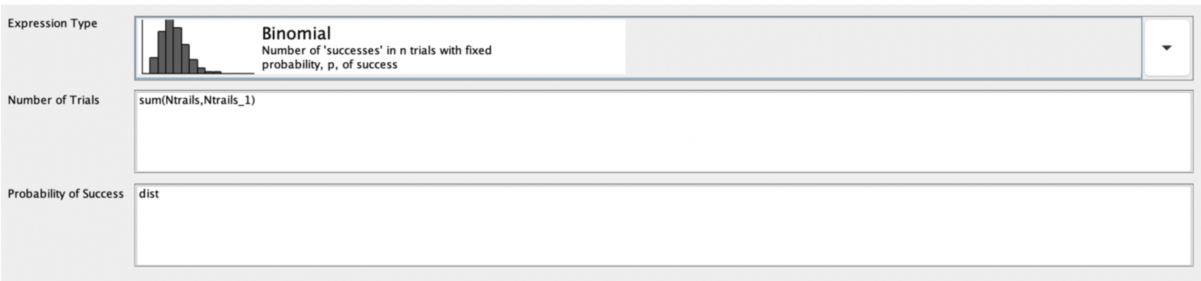
Prior distribution:



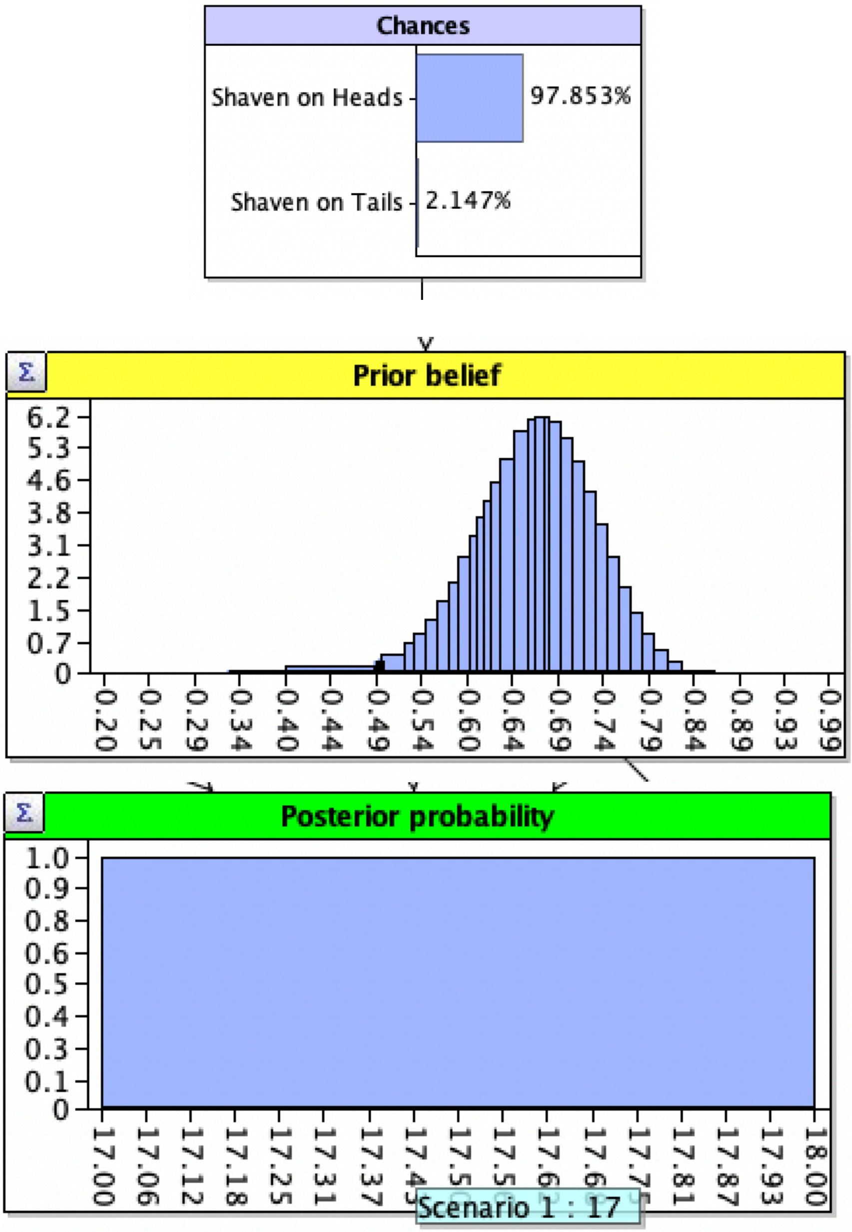


<u>Posterior probability:</u>

