Two continuous random variables

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Definition

A random pair (X, Y) is said to be continuous if there exists a positive and piecewise continuous function $f_{XY}(x, y)$ such that:

1. $\displaystyle\int_{\mathbb{R}} \int_{\mathbb{R}} f_{XY}(x, y) dxdy = 1.$
2. $\displaystyle\mathbb{P}\{X \in [a, b]\ ,\ Y \in [c, d]\} = \int_a^b \int_c^d f_{XY}(x, y)\, dxdy.$

The function $f_{XY}(x, y)$ is called "the joint probability density function associated to the continuous random pair(X, Y).

Note

1. Be careful the fact that both X and Y are continuous does'nt mean that the random pair (X, Y) is continuous !

2. X and Y are independent if and only if the joint probability density function of the random pair (X, Y) satisfies $f_{XY}(x, y) = f_X(x) \times f_Y(y)$.

Example

Consider the following function:

$$f(x,y) = \begin{cases} ce^{-(x+y)} & \text{for } 0 < x < y \\ 0 & \text{otherwise} \end{cases}$$

1. Determine c so that $f(x,y)$ is a joint density function.
2. Suppose that f is the joint density function of a random pair (X,Y), determine the probability that $X < 1$ and $Y < 2$.
3. Let $Z = X + Y$, determine its probability density function.

The Marginal probability density function

Exactly as in the discrete case the marginal distributions of X and Y can be obtained from the joint probability distribution of the random pair (X, Y) as follows:

$$f_X(x) = \int_{\mathbb{R}} f_{XY}(x, y)dy$$

$$f_Y(y) = \int_{\mathbb{R}} f_{XY}(x, y)dx.$$

Example

Reconsider the last example, and determine the marginal distributions.

The conditional density function

Definition

Let (X, Y) be a random pair with a joint density function, $f_{XY}(x, y)$, we define the conditional probability distribution of Y given $X = x$ by its probability density function:

$$f_{Y|X=x}(y) = \frac{f_{XY}(x, y)}{f_X(x)},$$

outside $\{f_X = 0\}$.

Note

1. By definition of the conditional distribution, we have, for any $B \subset \mathbb{R}$:

$$\mathbb{P}\{Y \in B | X = x\} = \int_B dy\, f_{Y|X=x}(y)$$

2. Note that the knowledge of f_X and of $f_{Y|X=x}$ for any x determine the joint density f_{XY} by using:

$$f_{XY}(x, y) = f_X(x) \times f_{Y|X=x}(y).$$

3. We have also a Bayes formula:
$f_{Y|X=x}(y) = \dfrac{f_{XY}(x, y)}{f_X(x)} = \dfrac{f_{X|Y=y}(x) \times f_Y(y)}{f_X(x)}$ and vice versa.

Example

Reconsider the last example and determine the two conditional distributions.

The random variable conditional expectation $\mathbb{E}(Y|X)$

Definition

Let (X, Y) be a continuous random pair, we define the conditional expectation of Y given $X = x$, $\mathbb{E}(Y|X = x)$ by:

$$\mathbb{E}(Y|X = x) = \int_{-\infty}^{+\infty} y f_{Y|X=x}(y)dy = \frac{1}{f_X(x)} \int_{-\infty}^{+\infty} y f_{XY}(x, y)dy.$$

Then we define the random variable conditional expectation $\mathbb{E}(Y|X)$ by:

$$\mathbb{E}(Y|X) = \varphi(X)$$

where

$$\varphi(x) = \mathbb{E}(Y|X = x).$$

The random variable conditional variance $\mathbb{V}(Y|X)$

Definition

We define the conditional variance of Y given X by:

$$\begin{aligned}\mathbb{V}(Y|X) &= \mathbb{E}(Y^2|X) - \mathbb{E}(Y|X)^2 \\ &= \psi(X)\end{aligned}$$

where

$$\begin{aligned}\psi(x) = \mathbb{V}(Y|X=x) &= \mathbb{E}\left[\left(Y - \mathbb{E}(Y|X=x)\right)^2 \middle| X = x\right] \\ &= \mathbb{E}(Y^2|X=x) - \mathbb{E}(Y|X=x)^2.\end{aligned}$$

Example

Reconsider again the last example and compute:

1. The conditional expectation $\mathbb{E}(Y|X)$.
2. The conditional variance $\mathbb{V}(Y|X)$.

The total expectation and variance theorems

Theorem

For any continuous random pair (X, Y) *we have:*

1. $$\mathbb{E}\Big(\mathbb{E}(Y|X)\Big) = \mathbb{E}(Y).$$

2. $$\mathbb{V}(Y) = \mathbb{V}\Big(\mathbb{E}(Y|X)\Big) + \mathbb{E}\Big(\mathbb{V}(Y|X)\Big).$$