LAIS, Lecture #7

Dfn 1 [four fundamental subspaces] $A \in \mathbb{R}^{m \times n}$

$\mathcal{R}(A), \mathcal{N}(A), \mathcal{R}(A^T), \mathcal{N}(A^T)$ $\boxed{=}$

Prp 2 $\mathbb{R}^m = \mathcal{R}(A) \oplus \mathcal{N}(A^T)$, $\mathcal{R}(A)^\perp = \mathcal{N}(A^T)$

Prf $x \in \mathcal{R}(A)^\perp \iff x \perp y \ \forall y \in \mathcal{R}(A)$

$\iff x \perp A\xi \ \forall \xi \in \mathbb{R}^n \iff$

$x \perp a_j \ \forall j \in [n]$ where $A = [a_1 \cdots a_n]$

$\iff x^T A = 0 \iff A^T x = 0 \iff x \in \mathcal{N}(A^T)$

$\boxed{=}$

Prp 3 The restriction of τ_{A^T} to $\mathcal{R}(A)$ induces an isomorphism $\mathcal{R}(A) \xrightarrow{\sim} \mathcal{R}(A^T)$

Prf $\tau_{A^T}|_{\mathcal{R}(A)} : \mathcal{R}(A) \longrightarrow \mathcal{R}(A^T)$

Take a basis $u_1, \dots, u_r$ of $\mathcal{R}(A)$, $r = \dim \mathcal{R}(A)$. $\exists\ \xi_i \in \mathbb{R}^n$ s.t. $u_i = A\xi_i \ \forall i \in [r]$. Now

$A^T A\xi_1, \dots, A^T A\xi_r \in \mathcal{R}(A^T)$.

$\sum_{i \in [r]} c_i A^T A \xi_i = 0 \implies$

$A^T \left(\sum_{i \in [r]} c_i A \xi_i \right) = 0$

$$\Rightarrow \sum_{i\in[r]} c_i A\xi_i \in \mathcal{R}(A)\cap\mathcal{N}(A^T)$$

$$\Rightarrow \sum_{i\in[r]} c_i A\xi_i = 0 \Rightarrow c_i = 0 \ \forall i$$

So the $A^TA\xi_i$, $i\in[r]$ are l.i. So $\dim\mathcal{R}(A)\le \dim\mathcal{R}(A^T)$. By symmetry $\dim\mathcal{R}(A)=\dim\mathcal{R}(A^T)$, so the $A^TA\xi_i$, $i\in[r]$ is a basis for $\mathcal{R}(A^T)$. So $\tau_{A^T}|_{\mathcal{R}(A)}$ is an isomorphism.

Rem 4 Prp 3 gives an alternative proof for $K=\mathbb{R}$ of the fact $\dim\mathcal{R}(A)=\dim\mathcal{R}(A^T)$ $\Leftrightarrow$ $\operatorname{rank}(A)=\operatorname{rank}(A^T)$

Rem 5 [Ai Jia] From Prp 3 we can also conclude $\mathcal{R}(A^TA)=\mathcal{R}(A^T)$

Prp 6 $A\in\mathbb{R}^{m\times n}$, $B\in\mathbb{R}^{n\times \ell}$

$$\operatorname{rank}(AB)=\operatorname{rank}(B)-\dim\mathcal{N}(A)\cap\mathcal{R}(B)$$

Prp 7 $\mathcal{B}(AA^T) = \mathcal{B}(A)$

Prf $\mathcal{B}(AA^T) \subseteq \mathcal{B}(A)$

Take $B = A^T$ in Prp 6:

$\mathrm{rank}(AA^T) = \mathrm{rank}(A^T) - \underbrace{\dim \mathcal{N}(A) \cap \mathcal{B}(A^T)}_{=0} = \mathrm{rank}(A^T)$

So $\mathrm{rank}(AA^T) = \mathrm{rank}(A^T) =$

$= \mathrm{rank}(A)$ So

$\mathrm{rank}(AA^T) = \mathrm{rank}(A) \Longleftrightarrow$

$\dim \mathcal{B}(AA^T) = \dim \mathcal{B}(A)$ $\boxed{\equiv}$

Prf [Prp 6] Apply rank + nullity theorem on $\tau_{AB} : \mathbb{R}^l \longrightarrow \mathbb{R}^m$

$l = \dim \mathcal{N}(AB) + \mathrm{rank}(AB)$

$\mathcal{N}(AB) = \{\xi \in \mathbb{R}^l : AB\xi = 0\}$

$= \{\xi \in \mathbb{R}^l : B\xi \in \mathcal{N}(A)\}$, $\mathcal{N}(B) \subset \mathcal{N}(AB)$

$= \mathcal{N}(B) \oplus \mathcal{C}$

$\tau_B|_{\mathcal{C}} : \mathcal{C} \xrightarrow{\sim} \mathcal{B}(B) \cap \mathcal{N}(A)$

$\tau_B|_{\mathcal{C}}$ is injective. Moreover $y \in \mathcal{B}(B) \cap \mathcal{N}(A)$

$\Rightarrow y = Bx$, $Ay = 0 \Rightarrow ABx = 0$

$\Rightarrow x \in \mathcal{N}(AB) \Rightarrow x = \underbrace{x_1}_{\in \mathcal{N}(B)} + \underbrace{x_2}_{\in \mathcal{C}}$

$\Rightarrow y = Bx_2 = \tau_B|_{\mathcal{C}}(x_2) \Rightarrow \tau_B|_{\mathcal{C}}$ is surjective

$\Rightarrow \dim \mathcal{C} = \dim \mathcal{B}(B) \cap \mathcal{N}(A)$

$l = \dim \mathcal{N}(B) + \dim \mathcal{B}(B) \cap \mathcal{N}(A) + \mathrm{rank}(AB)$

$\boxed{\equiv}$

$\mathcal{B}(B)$

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Prp 8 $\mathcal{B}(A) = \bigodot_{\lambda \neq 0} \mathcal{E}_{AA^T,\lambda}$

Prf $AA^T = U_A \Lambda_A U_A^T$

$\sigma_{AA^T} = \{\lambda_1 > \lambda_2 > \cdots > \lambda_s > \lambda_{s+1} \geq 0\}$

μ_i : multiplicity of λ_i

$U_i \in \mathbb{R}^{m \times \mu_i}$: orthonormal basis for $\mathcal{E}_{AA^T,\lambda_i}$

$U_A = [U_1 \cdots U_s \ U_{s+1}]$

$\Lambda_A = \mathrm{diag}(\underbrace{\lambda_1, \dots, \lambda_1}_{\mu_1}, \dots, \underbrace{\lambda_s, \dots, \lambda_s}_{\mu_s}, \underbrace{\lambda_{s+1}, \dots, \lambda_{s+1}}_{\mu_{s+1}})$

$$AA^T = \sum_{i \in [s+1]} \lambda_i U_i U_i^T = [U_1 \cdots U_s] \begin{bmatrix} \Lambda_1 & & \\ & \ddots & \\ & & \Lambda_s \end{bmatrix} [U_1 \cdots U_s]^T$$

$$\Rightarrow \mathcal{B}(AA^T) = \mathcal{B}([U_1 \cdots U_s]) = \bigoplus_{\lambda_i \neq 0} \mathcal{B}(U_i) = \bigoplus_{\lambda_i \neq 0} \mathcal{E}_{AA^T,\lambda_i}$$

