Carthage University
INSAT
Department of Mathematics & Computer Sciences

Random pair: Exercises' Sheet

Exercise 1: Let (X, Y) be a random pair and below is its contingency table:

$X \diagdown Y$	-1	0	1
1	1/8	1/4	1/8
2	1/4	1/8	1/8

1. Check whether the above table is a joint mass distribution function.
2. Compute $\mathbb{P}\{X + Y \leq 1\}$.
3. Determine the mass distribution function of $Z = X + Y$.
4. Determine the marginal mass distribution functions $\mathbb{P}_X$ and $\mathbb{P}_Y$.
5. Determine the random variables $\mathbb{E}(Y|X)$ and $\mathbb{V}(Y|X)$.
6. Compute the correlation coefficient ρ_{XY} and comment.
7. Compute the determination coefficient $R^2_{Y|X}$ and comment.

Exercise 2: Let X and Y be two discrete random variables such that:

X	-1	1
$\mathbb{P}_X$	1/3	2/3

Y	0	1
$\mathbb{P}_{Y\|X=-1}$	1/3	2/3

et

Y	0	1
$\mathbb{P}_{Y\|X=1}$	1/2	1/2

1. Determine the joint mass distribution function of the random pair (X, Y).
2. Determine the random variables $\mathbb{E}(Y|X)$ and $\mathbb{V}(Y|X)$.
3. Compute the correlation coefficient of X and Y, ρ_{XY}. Comment.
4. Compute the determination coefficient of X and Y, $R^2_{Y|X}$. Comment.

Exercise 3:
In the development of a new receiver for the transmission of digital information, each received bit is rated as acceptable, suspect, or unacceptable, depending on the quality of the received signal, with probabilities 0.9, 0.08, and 0.02, respectively. Assume that the ratings of each bit are independent. In the first four bits transmitted, let X denote the number of acceptable bits Y denote the number of suspect bits.

1. Determine the joint mass distribution function of the random pair (X, Y).

2. Determine the two marginal mass distribution function of X and Y.

3. Determine the random variables $\mathbb{E}(Y|X)$ and $\mathbb{V}(Y|X)$.

4. Compute the correlation coefficient of X and Y, ρ_{XY}. Comment.

5. Compute the determination coefficient of X and Y, $R^2_{Y|X}$. Comment.

<u>Exercise 4</u>: Let $f(x, y) = cxy$ for $0 < x < y < 1$ be a joint probability distribution function.

1. Determine the value of c.

2. Compute $\mathbb{P}\{X < \frac{1}{2}, Y < 1\}$.

3. Determine the marginal probability distributions of X and Y.

4. Compute $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

5. Determine the function $\varphi(x) = \mathbb{E}(Y|X = x)$ for each $x \in \mathbb{R}_+$ and deduce the conditional expectation random variable $\mathbb{E}(Y|X)$.

6. What about the strongness and the nature of the relationship between X and Y?

<u>Exercise 5</u>: Let $f(x, y) = ce^{-2x-3y}$ for $0 < y < x$ be a joint probability distribution function.

1. Determine the value of c.

2. Compute $\mathbb{P}\{X < 1, Y < 2\}$ and $\mathbb{P}\{0 < y < \frac{x}{2}\}$.

3. Determine the marginal probability distributions of X and Y.

4. Compute $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

5. Determine the function $\varphi(x) = \mathbb{E}(Y|X = x)$ for each $x \in \mathbb{R}_+$ and deduce the conditional expectation random variable $\mathbb{E}(Y|X)$.

6. What about the strongness and the nature of the relationship between X and Y?

<u>Exercise 6</u>: The conditional probability density of Y given $X = x$ is $f_{Y|X=x}(y) = xe^{-xy}$ for $0 < y$ and the marginal probability distribution of X is the uniform distribution $\mathcal{U}([0, 10])$.

1. Compute the probability $\mathbb{P}\{Y < 2|X = 2\}$.

2. Determine the function $\varphi(x) = \mathbb{E}(Y|X = x)$ for each $x \in [0, 10]$ and deduce the conditional expectation random variable $\mathbb{E}(Y|X)$.

3. Determine the marginal distribution of Y, f_Y.

4. What about the strongness and the nature of the relationship between X and Y?

<u>Exercise 7</u>: A popular clothing manufacturer receives Internet orders via two different routing systems. The time between orders for each routing system in a typical day is known to be exponentially distributed with a mean of 3.2 minutes. Both systems operate independently.

1. What is the probability that no orders will be received in a 5 minutes period? In a 10 minutes period?
2. What is the probability that both systems receive two orders between 10 and 15 minutes after the site is officially open for business?
3. Why is the joint probability distribution not needed to answer the previous questions?