LAIS, Lec #2

Dfn 1 A linear transformation is a function $\tau : \mathcal{U} \longrightarrow \mathcal{V}$, where $\mathcal{U}, \mathcal{V}$ are vector spaces over a field K, s.t. $\tau(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 \tau(u_1) + \alpha_2 \tau(u_2)$ $\forall \alpha_1, \alpha_2 \in K$, $\forall u_1, u_2 \in \mathcal{U}$. $\boxminus$

Rem 2 $\tau : \underset{\text{"source space"}}{\underbrace{\mathcal{U}}} \longrightarrow \underset{\text{"target space"}}{\mathcal{V}}$ $\boxminus$

Prp 3 $\tau : \mathcal{U} \longrightarrow \mathcal{V}$ l.t. $\mathcal{U}'$ is a subspace of $\mathcal{U}$, then $\underbrace{\tau(\mathcal{U}')}_{\{v \in \mathcal{V} : v = \tau(u), \text{ for some } u \in \mathcal{U}'\}}$ is a subspace of $\mathcal{V}$.

Prf $\alpha_1, \alpha_2 \in K$, $v_1, v_2 \in \tau(\mathcal{U}')$

goal: show $\alpha_1 v_1 + \alpha_2 v_2 \in \tau(\mathcal{U}')$

$\exists\, u_1, u_2 \in \mathcal{U}'$ s.t. $v_1 = \tau(u_1)$, $v_2 = \tau(u_2)$

$\alpha_1 v_1 + \alpha_2 v_2 = \alpha_1 \tau(u_1) + \alpha_2 \tau(u_2) =$

$\tau(\underbrace{\alpha_1 u_1 + \alpha_2 u_2}_{u_3 \in \mathcal{U}'}) \in \tau(\mathcal{U}')$ $\boxminus$

Rem 4 l.t. takes subspaces to subspaces, also called homomorphism of vector spaces

similar shape

category theory

morphism of vector spaces $\boxminus$

Dfn 5 ...

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denote...

Prp 3

of th...

Dfn 6

τ is

$\tau(u)$...

the ...

goes

of th...

Dfn 5 $\tau(U)$ is called the image of τ and denoted $\mathrm{im}(\tau)$. By Prp 3 it is a subspace of the target space.

Dfn 6 The Kernel of τ is $\mathrm{Ker}\,\tau = \{u \in U : \tau(u) = 0\}$. It is a subspace, the largest subspace that goes to the zero subspace of the target.

Prp 7 i) $A \in K^{m \times n}$ induces a l.t. $\tau_A : K^n \longrightarrow K^m$ by $\tau_A(u) = Au$. ii) Conversely every l.t. of the form $\tau : K^n \longrightarrow K^m$ must be $\tau = \tau_A$ for some $A \in K^{m \times n}$.

Prf i) easy to see that

$$A(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 A u_1 + \alpha_2 A u_2$$

$$\tau_A(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 \tau(u_1) + \alpha_2 \tau(u_2)$$

ii) $\tau : K^n \longrightarrow K^m$

we want to prove that $\exists A \in K^{m \times n}$ s.t. $\forall u \in K^n$ $\tau(u) = Au$.

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n

be

$m \times n$

u_2

$\tau(u_2)$

t

K^n

$A = [v_1 \cdots v_n]$

$v_i = \tau(\underset{\sim}{e_i})$

zeros everywhere
1 at entry i

$A = [\tau(e_1) \cdots \tau(e_n)] \in K^{m \times n}$

$u \in K^n$, $u = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \sum_{i \in [n]} c_i e_i$

$\tau(u) = \tau\left(\sum_{i \in [n]} c_i e_i\right) =$

$= \sum_{i \in [n]} c_i \tau(e_i) = \underbrace{[\tau(e_1) \cdots \tau(e_n)]}_{A} \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}_{u}$ ∎