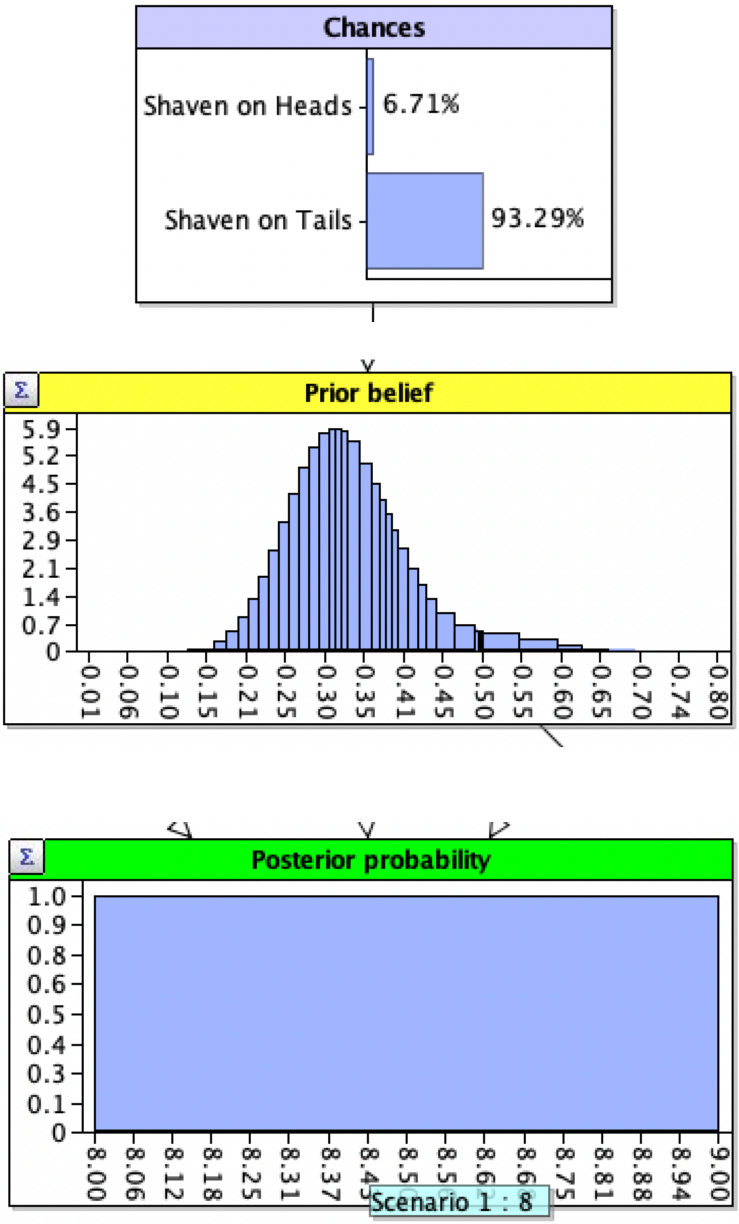




Posterior probability with scenario=17:

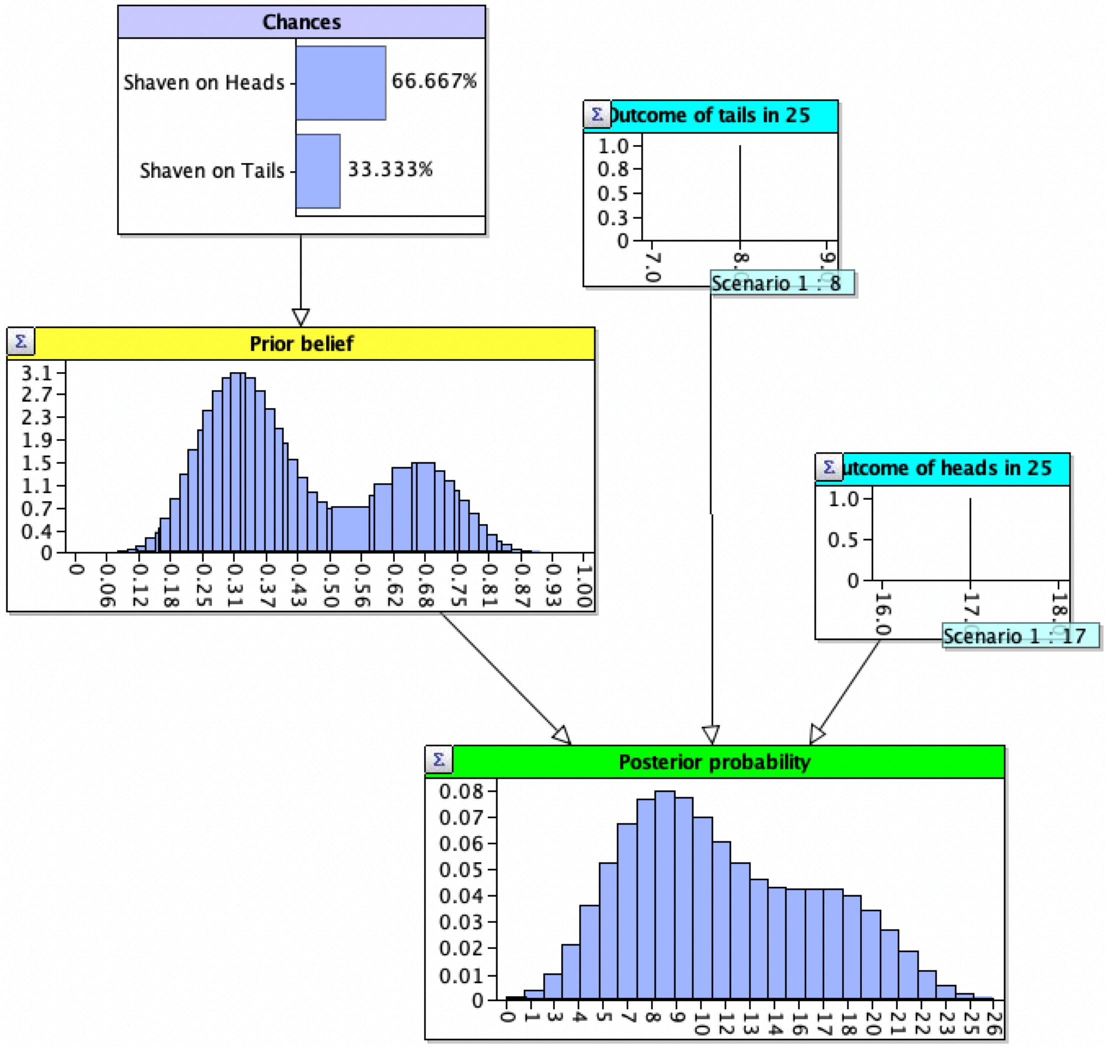


Posterior probability with scenario=8:

