LAIS, Lecture #13

Thm1 [cyclic-primary decomposition]

$A\in K^{n\times n}$, $m_A(x) = p_1^{l_1}(x)\cdots p_s^{l_s}(x)$

$\forall\, i\in[s]$ there exist unique integers $l_i = l_{i1}\ge l_{i2}\ge \cdots \ge l_{is_i}\ge 1$

s.t. $K^n = \bigoplus_{\substack{i\in[s]\\ j\in[s_i]}} V_{ij}$, where $V_{ij} = K[x]v_{ij}$

for some $v_{ij}\in K^n$ with $\mathrm{ann}(v_{ij}) = (p_i^{l_{ij}}(x))$

$\varphi_A : K[x] \longrightarrow \mathrm{End}(K^n)$

$p(x) \longmapsto \varphi_A(p(x)) = p(A)$

$v_{ij}\in N(p_i^{l_{ij}}(A))$

$v_{ij}\notin N(p_i^{l_{ij}-1}(A))$

Dfn2 $A\in K^{n\times n}$

The elementary divisors of A are $\{p_i^{l_{ij}}(x)\}_{\substack{i\in[s]\\ j\in[s_i]}}$

Thm3 [Rational Canonical Form]

A is similar to

$\mathrm{bdiag}\left(\mathrm{Comp}[p_i^{l_{ij}}(x)] : \substack{i\in[s]\\ j\in[s_i]}\right)$

block diagonal

Prf We have seen that a subspace $V \subset K^n$ is A-cyclic $\Leftrightarrow$ $\tau_A|_V$ admits a matrix representation $\mathrm{Comp}[q(x)]$. Specifically V is A-cyclic $\Leftrightarrow$ $V = K[x]v$ for some v and $\dim V =$ degree of the generator $q(x)$ of $\mathrm{ann}(v)$. Thus every V_{ij} admits a basis B_{ij} s.t.

$$\left[\tau_A|_{V_{ij}}\right]_{B_{ij},B_{ij}} = \mathrm{Comp}\left[p_i^{l_{ij}}(x)\right].$$

Now $B = \{B_{ij}\}_{\substack{i\in[s]\\ j\in[s_i]}}$ is a basis of K^n and

$$[\tau]_{B,B} = \mathrm{bdiag}\left(\mathrm{Comp}[p_i^{l_{ij}}(x)]_{\substack{i\in[s]\\ j\in[s_i]}}\right). \quad \blacksquare$$

Lem 4 $A \overset{\text{sim}}{\sim} A' \Rightarrow m_A(x) = m_{A'}(x)$

Prf $A' = SAS^{-1}$

$m_A(A) = 0 \Rightarrow$

$S\, m_A(A)\, S^{-1} = 0 \Rightarrow$

$m_A(SAS^{-1}) = 0 \Rightarrow$

$SA^iS^{-1} = (SAS^{-1})^i$

$m_A(A') = 0. \quad \blacksquare$

Lem 5 $A \overset{\text{sim}}{\sim} A'$ then A, A' have the same elementary divisors. $\square$

Prp 6 Similar matrices have the same rational canonical form. $\square$

Dfn 7 [characteristic polynomial]

$$P_A(x) = \det(xI - A) \quad \square$$

Prp 8 $P_{\mathrm{Comp}[q(x)]}(x) = q(x)$ $\square$

Thm 9 $P_A(x) = \prod_{\substack{i\in[s]\\ j\in[s_i]}} p_i^{l_{ij}}(x)$

Prf $A \overset{\text{sim}}{\sim} \mathrm{bdiag}\left(\mathrm{Comp}[p_i^{l_{ij}}(x)]\right)_{i\in[s], j\in[s_i]}$

$$P_A(x) = P_{\mathrm{bdiag}(\mathrm{Comp}[p_i^{l_{ij}}(x)], i\in[s], j\in[s_i])}(x)$$

$$= \prod_{\substack{i\in[s]\\ j\in[s_i]}} P_{\mathrm{Comp}[p_i^{l_{ij}}(x)]}(x) = \prod_{\substack{i\in[s]\\ j\in[s_i]}} p_i^{l_{ij}}(x) \quad \square$$

Thm 10 [Cayley-Hamilton]

$$P_A(A) = 0$$

Prf By Thm 9 $m_A(x) = \prod_{i\in[s]} p_i^{l_{i1}}(x)$ divides $P_A(x) = \prod_{\substack{i\in[s]\\ j\in[s_i]}} p_i^{l_{ij}}(x)$. $\square$

Dfn 11 [Jordan block]

$\lambda \in K$, ℓ positive integer

$$J(\lambda,\ell) = \begin{bmatrix} \lambda & & & \\ 1 & \ddots & & \\ & \ddots & \lambda & \\ & & 1 & \lambda \end{bmatrix} \in K^{\ell\times\ell}$$

Prp 12 $\mathrm{Comp}[(x-\lambda)^\ell]$ is similar to $J(\lambda,\ell)$.

