Chapter 2 Convex sets

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upcoming concepts

affine combination	convex combination	conic combination
affine set	convex set	convex cone
affine hull	convex hull	conic hull

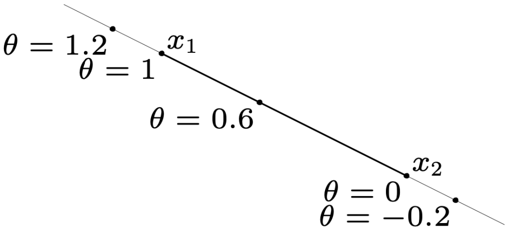
Affine set

affine combination of $x_1, \cdots, x_k \in \mathbb{R}^n$: points of the form

$$\theta_1 x_1 + \cdots + \theta_k x_k, \quad \text{where} \quad \theta_1 + \cdots + \theta_k = 1$$

line through x_1 and x_2: the set of all affine combinations of x_1 and x_2

$$\{x = \theta x_1 + (1-\theta) x_2 \mid \theta \in \mathbb{R}\}$$



affine set: $C \subseteq \mathbb{R}^n$ is affine if it contains the line through any pair of points in C

example

- the solution set of linear equations $\{x \mid Ax = b\}$ is an affine set
- conversely, every affine set can be expressed as the solution set of a system of linear equations

affine hull of $C \subseteq \mathbb{R}^n$: the set of all affine combinations of points in C

$$\textbf{aff}\, C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \cdots, x_k \in C, \theta_1 + \cdots + \theta_k = 1\}$$

facts

- the affine hull of C is the smallest affine set containing C
- if C is an affine set, then $\textbf{aff}\, C = C$

Convex set

convex combination of $x_1, \cdots, x_k \in \mathbb{R}^n$: points of the form

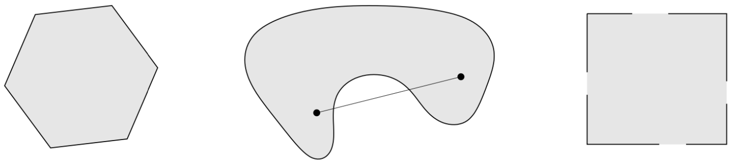
$$\theta_1 x_1 + \cdots + \theta_k x_k, \quad \text{where} \quad \theta_1, \cdots, \theta_k \geq 0 \quad \text{and} \quad \theta_1 + \cdots + \theta_k = 1$$

line segment between x_1 and x_2: the set of all convex combinations of x_1 and x_2

$$\{x = \theta x_1 + (1 - \theta) x_2 \mid 0 \leq \theta \leq 1\}$$

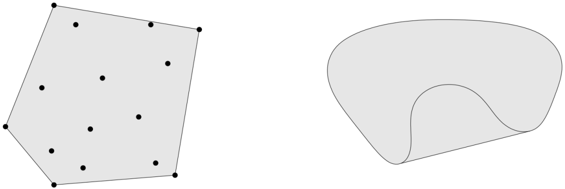
convex set: $C \subseteq \mathbb{R}^n$ is convex if contains line segment between any pair of points in C

examples (one convex, two nonconvex)



convex hull of $C \subseteq \mathbb{R}^n$: the set of all convex combinations of points in C

$$\textbf{conv}\, C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \cdots, x_k \in C; \theta_1, \cdots, \theta_k \geq 0; \theta_1 + \cdots + \theta_k = 1\}$$



facts

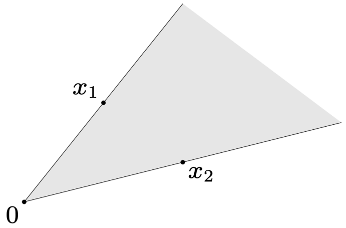
- the convex hull of C is the smallest convex set containing C
- if C is a convex set, then $\textbf{conv}\, C = C$

Convex cone

cone: $C \subseteq \mathbb{R}^n$ is a cone if $\theta x \in C$ for every $x \in C$ and $\theta \geq 0$.