Prp 8 $\tau : U \longrightarrow V$

$\dim \tau(U) \leq \dim U$

Prf Let $u_1, \ldots, u_n$ be a basis of U. Then $\tau(u_1), \ldots, \tau(u_n)$ is a spanning set of $\operatorname{im}(\tau)$: $v \in \operatorname{im}(\tau) \Rightarrow v = \tau(u)$ for some $u \in U$. Then

$u = \sum_{i \in [n]} \alpha_i u_i \Rightarrow$

$\tau(u) = \sum_{i \in [n]} \alpha_i \tau(u_i) \Rightarrow$

$\dim \tau(U) \leq n$. ∎

Dfn 9 $U = S \oplus T$

$\pi_{S,T} : U \longrightarrow S$

(recall $\forall\, u \in U$ $\exists!\, s \in S, t \in T$ s.t. $u = s + t$)

$\pi_{S,T}(u) = \underset{\sim}{s}$

component of u on S

"projection onto S along T" ∎

Prp 10 $\pi_{S,T}$ is a l.t.

Prf $\alpha_1, \alpha_2 \in K$

$u_1, u_2 \in U$

$\alpha_1 u_1 + \alpha_2 u_2 = s + t$

$u_1 = s_1 + t_1$

$u_2 = s_2 + t_2$

$m \times n$

$\sum_{i\in[n]} c_i e_i$

$=$

$\tau(e_1) \cdots \tau(e_n)] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

A $\quad$ u

$\blacksquare$

Prp 8 $\tau : U \to V$

$\dim \tau(U) \leq \dim U$

Prf Let $u_1, \ldots, u_n$ be a basis of U. Then $\tau(u_1), \ldots, \tau(u_n)$ is a spanning set of $im(\tau)$:

$v \in im(\tau) \Rightarrow v = \tau(u)$ for some $u \in U$. Then

$$u = \sum_{i\in[n]} \alpha_i u_i \Rightarrow$$

$$\tau(u) = \sum_{i\in[n]} \alpha_i \tau(u_i) \Rightarrow$$

$\dim \tau(U) \leq n$. $\blacksquare$

Dfn 9 $U = S \oplus T$

$\pi_{S,T} : U \to S$

(recall $\forall\, u \in U$ $\exists!\, s \in S, t \in T$ s.t. $u = s + t$)

$\pi_{S,T}(u) = \underbrace{s}_{\text{component of } u \text{ on } S}$

"projection onto S along T" $\blacksquare$

Prp 10 $\pi_{S,T}$ is a l.t.

Prf $\alpha_1, \alpha_2 \in K$

$u_1, u_2 \in U$

$\alpha_1 u_1 + \alpha_2 u_2 = s + t.$

$u_1 = s_1 + t_1$

$u_2 = s_2 + t_2$

goal: $s = \alpha_1 s_1 + \alpha_2 s_2$

$\alpha_1 u_1 + \alpha_2 u_2 = \alpha_1(s_1 + t_1) + \alpha_2(s_2 + t_2)$

$= \underbrace{(\alpha_1 s_1 + \alpha_2 s_2)}_{\in S} + \underbrace{(\alpha_1 t_1 + \alpha_2 t_2)}_{\in T}$ $\blacksquare$

Zixuan +1

Bi Kang +1

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Prp 11 i) $\operatorname{Ker} \Pi_{S,T} = T$

ii) $\operatorname{im} \Pi_{S,T} = S$

Prf i) $T \subset \operatorname{Ker} \Pi_{S,T}$

$\xi \in \operatorname{Ker} \Pi_{S,T} \Rightarrow \Pi_{S,T}(\xi) = 0$

$\xi = \underbrace{s}_{\in S} + \underbrace{t}_{\in T}$, $\Pi_{S,T}(\xi) = s$

$\Rightarrow s = 0 \Rightarrow \xi = t \Rightarrow \xi \in T$.

So $\operatorname{Ker} \Pi_{S,T} \subset T$.

ii)

$\operatorname{im} \Pi_{S,T} \subseteq S$.

