Continuous Random pair couple

DS:
TD
+
TD2 (all 7 exercises
check Ex3)

X, Y two r.v

$Y = \ell(x) + \text{Residual}$

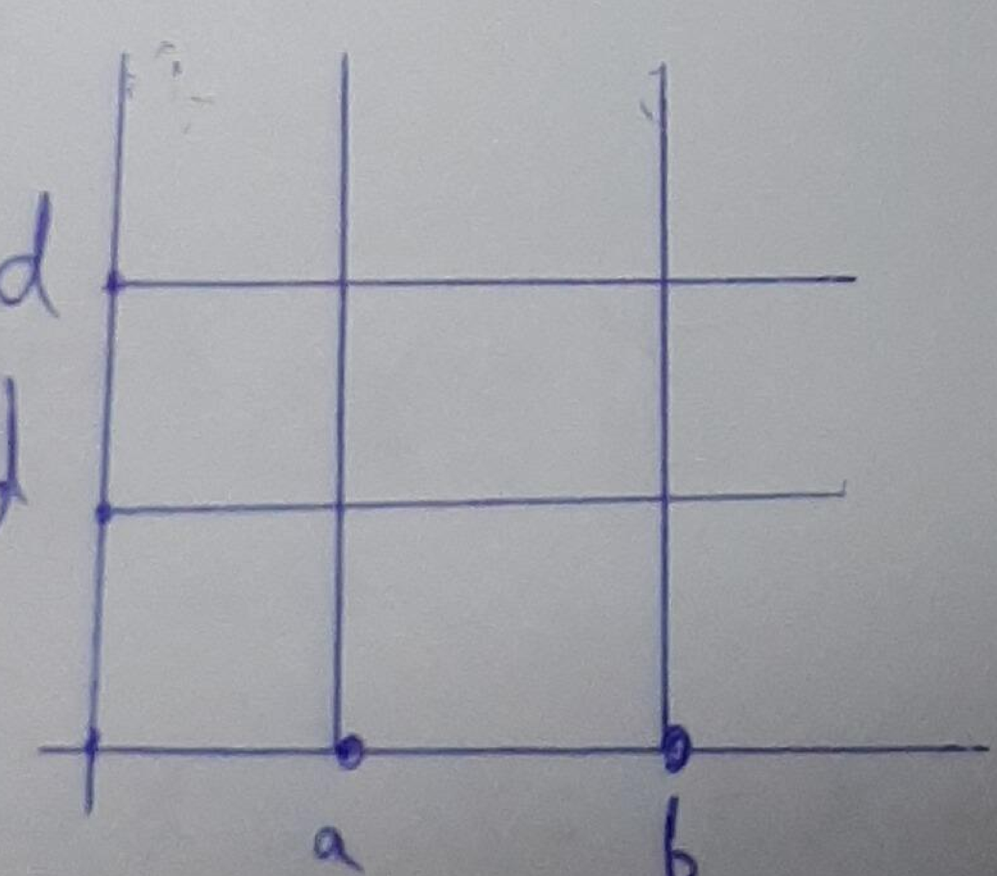
$\hookrightarrow \quad E(Y \mid X = x) = \ell(x)$

$$R^2_{xy} = \frac{V(E(Y \mid X))}{V(Y)}$$

Def: A random Pair (X, Y) is said to be continuous if there exists $0 \leq f_{xy}(x, y)$ that satisfies

i) $\int_{\mathbb{R}^2} f_{xy}(x,y)\,dx\,dy = 1$

ii) $\mathbb{P}\{X \in [a,b] \text{ and } Y \in [c,d]\} = \int_{[a,b]\times[c,d]} f_{xy}(x,y)\,dx\,dy$



