LAIS, Lecture #16

Lem 1 $\mathcal{V}, \mathcal{W}$ subspaces of $\mathbb{R}^n$. Then

$$\operatorname{codim}(\mathcal{V} \cap \mathcal{W}) \leq \operatorname{codim}(\mathcal{V}) + \operatorname{codim}(\mathcal{W})$$

Prf $\dim(\mathcal{V} \cap \mathcal{W}) =$

$$= \dim(\mathcal{V}) + \dim(\mathcal{W}) - \dim(\mathcal{V}+\mathcal{W})$$

$$\geq \dim(\mathcal{V}) + \dim(\mathcal{W}) - n$$

$$n - \dim(\mathcal{V} \cap \mathcal{W}) \leq -\dim(\mathcal{V}) - \dim(\mathcal{W}) + n + n$$

$\boxed{\equiv}$

Thm 2 [Interlacing-II]

$A \in \mathbb{R}^{n \times n}$, symmetric

$B \in \mathbb{R}^{r \times r}$, obtained by removing rows and columns indexed by $i_1, \dots, i_{n-r}$.

Then

$$\lambda_{i+n-r}(A) \underset{HW}{\leq} \lambda_i(B) \leq \lambda_i(A)$$

Prf $\lambda_i(A) = \min_{\dim \mathcal{V} = n-i+1} \max_{\substack{x \in \mathcal{V} \\ \|x\|_2 = 1}} x^T A x$

$$= \max_{\substack{x \in \mathcal{W} \\ \|x\|_2 = 1}} x^T A x \geq \max_{\substack{x \in \mathcal{W} \cap \operatorname{Span}(e_{i_1}, \dots, e_{i_{n-r}})^\perp \\ \|x\|_2 = 1}} x^T A x \geq$$

$\geq$

$=$

$=$

$=$

$$\geq \min_{\dim \mathcal{V}=n-i+1} \; \max_{\substack{x \in \mathcal{V} \cap \operatorname{Span}(e_{i_1},\dots,e_{i_{n-r}})^\perp \\ \|x\|_2=1}} x^T A x$$

$$= \min_{\substack{\mathcal{V}'=\mathcal{V} \cap \operatorname{Span}(e_{i_1},\dots,e_{i_{n-r}})^\perp \\ \dim \mathcal{V}=n-i+1}} \; \max_{\substack{x \in \mathcal{V}' \\ \|x\|_2=1}} x^T A x$$

$$= \min_{\substack{\mathcal{V}' \subseteq \mathbb{R}^r \\ \dim \mathcal{V}' \geq r-i+1}} \; \max_{\substack{z \in \mathcal{V}' \\ \|z\|_2=1}} z^T B z$$

$$= \min_{j=1,\dots,i} \left\{ \min_{\substack{\mathcal{V}' \subseteq \mathbb{R}^r \\ \dim \mathcal{V}'=r-j+1}} \; \max_{\substack{z \in \mathcal{V}' \\ \|z\|_2=1}} z^T B z \right\}$$

$$= \min_{j=1,\dots,i} \lambda_j(B) = \lambda_i(B) \quad \boxed{\equiv}$$

$x \geq$

Thm 3 $A \in \mathbb{R}^{n \times n}$, symmetric

$$\sum_{i \in [r]} \lambda_i(A) \geq \sum_{i \in [r]} a_{ii} \quad \forall r \in [n]$$

and equality for $r=n$.

Prf Induction on n.

If $n=1$, trivial.

Suppose $n>1$.

Let B be obtained by removing the last row and column of A. Then by Interlacing-II etric

$r \in [n]$

?.

$r < n$ ($r = n$, equality is easy)

$$\sum_{i\in[r]} \lambda_i(A) \overset{\text{Int-II}}{\geq} \sum_{i\in[r]} \lambda_i(B) \overset{\text{i.h.}}{\geq}$$

$$\sum_{i\in[r]} a_{ii} \quad \boxed{=}$$

Thm 4 $A \in \mathbb{R}^{n\times n}$, symmetric

$$\sum_{i\in[r]} \lambda_i(A) = \max_{\substack{U\in\mathbb{R}^{n\times r} \\ U^TU = I_r}} \operatorname{Trace}[U^TAU]$$

$$\lambda_1(A) = \max_{\|u\|_2=1} u^TAu \quad \text{(Rayleigh)}$$

Prf We first show that for any orthonormal $U \in \mathbb{R}^{n\times r}$ we have "$\geq$".