conic combination of $x_1, \cdots, x_k \in \mathbb{R}^n$: points of the form

$$\theta_1 x_1 + \cdots + \theta_k x_k, \quad \text{where} \quad \theta_1, \cdots, \theta_k \geq 0$$



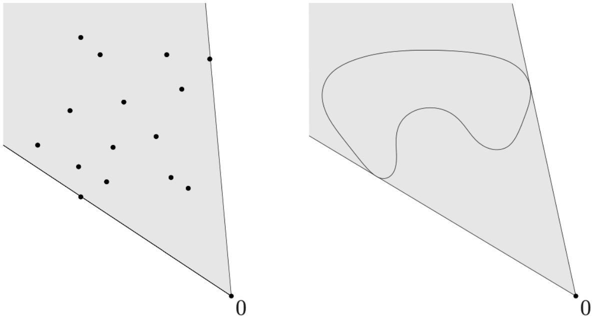
convex cone: $C \subseteq \mathbb{R}^n$ is a convex cone if it is convex and a cone

fact

C is a convex cone $\iff$ C contains all conic combinations of points in itself

conic hull of $C \subseteq \mathbb{R}^n$: the set of all conic combinations of points in C

$$\{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \cdots, x_k \in C; \theta_1, \cdots, \theta_k \geq 0\}$$



facts

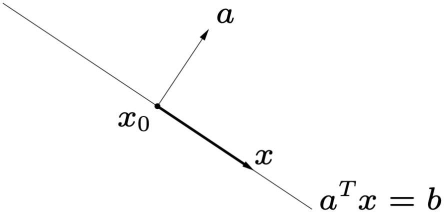
- the conic hull of C is the smallest convex cone containing C
- if C is a convex cone, then its conic hull is itself

a huge wave of zombies is approaching

- hyperplanes
- halfspaces
- Euclidean balls
- ellipsoids
- second-order cone (Lorentz cone)
- norm balls
- norm cones
- polyhedra
- positive semidefinite cone

Hyperplanes

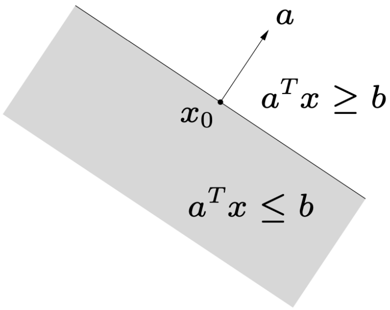
hyperplane: set of the form $\{x \mid a^T x = b\}$ ($a \in \mathbb{R}^n, a \neq 0, b \in \mathbb{R}$)



fact: hyperplanes are affine and convex

Halfspaces

halfspace: set of the form $\{x \mid a^T x \leq b\}$ ($a \in \mathbb{R}^n, a \neq 0, b \in \mathbb{R}$)



fact: halfspaces are convex

Euclidean balls

Euclidean ball with center x_c and radius r: two equivalent representations

- set of the form
$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\}$$
- set of the form
$$B(x_c, r) = \{x_c + ru \mid \|u\|_2 \leq 1\}$$

fact: Euclidean balls are convex

Ellipsoids

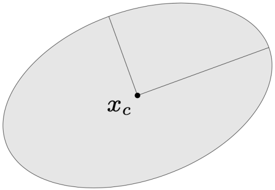
ellipsoid: two equivalent representations

- set of the form

$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\} \quad \text{with} \quad P \in \mathbb{S}^n_{++}$$

