LAIS, Lecture #18

Lem 1 [Matrix Inversion Lemma]
A, B, C, D matrices s.t. the matrix $A+BCD$ is defined. If A, C and $C^{-1}+DA^{-1}B$ are invertible, then

$$(A+BCD)^{-1} = \underbrace{A^{-1} - A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}}$$

Prf $(A+BCD)[\ \cdots\ ] =$

$$= I - B(C^{-1}+DA^{-1}B)^{-1}DA^{-1} + BCDA^{-1}$$
$$- BCDA^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}$$
$$= I + B[-(C^{-1}+DA^{-1}B)^{-1} + C - CDA^{-1}B(C^{-1}+DA^{-1}B)^{-1}]DA^{-1}$$
$$= I + B[\underbrace{-I + C(C^{-1}+DA^{-1}B) - CDA^{-1}B}_{=0}](C^{-1}+DA^{-1}B)^{-1}DA^{-1} \qquad \blacksquare$$

"square-root RLS adaptive filter"

standard RLS filter:

$$\gamma(i) = (1+\lambda^{-1}u_i P_{i-1} u_i^T)^{-1}$$
$$g_i = \lambda^{-1}\gamma(i) P_{i-1} u_i^T$$
$$w_i = w_{i-1} + g_i(d(i) - u_i w_{i-1})$$
$$P_i = \lambda^{-1} P_{i-1} - \frac{g_i g_i^T}{\gamma(i)}$$

Dfn 2 $A>0$, a square root of A is a matrix X s.t. $A = XX^T$. $\blacksquare$

Prp 3 $A > 0$, $A = U \Lambda U^T$ eigendecomposition, then $X = U \Lambda^{1/2}$ is square-root of A. ∎

Prp 4 $A > 0$, if X is a square-root of A, then XQ is also a square-root for any orthogonal matrix Q. ∎

Prp 5 [Schur complement]

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & \underbrace{D - CA^{-1}B}_{\Delta_A} \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

providing A is invertible ∎

Lem 6 $A \in \mathbb{R}^{n \times n}$, $A > 0$

$\iff S^T A S > 0$ $\forall$ invertible S. ∎

Lem 7 $M = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} > 0$

$\iff A > 0$ and $\Delta_A = D - B^T A^{-1} B > 0$

Prf By Lem 6 and Prp 5

$M > 0 \iff \begin{bmatrix} A & 0 \\ 0 & D - B^T A^{-1} B \end{bmatrix} > 0.$

$\iff A > 0$ and $D - B^T A^{-1} B > 0$. ∎

Thm 8 $M > 0$, there is a unique lower-triangular matrix L with positive elements in the diagonal s.t. $M = LL^T$. This L is called the Cholesky factor of M

Prf By induction on n, where $M \in \mathbb{R}^{n\times n}$. Write

$$M = \begin{bmatrix} \alpha & b^T \\ b & \underbrace{M'}_{(n-1)\times(n-1)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{b}{\alpha} & I_{n-1} \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \underbrace{\Delta_\alpha}_{M' - \frac{bb^T}{\alpha}} \end{bmatrix} \begin{bmatrix} 1 & b^T/\alpha \\ 0 & I_{n-1} \end{bmatrix}$$

By Lem 7, $\Delta_\alpha > 0$.

By i.h. $\Delta_\alpha = \underbrace{L_\Delta}_{\text{lower-triangular with } >0 \text{ in the diagonal}} L_\Delta^T$

$$= \begin{bmatrix} 1 & 0 \\ b/\alpha & I_{n-1} \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & L_\Delta L_\Delta^T \end{bmatrix} \begin{bmatrix} 1 & b^T/\alpha \\ 0 & I_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ b/\alpha & I_{n-1} \end{bmatrix} \begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & L_\Delta \end{bmatrix} \begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & L_\Delta^T \end{bmatrix} \begin{bmatrix} 1 & b^T/\alpha \\ 0 & I_{n-1} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \sqrt{\alpha} & 0 \\ b/\sqrt{\alpha} & L_\Delta \end{bmatrix}}_{\text{lower-triangular with positive elements in the diagonal}} \begin{bmatrix} \sqrt{\alpha} & b^T/\sqrt{\alpha} \\ 0 & L_\Delta^T \end{bmatrix} = LL^T$$

This shows the existence

Uniqueness: Suppose $M = LL^T = KK^T$ with K l.t. with >0 in the diagonal

$\Rightarrow K^{-1}L = K^T(L^T)^{-1}$

$\Rightarrow K^{-1}L = D$: diagonal

$\Rightarrow L = KD$

$LL^T = KK^T \Rightarrow$

$KDD^TK^T = KK^T$

$\Rightarrow DD^T = I$

$\Rightarrow D = I$.

