LAIS, Lecture #15

$A \in \mathbb{R}^{n\times n}$: symmetric

$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$

$\lambda_i(A) = +\infty \,,\ i<1$ | $Au_i = \lambda_i(A)u_i$

$\lambda_i(A) = -\infty \,,\ i>n$ | $i \in [n]$

Thm 1 [Courant-Fischer]

1) $\lambda_i(A) = \max\limits_{\dim \mathcal{V}=i} \ \min\limits_{\substack{x\in\mathcal{V}\\ \|x\|_2=1}} x^TAx$

2) $\lambda_i(A) = \min\limits_{\dim \mathcal{V}=n-i+1} \ \max\limits_{\substack{x\in\mathcal{V}\\ \|x\|_2=1}} x^TAx$ HW

Prf First we show that for any $\mathcal{V}$, with $\dim \mathcal{V}=i$,

$$\min_{\substack{x\in\mathcal{V}\\ \|x\|_2=1}} x^TAx \leq \lambda_i(A)$$

So let $\dim \mathcal{V}=i$.

Consider $\mathcal{W} = \mathrm{span}(u_i, \dots, u_n)$

$\forall\, \xi \in \mathcal{W} \,,\ \|\xi\|_2 = 1$

$\xi = \sum_{j=i}^{n} c_j u_j \,,\ \sum c_j^2 = 1$

$$\xi^TA\xi = \sum_{j=i}^{n} \lambda_j(A)\, c_j^2 \leq$$

$$\leq \lambda_i(A)\Big[\sum_{j=i}^{n} c_j^2\Big] = \lambda_i(A)$$

$\dim \mathcal{W} = n-i+1$

$\mathcal{V}\cap\mathcal{W}\neq 0$ because

$\dim\mathcal{V}+\dim\mathcal{W}=n+1$

so $\exists$ unit norm

$\xi\in\mathcal{V}\cap\mathcal{W}$. This

proves that

$$\min_{\substack{x\in\mathcal{V}\\ \|x\|_2=1}} x^TAx\le\lambda_i(A)$$

This proves

$$\max_{\dim\mathcal{V}=i}\ \min_{\substack{x\in\mathcal{V}\\ \|x\|_2=1}} x^TAx\le\lambda_i(A)$$

Let $\mathcal{V}^*=\mathrm{Span}(u_1,\dots,u_i)$

Equality is achieved

for $x=u_i$. $\blacksquare$

<u>Thm 2</u> [Weyl-I]

$A,B\in\mathbb{R}^{n\times n}$ symmetric

$$\lambda_i(A)+\lambda_n(B)\underset{\text{(HW)}}{\le}\lambda_i(A+B)\le\lambda_i(A)+\lambda_1(B)$$

<u>Prf</u> By Courant-Fischer

$$\lambda_i(A+B)=\max_{\dim\mathcal{V}=i}\ \min_{\substack{x\in\mathcal{V}\\ \|x\|_2=1}} x^T(A+B)x$$

$\forall\, x\in\mathbb{R}^n$, $x^T(A+B)x=x^TAx+x^TBx$

$\le x^TAx+\lambda_1(B)$ (α)

So for any $\mathcal{V}$

$$\min_{\substack{x\in\mathcal{V}\\ \|x\|_2=1}} x^T(A+B)x\le\min_{\substack{x\in\mathcal{V}\\ \|x\|_2=1}} x^TAx+\lambda_1(B).$$

$$\max_{\dim \mathcal{V}=i} \min_{\substack{x\in\mathcal{V}\\ \|x\|_2=1}} x^T(A+B)x$$

$$= \min_{\substack{x\in\mathcal{V}^*\\ \|x\|_2=1}} x^T(A+B)x \overset{(\alpha)}{\leq}$$

$$\leq \min_{\substack{x\in\mathcal{V}^*\\ \|x\|_2=1}} x^TAx + \lambda_1(B)$$

$$\leq \max_{\dim\mathcal{V}=i} \min_{\substack{x\in\mathcal{V}\\ \|x\|_2=1}} x^TAx + \lambda_1(B)$$

$$\overset{C\text{-}F}{\leq} \lambda_i(A) + \lambda_1(B). \quad \boxed{\equiv}$$

Thm 3 [Interlacing-I]

$\forall y \in \mathbb{R}^n, \forall \alpha \in \mathbb{R}$

$$\lambda_{i+1}(A) \underset{HW}{\leq} \lambda_i(A+\alpha yy^T) \leq \lambda_{i-1}(A)$$