$s \in S$, $\Pi_{S,T}(s) = s$ so

$S \subseteq \operatorname{im} \Pi_{S,T}$ $\boxed{\equiv}$

Rem 12 $\Pi_{S,T} : U \longrightarrow S$

Sometimes it is useful to think of $\Pi_{S,T}$ as

$\Pi_{S,T} : U \longrightarrow U$

$U \xrightarrow{\Pi_{S,T}} S \xhookrightarrow{\text{inclusion}} U$

endomorphism of U

inside

within

advantage: represented by a square matrix. $\boxed{\equiv}$

Thm 13 $\tau : V \longrightarrow V$

is a projection of the form $\rho_{S,T}$ for some S and T $\iff$ $\underbrace{\tau^2}_{\tau \circ \tau} = \tau$

Prf ($\Longrightarrow$) $\rho_{S,T}(\rho_{S,T}(u))$

$= \rho_{S,T}(\rho_{S,T}(s+t)) =$

$= \rho_{S,T}(s) = s = \rho_{S,T}(u)$

Rem 14 V: vector space

$\mathrm{End}(V) = \{\tau : V \to V\}$

it is a ring (non-commutative ring)

$\tau_1, \tau_2 \in \mathrm{End}(V)$

$\tau_1 + \tau_2 \in \mathrm{End}(V)$

$\tau_1 \circ \tau_2 \in \mathrm{End}(V)$

$\tau_2 \circ \tau_1 \in \mathrm{End}(V)$

matrix expression:

$K^{n \times n}$: it is a ring

Rem 14 V: vector space

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$\tau_2 \circ \tau_1 \in End(V)$

matrix expression:

$K^{n \times n}$: it is a ring ☺

($\Leftarrow$) suppose $\tau^2 = \tau$

goal: we need to find S, T s.t. $\tau = \rho_{S,T}$

Take $S = im(\tau)$

$T = Ker(\tau)$. We have to show $im(\tau) \cap Ker(\tau) = 0$

$\xi \in im(\tau) \cap Ker(\tau)$

$\xi \in Ker(\tau) \Rightarrow \tau(\xi) = 0$

$\xi \in im(\tau) \Rightarrow \xi = \tau(u)$ for some u

$0 = \tau(\xi) = \tau^2(u) = \tau(u) = \xi$

So $im(\tau) \oplus Ker(\tau)$ exists

and $im(\tau) \oplus Ker(\tau) \subseteq V$

($\Leftarrow$) suppose $\tau^2 = \tau$

goal: we need to find S, T s.t. $\tau = \sigma_{S,T}$

Take $S = \operatorname{im}(\tau)$

$T = \operatorname{Ker}(\tau)$. We have to show $\operatorname{im}(\tau) \cap \operatorname{Ker}(\tau) = 0$

$\xi \in \operatorname{im}(\tau) \cap \operatorname{Ker}(\tau)$

$\xi \in \operatorname{Ker}(\tau) \Rightarrow \tau(\xi) = 0$

$\xi \in \operatorname{im}(\tau) \Rightarrow \xi = \tau(u)$ for some u

$0 = \tau(\xi) = \tau^2(u) = \tau(u) = \xi$

So $\operatorname{im}(\tau) \oplus \operatorname{Ker}(\tau)$ exists

and $\operatorname{im}(\tau) \oplus \operatorname{Ker}(\tau) \subseteq U$

next goal: show

$\operatorname{im}(\tau) \oplus \operatorname{Ker}(\tau) = U$

Enough to prove

$U \subseteq \operatorname{im}(\tau) \oplus \operatorname{Ker}(\tau)$

Take $u \in U$

$u = \underbrace{\tau(u)}_{\in \operatorname{im}(\tau)} + \underbrace{u - \tau(u)}_{\in \operatorname{Ker}(\tau)}$.

