Chapter 11 Interior-point methods

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inequality constrained minimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \qquad i = 1, \ldots, m \\ & Ax = b \end{array}$$

general assumptions

- f_i convex and twice continuously differentiable
- $A \in \mathbb{R}^{p \times n}$ and $\mathbf{rank}\, A = p$
- p^* is finite and attained
- problem is strictly feasible: there exists $\tilde{x}$ with

$$\tilde{x} \in \mathbf{dom}\, f_0, \qquad f_i(\tilde{x}) < 0, \qquad i = 1, \ldots, m, \qquad A\tilde{x} = b$$

hence strong duality holds and dual optimum is attained

examples

- LP, QP, QCQP, GP
- entropy maximization with linear inequality constraints ($\mathcal{D} = \mathbf{R}^n_{++}$)

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n x_i \log x_i \\ \text{subject to} & Fx \preceq g \\ & Ax = b \end{array}$$

- differentiability may require reformulating the problem, e.g. piecewise-linear minimization or ℓ_∞-norm approximation via LP
- SDPs and SOCPs are better handled as problems with generalized inequalities

reformulation via indicator function

$$\begin{array}{ll} \text{minimize} & f_0(x) + \sum_{i=1}^{m} I_-(f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

where I_- is the indicator function of $\mathbb{R}_-$

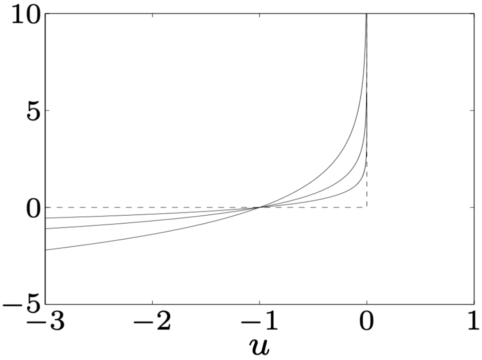
$$I_-(u) = \begin{cases} 0, & \text{if } u \leq 0 \\ \infty, & \text{if } u > 0 \end{cases}$$

Logarithmic barrier

approximation via logarithmic barrier

$$\begin{array}{ll} \text{minimize} & f_0(x) - \frac{1}{t}\sum_{i=1}^{m} \log(-f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

- an equality constrained problem
- for $t > 0$, the term $-(1/t)\log(-u)$ is a smooth approximation of I_-
- approximation improves as $t \to \infty$



- dashed line: function $I_-(u)$
- solid curves: function $-(1/t)\log(-u)$ for $t = 0.5, 1, 2$
- $t = 2$ gives the best approximation

logarithmic barrier function

$$\phi(x) = -\sum_{i=1}^{m} \log(-f_i(x)), \qquad \mathbf{dom}\,\phi = \{x \mid f_i(x) < 0,\ i = 1, \ldots, m\}$$

- convex function (follows from composition rule)
- twice continuously differentiable (can be easily computed)

$$\nabla \phi(x) = \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x)$$

$$\nabla^2 \phi(x) = \sum_{i=1}^{m} \frac{1}{f_i(x)^2} \nabla f_i(x) \nabla f_i(x)^T + \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla^2 f_i(x)$$

Central path

centering problem

$$\begin{array}{ll} \text{minimize} & tf_0(x) + \phi(x) \\ \text{subject to} & Ax = b \end{array}$$

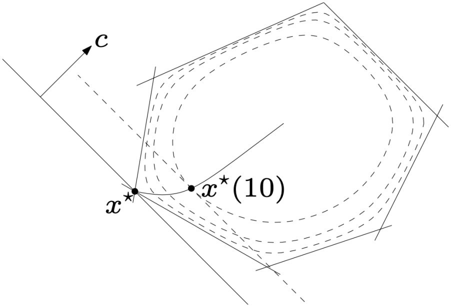
- assume it has a unique solution $x^*(t)$ for each $t > 0$
- the curve $\{x^*(t) \mid t > 0\}$ is called the central path
- there exists some w such that $(x = x^*(t), \nu = w)$ satisfies

$$t\nabla f_0(x) + \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x) + A^T \nu = 0, \qquad Ax = b$$

example central path for an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, 6 \end{array}$$

the hyperplane $c^T x = c^T x^*(t)$ is tangent to the level curve of ϕ through $x^*(t)$



dual points from central path

- by the optimality condition, $x^*(t)$ minimizes the Lagrangian

$$L(x, \lambda^*(t), \nu^*(t)) = f_0(x) + \sum_{i=1}^{m} \lambda_i^*(t) f_i(x) + \nu^*(t)^T (Ax - b)$$

  where we define $\lambda_i^*(t) = 1/\left(-t f_i(x^*(t))\right)$ and $\nu^*(t) = w/t$

