LAIS, Lecture #9

$A \in \mathbb{R}^{m\times n}$

$\sigma(AA^T)\setminus\{0\} = \sigma(A^TA)\setminus\{0\}$

$= \{\lambda_1 > \dots > \lambda_s\} \subset \mathbb{R}_{>0}$

v_{ij}, $j \in \mu_i$: o.n.b. of $\mathcal{E}_{A^TA,\lambda_i}$

$u_{ij} = \frac{1}{\sqrt{\lambda_i}} A v_{ij}$, $j \in [\mu_i]$: o.n.b. of $\mathcal{E}_{AA^T,\lambda_i}$

$U_i = [u_{i1} \dots u_{i\mu_i}]$ $\quad \bar{U} = [U_1 \dots U_s]$: o.n.b. of $\mathcal{B}(A)$

$V_i = [v_{i1} \dots v_{i\mu_i}]$ $\quad \bar{V} = [V_1 \dots V_s]$: o.n.b. of $\mathcal{B}(A^T)$

$A = \bar{U} \underbrace{\bar{\Sigma}}_{\text{square}} \bar{V}^T$, $\bar{\Sigma} = \mathrm{diag}(\sqrt{\lambda_1}, \dots, \underbrace{\sqrt{\lambda_s}, \dots, \sqrt{\lambda_s}}_{\mu_s})$

$A v_{ij} = \sqrt{\lambda_i}\, u_{ij}$

Thm 1 Multiplication by A induces an isomorphism $\mathcal{B}(A^T) \xrightarrow{\sim} \mathcal{B}(A)$ and the inverse map is

$$\sum_{i\in[s]} \frac{1}{\sqrt{\lambda_i}} v_{ij} u_{ij}^T .$$

Prf $\left(\sum_{i\in[s]} \frac{1}{\sqrt{\lambda_i}} v_{ij} u_{ij}^T\right) A v_{ij} =$

$$= \left(\sum_{\alpha\in[s]} \frac{1}{\sqrt{\lambda_\alpha}} u_{\alpha\beta} u_{\alpha\beta}^T\right) \sqrt{\lambda_i}\, u_{ij}$$

$$= \left(\frac{1}{\sqrt{\lambda_i}} v_{ij} u_{ij}^T\right) \sqrt{\lambda_i}\, u_{ij} = v_{ij} \quad \blacksquare$$

Dfn 2

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Prp 4

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AA^T

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Dfn 2 The matrix $\sum_{i \in [s]} \frac{1}{\sqrt{\lambda_i}} v_{ij} u_{ij}^T$ is called the Moore-Penrose pseudo-inverse, denoted A^+. ▢

Prp 4 Suppose $Ax = b$ is a linear system of equations. If it is consistent, then A^+b is the unique solution of minimal Euclidean norm.

Prf First check that $AA^+b = b$.

$AA^+b = (\bar{U}\bar{\Sigma}\bar{V}^T)(\bar{V}\bar{\Sigma}^{-1}\bar{U}^T)b$

$= \bar{U}\bar{U}^T b = \Pi_{\mathcal{B}(A), \mathcal{B}(A)^\perp}(b) = b$

because $b \in \mathcal{B}(A)$, since the system is consistent.

Let $A^+b + \xi$, $\xi \in \mathcal{N}(A)$ be any other solution.

$\|A^+b + \xi\|_2^2 = (A^+b + \xi)^T(A^+b + \xi)$

$= \|A^+b\|_2^2 + \|\xi\|_2^2 + 2\xi^T A^+ b$

By Lem 3 $\underbrace{\xi}_{\in \mathcal{N}(A)} \perp \underbrace{A^+b}_{\in \mathcal{B}(A^T)}$

So $\xi \neq 0 \Rightarrow$

$\|A^+b + \xi\|_2 > \|A^+b\|_2$. ▢

Dfn 5 A least-squares problem is a problem of the form

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2 \quad \text{(LS)}$$ ▢

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Pr

$\Rightarrow$

Lem 3 $\mathcal{R}(A^+) = \mathcal{R}(A^T)$

Prf $A^+ = \bar{V}\bar{\Sigma}^{-1}\bar{U}^T$

$\Rightarrow \mathcal{R}(A^+) \subset \mathcal{R}(\bar{V}) = \mathcal{R}(A^T)$

$A^+\bar{U} = \bar{V}\bar{\Sigma}^{-1} \Rightarrow$

$A^+\bar{U}\bar{\Sigma} = \bar{V} \Rightarrow$

$\mathcal{R}(\bar{V}) = \mathcal{R}(A^T) \subset \mathcal{R}(A^+)$ ▣

Prp 6 $x \in \mathbb{R}^n$ is a solution to (LS) $\Leftrightarrow$ $Ax = \underbrace{\Pi_{\mathcal{R}(A),\mathcal{R}(A)^\perp}(b)}_{b_A^{\parallel}}$