Prf $A = \mathrm{Comp}[(x-\lambda)^\ell] = \begin{bmatrix} 0 & & 0 & -c_\ell \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & -c_2 \\ 0 & & 1 & -c_1 \end{bmatrix}$

$$(x-\lambda)^\ell = \sum_{i=0}^{\ell} \underbrace{\binom{\ell}{i}(-\lambda)^{\ell-i}}_{c_{\ell-i}} x^i$$

K^ℓ has a basis

$e_1, \underbrace{Ae_1}_{e_2}, \underbrace{A^2e_1}_{e_3}, \ldots, \underbrace{A^{\ell-1}e_1}_{e_\ell}$

Consider

Jordan chain $\leftarrow$ $e_1, (x-\lambda)e_1, (x-\lambda)^2e_1, \ldots, (x-\lambda)^{\ell-1}e_1$

it is also a basis of K^ℓ.

We can see that by induction, by noting that

$\mathrm{span}(e_1, xe_1, \ldots, x^ie_1) = \mathrm{span}(e_1, (x-\lambda)e_1, \ldots, (x-\lambda)^ie_1)$.

Set $b_i = (x-\lambda)^{i-1}e_1$ $\forall i \in [\ell]$. $B = \{b_1, \ldots, b_\ell\}$

$[\tau_A]_{B,B}$

$$[Ab_i]_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{-th entry}$$

$Ab_i = xb_i = (x-\lambda+\lambda)b_i$

$= b_{i+1} + \lambda b_i \quad \forall i = 1, \ldots, \ell-1$

$Ab_\ell = xb_\ell = x(x-\lambda)^{\ell-1}e_1$

$= (x-\lambda+\lambda)(x-\lambda)^{\ell-1}e_1$

$= \underbrace{(x-\lambda)^\ell e_1}_{\to 0} + \lambda b_\ell = \lambda b_\ell$

Thm 13 Suppose $k = \bar{k}$
$m_A(x) = (x-\lambda_1)^{l_1} \cdots (x-\lambda_s)^{l_s}$,
elementary divisors
are $\{(x-\lambda_i)^{l_{ij}},\ \begin{matrix} i \in [s] \\ j \in [s_i] \end{matrix}\}$.
Then A is similar to
$\mathrm{bdiag}\left(J(\lambda_i, l_{ij}) : \begin{matrix} i \in [s] \\ j \in [s_i] \end{matrix}\right)$

Prf By Thm 3
$A \overset{\text{sim}}{\sim} \mathrm{bdiag}\left(\mathrm{Comp}[p_i^{l_{ij}}(x)]_{i,j}\right)$
Since $k = \bar{k}$ $\quad p_i(x) = (x-\lambda_i)$
By Prp 12 $\mathrm{Comp}[(x-\lambda_i)^{l_{ij}}] \overset{\text{sim}}{\sim}$
$J(\lambda_i, l_{ij})$ $\boxed{\equiv}$

Ex 14 $A_1 = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 & & \\ 1 & 2 & \\ & 1 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & & \\ 1 & 2 & \\ & & 2 \end{bmatrix}$

$p_{A_1}(x) = p_{A_2}(x) = p_{A_3}(x) = (x-2)^3$

$m_{A_1}(x) = x-2 \quad m_{A_2}(x) = (x-2)^3 \quad m_{A_3}(x) = (x-2)^2$

$k^3 = \mathrm{span}(e_1) \oplus \mathrm{span}(e_2) \oplus \mathrm{span}(e_3) \quad (A_1)$. $\boxed{\equiv}$

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Lem 15 $A = J(\lambda, \ell)$

$W_i = \{ v \in K^\ell : (x-\lambda)^i v = 0 \}$

$= \mathcal{N}\left((A-\lambda I)^i \right)$. Then

$W_1 \subseteq W_2 \subseteq \cdots \subseteq W_\ell = K^\ell$

$\dim W_i = i$

Prf $A - \lambda I = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$

It takes e_i to e_{i+1} $\forall\, i \in [\ell-1]$
and it takes e_ℓ to zero
Apply this inductively.

Ex 16 [Ferrers diagram]

partitions of integers

$3 = 3 = 2+1 = 1+1+1$

$$\begin{array}{ll} 3 & \cdot\ \cdot\ \cdot \\ 2+1 & \cdot\ \cdot \\ & \cdot \\ 1+1+1 & \cdot \\ & \cdot \\ & \cdot \end{array}$$

Dfn 11 [Jordan block]

$\lambda \in K$, l: positive integer

$$J(\lambda, l) = \begin{bmatrix} \lambda & & & \\ 1 & \ddots & & \\ & \ddots & \lambda & \\ & & 1 & \lambda \end{bmatrix} \in K^{l \times l}$$

Thm 17 $K = \overline{K}$, $A \in K^{n \times n}$

$\lambda \in K$ is an eigenvalue

$\alpha_i = \dim \mathcal{N}((A - \lambda I)^i) \quad i = 1, 2, 3, \ldots$

i) The geometric multiplicity of λ is α_1. This is also the number of Jordan blocks of the form $J(\lambda, *)$.

ii) The size of the largest Jordan block of λ is the smallest i for which $\alpha_i = \alpha_{i+1}$.

iii) build a Ferrers diagram whose i-th column has $\alpha_{i+1} - \alpha_i$ dots. ($\alpha_0 = 0$) Then the sizes of the Jordan blocks are the lengths of the rows of the diagram.