- the duality gap for the original problem associated to these values

$$g(\lambda^*(t), \nu^*(t)) = L(x^*(t), \lambda^*(t), \nu^*(t)) = f_0(x^*(t)) - m/t$$

  as a consequence

$$f_0(x^*(t)) - p^* \leq m/t$$

  which confirms the intuitive idea that $f_0(x^*(t)) \to p^*$ as $t \to \infty$

interpretation via KKT conditions

$x = x^*(t)$, $\lambda = \lambda^*(t)$, $\nu = \nu^*(t)$ satisfy

1. primal constraints $\quad f_i(x) \leq 0, \quad i = 1, \dots, m, \quad Ax = b$
2. dual constraints $\quad \lambda \succeq 0$
3. approximate complementary slackness $\quad -\lambda_i f_i(x) = 1/t, \quad i = 1, \dots, m$
4. gradient of Lagrangian with respect to x vanishes

$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + A^T \nu = 0$$

the only difference with KKT is that condition 3 replaces $\lambda_i f_i(x) = 0$

centering problem without equality constraints

$$\text{minimize} \qquad t f_0(x) - \sum_{i=1}^{m} \log(-f_i(x))$$

force field interpretation

- $t f_0(x)$ is potential of force field $F_0(x) = -t\nabla f_0(x)$
- $-\log(-f_i(x))$ is potential of force field $F_i(x) = (1/f_i(x))\,\nabla f_i(x)$

the forces balance at $x^*(t)$

$$F_0(x^*(t)) + \sum_{i=1}^{m} F_i(x^*(t)) = 0$$

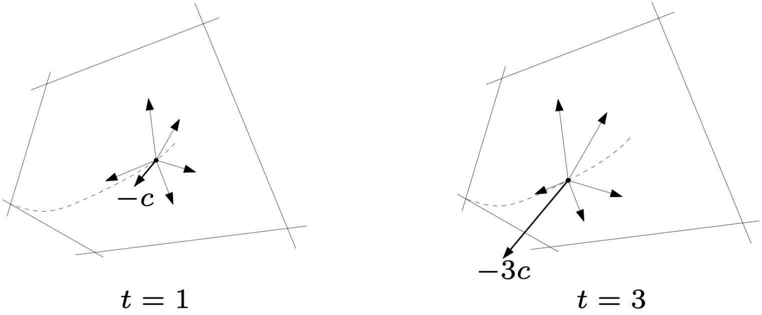
example

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \qquad i = 1, \ldots, m \end{array}$$

- objective force field is constant $F_0(x) = -tc$
- constraint force decays as inverse distance to constraint hyperplane

$$F_i(x) = \frac{-a_i}{b_i - a_i^T x}, \qquad \|F_i(x)\|_2 = \frac{1}{\mathbf{dist}(x, \mathcal{H}_i)}$$

where $\mathcal{H}_i = \{x \mid a_i^T x = b_i\}$



- a small LP example with $n = 2$ and $m = 5$
- the equilibrium position of the particle traces out the central path
- larger value of objective force moves the particle closer to the optimal point

Barrier method

given strictly feasible x, $t := t^{(0)} > 0$, $\mu > 1$, tolerance $\epsilon > 0$

repeat

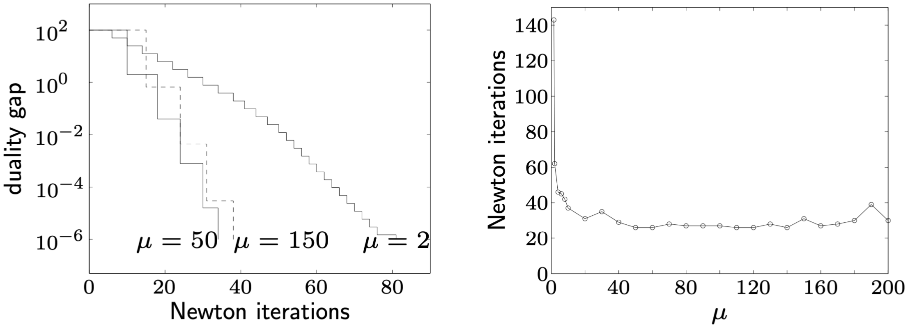
1. *Centering step.* Compute $x^*(t)$ by minimizing $tf_0 + \phi$ subject to $Ax = b$
2. *Update.* $x := x^*(t)$
3. *Stopping criterion.* **quit** if $m/t < \epsilon$
4. *Increase* t. $t := \mu t$