- set of the form

$$\{x_c + Au \mid \|u\|_2 \le 1\} \qquad \text{with } A \text{ square and nonsingular}$$



fact: ellipsoids are convex

Norm balls

norm: a function $\|\cdot\|$ satisfying

- $\|x\| \geq 0$, equality holds iff $x = 0$;
- $\|tx\| = |t| \cdot \|x\|$ for $t \in \mathbb{R}$
- $\|x + y\| \leq \|x\| + \|y\|$

norm ball with center x_c and radius r: set of the form

$$\{x \mid \|x - x_c\| \leq r\}$$

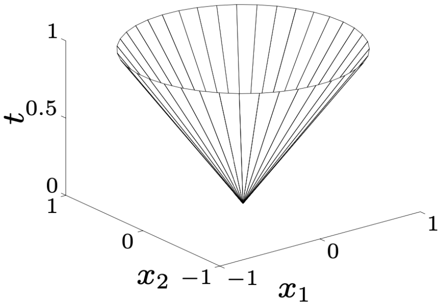
fact: norm balls are convex

Norm cones

norm cone: set of the form

$$\{(x, t) \mid \|x\| \leq t\}$$

Euclidean norm cone is also called second-order cone



fact: norm cones are convex cones

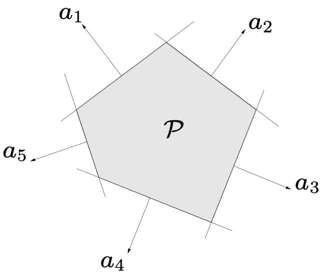
Polyhedra

polyhedron: solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \quad Cx = d$$

where $A \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{p \times n}$, $\preceq$ is componentwise inequality

i.e. polyhedra are intersections of finite number of halfspaces and hyperplanes;



fact: polyhedra are convex

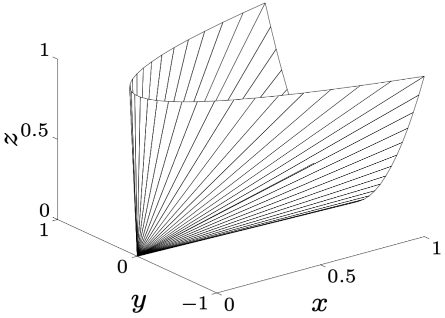
Positive semidefinite cone

- $\mathbb{S}^n$: set of symmetric $n \times n$ matrices
- $\mathbb{S}^n_+ = \{X \in \mathbb{S}^n \mid X \succeq 0\}$: set of symmetric positive semidefinite $n \times n$ matrices

$$X \in \mathbb{S}^n_+ \qquad \Longleftrightarrow \qquad z^T X z \geq 0 \text{ for all } z \in \mathbb{R}^n$$

- $\mathbb{S}^n_{++} = \{X \in \mathbb{S}^n \mid X \succ 0\}$: set of symmetric positive definite $n \times n$ matrices

fact: positive semidefinite cone $\mathbb{S}^n_+$ is a convex cone



Establishing convexity

practical methods for establishing convexity of a set C

1. apply definition

$$x_1, x_2 \in C, 0 \leq \theta \leq 1 \qquad \Longrightarrow \qquad \theta x_1 + (1-\theta)x_2 \in C$$

2. reconstruct C from known convex sets by operations preserving convexity:

   - intersection
   - affine functions
   - perspective function
   - linear-fractional functions

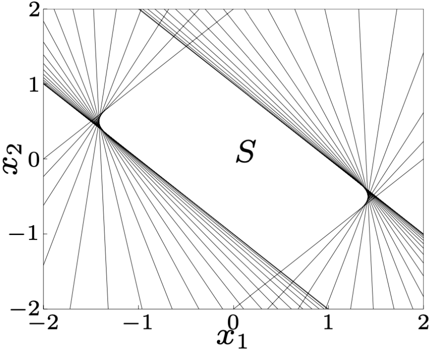
Intersection

the intersection of (any number of) convex sets is convex

example

$$S = \{x \in \mathbb{R}^m \mid |p_x(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

where $p_x(t) = x_1 \cos t + \cdots + x_m \cos mt$



Affine function

affine function $f : \mathbb{R}^n \to \mathbb{R}^m$ is of the form

$$f(x) = Ax + b \qquad \text{with } A \in \mathbb{R}^{m \times n} \text{ and } b \in \mathbb{R}^m$$