Done by Prp 7. ☺

Prp 9 $\mathbb{R}^m = \mathcal{B}(A) \odot \mathcal{N}(A^T)$

orthogonal direct sum

$$= \left(\bigodot_{\lambda_i \neq 0} \mathcal{E}_{AA^T,\lambda_i} \right) \odot \underbrace{\mathcal{N}(A^T)}_{\mathcal{N}(AA^T) = \mathcal{E}_{AA^T,0}}$$

$$\mathbb{R}^m = \underbrace{(\mathcal{E}_{AA^T,\lambda_1} \odot \cdots \odot \mathcal{E}_{AA^T,\lambda_s})}_{\mathcal{B}(A)} \odot \underbrace{\mathcal{N}(A^T)}_{\mathcal{E}_{AA^T,0}}$$

$$\uparrow \sim A$$

$$\mathbb{R}^n = \underbrace{(\mathcal{E}_{A^TA,\nu_1} \odot \cdots \odot \mathcal{E}_{A^TA,\nu_t})}_{\mathcal{B}(A^T)} \odot \underbrace{\mathcal{N}(A)}_{\mathcal{E}_{A^TA,0}}$$

$$\tau_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

Prp 10 $\sigma(AA^T) \setminus \{0\} = \sigma(A^TA) \setminus \{0\}$

Prf $0 \neq \lambda \in \sigma(AA^T) \Rightarrow \exists u \neq 0$
$AA^Tu = \lambda u \Rightarrow A^TA(A^Tu) = \lambda(A^Tu)$
$\|A^Tu\|_2^2 = \underbrace{\underbrace{\lambda}_{\neq 0}\underbrace{\|u\|_2^2}_{\neq 0}}_{\neq 0} \Rightarrow A^Tu \neq 0 \Rightarrow$
$\lambda \in \sigma(A^TA) \setminus \{0\}$.

So $\sigma(AA^T) \setminus \{0\} \leq \sigma(A^TA) \setminus \{0\}$. By symmetry we are done $\blacksquare$

Prp 11 [Refinement of Prp 3]
The restriction of τ_{A^T} on $\mathcal{E}_{AA^T,\lambda_i}$, $i \in [s]$, induces an isomorphism with $\mathcal{E}_{A^TA,\lambda_i}$.

Prf Take a basis $u_1, \ldots, u_{\mu_i}$ of $\mathcal{E}_{AA^T,\lambda_i}$. We will show that $A^Tu_1, \ldots, A^Tu_{\mu_i}$ is a basis for $\mathcal{E}_{A^TA,\lambda_i}$.

...onal
...sum

$A^T,0$

$AA^Tu_j = \lambda_i u_j \quad , j \in [\mu_i]$

$A^TA(A^Tu_j) = \lambda_i(A^Tu_j) \Rightarrow$

$A^Tu_1, \ldots, A^Tu_{\mu_i} \in \mathcal{E}_{A^TA,\lambda_i}$

Let's show they are l.i.

$\sum_j c_j A^Tu_j = 0 \Rightarrow$

$\sum c_j AA^Tu_j = 0 \Rightarrow$

$\sum c_j \underbrace{\lambda_i}_{\neq 0} u_j = 0 \Rightarrow$

$\sum c_j u_j = 0 \Rightarrow c_j = 0$

$\Rightarrow \dim \mathcal{E}_{AA^T,\lambda_i} \leq \dim \mathcal{E}_{A^TA,\lambda_i}$

By symmetry we are done ☺

$$\tau_A : \underbrace{\mathcal{E}_{A^TA,\lambda_1} \odot \cdots \odot \mathcal{E}_{A^TA,\lambda_s} \odot N(A)}_{\mathbb{R}^n} \xrightarrow{A} \underbrace{\mathcal{E}_{AA^T,\lambda_1} \odot \cdots \odot \mathcal{E}_{AA^T,\lambda_s} \odot N(A^T)}_{\mathbb{R}^m}$$

$$\mathcal{E}_{A^TA,\lambda_1} \xrightarrow{\sim} \mathcal{E}_{AA^T,\lambda_1} \qquad \mathcal{E}_{A^TA,\lambda_s} \xrightarrow{\sim} \mathcal{E}_{AA^T,\lambda_s}$$