LAIS, Lecture #5

Dfn 1 [orthogonal vectors]
$x, y \in \mathbb{R}^n$ are orthogonal,
$x \perp y$, if $x^T y = 0$. $\boxminus$

Dfn 2 [orthonormal matrix]
$U \in \mathbb{R}^{n \times s}$, $s \leq n$, is orthonormal
if $U^T U = I_s$. $\boxminus$

Dfn 3 [orthogonal matrix]
$O \in \mathbb{R}^{n \times n}$ is orthogonal, if
$O^T O = I_n$. $\boxminus$

Dfn 4 [orthogonal subspaces]
$\mathcal{V}, \mathcal{W}$ are orthogonal,
$\mathcal{V} \perp \mathcal{W}$, if $v \perp w$
$\forall v \in \mathcal{V}, w \in \mathcal{W}$. $\boxminus$

Dfn 5 [orthogonal complement]
$\mathcal{V} \subset \mathbb{R}^n$, $\mathcal{V}^\perp = \{x \in \mathbb{R}^n : \langle x \rangle \perp \mathcal{V}\}$ $\boxminus$

Prp 6 $\mathcal{V} \subseteq \mathbb{R}^n \Rightarrow \mathbb{R}^n = \mathcal{V} \oplus \mathcal{V}^\perp$. HW $\boxminus$

Ex 7 i) $U \in \mathbb{R}^{n \times s}$, $U^T U = I_s$
$\overset{?}{\Rightarrow} UU^T = I_n$? $(s < n)$
NO, $\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$

ii)
O
$\Rightarrow *U$
$s = \operatorname{Trace}(I$
$\Rightarrow$
$* 0 \neq \{$
Df

ii) $\mathcal{O} \in \mathbb{R}^{n\times n}$ is orthogonal

$\mathcal{O}^T\mathcal{O} = I_n \overset{?}{\Longrightarrow} \mathcal{O}\mathcal{O}^T = I_n$?

YES

$* \; UU^T = I_n \Rightarrow \mathrm{Trace}(UU^T) = \mathrm{Trace}(I_n)$

$\mathrm{Trace}(I_s) = \mathrm{Trace}(U^TU) = \mathrm{Trace}(UU^T) = \mathrm{Trace}(I_n) = n$

$\Rightarrow \; \Leftarrow$

$* \; 0 \neq \xi \in \mathcal{B}(U)^\perp \Rightarrow 0 = \underbrace{UU^T}_{UU^T = I_n}\xi = \xi$

Dfn 8 [orthogonal projections]

$(\mathbb{R}^n = \mathcal{S} \oplus \mathcal{T}, \; \Pi_{\mathcal{S},\mathcal{T}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n)$

$\Pi_{\mathcal{S},\mathcal{S}^\perp}$ is the orthogonal projection onto $\mathcal{S}$.

$\left(\begin{array}{l} \mathrm{im}\,\Pi_{\mathcal{S},\mathcal{T}} = \mathcal{S} \\ \mathrm{Ker}\,\Pi_{\mathcal{S},\mathcal{T}} = \mathcal{T} \end{array}\right)$ $\mathrm{Ker}\,\Pi_{\mathcal{S},\mathcal{S}^\perp} \perp \mathrm{im}\,\Pi_{\mathcal{S},\mathcal{S}^\perp}$ ☰

Thm 9 $\mathcal{V} \subseteq \mathbb{R}^n$, $\underbrace{B_\mathcal{V} = [v_1 \cdots v_d]}_{\text{basis for } \mathcal{V}} \in \mathbb{R}^{n\times d}$

$\Pi_{\mathcal{V},\mathcal{V}^\perp} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\Pi_{\mathcal{V},\mathcal{V}^\perp} = T_A, \; A = B_\mathcal{V}(B_\mathcal{V}^T B_\mathcal{V})^{-1} B_\mathcal{V}^T$

Pf $\Pi_{\mathcal{V},\mathcal{V}^\perp} = T_A, \; A \in \mathbb{R}^{n\times n}$

$A = [a_1 \cdots a_n], \; a_j = \Pi_{\mathcal{V},\mathcal{V}^\perp}(e_j)$

$e_j = \sum c_{ij} v_j + \underbrace{\xi}_{\in \mathcal{V}^\perp} \quad \forall j \in [n] \Rightarrow$

$e_j = B_\mathcal{V} c_j + \xi_j \Rightarrow a_j = B_\mathcal{V} c_j$

$B_\mathcal{V}^T e_j = B_\mathcal{V}^T B_\mathcal{V} c_j \Rightarrow c_j = (B_\mathcal{V}^T B_\mathcal{V})^{-1} B_\mathcal{V}^T e_j$

$(\mathrm{rank}(A^TA) = \mathrm{rank}(A))$

$$a_j = B_{\mathcal{V}} (B_{\mathcal{V}}^T B_{\mathcal{V}})^{-1} B_{\mathcal{V}}^T e_j$$

$$A = B_{\mathcal{V}} (B_{\mathcal{V}}^T B_{\mathcal{V}})^{-1} B_{\mathcal{V}}^T \quad \square$$