- the image of a convex set under f is convex

$$S \subseteq \mathbb{R}^n \text{ convex} \qquad \Longrightarrow \qquad f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbb{R}^m \text{ convex} \qquad \Longrightarrow \qquad f^{-1}(C) = \{x \in \mathbb{R}^n \mid f(x) \in C\} \text{ convex}$$

examples

- scaling and translation: if $S \subseteq \mathbb{R}^n$ is convex, $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}^n$, then

  $$\alpha S = \{\alpha x \mid x \in S\} \qquad \text{and} \qquad S + a = \{x + a \mid x \in S\}$$

  are convex
- projection: if $S \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is convex, then

  $$T = \{x_1 \in \mathbb{R}^m \mid (x_1, x_2) \in S \text{ for some } x_2 \in \mathbb{R}^n\}$$

  is convex

- solution set of linear matrix inequality

$$\{x \in \mathbb{R}^n \mid x_1 A_1 + \cdots + x_n A_n \preceq B\}$$

where $A_i, B \in \mathbb{S}^m$, is convex

proof

inverse image of the positive semidefinite cone under the affine function

$$f : \mathbb{R}^n \to \mathbb{S}^m, \qquad f(x) = B - (x_1 A_1 + \cdots + x_n A_n)$$

- hyperbolic cone

$$\left\{x \in \mathbb{R}^n \;\middle|\; x^T P x \leq \left(c^T x\right)^2, c^T x \geq 0\right\}$$

where $P \in \mathbb{S}_+^n$ and $c \in \mathbb{R}^n$, is convex

proof
inverse image of the second-order cone

$$\{(z,t) \mid z^T z \leq t^2, t \geq 0\}$$

under the affine function $f: \mathbb{R}^n \to \mathbb{R}^{n+1}$ given by $f(x) = (P^{1/2}x, c^T x)$

Perspective function

perspective function $P\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ given by

$$P(x,t) = x/t, \qquad \mathbf{dom}\, P = \mathbb{R}^n \times \mathbb{R}_{++} = \{(x,t) \mid t > 0\}$$

- images of convex sets under perspective function are convex
- inverse images of convex sets under perspective function are convex

Linear-fractional functions

linear-fractional function $f : \mathbb{R}^n \to \mathbb{R}^m$ given by

$$f(x) = \frac{Ax + b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

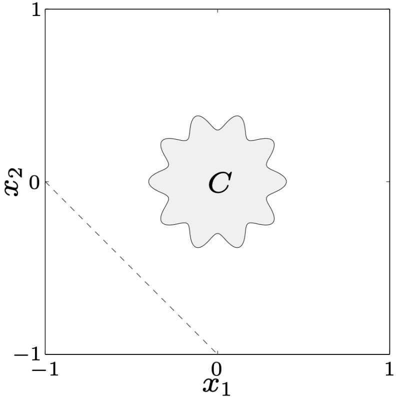
it is the composition of an affine function g and the perspective function P, where

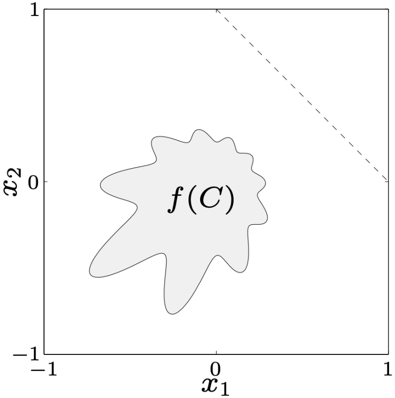
$$g(x) = \begin{bmatrix} A \\ c^T \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix}$$

- images of convex sets under linear-fractional functions are convex
- inverse images of convex sets under linear-fractional functions are convex

example

$$f(x) = \frac{x}{x_1 + x_2 + 1}, \qquad \textbf{dom}\, f = \{(x_1, x_2) \mid x_1 + x_2 + 1 > 0\}$$