remarks

- terminates with $f_0(x) - p^* \le \epsilon$
- centering usually done using Newton's method, starting at current x
- choice of μ involves a trade-off: larger μ means fewer outer (centering) iterations and more inner (Newton) iterations; typical values $10 \le \mu \le 20$
- several heuristics for choice of $t^{(0)}$

Examples

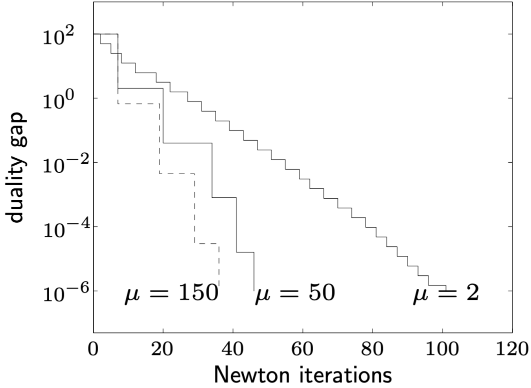
inequality form LP $(m = 100$ inequalities, $n = 50$ variables$)$



- starts with x on central path ($t^{(0)} = 1$, duality gap 100)
- terminates when $t = 10^8$ (gap 10^{-6})
- centering uses Newton's method with backtracking
- total number of Newton iterations not very sensitive for $\mu \geq 10$

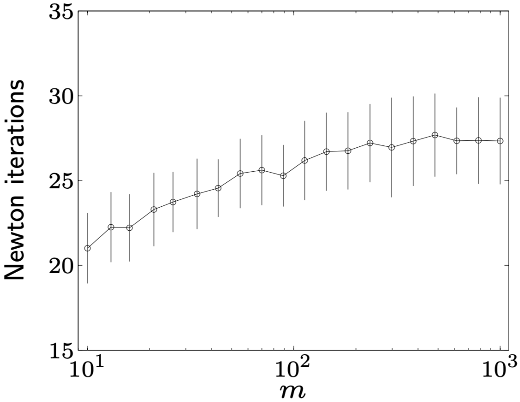
geometric program ($m = 100$ inequalities and $n = 50$ variables)

$$\begin{array}{ll} \text{minimize} & \log\left(\sum_{k=1}^{5} \exp\left(a_{0k}^T x + b_{0k}\right)\right) \\ \text{subject to} & \log\left(\sum_{k=1}^{5} \exp\left(a_{0k}^T x + b_{0k}\right)\right) \leq 0, \qquad i = 1, \ldots, m \end{array}$$



family of standard LPs $\quad (A \in \mathbb{R}^{m \times 2m})$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \succeq 0 \end{array}$$



- solve 100 randomly generated instances for each m between 10 and 1000
- number of iterations grows very slowly as m ranges over a 100 : 1 ratio

Convergence analysis

outer (centering) iterations number is exactly

$$\left\lceil \frac{\log\left(m/\epsilon t^{(0)}\right)}{\log \mu} \right\rceil$$

plus the initial centering step for computing $x^*\left(t^{(0)}\right)$

inner (Newton) iterations

$$\text{minimize} \qquad t f_0(x) + \phi(x)$$

see convergence analysis of Newton's method

- $t f_0 + \phi$ must have closed sublevel sets for $t \geq t^{(0)}$
- classical analysis requires strong convexity and Lipschitz condition
- analysis via self-concordance requires self-concordance of $t f_0 + \phi$

feasibility problem find x such that

$$\begin{array}{ll} f_i(x) \leq 0, & i = 1, \ldots, m \\ Ax = b \end{array}$$

phase I computes strictly feasible starting point for barrier method

Basic phase I method

basic phase I method (with optimal value $\bar{p}^*$)

