LAIS, Lecture #11

Prp 1 An A-invariant subspace $\mathcal{V}$ is A-cyclic $\Leftrightarrow$ $\mathcal{V} = k[x]v$ for some $v \in k^n$. $\square$

Ex 2 $Av = \lambda v$, $v \neq 0$

$\mathrm{Span}(v)$ is A-cyclic

$\mathrm{ann}(v) = (x - \lambda)$ $\square$

Ex 3 $A = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 0 & 0 & 1 \\ & & 1 & 0 & 0 \\ & & 0 & 1 & 0 \end{bmatrix}$

$\mathcal{V} = \langle e_3, e_4, e_5 \rangle$: A-invariant

$k^5 = \underbrace{\mathrm{Span}(e_1)}_{x-1} \oplus \underbrace{\mathrm{Span}(e_2)}_{x-1} \oplus \underbrace{\mathcal{V}}_{A\text{-cyclic}}$

$\mathcal{V} = \mathrm{Span}(e_3, \underbrace{e_4}_{Ae_3}, \underbrace{e_5}_{A^2e_3})$

$A^3 e_3 = e_3 \Rightarrow A^3 e_3 - e_3 = 0$

$\Leftrightarrow (x^3 - 1) e_3 = 0 \Rightarrow \mathrm{ann}(\mathcal{V}) = (x^3 - 1)$ $\square$

Dfn 4 [companion matrix]

$p(x)$: monic polynomial

$p(x) = x^n + c_1 x^{n-1} + \cdots + c_{n-1}x + c_n$

$$\mathrm{Comp}[p(x)] = \begin{bmatrix} 0 & 0 & & 0 & -c_n \\ 1 & 0 & & & -c_{n-1} \\ 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & & 0 & \\ & & & 1 & -c_1 \end{bmatrix}$$

HW5 What is the characteristic polynomial of $\mathrm{Comp}[p(x)]$? Give a proof.

Prp 6 V: A-invariant

V is A-cyclic $\iff$ $\exists$ a basis $\mathcal{B}$ of V s.t. $[\tau|_A]_{\mathcal{B},\mathcal{B}} = \mathrm{Comp}[p(x)]$ for some $p(x)$.

Prf ($\Rightarrow$) Suppose V is A-cyclic. By dfn $\exists\, v \in V$ s.t. $v, Av, A^2v, \ldots, A^{d-1}v$ is a basis of V. Set $b_i = A^{i-1}v$.

$\mathcal{B} = [b_1\ b_2 \cdots b_d]$,

$$[\tau|_A]_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} 0 & & O & -c_d \\ 1 & & & -c_{d-1} \\ 0 & 1 & & \\ & O & \ddots & \vdots \\ & & 1 & -c_1 \end{bmatrix}$$

$Ab_i = b_{i+1}$, $i < d$

$\mathrm{ann}(v) = (p(x))$

$p(x) = x^d + c_1 x^{d-1} + \cdots + c_d$

$p(x)\, v = 0 \Rightarrow$

$A^d v = -(c_1 A^{d-1} v + \cdots + c_d v)$

The converse for you. ⊟

Dfn 7 V: A-invariant

$p(x) \in k[x]$

$\underbrace{V^{(p)}}_{\text{subspace of } V} = \{ v \in V \mid p(x) v = 0 \}$ ⊟

Lem 8 V, W: A-invariant

$p(x) \in k[x]$. Suppose $V \oplus W$ exists. Then

$(V \oplus W)^{(p)} = V^{(p)} \oplus W^{(p)}$

Prf The inclusion $\supseteq$ is clear. Let's prove $\subseteq$:

$\underbrace{v}_{\in V} + \underbrace{w}_{\in W} \in (V \oplus W)^{(p)} \Rightarrow$

$p(x)(v+w) = 0 \Rightarrow$

$\underbrace{p(x) v}_{\in V} = - \underbrace{p(x) w}_{\in W}$

$\Rightarrow v \in V^{(p)}, w \in W^{(p)}$ ⊟

Lem ... V, W: A-invariant

$p(x) \in K[x]$. Suppose $V \oplus W$ exists. Then

$$(V \oplus W)^{(p)} = V^{(p)} \oplus W^{(p)}$$

Prf The inclusion $\supseteq$ is clear. Let's prove $\subseteq$:

$\underset{\in V}{v} + \underset{\in W}{w} \in (V \oplus W)^{(p)} \Rightarrow$

$p(x)(v+w) = 0 \Rightarrow$

$\underbrace{p(x)v}_{\in V} = \underbrace{-p(x)w}_{\in W}$

$\Rightarrow v \in V^{(p)}, w \in W^{(p)}$ ▣

Lem 10 Suppose $\mathrm{ann}(v) = (p^e(x))$

$$(K[x]v)^{(p)} = K[x]\, p^{e-1}(x)\, v$$

Prf The inclusion $\supseteq$ is clear. Let's prove $\subseteq$.

Suppose $f(x)v \in (K[x]v)^{(p)}$

$\Rightarrow p(x)f(x)v = 0$

$\Rightarrow p(x)f(x) \in \mathrm{ann}(v)$

$\Rightarrow p^e(x) \mid p(x)f(x)$

$\Rightarrow p^{e-1}(x) \mid f(x)$ ▣

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Lem 11 Suppose $\mathrm{ann}(v) = (p(x)^e)$

$\dim (k[x]v)^{(p)} = \deg p(x)$

Prf By Lem 10

$(k[x]v)^{(p)} = k[x]\, p(x)^{e-1} v$

$\dim (k[x]v)^{(p)} = \deg q(x)$

where $\mathrm{ann}(p(x)^{e-1} v) = (q(x))$.

$q(x) p(x)^{e-1} v = 0 \Rightarrow p \mid q$.

On the other hand $p \in \mathrm{ann}(p(x)^{e-1} v)$

$\Rightarrow q \mid p$. So $p \sim q$. $\boxed{\equiv}$

$p = \pm q$

Prp 12 $A \in k^{n\times n}$. Suppose

$m_A(x) = p(x)^{\ell}$,

$p(x)$: monic, irreducible.

There exist unique integers

$\ell = \ell_1 \geq \ell_2 \geq \dots \geq \ell_s \geq 1$ and

$v_i \in k^n$, $i \in [s]$, s.t.

$k^n = \bigoplus_{i\in[s]} k[x] v_i$ and $\mathrm{ann}(v_i) = (p(x)^{\ell_i})$

$A \in k^{n\times n}$

$m_A(x)$

$\lambda \in \sigma(A)$ $\Rightarrow x - \lambda \mid$

Step

HW,

Prf (Existence)

Step 1

First we show the existence of a $v \in K^n$ s.t. $\mathrm{ann}(v) = (p^{\ell}(x))$.
Take any $u \in K^n$. Then $p^{\ell}(x)u = 0$. If $\mathrm{ann}(u) = (g(x))$
$\Rightarrow g(x) \mid p^{\ell}(x)$, and since $p(x)$ is irreducible $g(x) = p^{\ell_u'}(x)$, $\ell_u' \le \ell$.
Suppose $\mathrm{ann}(u) = (p^{\ell_u'}(x))$, $\ell_u' < \ell$ $\forall u \in K^n$.
So $p^{\ell-1}(x) \in \mathrm{ann}(K^n) \Rightarrow\Leftarrow$

Step 2

So let $v_1 \in K^n$, $\mathrm{ann}(v_1) = (p^{\ell}(x))$.
Suppose $K[x]v_1$ has an A-invariant complement V: $K^n = K[x]v_1 \oplus V$.
Then $\dim V < n$. Also $\mathrm{ann}(V) = (p^{e}(x))$, $e \le \ell$.
So we are done by induction on n, by applying the statement to V.

$\left.^{\ell}(x)\right).$

ment

ℓ

Step 3 If $k^n = k[x]v_1$, then we are done. So suppose $k[x]v_1 \subsetneq k^n$. Set $U_0 = 0$. We will show by induction the existence of A-invariant subspace U_j s.t. $k[x]v_1 \cap U_j = 0$ and if $k[x]v_1 \oplus U_j \subsetneq k^n$ $\exists$ $U_{j+1} \supsetneq U_j$ s.t. $k[x]v_1 \cap U_{j+1} = 0$

Step 4 Suppose by induction $\exists$ A-invariant U_j s.t. $k[x]v_1 \cap U_j = 0$ and $k[x]v_1 \oplus U_j \subsetneq k^n$.

Take $u \in k^n \setminus k[x]v_1 \oplus U_j$.

Define $I = \{f(x) \in k[x] : f(x)u \in k[x]v_1 \oplus U_j\}$

I is an ideal, $I = (g(x))$

$p^{\ell}(x)u = 0 \in k[x]v_1 \oplus U_j \Rightarrow$

$g(x) = p^{\ell'}(x)$, $\ell' \leq \ell$

$\operatorname{ann}(v_i) = (p(x)^{l_i})$

$A \in \mathbb{C}^{n \times n}$

$m_A(x)$

$\lambda \in \sigma(A) \Longrightarrow x - \lambda \mid m_A(x)$

HW, Tuesday