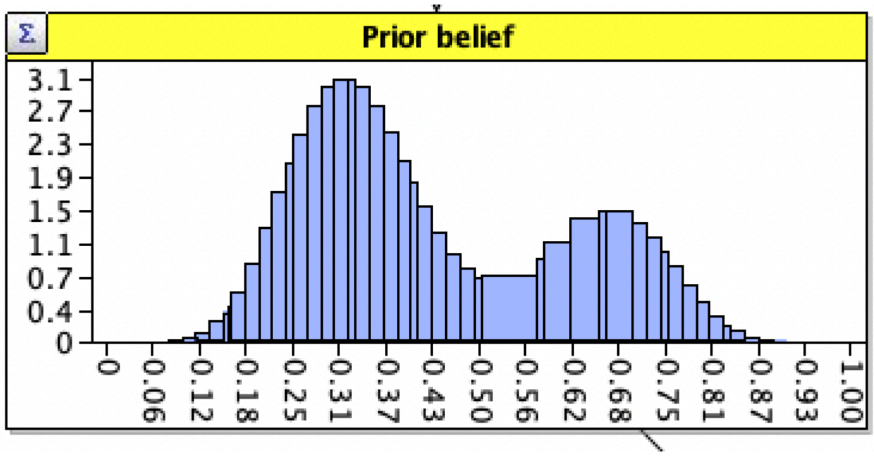


CASE-2, For parameters , shaven on heads = 10:20 and tails= 20:10

RISK MAP:

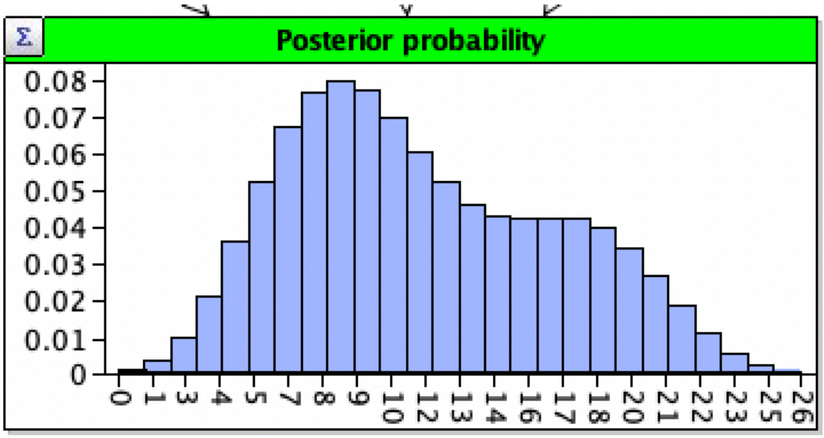


Prior distribution:

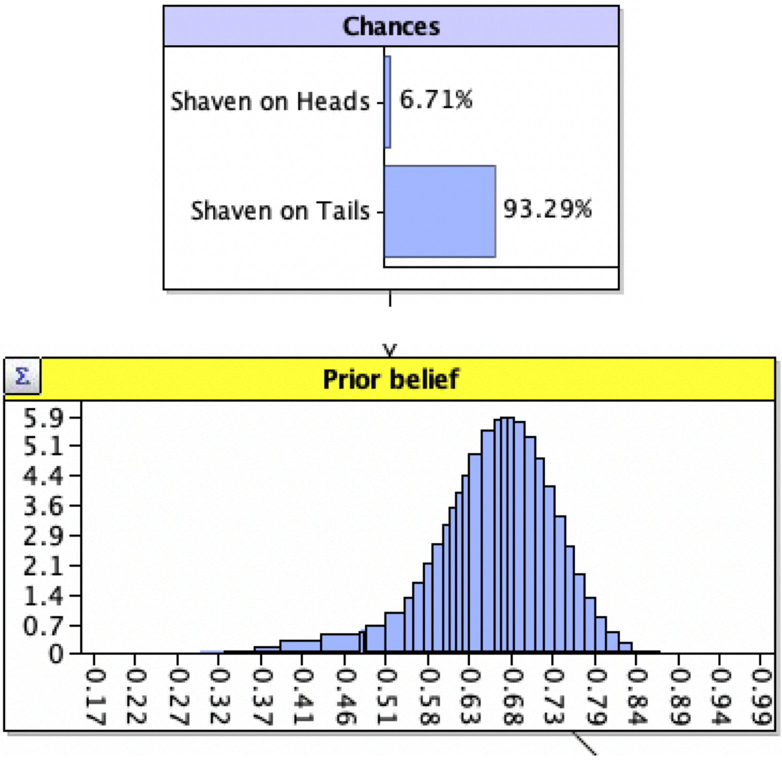


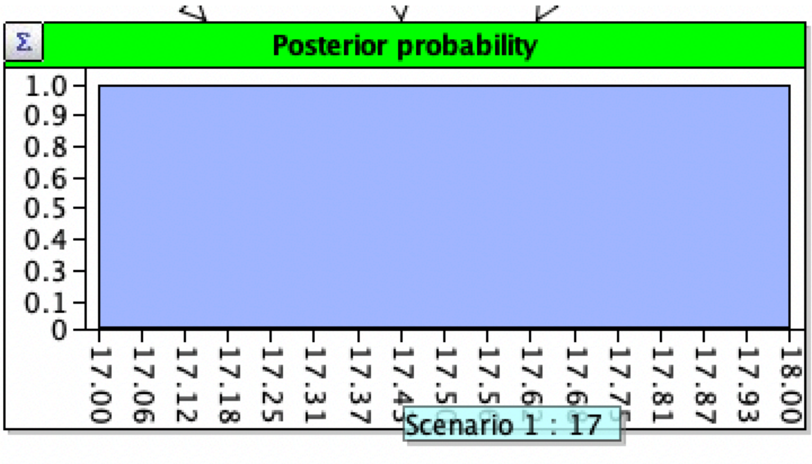
Chances	Shaven on Heads	Shaven on Tails
Expressions	Beta(10.0,20....	Beta(20.0,10....

<u>Posterior probability:</u>

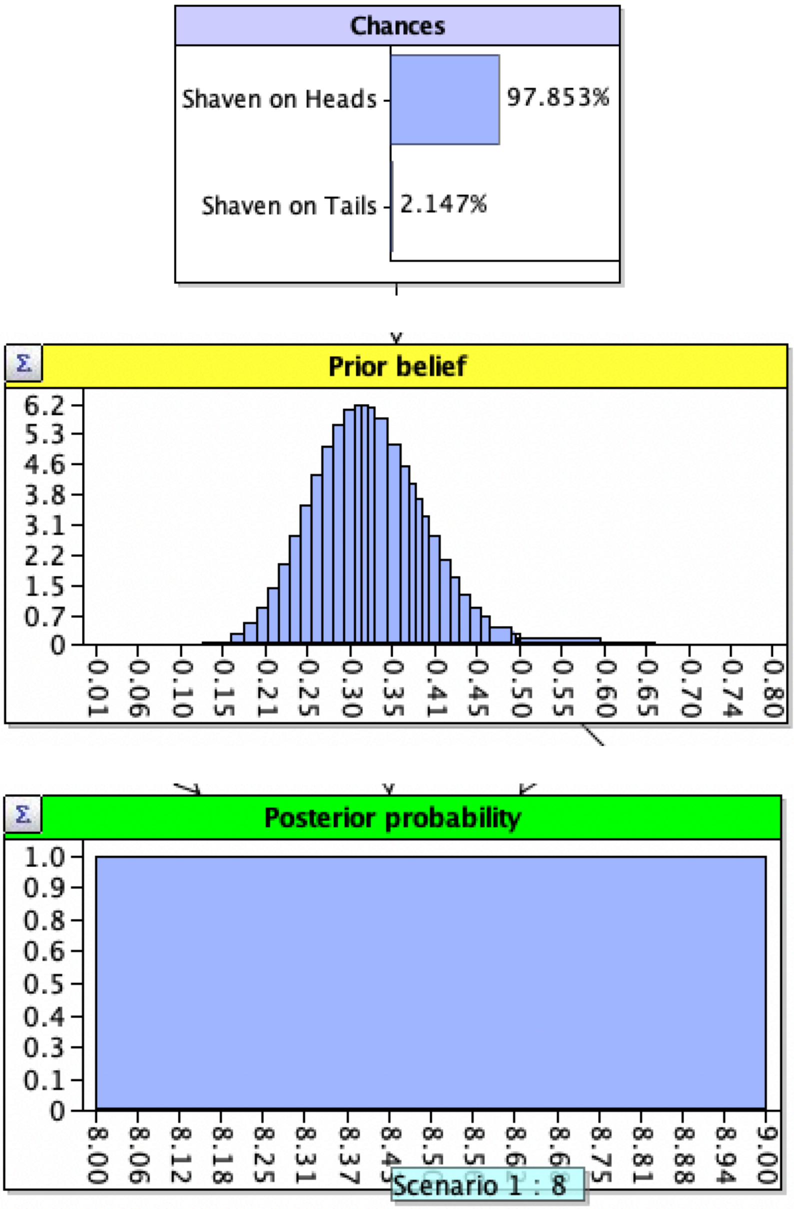


<u>Posterior probability with scenario=17:</u>





Posterior probability with scenario=8:



