LAIS, Lecture #8

Prp 1 Multiplication (Hw) by A^T induces an isomorphism $\mathbb{B}(A) \to \mathbb{B}(A^T)$. $\boxed{=}$

$$\mathbb{B}(A) = \bigoplus_{\substack{\lambda \in \sigma(AA^T)\\ \lambda \neq 0}} \mathcal{E}_{AA^T,\lambda}$$

Prp 2 $(\lambda \neq 0)$ Multiplication by A^T induces an isomorphism $\mathcal{E}_{AA^T,\lambda} \to \mathcal{E}_{A^TA,\lambda}$. The inverse map is $\frac{1}{\lambda}A$

Prf $\mathcal{E}_{AA^T,\lambda} \xrightarrow{A^T} \mathcal{E}_{A^TA,\lambda} \xrightarrow{\frac{1}{\lambda}A} \mathcal{E}_{AA^T,\lambda}$

$u \qquad A^Tu \qquad AA^Tu = \lambda u$

Prp 3 Multiplication by $\frac{1}{\sqrt{\lambda}}A^T$ induces an isomorphism $\mathbb{B}(A) \to \mathbb{B}(A^T)$ and the inverse map is $\frac{1}{\sqrt{\lambda}}A$. $\boxed{=}$

Lem 4 Multiplication by $\frac{1}{\sqrt{\lambda}}A^T$ takes an orthonormal basis of $\mathbb{B}(A)$ to an orthonormal basis of $\mathbb{B}(A^T)$

Prf Let $u_1, \dots, u_{\mu_\lambda}$ be an orthonormal basis of $\mathcal{E}_{AA^T,\lambda}$. Define $v_j = \frac{1}{\sqrt{\lambda}} A^T u_j$. Since $\frac{1}{\sqrt{\lambda}} A^T$ is an isomorphism $\mathcal{B}(A) \to \mathcal{B}(A^T)$ the v_j's are a basis of $\mathcal{B}(A^T)$. Check orthonormality:

$$v_j^T v_j = \left(\frac{1}{\sqrt{\lambda}} A^T u_j\right)^T \left(\frac{1}{\sqrt{\lambda}} A^T u_j\right)$$
$$= \frac{1}{\lambda} u_j^T A A^T u_j = \frac{1}{\lambda} u_j^T \lambda u_j$$
$$= u_j^T u_j = 1.$$

$$\underset{j \neq r}{v_j^T v_r} = \frac{1}{\lambda} u_j^T u_r = 0 \quad \square$$

Thm 5 [Singular Value Decomposition, SVD] $A \in \mathbb{R}^{m \times n}$, there exist orthogonal matrices $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$ and $\Sigma \in \mathbb{R}^{m \times n}$ of the form $\Sigma = \begin{bmatrix} \overline{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}$ with $\overline{\Sigma} = \operatorname{diag}(\underbrace{\sigma_1, \dots, \sigma_r}_{\text{singular values}})$, $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$, $r = \operatorname{rank}(A)$, s.t. $A = U \Sigma V^T$. Moreover

$$\underbrace{U}_{\substack{\text{left singular}\\ \text{vectors}}} = [\underbrace{\overline{U}}_{\substack{\text{o.n.b.}\\ \mathcal{B}(A)}} \ \underbrace{\tilde{U}}_{\substack{\text{o.n.b.}\\ \mathcal{N}(A^T)}}], \quad \underbrace{V}_{\substack{\text{right}\\ \text{singular}\\ \text{vectors}}} = [\underbrace{\overline{V}}_{\substack{\text{o.n.b.}\\ \mathcal{B}(A^T)}} \ \underbrace{\tilde{V}}_{\substack{\text{o.n.b.}\\ \mathcal{N}(A)}}]$$

$A = U\Sigma V^T = \bar{U}\bar{\Sigma}\bar{V}^T$

full-SVD　thin-SVD

Prf Recall $\sigma(AA^T)\setminus\{0\} = \sigma(A^TA)\setminus\{0\}$
$\{\lambda_1, \dots, \lambda_s\}$ with
$\lambda_1 > \dots > \lambda_s > 0$ and the
multiplicity of λ_i is μ_i.
Let u_{ij}, $j \in [\mu_i]$ be an
orthonormal basis for $\mathcal{E}_{A^TA,\lambda_i}$
$U_i = [u_{i1} \cdots u_{i\mu_i}] \in \mathbb{R}^{n\times\mu_i}$, $i \in [s]$
Let $U_{s+1} = [u_{s+1,1} \cdots u_{s+1,r}]$ be an
orthonormal basis for $\mathcal{N}(A)$
$U = [U_1 \cdots U_s\ U_{s+1}] \in \mathbb{R}^{n\times n}$ is
an o.n.b. of $\mathbb{R}^n$.

