LAIS, Lec #1

what: linear algebra and applications in information science
- → theory → spectral theory
- → applications
- → project

why: because learning very good linear algebra is one of the best investments you can make in your education

who: 2 × PhD

References
i) Carl Meyer's "Matrix Analysis and Applied Linear Algebra" (undergrad)
ii) Steven Roman Linear (graduate)
iii) lecture notes

Dfn 1 Vector Space over a field K ($\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$) is a set V with an operation $+$

$+ : V \times V \to V$

i) $\forall x, y \in V$, $x+y \in V$

ii) $(x+y)+z = x+(y+z)$ associativity

iii) $x+y = y+x$ commutative

iv) $\exists\, \underbrace{0}_{\text{identity with respect to } +} \in V$ s.t. $x+0 = x \;\forall x \in V$

v) $\forall x \in V \;\exists\, y \in V$ s.t. $x+y=0$
additive inverse $\underbrace{y}_{-x}$

V is a abelian group

V is an abelian group

together with an "action" of the field K on V

$K \times V \to V$

i) $\forall \alpha \in K, \forall x \in V$
$\alpha x \in V$

ii) $(\alpha + b)x = \alpha x + bx$

iii) $\alpha(x+y) = \alpha x + \alpha y$

iv) $(\alpha b)x = \alpha(bx)$

v) $1x = x$

ring homomorphism

$K \to \mathrm{End}(V)$

$V \xrightarrow{f} V$

$f(x+y) = f(x) + f(y)$

group homomorphism

$f, g \in \mathrm{End}(V)$

$g \circ f \in \mathrm{End}(V)$

$g + f \in \mathrm{End}(V)$

Ex 2 Examples of vector spaces

i) K^n ($\mathbb{R}^n$): coordinate vector space

$x \in K^n$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

ii) $K^{m \times n}$: set of $m \times n$ matrices with entries in K

$A, B \in K^{m \times n}$ $\quad A + B \in K^{m \times n}$

$\alpha \in K$, $\alpha A \in K^{m \times n}$

iii) S : set, $f : S \to K$

$f, g : S \to K$

$(f+g)(s) = f(s) + g(s)$

$\alpha \in K$

$f : S \to$

$(\alpha f)(s) =$

Dfn 3 Linear subspace

S: linear subspace of vector space V if S is a vecto

together with an "action" of the field K on V

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S: linear subspace of vector space V if S is a vector space on its own using the operations of V.

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Dfn 3 Linear subspace

S : linear subspace of vector space V if S is a vector space on its own using the operations of V. ⊟

Prp 4 S is a linear subspace of V if S is closed under linear combinations, i.e., if $\forall\, \alpha_1, \alpha_2 \in K$ $\forall\, x_1, x_2 \in S$ $\alpha_1 x_1 + \alpha_2 x_2 \in S$ ⊟ HW

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Ex5 [Subspaces]

i) $V = K^n$, fix $1 \le i_1 < \dots < i_\ell \le n$

$$S = \left\{ \underset{\sim}{x} \in V : x_{i_j} = 0 \ \forall \ j = 1, \dots, \ell \right\}$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

ii) $V = K^{n \times n}$

S: upper-triangular matrices

lower-triangular matrices

diagonal matrices

$$S_{\substack{\text{upper} \\ \text{triangular}}} = \left\{ \underset{\sim}{X} \in K^{n \times n} : j < i \Rightarrow x_{ij} = 0 \right\}$$

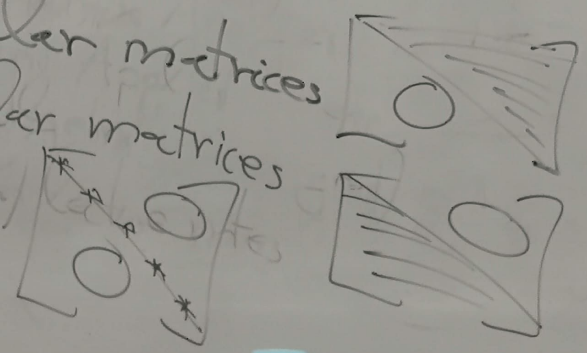
(x_{ij})

$$\underset{n \times n}{\underset{\sim}{A}} x = b \iff L \underbrace{U x}_{y} = b$$