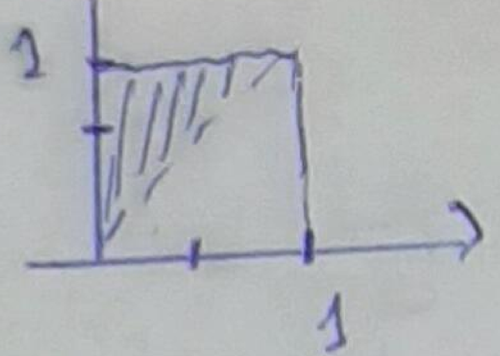
f_{xy} is called the joint density

Exemple: find c such that

$f(x,y) = c\,xy\,\mathbb{1}_{\{0 < x < y < 1\}}$ is a joint density

$$\{0 < x < y < 1\} = \{(x,y) \mid x \in [0,1] \text{ and } x < y \leq 1\}$$

$$= \{(x,y) \mid y \in [0,1] \text{ and } 0 < x < y$$



$$\int f(x,y)\,dx\,dy = 1 = c\int_0^1 dx\left[x\int_x^1 dy\, y\right] = c\int_0^1 dx\, x\left(\frac{1}{2} - \frac{1}{2}x^2\right)$$

$$1 = \frac{c}{2}\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$1 = \frac{c}{8} \Rightarrow \underline{c = 8}$$

Marginal Densities:

$$f_X(x) = \int_{\mathbb{R}} dy\, f_{XY}(x,y)$$

$$f_Y(y) = \int_{\mathbb{R}} dx\, f_{XY}(x,y)$$

$$\boxed{\begin{aligned}&\{(x,y) : x \in [0,1],\ x < y \leq 1\} \\ &= \mathbb{1}_{[0,1]}(x) \times \mathbb{1}_{[x,1]}(y)\end{aligned}}$$

exemple:

X, Y two r.v such that $f(x,y) = 8xy$

determine $f_X(x)$ and $f_Y(y)$

$$f_X(x) = \int_0^1 dy\, 8xy\, \mathbb{1}_{\{0<x<y<1\}}(x,y) = 8x\left(\int_x^1 y\,dy\right)\mathbb{1}_{[0,1]}(x)$$

$$= 4x(1-x^2)\,\mathbb{1}_{[0,1]}(x)$$

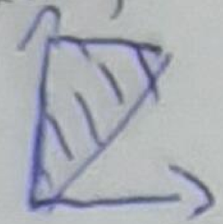
Conditional density function:

x fixée, $f_{Y|X=x} = \dfrac{f_{XY}(x,y)}{f_X(x)}$

exemple:

$$f_{Y|X=x} = \frac{8xy\,\mathbb{1}_\Delta}{4x(1-x^2)\,\mathbb{1}_{[0,1]}(x)} = \frac{2\cdot \mathbb{1}_{[0,1]}(x)}{1-x^2}\cdot y\cdot \mathbb{1}_{[x,1]}(y)$$

Remarque: (explication)

let $\Delta =$  $= \{(s,t) : 0 < s < t < 1\}$

$$f_{Y|X=x}(y) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{8xy\,\mathbb{1}_\Delta(x,y)}{4x(1-x^2)\,\mathbb{1}_{[0,1]}(x)} = \frac{2}{1-x^2}\,\mathbb{1}_{[0,1]}(x)\cdot y\cdot \mathbb{1}_{[x,1]}(y)$$

Conditional Expectation

$$e(x) = E(Y|X=x) = \int_{\mathbb{R}} y\, f_{Y|X=x}(y) = \frac{2}{1-x^2}\,\mathbb{1}_{[0,1]}(x)\int_x^1 y\,dy$$

$$= \mathbb{1}_{[0,1]}\, x$$

$$E(Y|X) = e(X) = \mathbb{1}_{[0,1]}(X)$$

Thm:

X, Y two r.v

then

i) $0 \leq \rho^2_{XY} \leq R^2_{XY} \leq 1$

ii) if $\rho^2_{XY} = 1 \Rightarrow \underbrace{Y}_{X,Y \text{ are linear}} = \underbrace{\beta}_{\text{slope}} X + \overset{\text{intercept}}{\alpha}$ for some $\alpha, \beta \in \mathbb{R}$ $(\exists \alpha, \beta \in \mathbb{R} \mid \ldots)$

iii) if $\rho^2 = R^2 \Leftrightarrow X, Y$ are linearly correlated

$\Leftrightarrow E(Y|X) = e(X) = \beta X + \alpha$

iv) if $R^2 = 0 \Rightarrow X$ and Y are not correlated

$\rho^2 = 0 \Rightarrow X$ and Y are not linearly correlated.