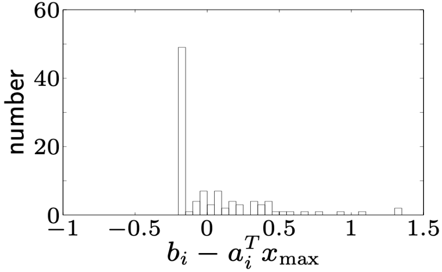
$$\begin{array}{ll} \text{minimize} & s \\ \text{subject to} & f_i(x) \leq s, \quad i = 1, \dots, m \\ & Ax = b \end{array}$$

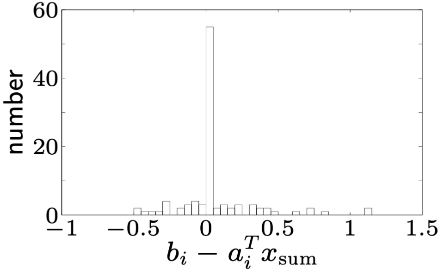
- if (x, s) feasible with $s < 0$, then x is strictly feasible for feasibility problem
- if $\bar{p}^* > 0$, then feasibility problem is infeasible
- if $\bar{p}^* = 0$ and not attained, then feasibility problem is infeasible
- if $\bar{p}^* = 0$ and attained, then feasibility problem is feasible, but not strictly

sum of infeasibilities phase I method

$$
\begin{array}{ll}
\text{minimize} & \mathbf{1}^T s \\
\text{subject to} & f_i(x) \leq s_i, \qquad i = 1, \ldots, m \\
& Ax = b \\
& s \succeq 0
\end{array}
$$

comparison of methods infeasible set of 100 linear inequalities in 50 variables





- left: basic phase I solution; satisfies 39 inequalities
- right: sum of infeasibilities phase I solution; satisfies 79 inequalities
- for infeasible problems, second method produces a solution that satisfies many more inequalities than first method

Example

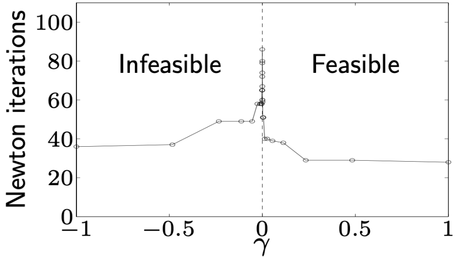
family of linear feasibility problems

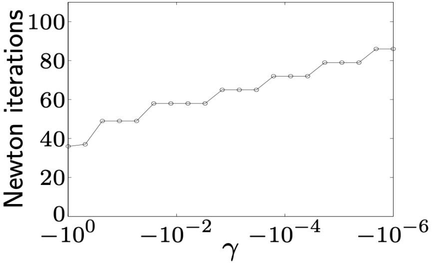
$$Ax \preceq b + \gamma \Delta b$$

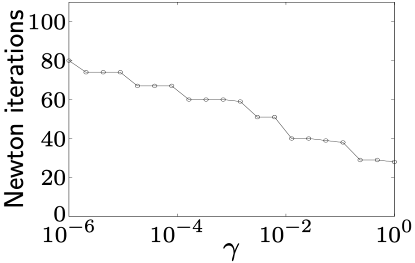
- data chosen to be strictly feasible for $\gamma > 0$, infeasible for $\gamma < 0$, feasible but not strictly feasible for $\gamma = 0$
- use basic phase I method, terminate when $s < 0$ (find a strictly feasible point) or when dual objective > 0 (produce a certificate of infeasibility)

conclusion

- cost of solving a convex feasibility problem using barrier method is modest when the problem is not close to the boundary between feasibility and infeasibility
- cost grows when the problem is very close to the boundary
- cost becomes infinite when the problem is exactly on the boundary







number of iterations roughly proportional to $\log(1/|\gamma|)$

Self-concordance assumptions

same general assumptions in this chapter plus

- sublevel sets (of f_0 on the feasible set) are bounded
- $tf_0 + \phi$ is self-concordant with closed sublevel sets for all $t \geq t^{(0)}$

the second condition above

- holds for LP, QP, QCQP
- may require reformulating the problem, e.g.