$\mathbb{R}^n = \mathcal{R}(A^T) \oplus \mathcal{N}(A)$

Define $v_{ij} = \frac{1}{\sqrt{\lambda_i}} A u_{ij}$ for $i \in [s]$
$V_i = [v_{i1} \cdots v_{i\mu_i}]$. By Lem 4
V_i is an orthonormal basis
for $\mathcal{E}_{AA^T,\lambda_i}$. For $i = s+1$ let
$V_{s+1} = [v_{s+1,1} \cdots v_{s+1,r'}]$ be an
o.n.b. for $\mathcal{N}(A^T)$. Then
$V = [V_1 \cdots V_s\ V_{s+1}]$ is an
o.n.b. of $\mathbb{R}^m$.
$\tau_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$
$[\tau_A]_{U,V}$, the (i,j) column of
this matrix is $[Au_{ij}]_V = \begin{cases} \sqrt{\lambda_i}\, e_{(i,j)}, & i \in [s] \\ 0 \end{cases}$

$$[\tau_A]_{U,V} = \begin{bmatrix} \sqrt{\lambda_1} & & & & & & \\ & \ddots & & & & & \\ & & \sqrt{\lambda_1} & & & & \\ & & & \ddots & & & \\ & & & & \sqrt{\lambda_s} & & \\ & & & & & \ddots & \\ & & & & & & \sqrt{\lambda_s} \\ & & & & & & & \ddots \end{bmatrix} =: \begin{bmatrix} \bar{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} = \Sigma$$

nk(A),

$\tilde{V}^T]$

o.n.b.
$\mathcal{N}(A)$

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$\mathcal{U}$: canonical basis of $\mathbb{R}^n$

$\mathcal{E}$: -||- $\mathbb{R}^m$

$[\tau_A]_{\mathcal{E},\mathcal{U}} = A$

$M_{\mathcal{U},\mathcal{E}} = U$

$[\tau_A]_{\mathcal{E},\mathcal{U}} = \underbrace{M_{\mathcal{V},\mathcal{U}}}_{V} \underbrace{[\tau_A]_{\mathcal{U},\mathcal{V}}}_{\Sigma} \underbrace{M_{\mathcal{E},\mathcal{U}}}_{U^T}$

$\Rightarrow A = V \Sigma U^T$

swap V and U ☺

Second proof of Thm 5:

Prf Consider the eigendecompositions

$AA^T = \bar{U} \bar{\Lambda} \bar{U}^T$,

$A^T A = \bar{W} \bar{\Lambda} \bar{W}^T$

$\bar{\Lambda} = \mathrm{diag}(\underbrace{\lambda_1, \dots, \lambda_1}_{\mu_1}, \dots, \underbrace{\lambda_s, \dots, \lambda_s}_{\mu_s})$

$\Rightarrow \bar{U}$: o.n.b. for $\mathcal{B}(A)$

$\bar{W}$: o.n.b. for $\mathcal{B}(A^T)$

$\exists B \in \mathbb{R}^{r \times n}_{\mathrm{rank}(A)}$ s.t.

(1) $A = \bar{U} B \Rightarrow A^T = \underbrace{B^T}_{n \times r} \bar{U}^T$

claim: the columns of B^T are a basis for $\mathcal{B}(A^T)$.

ξ

$A^T =$

$\xi = \sum_{j \in}$

prf of

$\Rightarrow \mathcal{B}$

$A^T \bar{U} =$

$\exists G$

$B^T = \bar{W}$

$A = \bar{U}$

$AA^T = \bar{U}$

mpositions

$\xi \in \mathcal{B}(A^T)$

$A^T = [\alpha_1 \cdots \alpha_m]$

$\xi = \sum_{j \in [m]} c_j \alpha_j = A^T c = B^T \underbrace{\bar{U}^T c}_{c'}$

prf of claim: $A^T = B^T \bar{U}^T$

$\Rightarrow \mathcal{B}(A^T) \subset \mathcal{B}(B^T)$

$A^T \bar{U} = B^T \Rightarrow \mathcal{B}(B^T) \subset \mathcal{B}(A^T)$.

$\exists$ $G \in \mathbb{R}^{r \times r}$ invertible s.t.

$B^T = \bar{W} G \overset{(1)}{\Longrightarrow}$

$A = \bar{U} G^T \bar{W}^T$ (2)

$AA^T = \bar{U} G^T G \bar{U}^T = \bar{U} \bar{\Lambda} \bar{U}^T$

$\Rightarrow G^T G = \bar{\Lambda}$

$\Rightarrow G = \underbrace{\Theta}_{\substack{r \times r \\ \text{orthogonal}}} \bar{\Lambda}^{1/2}$

$A = \bar{U} \bar{\Lambda}^{1/2} \Theta^T \bar{W}^T$

$= \bar{U} \bar{\Lambda}^{1/2} (\underbrace{\bar{W} \Theta}_{\bar{V}\text{: o.n.b. for } \mathcal{B}(A^T)})^T$

$A = \bar{U} \bar{\Lambda}^{1/2} \bar{V}^T$ ☺