3. QUESTION-3.4

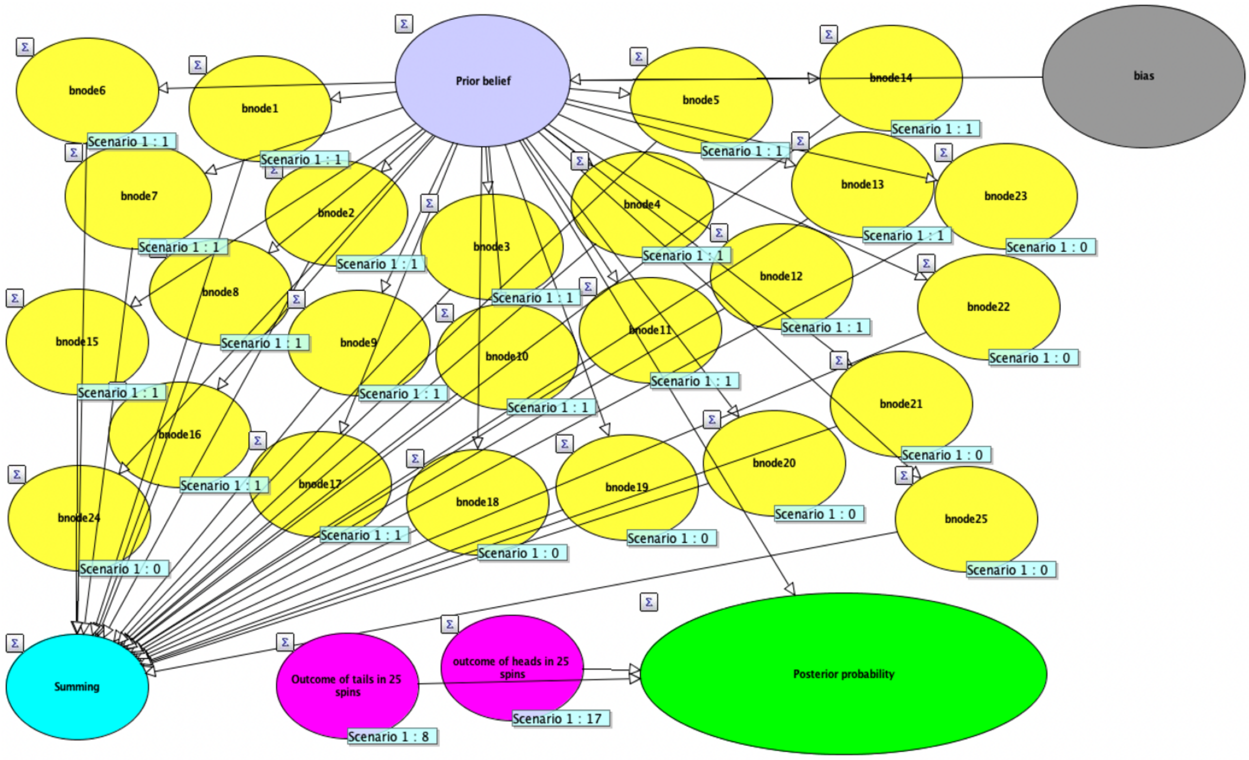
SOLUTION:

We have designed the model in the following way,

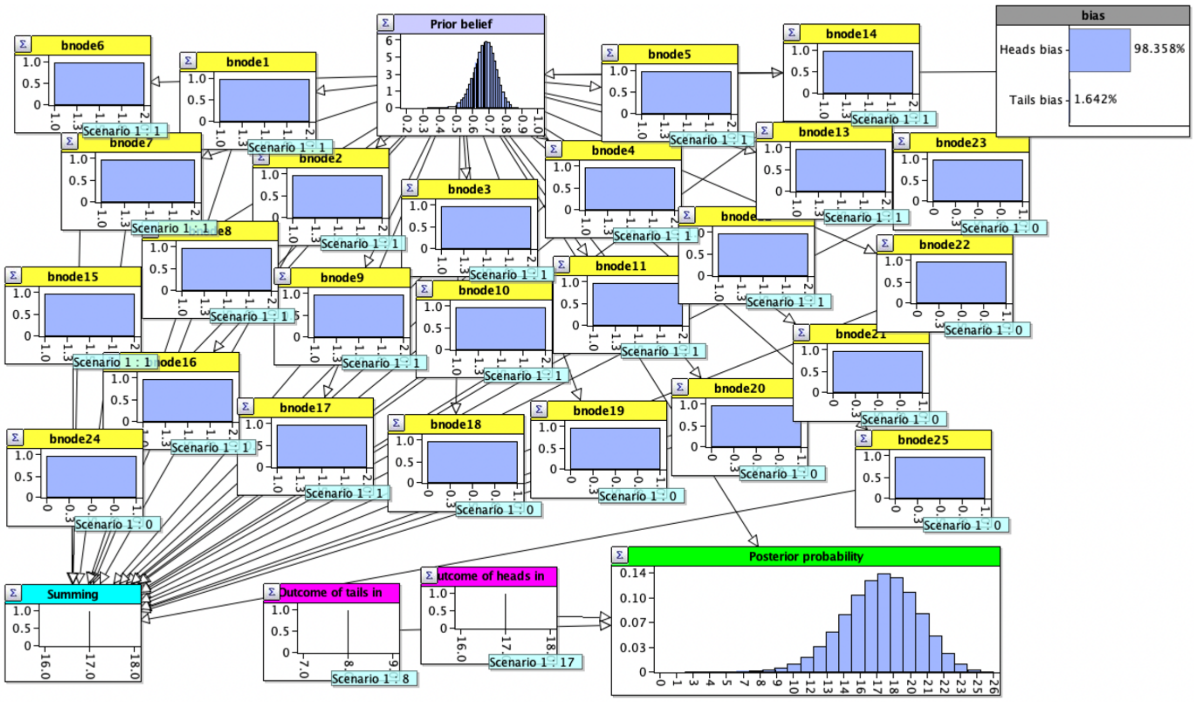
- As per the question there occurs 25 spins, which is the mixture of heads and tails
- There are 17 heads and 8 tails.
- Two nodes are created, one for number of times comes out as heads and other for tails.
- We assume the prior belief with the parameter 20:10 and 10:20
- 25 bernoulli series of trails are created instead of binomial and the values are passed to the prior distribution.
- And, finally posterior probability is calculated is the usual manner.
- Binomial gives better results than Bernoulli because, Bernoulli process the values one by one whereas binomial process in a single run so that it produces better results.

