LAIS, Lec #3

Dfn 1 $\tau: U \longrightarrow V$ is called isomorphism if "ισος" "μορφή" $Ker(\tau)=0$ and $im(\tau)=V$.

$\Leftrightarrow \tau$: injective $\quad \Leftrightarrow \tau$: surjective

When τ is an isomorphism we say U and V are isomorphic denoted $U \cong V$

Thm 1 i) $\tau: U \longrightarrow V$ is an isomorphism $\Leftrightarrow$ τ takes some basis of U to a basis of V ii) In case of i) then τ takes any basis of U to a basis of V.

Prf i) ($\Rightarrow$) Let $u_1, \dots, u_n$ be a basis of U. We will show $\tau(u_1), \dots, \tau(u_n)$ is a basis of V. Take $v \in V$. Then $\exists u \in U$ s.t. $v = \tau(u)$. Also $u = \sum_{i \in [n]} \alpha_i u_i$ $\Rightarrow v = \sum_{i \in [n]} \alpha_i \tau(u_i) \Rightarrow$ $\tau(u_1), \dots, \tau(u_n)$ is a spanning set of V.

Suppose $\sum \lambda_i \tau(u_i) = 0$

$\Rightarrow \tau\left(\sum \lambda_i u_i\right) = 0$

$\Rightarrow \sum \lambda_i u_i = 0 \Rightarrow \lambda_i = 0 \quad \forall i \in [n]$

($\Leftarrow$) Suppose $u_1, \ldots, u_n$ is a basis of U s.t. $\tau(u_1), \ldots, \tau(u_n)$ is a basis of V. injectivity:

Suppose $\tau(u) = 0$, $u = \sum \alpha_i u_i$

$\Rightarrow \sum \alpha_i \tau(u_i) = 0 \Rightarrow \alpha_i = 0 \Rightarrow u = 0$

surjectivity: $v \in V$, $\exists b_i \in K$ s.t.

$v = \sum b_i \tau(u_i) = \tau\left(\underbrace{\sum b_i u_i}_{u}\right) = \tau(u)$ $\boxed{\equiv}$

Ex 3 $\tau : K^5 \longrightarrow K^5$

$\tau(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (\alpha_5, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$

$\tau(e_i) = e_{i+1} \quad \forall i = 1, \ldots, 4$

$\tau(e_5) = e_1$. By Thm 2

τ is an isomorphism.

In fact τ is called a permutation, in fact a cycle of length 5.

$\tau^5 = id$ $\boxed{\equiv}$

$,\alpha_4)$

Thm 4 $\tau_A : K^n \longrightarrow K^n$ is an isomorphism $\Leftrightarrow$ A is invertible.

Prf ($\Rightarrow$) $e_1, \dots, e_n$ is a basis of K^n. By Thm 2 $\tau_A(e_1), \dots, \tau_A(e_n)$ is a basis of K^n. But $\tau_A(e_i) = a_i$ where $A = [a_1 \cdots a_n] \in K^{n\times n}$. So $a_1, \dots, a_n$ is a basis of K^n. So the columns of A are l.i. A is square $\Rightarrow$ A is invertible.

($\Leftarrow$) Suppose A invertible.
$\Rightarrow a_1, \dots, a_n$ are l.i.
$a_i = \tau_A(e_i)$ so the basis $e_1, \dots, e_n$ goes by τ_A to the basis $a_1, \dots, a_n$. Done by Thm 2. ☺

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Prp 5 If $\tau : U \to V$ is HW isomorphism, then τ^{-1} is a linear transformation. ☺

Thm 6 U, V : vector spaces
$u_1, \dots, u_n$ basis of U
$v_1, \dots, v_n$ elements of V
Then $\exists !$ linear transformation $\tau : U \to V$ s.t. $\tau(u_i) = v_i$.