Cor 10 $\mathcal{V} \subset \mathbb{R}^n$, $B_{\mathcal{V}} \in \mathbb{R}^{n \times d}$ orthonormal basis, then $\pi_{\mathcal{V},\mathcal{V}^\perp} = \tau_A$, $A = B_{\mathcal{V}} B_{\mathcal{V}}^T$ $\square$

geometric view

$$\pi_{\mathcal{V},\mathcal{V}^\perp} : \mathbb{R}^n \xrightarrow[\text{analysis step}]{B_{\mathcal{V}}^T} \mathbb{R}^{\dim \mathcal{V}} \xrightarrow[\text{synthesis step}]{B_{\mathcal{V}}} \mathbb{R}^n$$

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Prp 11 $\mathcal{V}, \mathcal{W} \subset \mathbb{R}^n$ HW

$\mathcal{V} \perp \mathcal{W} \iff B_{\mathcal{V}}^T B_{\mathcal{W}} = 0$

$\iff \Pi_{\mathcal{V},\mathcal{V}^\perp} \circ \Pi_{\mathcal{W},\mathcal{W}^\perp} = 0$

Dfn 12 [orthogonal resolution of the identity]

$id = \sum_{i\in[s]} \Pi_{\mathcal{V}_i,\mathcal{V}_i^\perp}$ s.t. $\mathcal{V}_i \perp \mathcal{V}_j$ $\forall i \neq j$

Prp 13 $id = \sum_{i\in[s]} \Pi_{\mathcal{V}_i,\mathcal{V}_i^\perp}$ is an orthogonal resolution of the identity $\iff \mathbb{R}^n = \bigoplus_{i\in[s]} \mathcal{V}_i$ and $\mathcal{V}_i^\perp = \bigoplus_{j\neq i} \mathcal{V}_j$

Prf ($\Longrightarrow$) $x_i \in \mathcal{V}_i \cap \left(\sum_{j\neq i} \mathcal{V}_j\right)$

$x_i = \sum_{j\neq i} x_j$, $x_j \in \mathcal{V}_j \overset{x_i^T(\ldots=\ldots)}{\Longrightarrow} x_i = 0$

So $\sum_{i\in[s]} \mathcal{V}_i = \bigoplus_{i\in[s]} \mathcal{V}_i$

$x \in \mathbb{R}^n$ $x = id(x) = \sum_{i\in[s]} \underbrace{\Pi_{\mathcal{V}_i,\mathcal{V}_i^\perp}(x)}_{\in \mathcal{V}_i}$

So $\mathbb{R}^n = \bigoplus_{i \in [s]} \mathcal{V}_i = \mathcal{V}_i \oplus \underbrace{\left(\bigoplus_{j \neq i} \mathcal{V}_j \right)}$

subspace complement
$\perp \mathcal{V}_i$ so it is $\mathcal{V}_i^{\perp}$

($\Leftarrow$) HW $\boxed{\Xi}$

Prp 14 $Q \in \mathbb{R}^{n \times n}$ orthogonal, every partition of its columns induces an orthogonal resolution of the identity.

Prf $Q = [u_1 \cdots u_n]$

$[n] = \underbrace{\dot{\bigcup}_{i \in [s]} J_i}_{\text{partition}}$, $J_i \subseteq [n]$

Q_{J_i}: column-submatrix of Q with columns indexed by J_i

$\mathcal{V}_i = \mathcal{B}(Q_{J_i})$

$\mathcal{V}_i \perp \mathcal{V}_j \quad \forall i \neq j$

$\mathbb{R}^n = \sum_{i \in [s]} \mathcal{V}_i$

$x \in \mathbb{R}^n$, τ_Q is surjective $\Rightarrow \exists \alpha \in \mathbb{R}^n$ s.t. $x = Q\alpha$

Let α_{J_i}, $i \in [s]$ be a conformable decomposition of α according to the partition. Then

$x = \sum_{i \in [s]} \underbrace{Q_{J_i} \alpha_{J_i}}_{\in \mathcal{V}_i}$ $\boxed{\Xi}$

$i \neq j$

is

$x \in \mathbb{R}^n$

be

decomposition

o the

T. ∎

Dfn 15 [spectrum of a matrix]

$A \in K^{n\times n}$

$\sigma_K(A) = \{\lambda \in K : \operatorname{rank}(A-\lambda I) < n\}$ ∎

$\Leftrightarrow \det(A-\lambda I) = 0$

Rem 16 $K \hookrightarrow \overline{K}$

$A \in K^{n\times n} \Longrightarrow A \in \overline{K}^{n\times n}$

$\sigma_K(A) \subset \sigma_{\overline{K}}(A)$ ∎

Rem 17 i) Is $\sigma_K(A)$ always non-empty? Yes:

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ no real eigenvalues

ii) Is $\sigma_{\overline{K}}(A)$ always non-empty? Yes.

Thm 18 $A \in \mathbb{R}^{n\times n}$

$A^T = A \Longrightarrow$

$\sigma_{\mathbb{R}}(A) = \sigma_{\mathbb{C}}(A)$

$\Leftrightarrow$

$\sigma_{\mathbb{C}}(A) \subseteq \mathbb{R}$

Prf $Au = \lambda u$ (1)

$\lambda \in \sigma_{\mathbb{C}}(A)$, $0 \neq u \in \mathbb{C}^n$

(1) $\Rightarrow u^* A u = \lambda u^* u$ (2)

(1) $u^* A = \lambda^* u^* \Rightarrow$

$u^* A u = \lambda^* u^* u$ (3)

(2),(3) $\Rightarrow \lambda \underbrace{u^* u}_{\neq 0} = \lambda^* u^* u$

$\Rightarrow \lambda = \lambda^*$

$\Rightarrow \lambda \in \mathbb{R}$. ∎