$M_{11} = L_{11}^2 = K_{11}^2 \Rightarrow L_{11} = K_{11}$

because $L_{11}, K_{11} > 0$ ☺

$\Rightarrow D_{11} = 1 \ldots$ HW

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"square-root RLS adaptive filter"

$P_i^{1/2} \equiv$ Cholesky factor of P_i

$$\underbrace{\begin{bmatrix} 1 & \lambda^{-1/2} u_i P_{i-1}^{1/2} \\ 0 & \lambda^{1/2} P_{i-1}^{1/2} \end{bmatrix}}_{(n+1)\times(n+1)} \underbrace{\Theta}_{\text{orthogonal}}$$

$$= \begin{bmatrix} \underset{>0}{\underbrace{x}_{1\times 1}} & \underbrace{0}_{1\times n} \\ \underbrace{y}_{n\times 1} & \underbrace{Z}_{n\times n} \end{bmatrix}$$

$n\times n$ lower-triangular with >0 in the diagonal

standard RLS filter:

$$\gamma(i) = \left(1 + \lambda^{-1} u_i P_{i-1} u_i^T\right)^{-1}$$

$$g_i = \lambda^{-1} \gamma(i) P_{i-1} u_i^T$$

$$w_i = w_{i-1} + g_i\left(d(i) - u_i w_{i-1}\right)$$

$$P_i = \lambda^{-1} P_{i-1} - \frac{g_i g_i^T}{\gamma(i)}$$

Rem 9

Such an Θ does exist, e.g. by "Givens rotations" or "Householder reflections" $\boxed{z}$

$$\Rightarrow \begin{bmatrix}1 & \lambda^{-1/2}u_iP_{i-1}^{1/2}\\ 0 & \lambda^{-1/2}P_{i-1}^{1/2}\end{bmatrix}\begin{bmatrix}1 & 0\\ \lambda^{-1/2}P_{i-1}^{1/2T}u_i^T & \lambda^{-1/2}P_{i-1}^{1/2T}\end{bmatrix}$$

$$= \begin{bmatrix}x & 0\\ y & Z\end{bmatrix}\begin{bmatrix}x & y^T\\ 0 & Z^T\end{bmatrix}$$

i) $x^2 = 1+\lambda^{-1}u_i \underbrace{P_{i-1}^{1/2}P_{i-1}^{1/2T}}_{P_{i-1}} u_i^T$

$\Rightarrow x^2 = 1+\lambda^{-1}u_iP_{i-1}u_i^T = \gamma(i)^{-1}$

$\Rightarrow \boxed{x=\gamma(i)^{-1/2}}$

ii) $xy^T = \lambda^{-1}u_iP_{i-1}^{1/2}P_{i-1}^{1/2T} = \lambda^{-1}u_iP_{i-1}$

$\Rightarrow \gamma(i)^{-1/2}y^T = \lambda^{-1}u_iP_{i-1}$

$\Rightarrow y^T = \gamma(i)^{1/2}\lambda^{-1}u_iP_{i-1} = \frac{g_i^T}{\gamma(i)^{1/2}}$

$\Rightarrow \boxed{y=\frac{g_i}{\gamma(i)^{1/2}}}$

iii) $yy^T+ZZ^T =$

$= \lambda^{-1/2}P_{i-1}^{1/2}\lambda^{-1/2}P_{i-1}^{1/2T} \Rightarrow$

$ZZ^T = \lambda^{-1}P_{i-1} - \frac{g_ig_i^T}{\gamma(i)}$

$\Rightarrow ZZ^T = P_i$

By Thm 8 $Z = P_i^{1/2}$

$$\begin{bmatrix}1 & \lambda^{-1/2}u_iP_{i-1}^{1/2}\\ 0 & \lambda^{-1/2}P_{i-1}^{1/2}\end{bmatrix}\Theta =$$

$$\begin{bmatrix}\gamma(i)^{-1/2} & 0\\ \frac{g_i}{\gamma(i)^{1/2}} & P_i^{1/2}\end{bmatrix}$$

"post-array"

$$W_i = W_{i-1} + \left(\frac{g_i}{\gamma(i)^{1/2}}\right)\left(\gamma(i)^{-1/2}\right)^{-1}\left(d(i) - u_iW_{i-1}\right)$$