LAIS, Lecture #12

Dfn 1 V: subspace of K^n, $p(x) \in K[x]$,
$V^{(p)} = \{v \in V : p(A)v = 0\}$. ∎

Lem2 V, W: A-invariant subspaces of K^n s.t. $V \cap W = 0$.
Then $(V \oplus W)^{(p)} = V^{(p)} \oplus W^{(p)}$. ∎
"$(\cdot)^{(p)}$ commutes with direct sums"

Lem3 Suppose $V = K[x]v$, $\operatorname{ann}(v) = (p^{\ell}(x))$. Then
$V^{(p)} = K[x]\,p^{\ell-1}(x)v$ and
$\dim V^{(p)} = \deg p(x)$. ∎

Prp 4 Suppose $m_A(x) = p^{\ell}(x)$,
$p(x)$: monic, irreducible, $\ell \geq 1$.
Then $\exists$ unique integers
$\ell = \ell_1 \geq \ell_2 \geq \cdots \geq \ell_s \geq 1$
s.t. $K^n = \bigoplus_{i \in [s]} K[x]v_i$
$\operatorname{ann}(v_i) = (p^{\ell_i}(x))$.

Prf $\exists\, v_1 \in K^n$ s.t.
$\operatorname{ann}(v_1) = (p^{\ell}(x))$.
$V_1 = K[x]v_1$. By induction on n it is enough to show that V_1 has an A-invariant complement.

It is
if
$V_1 \cap$
then
U_{j+1}
$V_1 \cap U$
So s
$\Rightarrow$
$I =$
$p^{\ell}(x$
$1 \leq \ell' \leq$

It is enough to show that if U_j is A-invariant, $V_1 \cap U_j = 0$ and $V_1 + U_j \subsetneq K^n$ then $\exists\, U_{j+1} \supsetneq U_j$ s.t. U_{j+1} is A-invariant and $V_1 \cap U_{j+1} = 0$.

So suppose $V_1 + U_j \subsetneq K^n$

$\Rightarrow \exists\, u' \in K^n \setminus V_1 + U_j$.

$I = \{ f(x) \mid f(x) u' \in V_1 + U_j \}$.

$p^{l}(x) \in I \Rightarrow I = (p^{l'}(x))$

$1 \le l' \le l$.

$p^{l'}(x) u' \in V_1 + U_j \Rightarrow$

$\exists\, \alpha(x) \in K[x],\ u'' \in U_j$ s.t.

$$p^{l'}(x) u' = \alpha(x) v_1 + \underbrace{u''}_{\in U_j}$$

Multiply both sides by $p^{l-l'}(x)$, $0 = p^{l-l'}(x) p^{l'}(x) u' = p^{l-l'}(x)\alpha(x) v_1 + p^{l-l'}(x) u'' \Rightarrow$

$$\underbrace{p^{l-l'}(x)\alpha(x) v_1}_{\in V_1} = -\underbrace{p^{l-l'}(x) u''}_{\in U_j} = 0$$

$\Rightarrow p^{l-l'}(x)\alpha(x) \in \operatorname{ann}(v_1)$

$\Rightarrow p^{l}(x) \mid p^{l-l'}(x)\alpha(x)$

$\Rightarrow p^{\ell'}(x) \mid \alpha(x) \Rightarrow$

$\exists\, b(x) \in k[x]$ s.t.

$\alpha(x) = p^{\ell'}(x)\, b(x)$.

Define

$U_{j+1} = U_j + k[x]\left(u' - b(x)v_1\right)$

Then $U_{j+1} \supsetneq U_j$ and

U_{j+1} is A-invariant.

Suppose $f \in V_1 \cap U_{j+1}$.

$\exists\, \gamma(x), \delta(x) \in k[x]$

$u''' \in U_j$ s.t.

$$f = \gamma(x) v_1 = u''' + \delta(x)\left(u' - b(x)v_1\right)$$

$$\delta(x)\, u' = \underbrace{\left[\gamma(x) + \delta(x) b(x)\right]}_{\in V_1} v_1 - \underbrace{u'''}_{\in U_j}$$

$\Rightarrow \delta(x) \in I \Rightarrow \delta(x) = \varepsilon(x)\, p^{\ell'}(x)$

for some $\varepsilon(x) \in k[x]$. Then

$$f = u''' + \varepsilon(x)\, p^{\ell'}(x)\left(u' - b(x) v_1\right)$$

$$\Rightarrow f = \underbrace{u'''}_{\in U_j} + \varepsilon(x)\underbrace{u''}_{\in U_j} \Rightarrow f \in U_j$$

Also $f \in V_1$. So $f \in V_1 \cap U_j \Rightarrow f = 0$

This proves the existence part.

Prp 4 Suppose $m_A(x) = p^{\ell}(x)$, $p(x)$: monic, irreducible, $\ell \geq 1$.

Then $\exists$ unique integers $\ell = \ell_1 \geq \ell_2 \geq \cdots \geq \ell_s \geq 1$

s.t. $K^n = \bigoplus_{i \in [s]} K[x] v_i$

$\mathrm{ann}(v_i) = (p^{\ell_i}(x))$.

Prf Uniqueness Part.

Suppose $K^n = \bigoplus_{j \in [t]} K[x] w_j$

$\mathrm{ann}(w_j) = (r^{\nu_j}(x))$

$\nu = \nu_1 \geq \nu_2 \geq \cdots \geq \nu_t \geq 1$

$r(x)$: monic, irreducible

$p^{\ell}(x) K^n = 0 \Rightarrow p^{\ell}(x) \in \mathrm{ann}(w_j)$

$\Rightarrow r(x) = p(x)$.

$K^n = \bigoplus_{i \in [s]} K[x] v_i = \bigoplus_{j \in [t]} K[x] w_j \Rightarrow$

$\bigoplus_{i \in [s]} (K[x] v_i)^{(p)} = \bigoplus_{j \in [t]} (K[x] w_j)^{(p)}$

$\dim (K[x] v_i)^{(p)} = \dim (K[x] w_j)^{(p)} \underset{\text{Lem. 3}}{=} \deg p(x) \ \forall i, j$

$\Rightarrow s \cdot \deg p(x) = t \cdot \deg p(x) \Rightarrow s = t$.

If $\nu_1 = \ell_1 = 1$, we are done. Otherwise

$(\ell_1, \ell_2, \ldots, \ell_s) = \underbrace{(\ell_1, \ldots, \underset{>1}{\ell_{s'}}, 1, \ldots, 1)}_{s}$

$\underbrace{(\nu_1, \ldots, \underset{>1}{\nu_{t'}}, 1, \ldots, 1)}_{s}$

$p(x) \ \forall i,j$

So $p(x)K[x]v_i = 0 \ \forall i > l_{s'}$

$p(x)K[x]w_j = 0 \ \forall j > r_{t'}$

$$p(x)K^n = \bigoplus_{i\in[l_{s'}]} p(x)K[x]v_i$$

$$= \bigoplus_{j\in[r_{t'}]} p(x)K[x]w_j$$

$$\operatorname{ann}\left(p(x)K[x]v_i\right) = \left(p(x)^{l_i-1}\right)$$

$$\operatorname{ann}\left(p(x)K[x]w_j\right) = \left(p(x)^{r_j-1}\right)$$

$$\operatorname{ann}\left(p(x)K^n\right) = \left(p(x)^{l-1}\right)$$

Now apply the statement on $p(x)K^n$ by induction on l (recall the case $l=1$ is proved). We get $l_{s'} = r_{t'}$ and $l_i = r_i \ \forall i \in [l_{s'}]$. ∎

Prp 5 ← Primary Decomposition

$$m_A(x) = p_1^{l_1}(x)\cdots p_s^{l_s}(x)$$

$p_i(x)$: monic, irreducible

Define $V_i = \dfrac{m_A(x)}{p_i^{l_i}(x)} K^n$.

Then V_i is A-invariant, $\operatorname{ann}(V_i) = \left(p_i^{l_i}(x)\right)$ and

$$K^n = \bigoplus_{i\in[s]} V_i.$$

t'

Now apply the statement on $p(x)K^n$ by induction on l (recall the case $l=1$ is proved). We get $l_{s'} = r_{t'}$ and $l_i = r_i \ \forall i \in [l_{s'}]$. ☺

Prp 5 ← Primary Decomposition

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Define $V_i = \dfrac{m_A(x)}{p_i^{l_i}(x)} K^n$.

Then V_i is A-invariant, $\operatorname{ann}(V_i) = (p_i^{l_i}(x))$ and

$K^n = \bigoplus_{i \in [s]} V_i$.

Moreover, if $K^n = \bigoplus_{j \in [t]} W_j$ with $\operatorname{ann}(W_j) = (r_j^{v_j}(x))$, $r_j(x)$ monic, irreducible and the r_j's are distinct, then $s = t$ and up to a re-numbering of the r_j's, we have $p_j(x) = r_j(x)$, $l_j = v_j$ $\forall j \in [s]$ and $W_j = V_j \ \forall j \in [s]$.

Prf $p_i^{l_i}(x) \in \operatorname{ann}(V_i) \Rightarrow$

$\operatorname{ann}(V_i) = (p_i^{l_i'}(x))$, $l_i' \le l_i$.

If $l_i' < l_i$, then $\dfrac{m_A(x)}{p_i^{l_i}(x)} p^{l_i'}(x) K^n = 0$

$\Rightarrow \dfrac{m_A(x)}{p_i^{l_i - l_i'}(x)} K^n = 0$, contradiction.

So $l_i' = l_i$.

Decomposition

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Let's show $K^n = \bigoplus_{i\in[s]} V_i$.

$v \in V_j \cap \left(\sum_{i\neq j} V_i\right)$

$p_j^{l_j}(x)\, v = 0$

$p_i^{l_i}(x)\, V_i = 0 \Rightarrow \left(\prod_{i\neq j} p_i^{l_i}(x)\right)\left(\sum_{i\neq j} V_i\right) = 0$

$\left(\prod_{i\neq j} p_i^{l_i}(x)\right) v = 0$

$\Rightarrow v = 0$ because $\underbrace{\left(p_j^{l_j}(x), \prod_{i\neq j} p_i^{l_i}(x)\right)}_{= \gcd\left(p_j^{l_j}(x), \prod_{i\neq j} p_i^{l_i}(x)\right) = 1} \subset \operatorname{ann}(v) \Rightarrow 1 \in \operatorname{ann}(v)$

Prp 4 Suppose $m_A(x) = p^{l}(x)$, $p(x)$: monic, irreducible, $l \geq 1$. Then $\exists$ unique integers $l = l_1 \geq l_2 \geq \dots \geq l_s \geq 1$ s.t. $K^n = \bigoplus_{i\in[s]} K[x]v_i$, $\operatorname{ann}(v_i) = (p^{l_i}(x))$. ■

$m_A(x) = p^{\ell}(x)$,
irreducible, $\ell \geq 1$.
unique integers
$\cdots \geq \ell_s \geq 1$
$k[x]v_i$
$)$. ▨

$\mathrm{ann}(v)$

$(x))=1 \Rightarrow 1 \in \mathrm{ann}(v)$

HW: prove the uniqueness part of Prp 5. ▨

Thm 6 $A \in k^{n \times n}$
$m_A(x) = p_1^{\ell_1}(x) \cdots p_s^{\ell_s}(x)$.
For every $i \in [s]$ there exist unique integers
$\ell_i = \ell_{1i} \geq \ell_{2i} \geq \cdots \geq \ell_{s_i i} \geq 1$
s.t.
$$k^n = \bigoplus_{\substack{i \in [s] \\ j \in [s_i]}} k[x] v_{ij}$$
with $\mathrm{ann}(v_{ij}) = (p_i^{\ell_{ij}}(x))$
for some $v_{ij} \in k^n$. ▨