Prf $\lambda_i(A+\alpha yy^T) = \max_{\dim\mathcal{V}=i} \min_{\substack{x\in\mathcal{V}\\ \|x\|_2=1}} x^T(A+\alpha yy^T)x$

$$\leq \max_{\dim\mathcal{V}=i} \min_{\substack{x\in\mathcal{V}, x\perp y\\ \|x\|_2=1}} x^TAx$$

$$= \max_{\dim\mathcal{V}=i} \min_{\substack{x\in\mathcal{V}\cap \mathrm{Span}(y)^\perp\\ \|x\|_2=1}} x^TAx$$

$$= \max_{\substack{\mathcal{V}'\\ \mathcal{V}'=\mathcal{V}\cap\mathrm{Span}(y)^\perp\\ \dim\mathcal{V}=i}} \min_{\substack{x\in\mathcal{V}'\\ \|x\|_2=1}} x^TAx =$$

$$\leq \max_{j=i-1,i} \left[\max_{\dim\mathcal{V}'=j} \min_{\substack{x\in\mathcal{V}'\\ \|x\|_2=1}} x^TAx \right] = \max\{\lambda_{i-1}(A), \lambda_i(A)\}$$

$$= \lambda_{i-1}(A). \quad \boxed{\equiv}$$

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Thm 4 B: symmetric of rank r

$$\lambda_{i+r}(A) \le \lambda_i(A+B) \le \lambda_{i-r}(A)$$

Prf $B = U \underbrace{M}_{\substack{r\times r\\ \text{diagonal}}} U^T = \sum_{j=1}^{r} \mu_j v_j v_j^T$

$M = \operatorname{diag}(\mu_1, \dots, \mu_r)$ $\quad U = [v_1 \dots v_r]$

Apply Interlacing-I r times. ☺

Thm 5 [Weyl-II]

B: symmetric $\forall i, j \in [n]$:

$$\lambda_{i+j-1}(A+B) \le \lambda_i(A) + \lambda_j(B) \underbrace{\le}_{HW} \lambda_{i+j-n}(A+B)$$

Prf We first prove the statement for $A, B \ge 0$. $A^{(i-1)} = \sum_{\alpha \in [i-1]} \lambda_\alpha(A) u_\alpha u_\alpha^T$

Let $B = \sum_{b \in [n]} \lambda_b(B) v_b v_b^T$ be a spectral resolution of B. Define $B^{(j-1)} = \sum_{b \in [j-1]} \lambda_b(B) v_b v_b^T$

Interlacing-I

$$\lambda_{i+j-1}(A+B) \le \lambda_1(A-A^{(i-1)}+B-B^{(j-1)})$$

$$\overset{\text{Weyl-I}}{\le} \lambda_1(A-A^{(i-1)}) + \lambda_1(B-B^{(j-1)})$$

$$= \lambda_1\left(\sum_{\alpha=i}^{n} \lambda_\alpha(A) u_\alpha u_\alpha^T\right) + \lambda_1\left(\sum_{\beta=j}^{n} \lambda_\beta(B) v_\beta v_\beta^T\right)$$

$$\overset{\substack{A\ge 0\\ B\ge 0}}{=} \lambda_i(A) + \lambda_j(B)$$

Now for any symmetric A, B.
Pick $\delta > 0$ large enough so that $A+\delta I, B+\delta I \ge 0$. So

$$\lambda_{i+j-1}(A+\delta I+B+\delta I) \le \lambda_i(A+\delta I) + \lambda_j(B+\delta I) \Rightarrow$$

$$\lambda_{i+j-1}(A+B) + 2\delta \le \lambda_i(A) + \delta + \lambda_j(B) + \delta \quad \boxed{\equiv}$$

Thm 6 [Interlacing-II]
Let $B \in \mathbb{R}^{r\times r}$ be obtained by removing rows and columns of A indexed by $i_1, \dots, i_{n-r} \in [n]$

$$\lambda_{i+n-r}(A) \le \lambda_i(B) \le \lambda_i(A) \quad \boxed{\equiv}$$

Cor 7 $\forall\ i \in [n]$

$$\lambda_n(A) \le a_{ii} \le \lambda_1(A)$$

Prf $B = a_{ii}$ is obtained by removing $n-1$ rows and columns. Also $a_{ii} = \lambda_1(B)$
Now apply interlacing-II with $r=1$. $\boxed{\equiv}$

obtained

and columns

$\ldots, i_{n-r} \in [n]$.

$\lambda_i(A)$ ∎

$\lambda_1(A)$

tained

rows and

$= \lambda_1(B)$

Cor 8 $A \geq 0$. Then $\forall\, i \in [n]$ $a_{ii} \geq 0$.

Prf By Cor 7

$0 \leq \lambda_n(A) \leq a_{ii}$ ∎