LAIS, Lec#4

Thm1

$$\begin{tikzcd} U \arrow[r, "\tau"] \arrow[d, "\varphi_B"] & V \arrow[d, "\varphi_C"] \\ K^n \arrow[r, "{[\tau]_{B,C}}"] & K^m \end{tikzcd}$$

$B = \{u_1, \dots, u_n\}$

$[\tau]_{B,C} = \left[\, [\tau(u_1)]_C, \dots, [\tau(u_n)]_C \right] \in K^{m\times n}$ $\boxed{E}$

Prp2 $A \in K^{m\times n}$, $\tau_A : K^n \longrightarrow K^m$

HW $\mathcal{E}$: canonical basis of K^n $\quad A = [\tau_A]_{\mathcal{E},\mathcal{O}}$

$\mathcal{O}$: —||— K^m $\boxed{E}$

Prp3 $\underset{B}{U} \xrightarrow{\tau} \underset{C}{V} \xrightarrow{\sigma} \underset{D}{W}$

HW $[\sigma \circ \tau]_{B,D} = [\sigma]_{C,D} \; [\tau]_{B,C}$

matrix multiplication $\boxed{E}$

Change of basis

U, B, B': two bases

$$\begin{tikzcd} u \in U \arrow[r, "\varphi_B"] \arrow[d, "\varphi_{B'}"] & K^n \; [u]_B \\ K^n & \end{tikzcd}$$

$[u]_{B'} = \varphi_{B'} \circ \varphi_B^{-1}([u]_B)$

$\underbrace{\varphi_{B'} \circ \varphi_B^{-1} : K^n \longrightarrow K^n}_{\tau_A \text{ for some } A}$

$A = [a_1 \cdots a_n] = M_{B,B'}$

$a_j = \underbrace{\varphi_{B'} \circ \varphi_B^{-1}(e_j)}_{u_j} = [u_j]_{B'}$

Prp 4 $M_{B,B'}$ is invertible $\forall$ bases B, B'.

Prf $M_{B,B'} \cdot M_{B',B} = I_n$ ▤

Prp 5 Every invertible matrix is a change of basis matrix.

Prf $A \in K^{n \times n}$ invertible

$A = [a_1 \cdots a_n] = M_{B,\mathcal{E}}$

$\mathcal{E}$: canonical basis of K^n

$B = \{a_1, \ldots, a_n\}$ ▤

Prp 6 $B = \{u_1, \ldots, u_n\}$

$M_{\mathcal{E},B} = A^{-1}$, $A = [u_1 \cdots u_n]$ ▤

Prp 7 A: invertible $n \times n$, B: basis of K^n. There exists basis B' s.t. $A = M_{B,B'}$.

$^{n\times n}$ invertible

$\cdot a_n] = M_{B,\mathcal{E}}$

cal basis of K^n

$\ldots, a_n\}$

$= \{u_1, \ldots, u_n\}$

A^{-1}, $A = [u_1 \cdots u_n]$

: invertible $n \times n$,

f K^n. There exists

s.t. $A = M_{B,B'}$.

Prf $B = \{u_1, \ldots, u_n\}$

$A^{-1} = [\xi_1 \cdots \xi_n]$

$$u'_j = \sum_{i \in [n]} \underbrace{\xi_{ij}}_{i\text{-th entry of } \xi_j} u_j$$

The u'_j's are a basis because $\varphi_B(u'_j) = \xi_j$

$u'_j = \varphi_B^{-1}(\xi_j)$

$A^{-1} = M_{B',B}$

$B' = \{u'_1, \ldots, u'_n\}$

$\Rightarrow A = M_{B,B'}$

Thm 8 $\underset{B,B'}{U} \xrightarrow{\tau} \underset{C,C'}{V}$

HW $[\tau]_{B',C'} = M_{C,C'} [\tau]_{B,C} M_{B',B}$ Kc +1

Mac +1

Dfn 9 $A, B \in K^{m\times n}$ are equivalent if $\exists$ invertible matrices $P \in K^{m\times m}$, $Q \in K^{n\times n}$ s.t. $B = PAQ$.

Prp 10 Equivalence of matrices is an equivalence relation.

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S

$R \subset S \times S$

$(a,a) \in R \ \forall a$

$(a,b) \in R \Rightarrow (b,a) \in R$

$(a,b) \in R, (b,c) \in R$
$\Rightarrow (a,c) \in R$

Prp 3

$[\sigma \circ \tau$

Cha

U

$u \in U$

k

$[u$

LAIS, Lec#4

Prp 11 A, B are equivalent $\iff$ they represent the same linear transformation $\tau : V \to V$ but in possibly different bases. $\equiv$ HW

Thm 12 $A \in K^{m \times n}$, $\operatorname{rank}(A) = r$. Then A is equivalent to the matrix

$$J_r = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Prf $\tau_A : K^n \longrightarrow K^m$

Let $u_1, \dots, u_r$ be a basis of H, where $K^n = H \oplus \operatorname{Ker}(\tau)$, let $u_{r+1}, \dots, u_n$ be a basis for $\operatorname{Ker}(\tau)$. Then $v_1 = Au_1, \dots, v_r = Au_r$ is a basis for $\operatorname{im}(\tau_A)$. Extend $v_1, \dots, v_r$ to a basis $v_1, \dots, v_r, v_{r+1}, \dots, v_m$ of K^m. Take $B = \{u_1, \dots, u_n\}$, $C = \{v_1, \dots, v_m\}$.

$$[\underbrace{\tau_A(u_j)}_{Au_j}]_C = \begin{cases} e_j, & j \in [r] \\ 0, & \text{otherwise} \end{cases} \quad \equiv$$

Dfn 13 $A, B \in K^{n\times n}$
are similar if
$\exists$ invertible $S \in K^{n\times n}$
s.t. $B = S A S^{-1}$.
By what we saw $S = M_{C,C'}$

$$B = M_{C,C'} A M_{C',C}$$

similarity is a finer
equivalence relation
than matrix equivalence.

LAIS, Lec#4

Prp 11 A, B are equivalent $\iff$ they represent the same linear transformation $\tau: V \to V$ but in possibly different bases. $\boxed{\equiv}$ HW

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$$[\underbrace{\tau_A(u_j)}_{Au_j}]_{\mathcal{C}} = \begin{cases} e_j, & j \in [r] \\ 0, & \text{otherwise} \end{cases} \quad \boxed{\equiv}$$

spectral theory
of symmetric
matrices

$\downarrow$

Singular Value
Decomposition

$\downarrow$

Square matrices

$\ldots e_n\}$

$]$

u_j

entry

basis

$) = \xi_j$

$= \varphi_B^{-1}(\xi_j)$

Thm 8 $U \xrightarrow{\tau} V$

B, B' $\quad C, C'$

HW $[\tau]_{B',C'} = \underbrace{M_{C,C'}}_{P}\,\underbrace{[\tau]_{B,C}}_{A}\,\underbrace{M_{B',B}}_{Q}$ ∎

Ke +1

Mao +1

Bi Kang +1

Zixuan +1

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Prp 10 Equivalence of matrices is an equivalence relation ∎ HW