Prf $\|Ax-b\|_2^2 =$

$= \|Ax - \underbrace{b_A^{\parallel}}_{\in\mathcal{R}(A)} - \underbrace{b_A^{\perp}}_{\in\mathcal{R}(A)^\perp}\|_2^2 = \|Ax - b_A^{\parallel}\|_2^2 + \|b_A^{\perp}\|_2^2$

x is a solution

$\Leftrightarrow Ax = b_A^{\parallel}$ ▣

Thm 7 A^+b is the unique solution to (LS) of minimal Euclidean norm.

Prf By Prp 6 any solution must satisfy $Ax = b_A^{\parallel}$. By Prp 4 $A^+b_A^{\parallel}$ is the unique solution of minimal Euclidean norm

Now $A^+b_A^{\parallel} = A^+b$

because i) $b_A^{\perp} \in \mathcal{N}(A^T) = \mathcal{N}(A^+)$

ii) $A^+b = \bar{V}\bar{\Sigma}^{-1}\bar{U}^Tb$

$= \bar{V}\bar{\Sigma}^{-1}\bar{U}^T\bar{U}\bar{U}^Tb$

$= \bar{V}\bar{\Sigma}^{-1}\bar{U}^Tb_A^{\parallel}$

$= A^+b_A^{\parallel}$ ▣

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Rem 8 For more on SVD and spectral theory of symmetric matrices, see Horn + Johnson, "Matrix Analysis". $\square$

two extremes

- symmetric matrices ($\mathbb{R}, \mathbb{C}$)
  spectral decomposition
- general square matrices
  K
- rectangular matrices ($\mathbb{R}, \mathbb{C}$)
  SVD

"goal: $A \in K^{n \times n}$, $\tau_A : K^n \to K^n$
decompose K^n into smaller pieces where A acts very simply"

Dfn 9 A ring homomorphism $\varphi : R \to T$ is a map s.t.

$$\varphi(r_1 + r_2) = \varphi(r_1) + \varphi(r_2)$$
$$\varphi(r_1 r_2) = \varphi(r_1)\varphi(r_2)$$
$$\varphi(1_R) = 1_T . \square$$

Rem 10 Recall we have a ring homomorphism

$$K \to \mathrm{End}(K^n)$$
$$c \mapsto (v \mapsto cv) \quad \square$$

$\to k^n$

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Prp 11 Fix $A \in k^{n \times n}$.
We have a ring HW
homomorphism
$$\varphi_A : k[x] \longrightarrow \mathrm{End}(k^n)$$
defined by
$$p(x) = \sum_{i=0}^{\ell} \underbrace{c_i}_{\in k} x^{\ell - i} \in k[x]$$
$$\varphi_A(p(x)) = \sum_{i=0}^{\ell} c_i A^{\ell - i} \in \mathrm{End}(k^n)$$
where $A^0 := I$ ⊟

Rem 12 "k^n is a finitely generated torsion module over $k[x]$." (Roman, Ch. 7)

Dfn 13 An ideal of $k[x]$ is a subset $I \subset k[x]$ which is closed under subtraction ($p, q \in I$, $p - q \in I$) and closed under multiplication by elements of $k[x]$ ($\forall p \in I, \forall r \in k[x]$, $rp \in I$) ⊟

Dfn 14 $p_1, \dots, p_s \in k[x]$.
The ideal generated by p_i, $i \in [s]$ is
$$(p_1, \dots, p_s) = \{ r_1 p_1 + \dots + r_s p_s,\ r_i \in k[x],\ i \in [s] \}$$ ⊟

$n \times n$

ng HW

nd(k^n)

$k[x]$

End(k^n)

finitely

module

n, Ch. 7)

Dfn 13 An ideal of $k[x]$ is a subset $I \subset k[x]$ which is closed under subtraction ($p, q \in I$, $p - q \in I$) and closed under multiplication by elements of $k[x]$ ($\forall p \in I$, $\forall r \in k[x]$, $rp \in I$)

Dfn 14 $p_1, \dots, p_s \in k[x]$. The ideal generated by p_i, $i \in [s]$ is $(p_1, \dots, p_s) = \{ r_1 p_1 + \dots + r_s p_s,\ r_i \in k[x],\ i \in [s] \}$.

Dfn 15 Let I be an ideal. We say $\{p_i\}_{i \in S}$ (S could be infinite) are generators of I, if $\forall p \in I$, $\exists$ finite subset of S, $S' \subset S$, s.t. $p = \sum_{i \in S'} r_i p_i$, for $r_i \in k[x]$, $i \in S'$.