We have shown

$U = \operatorname{im}(\tau) \oplus \operatorname{Ker}(\tau)$

$u \in U$, $u = \underset{\in \operatorname{im}(\tau)}{s} + \underset{\in \operatorname{Ker}(\tau)}{t}$

we have to show

$\tau(u) = s$

$\tau(u) = \tau(s+t) = \tau(s) + \tau(t)$

$= \tau(s) = s$

$s \in \operatorname{im}(\tau) \Rightarrow s = \tau(x) \Rightarrow \tau(s) = \tau^2(x) = \tau(x) = s$

l: show

$im(\tau) \oplus Ker(\tau) = V$

h to prove

$= im(\tau) \oplus Ker(\tau)$

ke $u \in V$

$\underbrace{\tau(u)}_{\in im(\tau)} + \underbrace{u - \tau(u)}_{\in Ker(\tau)}$.

have shown

$= im(\tau) \oplus Ker(\tau)$

V, $u = \underbrace{s}_{\in im(\tau)} + \underbrace{t}_{\in Ker(\tau)}$

have to show

$u) = s$

$(u) = \tau(s+t) = \tau(s) + \tau(t)$ ∎

$\tau(s) = s$

$im(\tau) \Rightarrow s = \tau(x) \Rightarrow \tau(s) = \tau^2(x) = \tau(x) = s$

Dfn 15 $f: S \to T$

function of sets is called

i) injective ("1-1") if $f(s) = f(s') \Rightarrow s = s'$

ii) surjective ("onto") if $\forall t \in T$ $\exists s \in S$ s.t. $t = f(s)$.

iii) bijective, if it is both injective and surjective. ∎

Prp 16 $\tau: U \to V$ is injective $\Leftrightarrow Ker(\tau) = 0$

Prf $(\Rightarrow)$ suppose τ is injective. Let $\xi \in Ker(\tau)$

$\Rightarrow \tau(\xi) = 0 = \tau(0)$

$\Rightarrow \xi = 0$.

$(\Leftarrow)$ Suppose $Ker(\tau) = 0$.

Suppose $\tau(u_1) = \tau(u_2)$

$\Rightarrow \tau(u_1 - u_2) = 0$

$\Rightarrow u_1 - u_2 \in Ker\tau$

$\Rightarrow u_1 - u_2 = 0$. ∎

Zixuan	+1
Bi Kang	+1
Shijie	+1
Ai Jia	+1
Ke	+1/2
Tao	+1
Jun Feng	+1/2

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Prp 17 $A \in K^{m\times n}$
Then $\tau_A \cdot K^n \to K^m$
is injective $\Leftrightarrow$
the columns of A are l.i.

Prf ($\Rightarrow$) Suppose τ_A is
injective $A = [v_1\ v_2 \dots v_n]$
Suppose $\sum_{i\in[n]} \alpha_i v_i = 0 \Leftrightarrow$
$A\xi = 0$, $\xi = \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{bmatrix} \Rightarrow$
$\tau_A(\xi) = 0$ By Prp 16, $Ker(\tau_A) = 0$
So $\xi = 0 \Rightarrow$ the v_i's are l.i.

($\Leftarrow$) Suppose the
v_i's are l.i.
By Prp 16, enough
to show $Ker(\tau_A) = 0$.
Take $\xi \in Ker(\tau_A)$
$\Rightarrow \tau_A(\xi) = A\xi = 0$
$\Rightarrow \sum_{i\in[n]} \alpha_i v_i = 0$, $\xi = \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{bmatrix}$
$\Rightarrow$ all α_i are zero
(since the v_i's are l.i.)
$\Rightarrow \xi = 0 \Rightarrow Ker(\tau_A) = 0$.
$\boxed{\equiv}$

Thm 13 τ
is a projec
the form
some S a
$\tau^2 = \tau$
$\tau \circ \tau$
Prf ($\Rightarrow$) $\wp$
$= \wp_{S,T}(\wp_{S,}$
$= \wp_{S,T}(s) =$