Pf For $u \in U$ $\exists ! \alpha_i \in K$ s.t. $u = \sum \alpha_i u_i$. Now define $\tau : U \to V$ by $\tau(u) = \sum \alpha_i v_i$

It is well-defined because the α_i are unique

$g_1, g_2 \in U$, $c_1, c_2 \in K$

$$\tau(c_1 g_1 + c_2 g_2) = \sum_i (c_1 \alpha_{1i} + c_2 \alpha_{2i}) v_i$$
$$= c_1 \tau(g_1) + c_2 \tau(g_2)$$

$g_1 = \sum \alpha_{1i} u_i$

$g_2 = \sum \alpha_{2i} u_i$

$c_1 g_1 + c_2 g_2 = \sum (c_1 \alpha_{1i} + c_2 \alpha_{2i}) u_i$

$\Rightarrow \tau$ is linear transformation

Uniq

$\exists$

$v_i = \sigma($

$u \in U$

$\sigma(u) =$

Uniqueness: Suppose

$\exists\ \sigma: U \longrightarrow V$ s.t.

$v_i = \sigma(u_i) = \tau(u_i)\ \forall i \in [n]$

$u \in U,\ u = \sum \alpha_i u_i$

$\sigma(u) = \sum \alpha_i \sigma(u_i) = \sum \alpha_i \tau(u_i) = \tau(u)$

$\boxed{\equiv}$

Thm 7 Two vector spaces are isomorphic $\Longleftrightarrow$ they have the same dimension.

Prf ($\Longrightarrow$) Suppose $\tau: U \xrightarrow{\sim} V$ is an isomorphism. Let $u_1, \ldots, u_n$ be a basis of U, so $\dim U = n$.

By Thm 2 $\tau(u_1), \ldots, \tau(u_n)$ is a basis of V. So $\dim V = n$.

($\Longleftarrow$) Suppose $\dim U = \dim V$. Let $u_1, \ldots, u_n$ be a basis of U. Let $v_1, \ldots, v_n$ be a basis of V. By Thm 6 $\exists!$ $\tau: U \longrightarrow V$ s.t. $\tau(u_i) = v$

But τ takes a basis of U to a basis of V, so by Thm

τ is isomorphism.

is

V_i

U

hm2

Thm 8 ("rank+nullity thm")

$\tau : U \longrightarrow V$

$\mathrm{Ker}(\tau)$ $\quad$ $\mathrm{im}(\tau)$

$$\dim U = \dim \mathrm{Ker}(\tau) + \dim \mathrm{im}(\tau)$$

Prf Let H be any complement of $\mathrm{Ker}(\tau)$ in U, that is $U = \mathrm{Ker}(\tau) \oplus H$. Since the dimension is additive on direct sums ($\dim \bigoplus V_i = \sum \dim V_i$ HW)

$$\dim U = \dim \mathrm{Ker}(\tau) + \dim H$$

If we can show $H \cong \mathrm{im}(\tau)$, then we are done by Thm 7.

Define $\varphi : H \longrightarrow \mathrm{im}(\tau)$ by $\varphi = \tau|_H$, i.e.

$\varphi(h) = \tau(h)$. φ is l.t. because $\varphi(c_1h_1 + c_2h_2) =$ $h_1, h_2 \in H$

$= \tau(c_1h_1 + c_2h_2) =$

$= c_1\tau(h_1) + c_2\tau(h_2)$

$= c_1\varphi(h_1) + c_2\varphi(h_2)$.

We will show φ is isomorphism. Injectivity: suppose $\underbrace{\tau(h)}_{\in H} = 0 \Rightarrow h \in \mathrm{Ker}(\tau) \Rightarrow h = 0$

$\mathrm{Ker}(\tau) \cap H = 0$

Surjectivity: $v \in \mathrm{im}(\tau)$, we want to show $\exists h \in H$ s.t. $v = \varphi(h)$.

$\exists u \in U$ s.t. $v = \tau(u)$. $u = \underbrace{\xi}_{\in \mathrm{Ker}(\tau)} + \underbrace{h}_{\in H}$

$\Rightarrow v = \tau(u) = \underbrace{\tau(\xi)}_{=0} + \tau(h) = \tau(h)$

$= \varphi(h)$ ∎

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Define $\varphi: \mathcal{U} \longrightarrow \operatorname{im}(\tau)$

by $\varphi = \tau|_{\mathcal{U}}$, i.e.

$\varphi(h) = \tau(h)$. φ is l.t.

because $\varphi(c_1h_1 + c_2h_2) \overset{h_1,h_2 \in \mathcal{U}}{=}$

$= \tau(c_1h_1 + c_2h_2) =$

$= c_1\tau(h_1) + c_2\tau(h_2)$

$= c_1\varphi(h_1) + c_2\varphi(h_2)$.