Proper cones

proper cone: a cone $K \subseteq \mathbb{R}^n$ satisfying

- K is convex
- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line, or equivalently, $\pm x \in K \implies x = 0$)

examples

- nonnegative orthant $K = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \cdots, n\}$
- positive semidefinite cone $K = \mathbb{S}^n_+$
- nonnegative polynomials on $[0, 1]$

$$K = \{c \in \mathbb{R}^n \mid c_1 + c_2 t + \cdots + c_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$$

Generalized inequalities

generalized inequality on $\mathbb{R}^n$ defined by a proper cone $K \subseteq \mathbb{R}^n$

$$x \preceq_K y \quad \Longleftrightarrow \quad y - x \in K$$
$$x \prec_K y \quad \Longleftrightarrow \quad y - x \in \textbf{int}\, K$$

examples (same for $\prec$, $\succeq$, $\succ$)

- componentwise inequality ($K = \mathbb{R}^n_+$)

$$x \preceq_{\mathbb{R}^n_+} y \quad \Longleftrightarrow \quad x_i \le y_i, \quad i = 1, \cdots, n$$

- symmetric matrix inequality ($K = \mathbb{S}^n_+$)

$$X \preceq_{\mathbb{S}^n_+} Y \quad \Longleftrightarrow \quad Y - X \text{ positive semidefinite}$$

these two types are so common that we drop the subscript in $\preceq_K$

properties (same for $\prec$, $\succeq$, $\succ$)

- many properties of $\preceq_K$ are similar to $\leq$ on $\mathbb{R}$, e.g.

$$x \preceq_K y, \quad u \preceq_K v \qquad \Longrightarrow \qquad x + u \preceq_K y + v$$

- not always a linear ordering, namely, it could happen that $x \not\preceq_K y$ and $y \not\preceq_K x$

Minimum and minimal elements

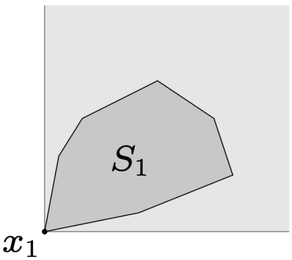
$x \in S$ is the **minimum element** of S with respect to $\preceq_K$ if

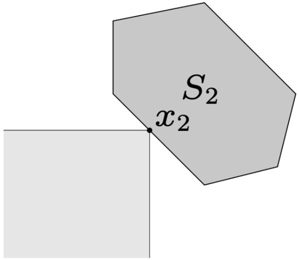
$$y \in S \qquad \Longrightarrow \qquad x \preceq_K y$$

$x \in S$ is the **minimal element** of S with respect to $\preceq_K$ if

$$y \in S, \quad y \preceq_K x \qquad \Longrightarrow \qquad y = x$$

example for $K = \mathbb{R}^2_+$



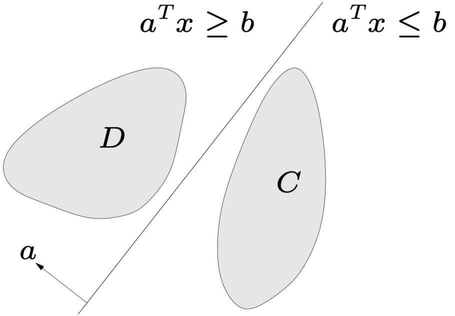


x_1 is the minimum element of S_1 x_2 is a minimal element of S_2

Separating hyperplane theorem

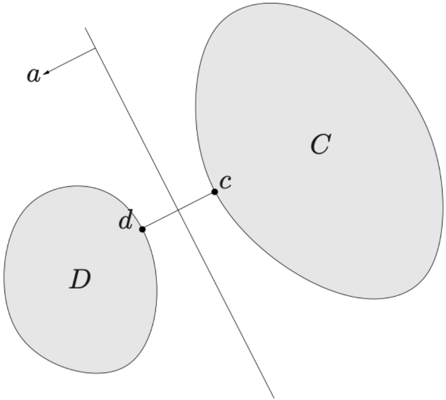
if C and D are nonempty disjoint convex sets, then there exist $a \neq 0$ and b such that

