LAIS, Lecture #6

Dfn 1 [spectrum]

$A \in K^{n \times n}$, $\sigma_{\mathbb{F}}(A) = \{\lambda \in \mathbb{F} : \operatorname{rank}_{\mathbb{F}}(A - \lambda I) < n\}$

$K \xhookrightarrow{\text{field extension}} \mathbb{F}$ ∎

usually $\mathbb{F} = K, \overline{K}$

Dfn 2 [eigenspace]

$\lambda \in K$, $\mathcal{E}_{A,\lambda} = \mathcal{N}(A - \lambda I)$ ∎

Dfn 3 [algebraic multiplicity]

$\lambda \in \sigma(A)$, the algebraic multiplicity of λ, denoted by α_λ, is the largest r s.t. $(x-\lambda)^r$ divides $\det(A - xI)$. ∎

Dfn 4 [geometric multiplicity]

$\lambda \in \sigma(A)$, $\gamma_\lambda = \dim \mathcal{E}_{A\lambda}$ ∎

Thm 5 $A \in \mathbb{R}^{n \times n}$, $A^T = A$

$\Rightarrow \sigma_{\mathbb{R}}(A) = \sigma_{\mathbb{C}}(A)$ $\left(\Leftrightarrow \sigma_{\mathbb{C}}(A) \subset \mathbb{R}\right)$ ∎

Prf Induction on n.

For $n=1$, trivial.

Now for $n>1$.

$$n\left\{\underbrace{\begin{bmatrix} * & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{bmatrix}}_{n}\right. = \begin{bmatrix} \alpha & \\ & U'\Lambda'U'^T \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \\ & U' \end{bmatrix} \begin{bmatrix} \alpha & \\ & \Lambda \end{bmatrix} \begin{bmatrix} 1 & \\ & U \end{bmatrix}^T$$

$$\begin{bmatrix} * & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & & * & \end{bmatrix} = QAQ^T$$

Prp 6 $A \in \mathbb{R}^{n\times n}$, $A^T = A$

$\lambda \neq \mu \in \sigma(A) \Rightarrow \mathcal{E}_{A,\lambda} \perp \mathcal{E}_{A,\mu}$

$$\sum_{\lambda\in\sigma(A)} \mathcal{E}_{A,\lambda} = \bigoplus_{\lambda\in\sigma(A)} \mathcal{E}_{A,\lambda} = \mathbb{R}^n$$

Prf (1) $Au = \lambda u$, $u \neq 0$

(2) $Av = \mu v$

(1) $v^T A u = \lambda v^T u$ (3)

(2) $u^T A v = \mu u^T v \Rightarrow v^T A u = \mu v^T u$ (4)

(3),(4) $\Rightarrow \lambda v^T u = \mu v^T u$

$\Rightarrow (\lambda - \mu) v^T u = 0 \Rightarrow v \perp u$ ∎

Dfn 7 $A \in \mathbb{R}^{n\times n}$, orthogonally diagonalizable, if $A = \underset{\text{orthogonal}}{U}\,\underset{\text{diagonal}}{\Lambda}\,U^T$ ∎

Rem 8 $A = U \Lambda U^T$ (1)

$U = [u_1 \cdots u_n]$

$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$

(1) $\Rightarrow AU = U\Lambda$

j-th column

$Au_j = \lambda_j u_j$

$\Rightarrow u_j$'s eigenvectors

$u_j \perp u_i \;\; \forall i \neq j$ ∎

Thm 9 $A \in \mathbb{R}^{n\times n}$

$A^T = A \Rightarrow$

A is orthogonally diagonalizable

(1)

Prf Induction on n.

For $n=1$, trivial.

Now for $n>1$.

Let $u \in \mathbb{R}^n$ be an ($\|u\|_2=1$) eigenvector with eigenvalue $\lambda \in \mathbb{R}$.

$Au=\lambda u$. Let $V \in \mathbb{R}^{n\times(n-1)}$ have an orthonormal basis in its columns for $\mathrm{span}(u)^\perp$, $V^TV=I_{n-1}$.

So $[u\ V] \in \mathbb{R}^{n\times n}$ is orthogonal.

$$\underbrace{\begin{bmatrix} u^T \\ V^T \end{bmatrix}}_{O} A\,[u\ V] = \begin{bmatrix} u^TAu & \underbrace{u^TAV}_{\to 0} \\ \underbrace{V^TAu}_{\lambda \underbrace{V^Tu}_{=0}} & V^TAV \end{bmatrix}$$

$$\overset{\text{i.h.}}{=} \begin{bmatrix} \lambda & 0 \\ 0 & \underbrace{\Phi}_{\substack{(n-1)\times(n-1) \\ \text{orthogonal}}} \underbrace{M}_{\substack{(n-1)\times(n-1) \\ \text{diagonal}}} \Phi^T \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \\ & \Phi \end{bmatrix} \begin{bmatrix} \lambda & \\ & M \end{bmatrix} \begin{bmatrix} 1 & \\ & \Phi^T \end{bmatrix}$$

$\Rightarrow$

$$A = \underbrace{[u\ V] \begin{bmatrix} 1 & \\ & \Phi \end{bmatrix}}_{U \text{ orthogonal}} \underbrace{\begin{bmatrix} \lambda & \\ & M \end{bmatrix}}_{\Lambda} \begin{bmatrix} 1 & \\ & \Phi \end{bmatrix}^T [u\ V]^T$$

$\underbrace{O_1, O_2}_{\text{orthogonal}}$ $(O_1O_2)^T(O_1O_2) = I$

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Cor 10 $A \in \mathbb{R}^{n \times n}, A^T = A$ HW

$\alpha_\lambda = \gamma_\lambda \; \forall \lambda \in \sigma(A)$.

