## LAIS, Lecture #14

**Dfn 1** A matrix $A \in K^{n\times n}$ is called diagonalizable if it is similar to a diagonal matrix, i.e. if $\exists$ invertible $S \in K^{n\times n}$ and diagonal $\Lambda \in K^{n\times n}$ s.t. $A = S\Lambda S^{-1}$. ⊡

$\operatorname{spectrum}_K(A) = \{\lambda_1, \ldots, \lambda_s\}$

$\mu_\alpha(\lambda_i)$: algebraic multiplicity

$\mu_\gamma(\lambda_i)$: geometric multiplicity

**Thm 2** $A \in K^{n\times n}$. TFAE:

1) $A$ is diagonalizable
2) $A$ is similar to $\operatorname{diag}(\underbrace{\lambda_1, \ldots, \lambda_1}_{\mu_\alpha(\lambda_1)}, \ldots, \underbrace{\lambda_s, \ldots, \lambda_s}_{\mu_\alpha(\lambda_s)})$
3) There are $n$ l.i. eigenvectors.
4) $K^n = \mathcal{E}_{A,\lambda_1} \oplus \cdots \oplus \mathcal{E}_{A,\lambda_s}$
5) $\mu_\gamma(\lambda_i) = \mu_\alpha(\lambda_i) \ \forall i$
6) The elementary divisors of $A$ are of the form $x - \lambda_i$
7) $m_A(x) = (x-\lambda_1)\cdots(x-\lambda_s)$

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$\text{spectrum}_K(A) = \{\lambda_1, \dots, \lambda_s\}$

$\mu_a(\lambda_i)$: algebraic multiplicity

$\mu_g(\lambda_i)$: geometric multiplicity

**Thm 2** $A \in K^{n\times n}$ TFAE:

1) $A$ is diagonalizable
2) $A$ is similar to $\operatorname{diag}(\underbrace{\lambda_1, \dots, \lambda_1}_{\mu_a(\lambda_1)}, \dots, \underbrace{\lambda_s, \dots, \lambda_s}_{\mu_a(\lambda_s)})$
3) There are $n$ l.i. eigenvectors.
4) $K^n = \mathcal{E}_{A,\lambda_1} \oplus \dots \oplus \mathcal{E}_{A,\lambda_s}$
5) $\mu_g(\lambda_1) + \dots + \mu_g(\lambda_s) = n$
6) The elementary divisors of $A$ are of the form $x - \lambda_i$
7) $m_A(x) = (x-\lambda_1)\cdots(x-\lambda_s)$

Prf 1) => (2) clear by definition 2) => 3)

$A = S \Lambda S^{-1} \Rightarrow$

$S = [u_{11} \cdots u_{1\mu_a(\lambda_1)} \cdots u_{s1} \cdots u_{s\mu_a(\lambda_s)}]$

$AS = S\Lambda \Rightarrow Au_{ij} = \lambda_i u_{ij}$

thus the $u_{ij}$'s are eigenvectors, $n$ of them. They are l.i. because $S$ is invertible

3) => 4) We already know

HW $\sum_{i \in [s]} \mathcal{E}_{A,\lambda_i} = \bigoplus_{i \in [s]} \mathcal{E}_{A,\lambda_i}$. Now

$\dim \bigoplus_{i \in [s]} \mathcal{E}_{A,\lambda_i} = \sum_{i \in [s]} \dim \mathcal{E}_{A,\lambda_i}$

and its dimension will be $n$.

4) => 5) $m_A(x) = (x-\lambda_1)^{l_1} \cdots (x-\lambda_s)^{l_s} q_1^{\mu_1}(x) \cdots q_t^{\mu_t}(x)$ with $q_i(x)$ distinct irreducible of degree $>1$ over $K$. The elementary divisors of $A$ are $(x-\lambda_1)^{l_{11}}, \ldots, (x-\lambda_1)^{l_{1s_1}}$ $\cdots (x-\lambda_s)^{l_{s1}} \cdots (x-\lambda_s)^{l_{ss}}$, $q_1^{\mu_{11}}(x), \cdots, q_1^{\mu_{1t_1}}(x)$ $\cdots q_t(x)^{\mu_{t1}} \cdots q_t(x)^{\mu_{tt_t}}$. $A$ is similar to $\mathrm{bdiag}(\mathrm{Comp}[(x-\lambda_1)^{l_{11}}], \ldots, \mathrm{Comp}[q_t^{\mu_{tt_t}}(x)]$

and $\mathrm{Comp}[(x-\lambda)^l] \overset{\text{sim}}{\sim} J(\lambda, l)$.

$\mu_a(\lambda_i)$ is the sum of the sizes of Jordan blocks of $\lambda_i$. $\mu_g(\lambda)$ is the the number of such blocks.

$= (x-\lambda_1)^{\ell_1} \cdots (x-\lambda_s)^{\ell_s} q_1^{\mu_1}(x) \cdots q_t^{\mu_t}(x)$

-t irreducible of degree

elementary divisors

$\lambda_1)^{\ell_{11}}, \ldots, (x-\lambda_1)^{\ell_{1s_1}}$

$-\lambda_s)^{\ell_{ss}}, \; q_1^{\mu_{11}}(x), \ldots, q_1^{\mu_{1t_1}}(x)$

$(x)^{\mu_{tt_t}}$. $A$ is similar

$p[(x-\lambda_1)^{\ell_{11}}], \ldots, \mathrm{Comp}[q_t^{\mu_{tt_t}}(x)])$

$(x-\lambda)^{\ell}] \overset{\text{sim}}{\sim} J(\lambda, \ell)$.

e sum of the sizes of

-s of $\lambda_i$. $\mu_g(\lambda)$ is the

such blocks.