Extend U to an orthonormal bases $W = [U\ V]$ of $\mathbb{R}^n$.

$$\sum_{i\in[r]} \lambda_i(A) \overset{A \approx W^TAW}{=} \sum_{i\in[r]} \lambda_i(W^TAW)$$

$$\overset{\text{Int-II}}{\geq} \sum_{i\in[r]} \lambda_i(U^TAU) = \operatorname{Trace}[U^TAU]$$

Equality is achieved for $U = [u_1 \cdots u_r]$. $\boxed{=}$

Thm 5 $A, B \in \mathbb{R}^{n\times n}$ symmetric

$$\sum_{i\in[r]} \lambda_i(A) + \sum_{i\in[r]} \lambda_i(B) \geq \sum_{i\in[r]} \lambda_i(A+B)$$

Prf from Thm 4. $\boxed{=}$

LAIS, Lecture #16

Thm 6 [Hadamard's inequality]

$A \in \mathbb{R}^{n\times n}$, $A \geq 0$. Then

$\det(A) \leq \prod_{i\in[n]} a_{ii}$

Prf $0 \overset{A\geq 0}{\leq} \lambda_n(A) \overset{\text{Int-II}}{\leq} a_{ii}$

So $\prod_{i\in[n]} a_{ii} \geq 0$. If rank$(A)<n$,

$\lambda_n(A)=0 \Rightarrow \det(A)=0 \Rightarrow$

$0=\det(A) \leq \prod_{i\in[n]} a_{ii}$.

So suppose rank$(A)=n$.

Then $\lambda_n(A)>0$ so

$a_{ii}>0 \ \forall i\in[n]$.

Define $D=\mathrm{diag}\left(\frac{1}{\sqrt{a_{11}}},\dots,\frac{1}{\sqrt{a_{nn}}}\right)$

$$\det(DAD)=\frac{\det(A)}{\prod_{i\in[n]} a_{ii}} =$$

$$=\prod_{i\in[n]}\lambda_i(DAD) \leq$$

arithmetic-geometric mean inequality

$$\leq \left(\frac{1}{n}\sum_{i\in[n]}\lambda_i(DAD)\right)^n =$$

$$=\left(\frac{1}{n}\mathrm{Trace}[DAD]\right)^n = 1 \quad \boxed{\equiv}$$

$\geq$

$=$

$=$

$=$

$\geq$

LAIS, Lecture #16

Signals + Systems

Signal Processing

Control Systems

Adaptive Filtering

Dfn 1 A discrete-time signal is a function $s: \mathbb{Z} \to \mathbb{R}$.

Dfn 2 A filter of order n is a linear function $w: \mathbb{R}^n \to \mathbb{R}$

Dfn 3 u: signal

regressor of order n

$$u_i = [u(i)\ u(i-1) \dots u(i-n+1)] \in \mathbb{R}^{1\times n}$$

Prb 4 [estimation]

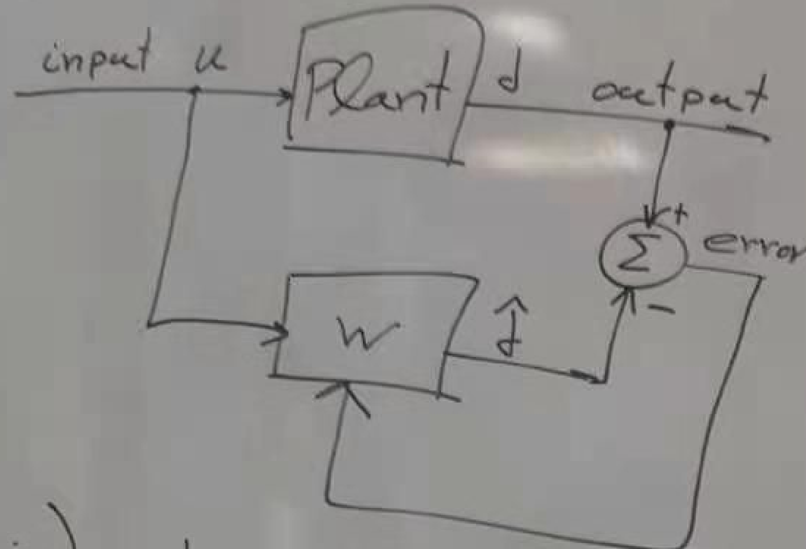
Given signals d, u find a filter $w \in \mathbb{R}^{n\times 1}$ s.t. the signal $\hat{d}(i) = u_i w$ estimates the signal $d(i)$ $\forall i$.

