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Dependence structure and portfolio risk in Indian foreign exchange market: A GARCH-EVT-Copula approach

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ABSTRACT

The study investigates dependence structure and estimates portfolio risk on data from foreign exchange market in India. We specify both marginal models for the foreign exchange returns and a joint model for the dependence. We employ the AR-t-GARCH-EVT models for the marginal distribution of each of five currency returns series. For the joint model, we choose seven copulas with different dependence structure such as Gaussian, Frank, Clayton, Gumbel, BB1, BB2 and BB7 copulas. Using LL, AIC, and BIC values we find BB1 as the best fitted copula. The evidence of tail dependence coefficients suggests that currency markets are more likely to boom together than to crash together. Portfolio risk is measured using VaR and CVaR and global minimum risk portfolio is selected based on efficient frontiers. The evidences have direct implications for investors and risk managers during extreme currency market movements.

1. Introduction

Managing market risk associated with financial assets has become a crucial issue for financial institutions, regulators, and portfolio managers. One of the most popular and widely used tools for measuring market risk is Value at Risk (VaR). VaR is defined as the amount of maximum loss in a portfolio under given probability over a certain time period. VaR is considered as a benchmark for measuring market risk, which is the main reason that it can summarize risks in a single number. Additionally, conditional VaR (CVaR) is a supplement to VaR, and it measures the expected loss, given that the loss is greater than or equal to the VaR at certain confidence levels.

VaR and CVaR estimations are not difficult to estimate if only a single asset in a portfolio is owned. However, the estimation for the same becomes difficult in case of a portfolio of assets. One of the main difficulties in estimating portfolio VaR and CVaR is to model the co-movement of returns, i.e., the dependence structure, especially because VaR and CVaR are concerned with the tail of the distribution. Traditionally, the interdependence between the returns is measured using the Pearson's correlation coefficient, which works from the assumptions that financial assets are normally distributed and that the relationships between financial assets are linear. However, empirical studies have shown that correlations between assets returns are both non-linear and time-varying. Specifically, most return distributions show asymmetric downside and upside movements as well as fat tails. All this being the case, finding a more apt approach to modeling dependencies between asset returns has become a significant challenge in risk management. Other complications refer directly to stylized facts related to the joint multivariate distributional characteristics of asset returns, in particular the departure from Gaussian distribution and tail dependence (or extreme co-movement). Solutions for handling these problems lie in the application of extreme value theory (EVT) with copula functions. The combined approach deals essentially with the extreme dependence structure of large (negative or positive) asset returns, all in multivariate frameworks.

As referred earlier, VaR and CVaR typically deals with the low probability events in the tails of asset return distribution. EVT focuses directly on the tails and therefore could potentially give us better estimates and forecasts of risk. But applying the EVT to the return series is inappropriate as they are not independently and identically distributed (iid). Following the approach of McNeil and Frey (2000), we will first use a generalized autoregressive conditional heteroscedasticity (GARCH) model to fit the return series and then apply EVT to the innovations rather than to the return series. While the GARCH-EVT model is applied to draw the marginal distributions, the multivariate dependence structure between markets is modeled by parametric family of extreme value copulas which

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are perfectly suitable for non-normal distribution and nonlinear dependences. The combined GARCH-EVT-Copula framework thus become the natural choice for estimating portfolio VaR and CVaR.

Since Embrechts, Mcneil, and Straumann (2002) identified the limitations of correlation-based models and noted the relative advantages of copula model, many researchers have started using copula to directly model the dependence structure across financial markets. Works along this line include Chollete et al. (2006), Hu (2006), and Mashal and Zeevi (2002), who report asymmetric extreme dependence between equity returns, i.e., the stock markets crash together but do not boom together. Hotta, Lucas, and Palaro (2008) use a mixed model with the conditional copula and multivariate GARCH to estimate the VaR of a portfolio composed of NASDAQ and S&P500 indices. Jondeau and Rockinger (2006) took normal GARCH based copula for the VaR estimation of a portfolio composed of international equity indices. Fernandez (2008) illustrates how tail-dependency tests may be misleading as a tool to select a copula that closely mimics the dependency structure of the data, using US stock data. Huang, Lee, Liang, and Lin (2009) use the GARCH-Copula model in the estimation of VaR of a portfolio comprising NASDAQ and TAIEX and observe that compared with traditional models, the copula model captures the VaR more successfully.