So $\mu_g(\lambda_i) \le \mu_a(\lambda_i)$

$K^n = \bigoplus_{i \in [s]} E_{A,\lambda_i} \Rightarrow$

$n = \sum_{i \in [s]} \underbrace{\mu_g(\lambda_i)}_{\dim E_{A,\lambda_i}} \le$

$\le \sum_{i \in [s]} \mu_a(\lambda_i) \le n$

So $\mu_g(\lambda_i) = \mu_a(\lambda_i) \; \forall i$

4) $\Rightarrow$ 5)

$$n = \sum_{i\in[s]} \dim \mathcal{E}_{A,\lambda_i} = \sum_{i\in[s]} \mu_g(\lambda_i)$$

5) $\Rightarrow$ 6) In the rational canonical form there will be $n$ Jordan blocks (one l.i. eigenvector per Jordan block). Since $A$ is $n\times n$, all Jordan blocks must have size $1\times 1$ and also there are no other elementary divisors beyond $x-\lambda_i$'s

6) $\Rightarrow$ 7) clear

7) $\Rightarrow$ 1) the rational canonical form of $A$ is block diagonal with blocks $\underbrace{\mathrm{Comp}[x-\lambda_i]}_{=[\lambda_i]}$ so it is a diagonal matrix.

7) ⇒ 1) the rational canonical form of A is block diagonal with blocks $\underbrace{\mathrm{Comp}[x-\lambda_i]}_{=[\lambda_i]}$

so it is a diagonal matrix.

$\lambda_i$'s

Dfn 3 [Gersgorin disks]

i-th row Gersgorin disk of $A \in \mathbb{C}^{n\times n}$

$$G_i(A) = \{z \in \mathbb{C} \mid |z-a_{ii}| \le \sum_{j\neq i} |a_{ij}|\}$$

The j-th column Gersgorin disk is

$$G^j(A) = \{z \in \mathbb{C} \mid |z-a_{jj}| \le \sum_{i\neq j} |a_{ij}|\}$$

Gersgorin region of A

$$\underbrace{\left(\bigcup_i G_i(A)\right) \cap \left(\bigcup_j G^j(A)\right)}_{G(A)}$$

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n disk

$a_{ii}| \le \sum_{j\neq i} |a_{ij}|\}$

The j-th column Gersgorin disk is

$$G^j(A) = \{z \in \mathbb{C} \mid |z - a_{jj}| \le \sum_{i \neq j} |a_{ij}|\}$$

Gersgorin region of A

$$\left(\bigcup_i G_i(A)\right) \cap \left(\bigcup_j G^j(A)\right)$$

$$G(A) \quad \boxed{=}$$

**Thm 4** [Gersgorin disk thm]

$A \in \mathbb{C}^{n\times n}$,

$\mathrm{spectrum}_{\mathbb{C}}(A) \subseteq G(A)$

Prf Say $Au = \lambda u$. We may take $\|u\|_\infty = 1$. Let $i \in [n]$ be s.t. $u_i = 1$.

$$\sum_{j=1}^{n} a_{ij} u_j = \lambda u_i \Rightarrow$$

$$\sum_{j \neq i} a_{ij} u_j = (\lambda - a_{ii}) u_i = \lambda - a_{ii}$$

$$\Rightarrow |\lambda - a_{ii}| = \left|\sum_{j\neq i} a_{ij} u_j\right| \le \sum_{j \neq i} |a_{ij}||u_j| \le \sum_{j\neq i} |a_{ij}|$$

$$\Rightarrow \lambda \in G_i(A)$$

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$\text{spectrum}_{\mathbb{C}}(A) = \text{spectrum}_{\mathbb{C}}(A^T)$

So $A^T v = \lambda v$ for some $v \neq 0$.

So $\lambda \in G^j(A)$ for some $j$. ∎

Dfn 5 $A \in \mathbb{C}^{n \times n}$ is strictly diagonally dominant if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ $\forall i$. ∎

Prp 6 If $A$ is strictly diagonally dominant, then $A$ is invertible.

Prf $A$ is non-invertible $\iff$ $A$ has a zero eigenvalue $\Rightarrow$ $|a_{ii}| \leq \sum_{j \neq i} |a_{ij}|$ for some $i$. ∎

"Variational and interlacing theorems of eigenvalues for symmetric matrices"

$A \in \mathbb{R}^{n\times n}$, symmetric

$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$

$\lambda_i(A) = +\infty$, $i<1$

$\lambda_i(A) = -\infty$, $i>n$

$U = [u_1 \cdots u_n]$ o.n. basis of eigenvectors

$Au_i = \lambda_i(A) u_i$

**Thm 7** $\lambda_1(A) = \max\limits_{\|x\|_2=1} x^T A x$

$\lambda_n(A) = \min\limits_{\|x\|_2=1} x^T A x$ HW

Prf [first]

$A = \underbrace{U \Lambda U^T}_{\text{spectral decomposition}}$

$x^T A x = x^T U \Lambda \underbrace{U^T x}_{y}$

$= y^T \Lambda y =$

$= \sum\limits_{i\in[n]} \lambda_i(A) y_i^2 \leq \lambda_1(A) \|y\|_2^2$

when $\|x\|_2=1$. $= \lambda_1(A)$

$x^T A x$ So $\max_{\|x\|_2=1} x^T A x \le \lambda_1(A)$

HW

$u_1^T A u_1 = \lambda_1(A)$ ▯

**Thm 8** [Courant-Fischer]

$\forall i$

$$\lambda_i(A) = \max_{\dim V = i} \min_{\substack{x \in V \\ \|x\|_2 = 1}} x^T A x$$ ▯