We will show φ is isomorphism. Injectivity:

suppose $\tau(\underbrace{h}_{\in \mathcal{U}}) = 0 \Rightarrow h \in \operatorname{Ker}(\tau) \Rightarrow h = 0$

surjectivity: $v \in \operatorname{im}(\tau)$, we want to show $\exists h \in \mathcal{U}$ s.t. $v = \varphi(h)$.

$\exists u \in V$ s.t. $v = \tau(u)$. $u = \underbrace{\xi}_{\in \operatorname{Ker}(\tau)} + \underbrace{h}_{\in \mathcal{U}}$

$\Rightarrow v = \tau(u) = \underbrace{\tau(\xi)}_{=0} + \tau(h) = \tau(h)$

$= \varphi(h)$ $\square$

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$\operatorname{Ker}(\tau) \cap \mathcal{U} = 0$

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U: vector space, $\dim U = n$

$\dim K^n = n$. By Thm 7

$U \cong K^n$. Let $\{u_1, \dots, u_n\} = B_U$

be any basis of U and $E = \{e_1, \dots, e_n\}$

the canonical basis of K^n. By Thms

6 and 2, $\exists!$ isomorphism

$\tau_{B_U}: U \to K^n$ s.t. $\tau_{B_U}(u_i) = e_i \ \forall i \in [n]$

$u \in U$, $u = \sum_{i\in[n]} \alpha_i u_i$, $\underbrace{\tau_{B_U}(u)}_{= [u]_{B_U}} = \sum \alpha_i e_i = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$

$$\begin{tikzcd} U \arrow[r, "\tau"] \arrow[d, "\varphi_{B_U}"'] & V \arrow[d, "\varphi_{B_V}"] \\ K^n \arrow[r, "\sigma"] & K^m \end{tikzcd}$$

"commutative diagram"

$\sigma = \varphi_{B_V} \circ \tau \circ \varphi_{B_U}^{-1} = \tau_A$

for some $A = [a_1 \cdots a_n]$

$\tau: K^n \to K^m$

$\tau = \tau_A$, $A = [a_1 \cdots a_n]$

$a_i = \tau(e_i)$

$B_U = \{u_1, \dots$

$B_V = \{v_1, \dots$

$a_i = \sigma(e_i) =$

$= \varphi_{B_V} \circ$

$= \varphi_{B_V}($

$A = [\,[$

$= [\tau$

$A\,[u]_{B_U} =$

$u \in U$

$\tau(u) \in V$

$\forall i \in [n].$

$\alpha_i e_i = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$

$$\begin{tikzcd} U \arrow[r, "\tau"] \arrow[d, "\varphi_{B_U}"] & V \arrow[d, "\varphi_{B_V}"] \\ K^n \arrow[r, "\sigma"] & K^m \end{tikzcd}$$

"commutative diagram"

$\sigma = \varphi_{B_V} \circ \tau \circ \varphi_{B_U}^{-1} = \tau_A$

for some $A = [\alpha_1 \cdots \alpha_n]$

$\tau : K^n \longrightarrow K^m$

$\tau = \tau_A,\ A = [\alpha_1 \cdots \alpha_n]$

$\alpha_i = \tau(e_i)$

$B_U = \{u_1, \ldots, u_n\}$

$B_V = \{v_1, \ldots, v_m\}$

$\alpha_i = \sigma(e_i) = \varphi_{B_V} \circ \tau \circ \underbrace{\varphi_{B_U}^{-1}(e_i)}_{u_i}$

$= \varphi_{B_V} \circ \tau(u_i)$

$= \varphi_{B_V}(\tau(u_i)) = [\tau(u_i)]_{B_V}$

$A = \big[\, [\tau(u_1)]_{B_V} \cdots [\tau(u_n)]_{B_V} \,\big]$

$= [\tau]_{B_U, B_V}$

$A\,[u]_{B_U} = [\tau(u)]_{B_V}$

$u \in U$

$\tau(u) \in V$

By Thm2 $\tau(u_1), \ldots$

is a basis of V. S

$\dim V = n. (\Leftarrow)$

Suppose $\dim U =$

Let $u_1, \ldots, u_n$ be a

of U. Let $v_1, \ldots, v_n$ b

of V. By Thm6

$\tau : U \longrightarrow V$ s.t.

But τ takes a b

to a basis of V, s

τ is isomorphi