$\mathbb{R}^{1\times n}$

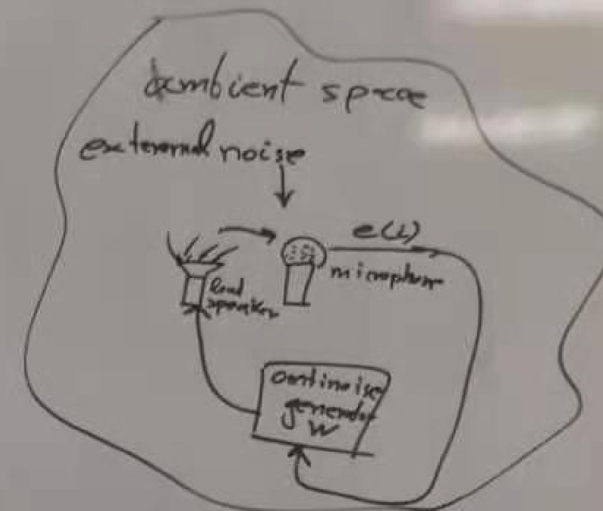
$\in \mathbb{R}^{n\times 1}$

Applications:

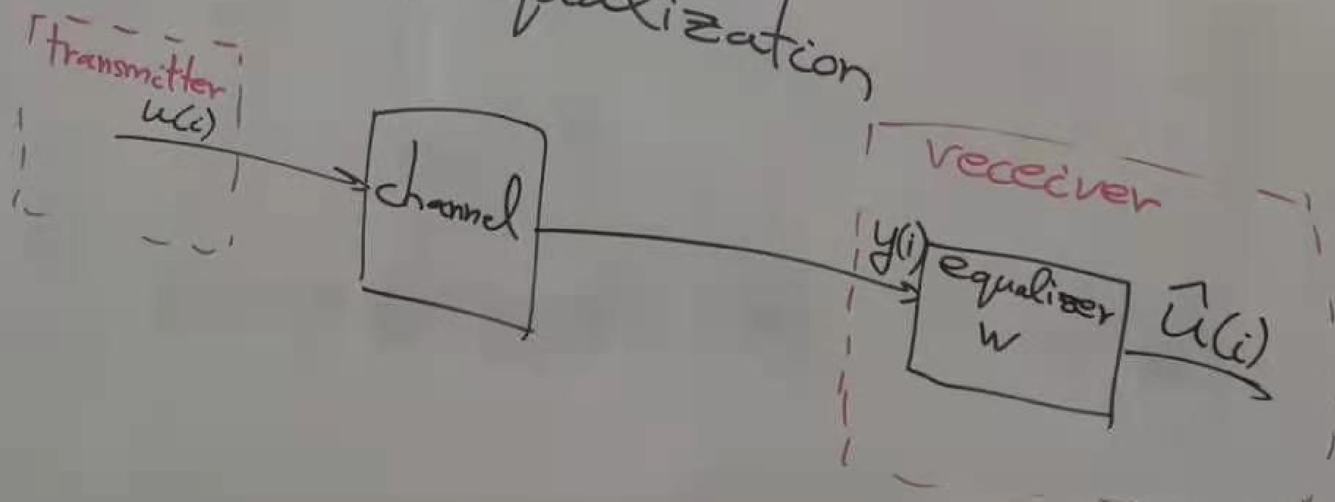
i) system identification



iii) active noise control



ii) channel equalization



Recursive-Least-Squares
Adaptive Filter
[$\mathcal{O}(n^2)$ complexity]

Rem trade-off
between accuracy and
complexity of AF.
E.g. LMS-AF has
$\mathcal{O}(n)$ but is not as
accurate as RLS. ☺

"batch filter"
non-adaptive

$$\min_{w \in \mathbb{R}^n} \sum_{j=0}^{N} (d(j) - u_j w)^2$$

$$\Longleftrightarrow$$

$$\min_{w \in \mathbb{R}^n} \| y_N - \mathcal{H}_N w \|_2 \quad (*)$$

$$\underbrace{y_N = \begin{bmatrix} d(0) \\ \vdots \\ d(N) \end{bmatrix}}_{\in \mathbb{R}^{N+1}}, \quad \mathcal{H}_N = \underbrace{\begin{bmatrix} u_0 \\ \vdots \\ u_N \end{bmatrix}}_{\mathbb{R}^{(N+1) \times n}}$$

the m
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$(d(j) - u_j w)^2$

$- \mathcal{U}_N w\|_2 \quad (*)$

$, \; \mathcal{U}_N = \begin{bmatrix} u_0 \\ \vdots \\ u_N \end{bmatrix}$

The most economic filter is $w_N = \mathcal{U}_N^{\dagger} y_N$.

If $\mathcal{U}_N$ has full-column rank, then the solution to (*) is unique:

$$w_N = (\mathcal{U}_N^T \mathcal{U}_N)^{-1} \mathcal{U}_N^T y_N.$$