CASE-1, For parameters, biased heads = 20:10 and tails= 10:20

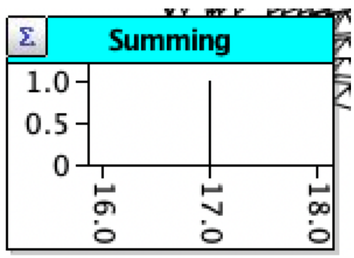
RISK MAP:

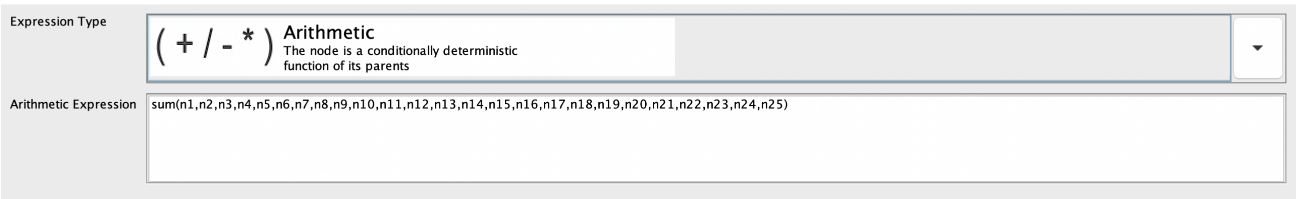


RISK MAP WITH GRAPHS:

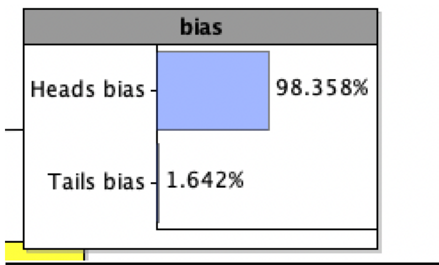


Summing the Bernoulli trails:



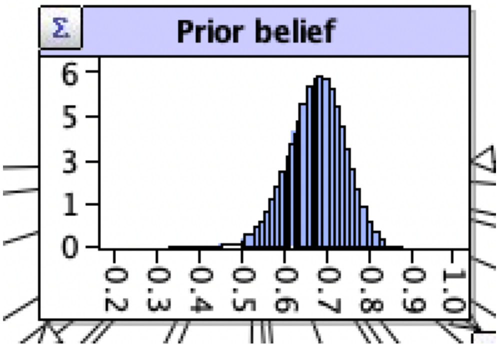


Chances of bias over heads or tails:



Heads bias	0.6666667
Tails bias	0.33333334

Prior distribution:



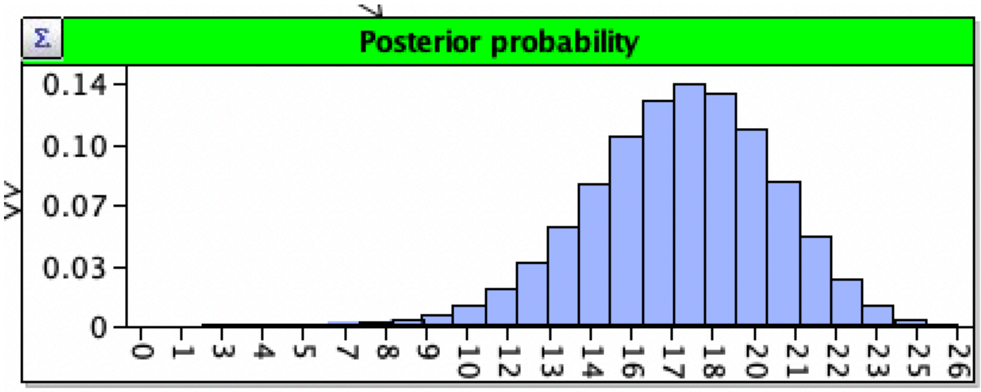
bias	Heads bias	Tails bias
Expressions	Beta(20,10,0....	Beta(10.0,20....

Expression Type	Beta A potentially non-symmetric distribution over a finite range
Alpha	20.0
Beta	10.0
Lower Bound	0.0
Upper Bound	1.0

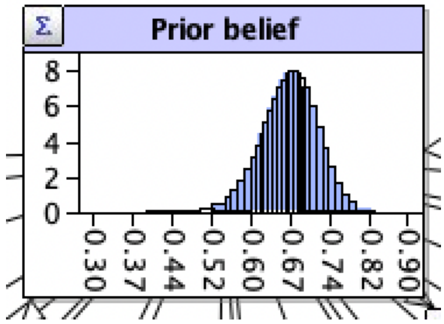
Expression Type	Beta A potentially non-symmetric distribution over a finite range
Alpha	10.0
Beta	20.0
Lower Bound	0.0
Upper Bound	1.0

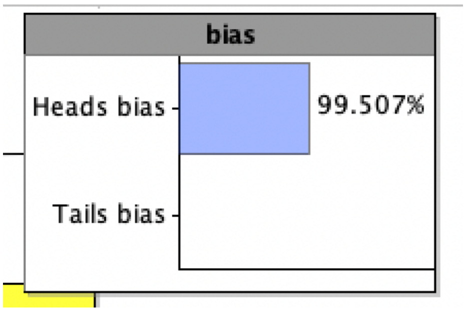
Posterior probability:

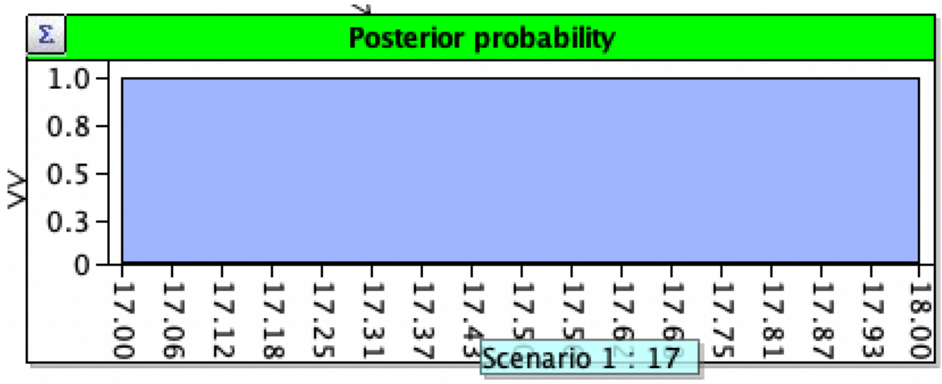
Expression Type	Binomial Number of 'successes' in n trials with fixed probability, p, of success
Number of Trials	sum(Ntrails,Ntrails_1)
Probability of Success	prior



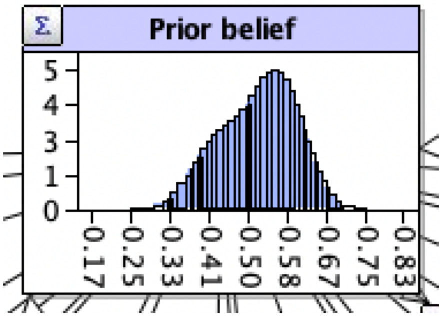
<u>Posterior probability with scenario=17:</u>

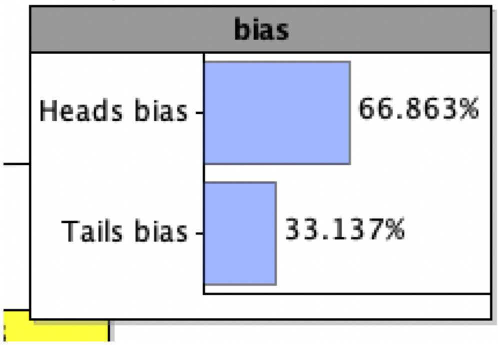


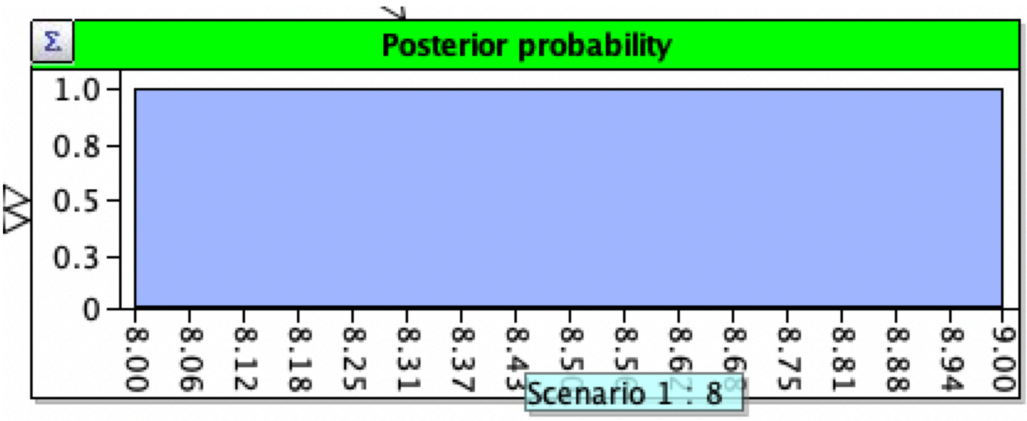




Posterior probability with scenario=8:

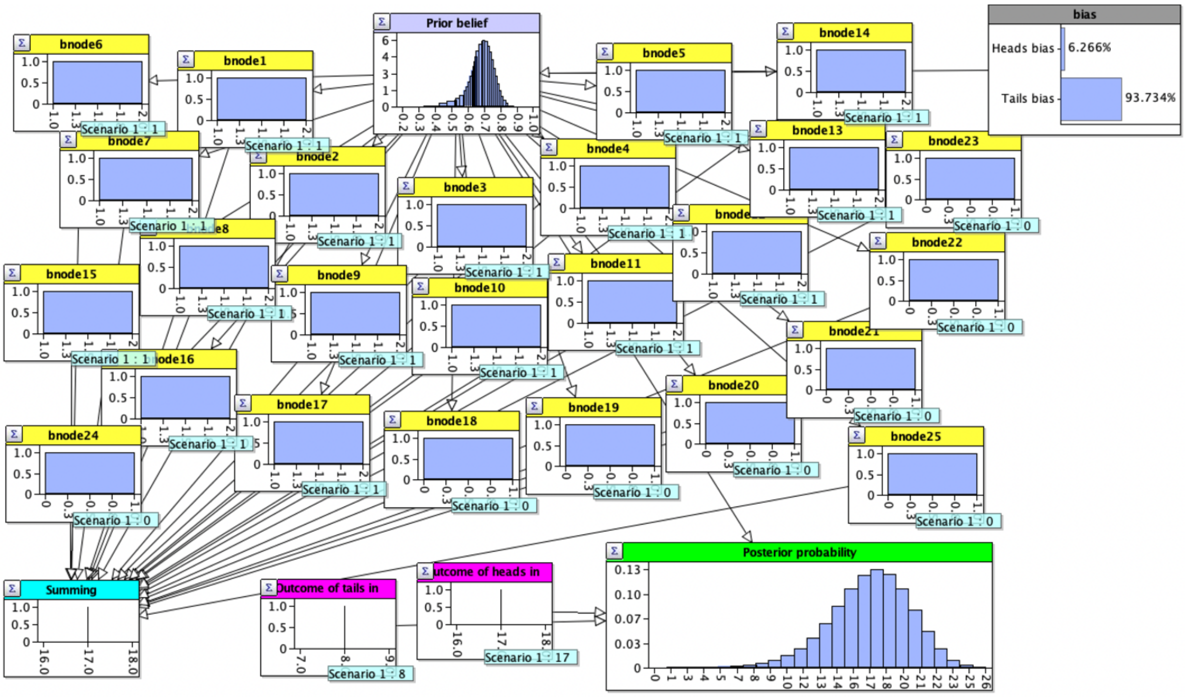




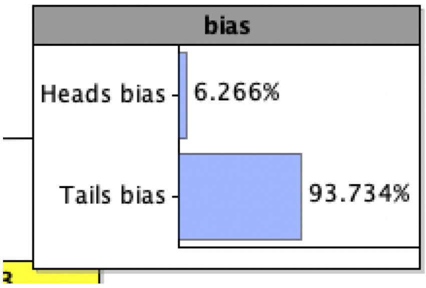


CASE-2, For parameters, biased heads = 10:20 and tails= 20:10

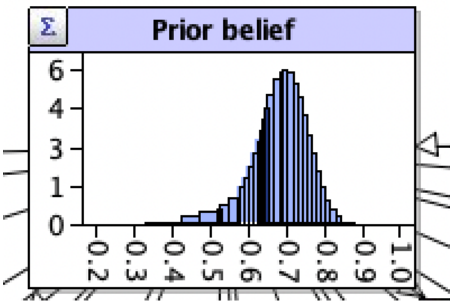
RISK MAP:



Chances of bias over heads or tails:

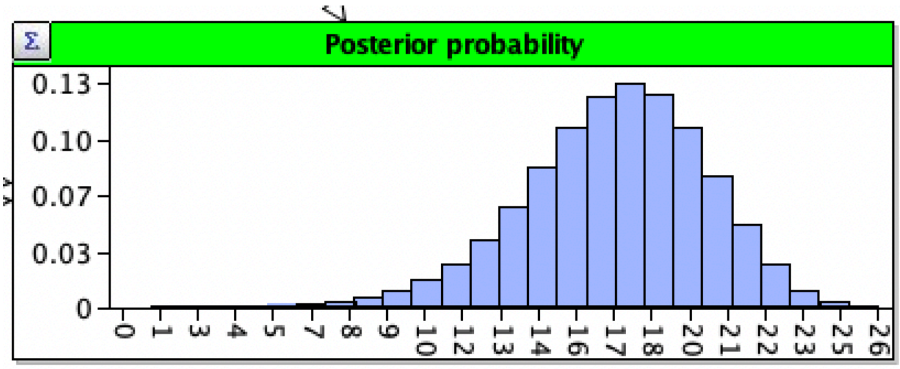


Prior distribution:

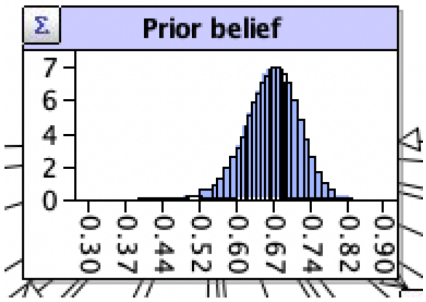


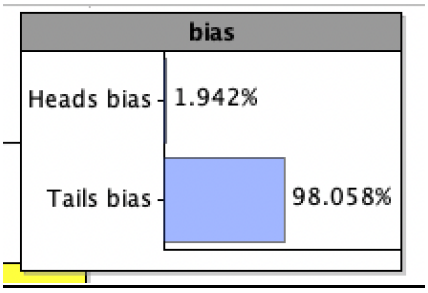
bias	Heads bias	Tails bias
Expressions	Beta(10.0,20....	Beta(20.0,10....

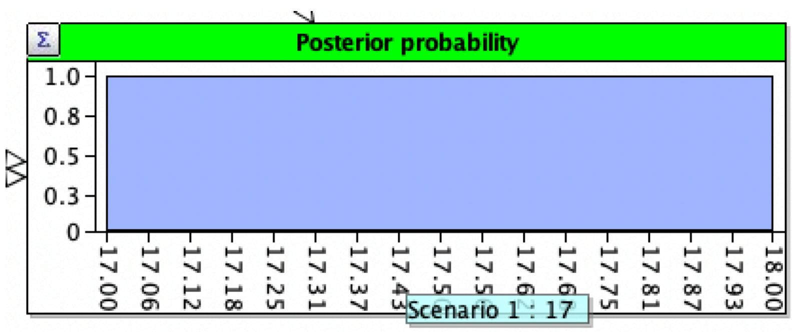
Posterior probability:



Posterior probability with scenario=17:







Posterior probability with scenario=8: