LAIS, Lecture #20

"Low-rank matrix completion"
(bounded rank)

Dfn 1 [space of low-rank matrices] $M(D\times N, r) = \{X \in \mathbb{R}^{D\times N} : \mathrm{rank}(X) \leq r\}$ ☰

Rem 2 $M(D\times N, r)$ is not a linear space. ☰

Dfn 3 Z: $D\times N$ matrix of variables over $\mathbb{R}$ (algebraically independent elements over $\mathbb{R}$)

$\mathbb{R}[Z]$: polynomial ring in variables $Z = (z_{ij})$ and coefficients in $\mathbb{R}$.

$I \subset [D]$, $\#I = r+1$

$J \subset [N]$, $\#J = r+1$

$p_{I,J}(Z) = \det(Z_{I,J})$ ☰

Prp 4 $\mathcal{X} \in \mathbb{R}^{D\times N}$

$\mathcal{X} \in M(D\times N, r)$

$\iff p_{I,J}(\mathcal{X}) = 0$

$\forall\, I, J$ as in Dfn 3. $\square$

Dfn 5 An algebraic variety of $\mathbb{R}^n$ is the common root locus of a set of polynomials in n variables with coefficients in $\mathbb{R}$. $\square$

Cor 6 $M(D\times N, r)$ is an algebraic variety determined by $\binom{D}{r+1}\binom{N}{r+1}$ polynomials of degree $r+1$. $\square$

Prp 7 Let $S \subseteq \mathbb{R}^n$ be a linear subspace. Then S is an algebraic variety.

Prf Let $b_1, \dots, b_c$ be a basis for $S^\perp$. Let $p_i(x) = b_i^T x$, where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a set of variables over $\mathbb{R}$. Then $\xi \in S \iff p_i(\xi) = 0 \ \forall i \in [c]$. $\square$

Dfn 8 [matrix coordinate projection]

$\Omega \subseteq [D]\times[N]$

$\mathbb{R}^{\Omega}$ is defined as the set of $D\times N$ matrices with support in Ω ($x_{ij}=0 \ \forall\ (i,j)\notin\Omega$).

$\mho_\Omega : M(D\times N, r) \longrightarrow \mathbb{R}^{\Omega}$

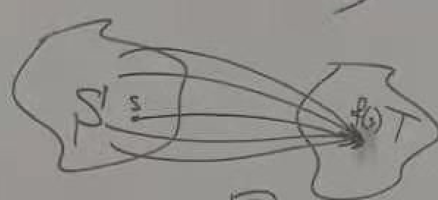
$$(\mho_\Omega(X))_{ij} = \begin{cases} x_{ij}, & \text{if } (i,j)\in\Omega \\ 0, & \text{if } (i,j)\notin\Omega \end{cases}$$

Dfn 9 [fiber]

$f : S \longrightarrow T$

a map of sets

for $s\in S$, the fiber of f over s is $f^{-1}(f(s)) = \{s'\in S : f(s')=f(s)\}$



Dfn 10 [low-rank matrix completion]

Let $X^* \in M(D\times N, r)$

Given $\mho_\Omega(X^*)$

we want to find an element of the fiber $\mho_\Omega^{-1}(\mho_\Omega(X^*))$

Any matrix $Y \in \mho_\Omega^{-1}(\mho_\Omega(X^*))$ is called a completion of X^* in $M(D\times N, r)$.

coordinate

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matrices

in Ω

$\in \Omega)$

$,r) \longrightarrow \mathbb{R}^{\Omega}$

x_{ij}, if $(i,j) \in \Omega$

0, if $(i,j) \notin \Omega$

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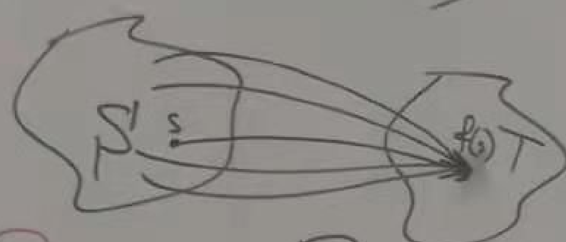

Rem 10.5 [fibers of linear maps]

$\tau: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\xi \in \mathbb{R}^n$, given $\tau(\xi) \in \mathbb{R}^m$

$\tau \cong A \in \mathbb{R}^{m \times n}$

$Ax = \tau(\xi)$: fiber

Dfn 10 [low-rank matrix completion]

Let $X^* \in M(D \times N, r)$

Given $\Omega_\Omega(X^*)$

we want to find an element of the fiber $\Omega_\Omega^{-1}(\Omega_\Omega(X^*))$

Any matrix $Y \in \Omega_\Omega^{-1}(\Omega_\Omega(X^*))$ is called a completion of X^* in $M(D \times N, r)$.

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Prb 11 For what Ω is $\pi_\Omega^{-1}(\pi_\Omega(x^*))$ a finite set? ⊜

Prb 12 For what Ω is $\pi_\Omega^{-1}(\pi_\Omega(x^*)) = \{x^*\}$? ⊜

$\pi_\Omega(x^*) \in \mathbb{R}^{10\times 10}$

$x^* \in \mathbb{R}^{10\times 10}$, $\operatorname{rank}(x^*) = 4$

$P_{I,J}(Z) = 0$

$z_{ab} \mapsto x_{ab}$ if $(a,b) \in \Omega$

$BC = \begin{bmatrix} b_1 \\ \vdots \\ b_D \end{bmatrix} [c_1 \cdots c_D]$

row vectors

column

$b_i c_j = x_{ij}$ if $(i,j) \in \Omega$

Dfn 13 [vector coordinate projection] Let $\omega \subseteq [D]$ define $\pi_\omega : \mathbb{R}^D \longrightarrow \mathbb{R}^{\#\omega}$ by $\pi_\omega(\xi) = (\xi_i)_{i\in\omega}$. ⊜

Dfn 14 [vector completion] Let $S \subseteq \mathbb{R}^D$, $\xi \in S$, $\omega \subseteq [D]$. Given $\pi_\omega(\xi)$ and S find ξ. ⊜

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Prp 15 [vector completion]

$S \subseteq \mathbb{R}^D$, $\dim S = r$

$\omega \subseteq [D]$, $\xi \in S$.

$S_{\Omega_\omega(\xi)} = \{ y \in S : \Omega_\omega(y) = \Omega_\omega(\xi) \}$

If $\dim \Omega_\omega(S) = r$, then

$S_{\Omega_\omega(\xi)} = \{\xi\}$ and $\# S_{\Omega_\omega(\xi)} = +\infty$

otherwise.

Prf Let $B = [b_1 \cdots b_r] \in \mathbb{R}^{D \times r}$

be a basis of S. Then

$\Omega_\omega(B) = [\Omega_\omega(b_1) \cdots \Omega_\omega(b_r)] \in \mathbb{R}^{\#\omega \times r}$

is a spanning set for $\Omega_\omega(S)$

$y \in S_{\Omega_\omega(\xi)} \iff$

$\Omega_\omega(y) = \Omega_\omega(\xi)$

$y \in S \Rightarrow \exists c \in \mathbb{R}^r$

s.t. $y = Bc$

$\iff \Omega_\omega(\xi) = \Omega_\omega(Bc)$

$\iff \Omega_\omega(\xi) = \Omega_\omega(B)c$

Now $\xi = Bc'$, $c' \in \mathbb{R}$

$\iff \Omega_\omega(B)c' = \Omega_\omega(B)c$

$\iff \Omega_\omega(B)(c - c') = 0$

$\iff c - c' \in \mathcal{N}(\Omega_\omega(B))$

$\mathcal{N}(\Omega_\omega(B)) = 0 \iff$

$\dim \Omega_\omega(S) = r$ $\boxed{\equiv}$

Cor 16 $S \subseteq \mathbb{R}^D$, $\xi \in S$
$\dim S = r$, $\omega \subseteq [D]$
If $\# \omega < r$, then there are infinitely many $y \in S$ s.t.
$\Omega_\omega(y) = \Omega_\omega(\xi)$. ☺

Dfn 17 $\Omega \subseteq [D] \times [N]$
$$\Omega = \bigcup_{j \in [N]} \omega_j \times \{j\}$$ for some $\omega_j \subseteq [D]$.
$$\Omega = \bigcup_{i \in [D]} \psi_i \times \{i\}$$
for some $\psi_i \subseteq [N]$
☺

Dfn 18 [Subspace version of low-rank matrix completion] $\operatorname{rank}(X^*) = r$
$S^* = \mathcal{B}(X^*)$
$X^* = [x_1^* \dots x_N^*]$
The problem is to do vector completion for every $\Omega_{\omega_j}(x_j^*)$.
That is, for every $j \in [N]$ we want to find $y_j \in S^*$ s.t.
$\Omega_{\omega_j}(x_j^*) = \Omega_{\omega_j}(y_j)$ ☺

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Prp 19 $\mathcal{X}^*, \mathcal{S}^*$ as in Dfn 18.
If $\dim \Omega_{\omega_j}(\mathcal{S}^*) = r$, then the data $\Omega_{\Omega}(\mathcal{X}^*), \mathcal{S}^*$ uniquely determine $\mathcal{X}^*$.

Prf Let $B^* \in \mathbb{R}^{D\times r}$ be a basis of $\mathcal{S}^*$.
For every $j \in [N]$ solve $\Omega_{\omega_j}(B^*)c_j = \Omega_{\omega_j}(x_j^*)$

$$c_j = \underbrace{\left(\Omega_{\omega_j}(B^*)^T\,\Omega_{\omega_j}(B^*)\right)^{-1}\Omega_{\omega_j}(B^*)^T}_{\Omega_{\omega_j}(B^*)^\dagger}\,\Omega_{\omega_j}(x_j^*)$$

Define $y_j = B^* c_j$. Claim $y_j = x_j^*$.
Write $x_j^* = B^* c_j^* \Rightarrow \Omega_{\omega_j}(y_j) = \Omega_{\omega_j}(B^*)\,c_j$

$$\Omega_{\omega_j}(y_j) = \Omega_{\omega_j}(B^*)\left(\Omega_{\omega_j}(B^*)^T\,\Omega_{\omega_j}(B^*)\right)^{-1}\Omega_{\omega_j}(B^*)^T\,\Omega_{\omega_j}(x_j^*)$$

$$\Rightarrow \Omega_{\omega_j}(y_j) = \Omega_{\omega_j}(x_j^*)$$

$$\overset{\text{Prp 15}}{\Longrightarrow} y_j = x_j^* \quad ∎$$