$$a^T x \leq b \quad \text{for} \quad x \in C, \qquad a^T x \geq b \quad \text{for} \quad x \in D$$



the hyperplane $\{x \mid a^T x = b\}$ is called a **separating hyperplane**

proof of separating hyperplane theorem



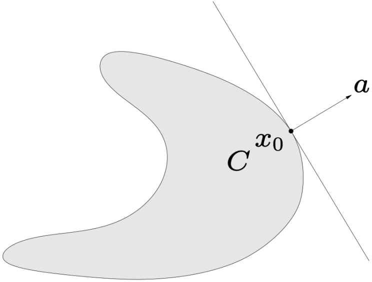
- strict separation requires additional assumptions (e.g. point and closed convex set)
- converse separating theorem requires additional assumptions (e.g. one set is open)

Supporting hyperplane theorem

supporting hyperplane to a set C at a boundary point x_0 is a hyperplane

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$, such that $a^T x \leqslant a^T x_0$ for all $x \in C$



if C is convex, then supporting hyperplane exists at every boundary point of C

Dual cones

dual cone of a cone K

$$K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$$

examples

$$\begin{aligned}
K = \mathbb{R}^n_+ \quad &\Longrightarrow \quad K^* = \mathbb{R}^n_+ \\
K = \mathbb{S}^n_+ \quad &\Longrightarrow \quad K^* = \mathbb{S}^n_+ \\
K = \{(x,t) \mid \|x\|_2 \leq t\} \quad &\Longrightarrow \quad K^* = \{(x,t) \mid \|x\|_2 \leq t\} \\
K = \{(x,t) \mid \|x\|_1 \leq t\} \quad &\Longrightarrow \quad K^* = \{(x,t) \mid \|x\|_\infty \leq t\}
\end{aligned}$$

first three examples are **self-dual** cones

Dual generalized inequalities

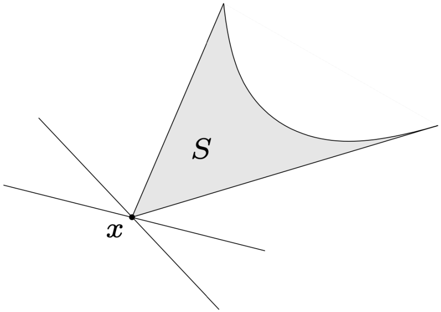
assume K is a proper cone, then K^* is also a proper cone

hence K^* also defines generalized inequalities

$$y \succeq_{K^*} 0 \qquad \Longleftrightarrow \qquad y^T x \geq 0 \text{ for all } x \succeq_K 0$$

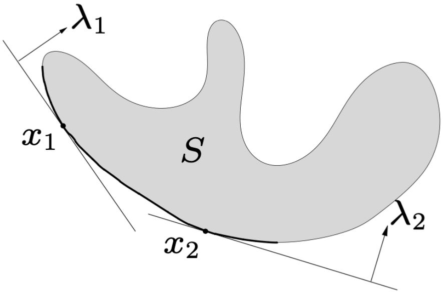
Dual characterization of minimum element

x is the minimum element of S with respect to $\preceq_K$ $\iff$ x is the unique minimizer of $\lambda^T z$ over S for each $\lambda \succ_{K^*} 0$



Dual characterization of minimal element

x is a minimal element of S with respect to $\preceq_K$ $\Longleftarrow$ x minimizes $\lambda^T z$ over S for some $\lambda \succ_{K^*} 0$



x is a minimal element of a convex set S with respect to $\preceq_K$ $\Longrightarrow$ x minimizes $\lambda^T z$ over S for some nonzero $\lambda \succeq_{K^*} 0$

Optimal production frontier

- different production methods use different amounts of resources $x \in \mathbb{R}^n$
- production set P: resource vectors x for all possible production methods
- efficient (Pareto optimal) methods correspond to resource vectors x that are minimal with respect to $\mathbb{R}^n_+$

example for $n = 2$: x_1, x_2, x_3 are efficient; x_4, x_5 are not