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n x_i \log x_i \\ \text{subject to} & Fx \preceq g \end{array} \quad \Longrightarrow \quad \begin{array}{ll} \text{minimize} & \sum_{i=1}^n x_i \log x_i \\ \text{subject to} & Fx \preceq g, \quad x \succeq 0 \end{array}$$

- needed for complexity analysis; barrier method works even when self-concordance assumption does not apply

Newton iterations per centering step

general result for closed strictly convex self-concordant function f

$$\#\ \text{Newton iterations} \leq \frac{f(x) - p^\star}{\gamma} + c$$

where γ and c are constants depending only on Newton algorithm parameters

barrier method effort of computing $x^+ = x^\star(\mu t)$ starting at $x = x^\star(t)$

$$\#\ \text{Newton iterations} \leq \frac{\mu t f_0(x) + \phi(x) - \mu t f_0(x^+) - \phi(x^+)}{\gamma} + c$$

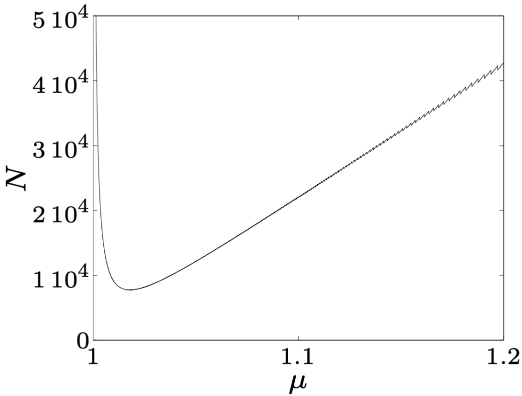
deriving an upper bound with $\lambda = \lambda^*(t)$ and $\nu = \nu^*(t)$

$$
\begin{aligned}
& \mu t f_0(x) + \phi(x) - \mu t f_0(x^+) - \phi(x^+) \\
&= \mu t f_0(x) - \mu t f_0(x^+) + \sum_{i=1}^{m} \log(-\mu t \lambda_i f_i(x^+)) - m \log \mu \\
&\leq \mu t f_0(x) - \mu t f_0(x^+) - \mu \sum_{i=1}^{m} \lambda f_i(x^+) - m - m \log \mu \\
&\leq \mu t f_0(x) - \mu t g(\lambda, \nu) - m - m \log \mu \\
&= m(\mu - 1 - \log \mu)
\end{aligned}
$$

Total number of Newton iterations

total number of Newton steps in barrier method excluding initial centering step

$$\#\text{ Newton iterations } \leq N = \left\lceil \frac{\log\left(m/\epsilon t^{(0)}\right)}{\log \mu} \right\rceil \left(\frac{m(\mu - 1 - \log \mu)}{\gamma} + c \right)$$



- figure shows N for typical values of γ, c, $m = 100$, $m/\epsilon t^{(0)} = 10^5$
- confirms trade-off in choice of μ
- in practice, number of iterations is in the tens; not very sensitive for $\mu \geq 10$

polynomial-time complexity of barrier method

- we choose $\mu = 1 + 1/\sqrt{m}$, which approximately optimizes worst-case complexity
- for such μ simple calculation shows $N = O\left(\sqrt{m}\log\left(m/\epsilon t^{(0)}\right)\right)$
- number of Newton iterations for fixed gap reduction is $O(\sqrt{m})$
- multiply with cost of one Newton iteration (a polynomial function of problem dimensions) to get bound on number of flops
- in practice we choose μ fixed (between 10 and 20)

minimization with generalized inequalities

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \preceq_{K_i} 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

assumptions

- f_0 convex function
- $f_i : \mathbb{R}^n \to \mathbb{R}^{k_i}$ convex with respect to proper cones $K_i \subset \mathbb{R}^{k_i}$ for $i = 1, \ldots, m$
- all f_i twice continuously differentiable
- $A \in \mathbb{R}^{p \times n}$ with $\mathbf{rank}\, A = p$
- $p^\star$ is finite and attained
- problem is strictly feasible, hence strong duality holds and dual optimum is attained

examples of greatest interest SOCP, SDP

Logarithmic barrier and central path

generalized logarithm for a proper cone

function $\psi\colon \mathbb{R}^q \to \mathbb{R}$ is a generalized logarithm for a proper cone $K \subseteq \mathbb{R}^q$ if

1. $\mathbf{dom}\,\psi = \mathbf{int}\,K$
2. ψ is concave, closed, twice continuously differentiable
3. $\nabla^2\psi(y) \prec 0$ for $y \succ_K 0$
4. there exists a constant $\theta > 0$ (degree of ψ) such that for $y \succ_K 0$ and $s > 0$

$$\psi(sy) = \psi(y) + \theta \log s$$

properties $\quad \nabla\psi(y) \succeq_{K^*} 0 \quad$ and $\quad y^T\nabla\psi(y) = \theta \quad$ for any $\quad y \succ_K 0$

examples

- nonnegative orthant $K = \mathbb{R}^n_+$

$$\psi(y) = \sum_{i=1}^{n} \log y_i, \qquad (\theta = n)$$

$$\nabla\psi(y) = (1/y_1, \ldots, 1/y_n), \qquad y^T\nabla\psi(y) = n$$

- positive semidefinite cone $K = \mathbb{S}^n_+$

$$\psi(Y) = \log\det Y, \qquad (\theta = n)$$

$$\nabla\psi(Y) = Y^{-1}, \qquad \mathbf{tr}(Y\nabla\psi(Y)) = n$$

- second-order cone $K = \{y \in \mathbb{R}^{n+1} \mid (y_1^2 + \cdots + y_n^2)^{1/2} \le y_{n+1}\}$

$$\psi(y) = \log\left(y_{n+1}^2 - y_1^2 - \cdots - y_n^2\right), \qquad (\theta = 2)$$

$$\nabla\psi(y) = \frac{2}{y_{n+1}^2 - y_1^2 - \cdots - y_n^2} \begin{bmatrix} -y_1 \\ \vdots \\ -y_n \\ y_{n+1} \end{bmatrix}, \qquad y^T \nabla\psi(y) = 2$$

logarithmic barrier function for $f_1(x) \preceq_{K_1} 0, \; \ldots, \; f_m(x) \preceq_{K_m} 0$

$$\phi(x) = -\sum_{i=1}^{m} \psi_i(-f_i(x)),$$
$$\mathbf{dom}\,\phi = \{x \mid f_i(x) \prec_{K_i} 0, \; i = 1, \ldots, m\}$$

- ψ_i is generalized logarithm for K_i with degree θ_i
- ϕ is convex and twice continuously differentiable

central path

- $\{x^*(t) \mid t > 0\}$ where $x^*(t)$ solves

$$\begin{array}{ll} \text{minimize} & t f_0(x) + \phi(x) \\ \text{subject to} & Ax = b \end{array}$$

- $x = x^*(t)$ if there exists $w \in \mathbb{R}^p$ such that

$$t\nabla f_0(x) + \sum_{i=1}^{m} Df_i(x)^T \nabla \psi_i(-f_i(x)) + A^T w = 0$$

where $Df_i(x) \in \mathbb{R}^{k_i \times n}$ is derivative (Jacobian) matrix of f_i at x

dual points on central path

- $x^*(t)$ minimizes Lagrangian $L(x, \lambda^*(t), \nu^*(t))$, where

$$\lambda_i^*(t) = \frac{1}{t} \nabla \psi_i(-f_i(x^*(t))), \qquad \nu^*(t) = \frac{w}{t}$$

- $\lambda_i^*(t) \succ_{K_i^*} 0$ from properties of ψ_i, therefore duality gap

$$f_0\left(x^*(t)\right) - g\left(\lambda^*(t), \nu^*(t)\right) = \frac{1}{t} \sum_{i=1}^{m} \theta_i$$

example SDP with $F_i, G \in \mathbb{S}^p$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F(x) = \sum_{i=1}^{n} x_i F_i + G \preceq 0 \end{array}$$

- logarithmic barrier: $\phi(x) = \log\det(-F(x)^{-1})$
- central path: $x^*(t)$ minimizes $tc^T x - \log\det(-F(x))$, hence

$$tc_i - \mathbf{tr}(F_i F(x^*(t))^{-1}) = 0, \qquad i = 1, \ldots, n$$

- dual point on central path: $Z^*(t) = -(1/t)F(x^*(t))^{-1}$ is feasible for

$$\begin{array}{lll} \text{maximize} & \mathbf{tr}(GZ) & \\ \text{subject to} & \mathbf{tr}(F_i Z) + c_i = 0, & i = 1, \ldots, n \\ & Z \succeq 0 & \end{array}$$

- duality gap on central path: $c^T x^*(t) - \mathbf{tr}(GZ^*(t)) = p/t$

Barrier method

given strictly feasible x, $t := t^{(0)} > 0$, $\mu > 1$, tolerance $\epsilon > 0$

repeat

1. *Centering step.* Compute $x^*(t)$ by minimizing $tf_0 + \phi$ subject to $Ax = b$
2. *Update.* $x := x^*(t)$
3. *Stopping criterion.* **quit** if $\left(\sum_i \theta_i\right)/t < \epsilon$
4. *Increase* t. $t := \mu t$

remarks

- only difference is duality gap m/t on central path is replaced by $\sum_i \theta_i/t$
- number of outer iterations

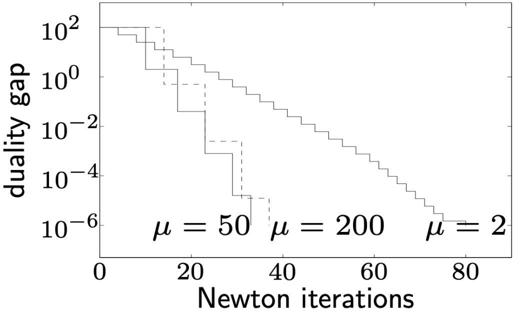
$$\left\lceil \frac{\log\left(\left(\sum_i \theta_i\right)/\left(\epsilon t^{(0)}\right)\right)}{\log \mu} \right\rceil$$

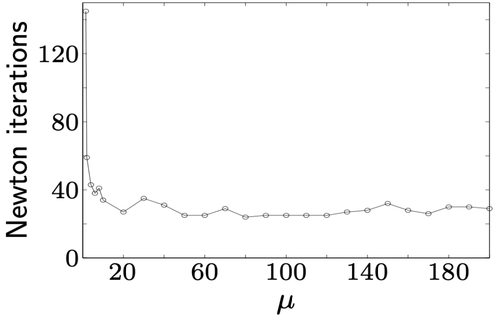
- complexity analysis via self-concordance applies to SDP and SOCP

Examples

SOCP (50 variables, 50 SOC constraints in $\mathbb{R}^6$)

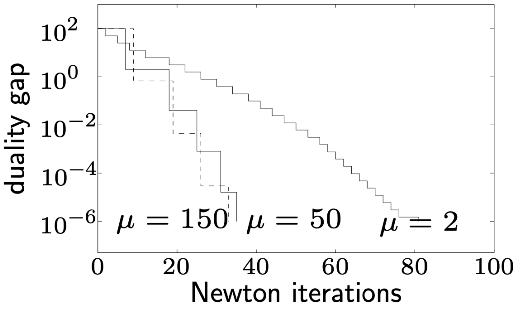
$$
\begin{array}{ll}
\text{minimize} & f^T x \\
\text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \qquad i = 1, \ldots, m
\end{array}
$$

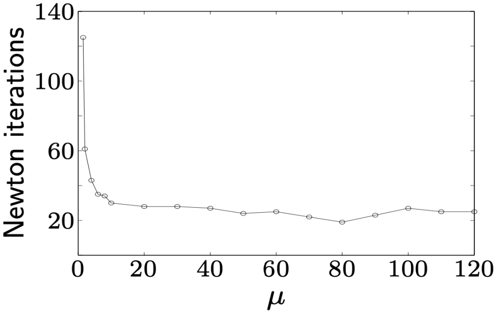




SDP $(100$ variables, LMI constraints in $\mathbb{S}^{100})$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \sum_{i=1}^{n} x_i F_i + G \preceq 0 \end{array}$$

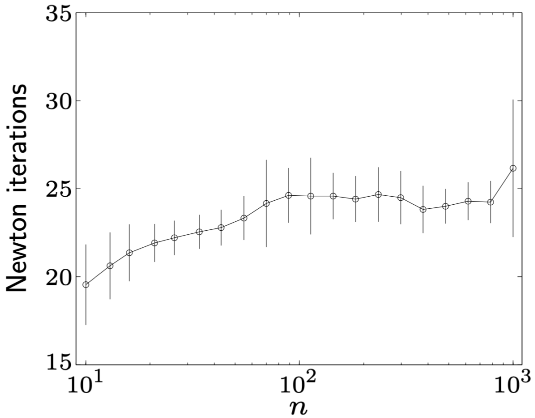




family of SDPs $\quad (A \in \mathbb{S}^n,\ x \in \mathbb{R}^n)$

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T x \\ \text{subject to} & A + \mathbf{diag}(x) \succeq 0 \end{array}$$

solve 100 randomly generated instances for each n between 10 and 1000



Primal-dual interior-point methods

more efficient than barrier method when high accuracy is needed

- update primal and dual variables at each iteration; no distinction between inner and outer iteration
- often exhibit superlinear asymptotic convergence
- search directions can be interpreted as Newton directions for modified KKT conditions
- can start at infeasible points
- cost per iteration same as barrier method