$A = LU$

$Ly = b$

$Ux = y$



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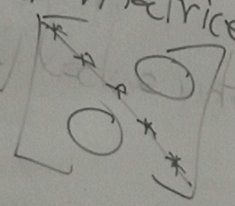
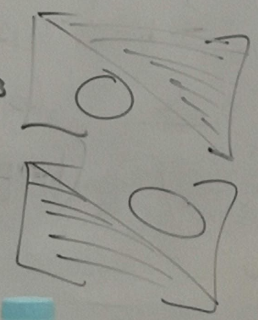
S: upper-triangular matrices

lower-triangular matrices

diagonal matrices





$$S_{\substack{\text{upper} \\ \text{triangular}}} = \left\{ \underset{\sim}{X} \in K^{n \times n} : j < i \Rightarrow x_{ij} = 0 \right\}$$

$$(x_{ij})$$

$$S_{\text{Diagonal}} = S_{\text{upper-triangular}} \cap S_{\text{lower-triangular}}$$

iv) symmetric matrices

X : symmetric $\Leftrightarrow x_{ij} = x_{ji} \ \forall \ i, j$

$$\underset{n \times n}{\underset{\sim}{A}} x = b \Leftrightarrow L \underset{y}{U x} = b$$

$$A = LU$$

$$Ly = b$$

$$Ux = y$$

$$X \subset \mathbb{R}^{m \times n}$$

$$X \quad X^T$$

V is abelian group

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Ex 5 [Subspaces]

$V = K^n$, fix $1 \le i_1 < \dots < i_\ell \le n$

$= \{ \underset{\sim}{x} \in V : x_{i_j} = 0 \ \forall \ j = 1, \dots, \ell \}$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

ii) $V = K^{n \times n}$

S: upper-triangular matrices

lower-triangular matrices

diagonal matrices

$S_{\text{upper triangular}} = \{ \underset{\sim}{X} \in K^{n \times n} : j < i \Rightarrow x_{ij} = 0 \}$

(x_{ij})

$S_{\text{Diagonal}} = S_{\text{upper-triangular}} \cap S_{\text{lower-triangular}}$

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X: symmetric $\Leftrightarrow x_{ij} = x_{ji} \ \forall \ i, j$

$\Leftrightarrow X = X^T$

$\underset{n \times n}{A} x = b \Leftrightarrow L \underbrace{U x}_{y} = b$

$A = LU$

$Ly = b$

$Ux = y$

$X \in \mathbb{R}^{m \times n}$

Y X X^T

X, Y: symmetric $n \times n$

$X + Y \in K^{n \times n}$

$(X+Y)^T = X^T + Y^T = X + Y$

v) skew-
matr
$x_{ij} =$

Prp 6

$j < i \Rightarrow x_{ij} = 0\}$

$\text{ular} \cap S_{\text{lower-triangular}}$

matrices

$\Leftrightarrow x_{ij} = x_{ji} \ \forall i,j$

X^T

$X \in \mathbb{R}^{m \times n}$

$Y \quad X \quad X^T$

X, Y: symmetric $n \times n$

$\text{ce } X + Y \in K^{n \times n}$

$(X+Y)^T = X^T + Y^T = X + Y$

v) skew-symmetric matrices

$x_{ij} = -x_{ji} \ \forall i,j$

Prp 6 $K^{n \times n} = S_{\text{symmetric}} + S_{\text{skew-symmetric}}$ HW $\square$

Dfn 7 linear equations: $Ax = 0$

Prp 8 Every linear subspace is the solution set of a homogeneous system of linear equations $Ax = 0$

↳ we will see prf later

$\in C(\ )$

Ex 2 Examples

vector space

i) $K^n(\mathbb{R}^n)$: co

$x \in K$

ii) K^n

matric

A

$f, g \in C_r$

$g \circ f \in C_n$

$g + f \in C_n$

Kew-symmetric
matrices

$x_{ij} = -x_{ji} \quad \forall i,j$

6 $K^{n\times n} = S_{symmetric} + S_{skew-symmetric}$ HW

n 7 linear equations: $Ax = 0$

p 8 Every linear subspace
the solution set of a
omogeneous system of linear
equations $Ax = 0$
↳ we will see prf later

vector space ⟶ subspaces
↓
new subspaces

S_1, S_2 : subspaces

$S_1 \cap S_2$: subspace

largest linear subspace HW
contained both in S_1, S_2

Dfn 9 $S_i,\ i \in [l] = \{1, \dots, l\}$
sum of the S_i is

$$\sum_{i\in[l]} S_i = \left\{ \sum_{i\in[l]} \alpha_i x_i : \begin{array}{l} \forall \alpha_i \in K \\ \forall x_i \in S_i \end{array} \right\}$$

smallest linear subspace
that contains all S_is

Q: Is $S_1 \cup S_2$ a subspace?