"geometric view"

$\lambda_1 > \lambda_2 > \cdots > \lambda_s$, $\#\sigma(A) = s$

$U_i \in \mathbb{R}^{n \times \gamma_i}$: orthonormal basis for $\mathcal{E}_{A,\lambda_i}$

$$A = [U_1 \cdots U_s] \begin{bmatrix} \underbrace{\mathrm{diag}(\lambda_1, \ldots, \lambda_1)}_{\gamma_{\lambda_1} \times \gamma_{\lambda_1}} & & \\ & \ddots & \\ & & \underbrace{\mathrm{diag}(\lambda_s, \ldots, \lambda_s)}_{\gamma_{\lambda_s} \times \gamma_{\lambda_s}} \end{bmatrix} [U_1 \cdots U_s]^T$$

$$\begin{bmatrix} \Lambda_1 & & \\ & \ddots & \\ & & \Lambda_s \end{bmatrix}$$

$\Lambda_i = \mathrm{diag}(\lambda_i, \ldots, \lambda_i) : \gamma_{\lambda_i} \times \gamma_{\lambda_i}$

"spectral resolution of A"

$$\Rightarrow A = \sum_{i \in [s]} \lambda_i U_i U_i^T$$

the set of eigenspaces of A induces an orthogonal resolution of the identity

τ_A

D

i)

ii)

iii)

iv)

$$\begin{tikzcd}
 & \mathbb{R}^{r_1} \arrow[r, "\lambda_1"] & \mathbb{R}^{r_1} \arrow[dr, "U_1"] & \\
\tau_A : \mathbb{R}^n \arrow[ur, "U_1^T"] \arrow[dr, "U_s^T"'] & \times \; \vdots \; \times & \times \; \vdots \; \times & \Sigma\, \mathbb{R}^n \\
 & \mathbb{R}^{r_s} \arrow[r, "\lambda_s"] & \mathbb{R}^{r_s} \arrow[ur, "U_s"'] &
\end{tikzcd}$$

Dfn 11 $A^T = A$

i) positive-semidefinite $\sigma(A) \subset \mathbb{R}_{\geq 0}$ $(A \geq 0)$

ii) positive-definite $\sigma(A) \subset \mathbb{R}_{>0}$

iii) negative-definite $\sigma(A) \subset \mathbb{R}_{<0}$

iv) indefinite $\sigma(A) \not\subset \mathbb{R}_{>0}$ and $\sigma(A) \not\subset \mathbb{R}_{\leq 0}$ $\boxed{\equiv}$

Thm 12 $A \in \mathbb{R}^{n \times n}$, $A^T = A$

The following are equivalent:

i) $A \geq 0$

ii) $x^T A x \geq 0 \;\; \forall x \in \mathbb{R}^n$

iii) $\exists$ some $B \in \mathbb{R}^{n \times r}$ for some r s.t. $A = BB^T$.

Prf i) $\Rightarrow$ ii)

$$x^T A x = x^T \Big(\sum_{i \in [s]} \lambda_i U_i U_i^T \Big) x$$

$$= \sum_{i \in [s]} \lambda_i \, x^T U_i U_i^T x$$

$$= \sum_{i \in [s]} \lambda_i \, \| U_i^T x \|_2^2 \geq 0$$

ii) =

A

$0 \leq U^T$

i) =

$= U$

iii)

x^T

$^{n\times n}, A^T = A$

are

$\forall x \in \mathbb{R}^n$

$B \in \mathbb{R}^{n\times r}$

r s.t.

$\sum \lambda_i U_i U_i^T \Big) x$

$\underline{ii) \Rightarrow i)}$ $\lambda \in \sigma(A)$

$Au = \lambda u \quad , \|u\|_2 = 1$

$0 \le u^T A u = \lambda$

$i) \Rightarrow iii)$ $A = U \Lambda U^T =$

$= \underbrace{U \Lambda^{1/2}}_{B} \underbrace{\Lambda^{1/2} U^T}_{B^T}$

$\underline{iii) \Rightarrow i)}$ $\Longleftrightarrow iii) \Rightarrow ii)$

$x^T A x = x^T B B^T x = \|B^T x\|_2^2 \ge 0$

$\boxed{\equiv}$

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of A

$\Rightarrow A = \sum_{i \in [s]} \lambda_i v_i v_i^T$

the set of eigensp

of A induces an

orthogonal resoluti

of the identity

Dfn 13 $\begin{bmatrix} \text{four fundamental} \\ \text{subspaces of } A \in \mathbb{R}^{m \times n} \end{bmatrix}$

$\mathcal{R}(A), \mathcal{N}(A^T) \subset \mathbb{R}^m$

$\mathcal{R}(A^T), \mathcal{N}(A) \subset \mathbb{R}^n$ $\boxed{\equiv}$

Prp 14 $\mathcal{R}(A)^\perp = \mathcal{N}(A^T)$

In particular $\mathbb{R}^m = \mathcal{R}(A) \oplus \mathcal{N}(A^T)$

Prf $x \perp \mathcal{R}(A) \iff x \perp Ay \ \forall y \in \mathbb{R}^n$

$\iff x \in \mathcal{N}(A^T)$ $\boxed{\equiv}$

... HW