$\alpha_1, \alpha_2 \in K \quad \forall x_1, x_2$
$\alpha_1 x_1 + \alpha_2 x_2 \in S$

vector space $\longrightarrow$ subspaces

$\downarrow$

new subspaces

S_1, S_2 : subspaces

$S_1 \cap S_2$: subspace

largest linear subspace contained both in S_1, S_2 HW

Dfn 9 $S_i,\ i \in [\ell] = \{1, \dots, \ell\}$

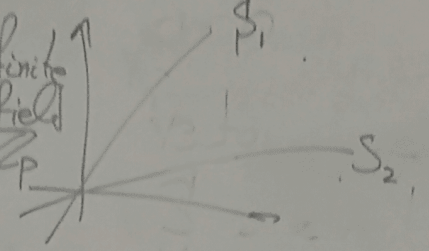
sum of the S_i is

$$\sum_{i \in [\ell]} S_i = \left\{ \sum_{i \in [\ell]} \alpha_i x_i : \forall \alpha_i \in K,\ \forall x_i \in S_i \right\} \ominus$$

smallest linear subspace that contains all S_is

Q: Is $S_1 \cup S_2$ a subspace?

cryptography, coding theory } finite field $\mathbb{Z}_p$



Jia Qi +1
Bi Kang +1
Tao +1

Prp 10 K : infinite

$S_i \subset K^n$

$S_i,\ i \in [\ell]$: linear subspaces of K^n. Then $\bigcup_{i \in [\ell]} S_i \subsetneq K^n$ $\ominus$ HW

Dfn 11 [direct sum, independent subspaces]

$S_i,\ i \in [\ell]$, $S_i \subset V$

$\sum_{i \in [\ell]} S_i$ is a direct sum if $\forall i$ $S_i \cap \left(\sum_{j \neq i} S_j \right) = 0$ $\ominus$

$\iff$ subspaces are independent

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Ex 12 [direct sum, independent subspaces]

i) two distinct lines in $\mathbb{R}^2$ are independent

ii) three lines in $\mathbb{R}^2$ not independent

iii) three lines in $\mathbb{R}^3$ are independent $\Leftrightarrow$ they are not coplanar

iv) two planes in $\mathbb{R}^3$ not independent

Dfn 13 [l.i. vectors]

$v_1, \dots, v_\ell \in V$ are l.i. if

whenever $\sum_{i \in [\ell]} \alpha_i v_i = 0$

then $\alpha_i = 0 \ \forall i \in [\ell]$ ⊟

Prp 14 The vectors HW

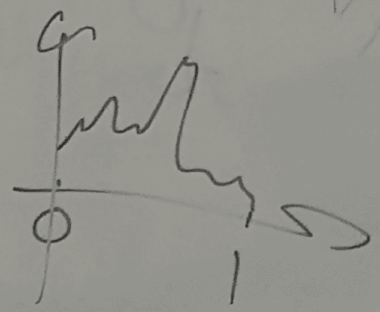
$v_1, \dots, v_\ell \in V$ are l.i.

$\iff$ the subspaces $S_i = \mathrm{span}(v_i)$

$i \in [\ell]$ are independent. ⊟

Dfn 15 The dimension of V is the size of a maximal l.i. set in V = size of a minimal spanning set of V. Such a set is called a basis of V.

$\underbrace{K^n}_{\dim = n}, \underbrace{K^{m \times n}}_{\dim = mn}$



$f : [0,1] \to \mathbb{R}$

Thm 17 [Grassmann Formula] HW

S, T: linear subspaces of V ($\dim V < +\infty$)

$$\dim(S+T) + \dim(S \cap T) = \dim(S) + \dim(T)$$

Thm 16 [complement]

$S \subset V$ then $\exists\, T \subset V$ s.t. $V = S \oplus T$

i) $V = S + T$

ii) $S \cap T = 0$

Prf Induction on $\operatorname{codim} S = \dim V - \dim S = c$

If $c = 0$ then $S = V$, $T = 0$. $V = S \oplus 0$

If $c > 0$ then $S \subsetneq V \Rightarrow \exists\, w \in V \setminus S$

$S' = S \oplus \operatorname{span}(w)$ now $\operatorname{codim} S' = \operatorname{codim} S - 1$

By induction $\exists\, T' \subset V$ s.t. $V = S' \oplus T'$

$$= S \oplus \underbrace{[\operatorname{span}(w) \oplus T']}_{T} \quad \boxtimes$$

Lem 18

Complements are not unique $\boxtimes$

Jia Qu +1

Bu Kang +1

Tao +1

Ke +1

Ai Jia +1

Junteng +1