The studies mentioned above have focused on the equity portfolio risk using the copula model. There are some studies which have investigated dependence structure and estimated risk of foreign exchange portfolio using VaR and CVaR based on GARCH-EVT-Copula model. For example, Patton (2001) applies copula to the modeling of the time-varying joint distribution of the Mark/Dollar and Yen/Dollar exchange rate returns, and finds that the conditional dependence between these exchange rates is asymmetric. Patton (2006) considers an extension of the theory of copulas to allow for conditional dependence structure of these exchange rates, and shows that the Mark/Dollar and Yen/Dollar exchange rates are more correlated when they are depreciating against the Dollar than when they are appreciating. Tursunalieva and Silvapulle (2007) extends Patton's work and shows that the SJC-copula captures the tail dependence between the exchange rates in pre-euro and post-transition periods, while the rotated Gumbel copula captures the dependence during the transition period. Wang, Chen, Jin, and Zhou (2010) apply GARCH-EVT-Copula model to study the risk of foreign exchange portfolio using data from Chinese market.

Incidentally, all of the above studies have been conducted on financial data taken from various developed and developing countries other than India. Curiously enough, to the best of our knowledge, there is no such study using financial data from India which has a huge potentiality to attract foreign investors to diversify their international portfolios. This paper tries to fill this gap by doing a similar study on financial data from India. Here we endeavor to investigate the dependence structure and estimate portfolio risk on data from foreign exchange market in India by using GARCH-EVT-Copula. The findings of the study have direct implications for investors and risk players in foreign exchange markets in India. We use VaR and CVaR for portfolio risk analysis.

To study the different foreign exchange rates dependence, we specify both the marginal models for the currency returns and a joint model for the dependence. We model the marginal distributions using AR-t-GARCH-EVT approach. More clearly we first use an autoregressive (AR) model with an appropriate GARCH errors following a Student-*t* distribution to model the volatility of return series in order to generate standardized residuals which should be iid. Then we model the standardized innovation with EVT approach. For the joint model, we choose seven copulas with different dependence structure: Gaussian copula; Frank copula, Clayton copula, Gumbel copula and BB1, BB2 and BB7 copulas.

The remainder of the paper is structured as follows. Section 2 presents the methodology which explains construction of GARCH-EVT-Copula model and details techniques of portfolio risk analysis. In Section 3, we describe the data and report the empirical findings. Section 4 explains implications and policy lessons. Finally Section 5 concludes the study.

2. Construction of GARCH-EVT-Copula model

In this section first we discuss briefly the copula function, tail dependence and some copula models to be used in the study. Secondly, we explain the marginal distributions of AR-t-GARCH-EVT model. Finally, we present the techniques of portfolio risk analysis.

2.1. Copula function, tail dependence and copula models

2.1.1. The Copula function

Since our study would deal with bivariate data, we define the copula in bivariate form. A bivariate copula function $C(u_1, u_2)$ is defined as a cumulative distribution function for a bivariate vector with support in $[0, 1]''$ and uniform marginal. Denoting (U_1, U_2) the corresponding bivariate vector, the copula function is defined as

$$C(u_1, u_2) = P(U_1 \le u_1, U_2 \le u_2). \tag{1}$$

If we now select arbitrary marginal distribution functions,

$$F_i(x_i) = P(X_i \le x_i) \quad \text{for } i = 1, 2 \tag{2}$$

and use the transformations $U_i = F_i(X_i)$, one can easily check that the function

$$F_i(x_1, x_2) = C[F_1(x_1), F_2(x_2)] \tag{3}$$

defines a new bivariate distribution, evaluated at x_1, x_2 with marginal F_i, $i = 1, 2$. Sklar (1973) shows the converse, namely, that any bivariate distribution function F can be written in terms of its marginal using a copula representation, as in Eq. (3). Moreover, if we assume that the marginal distribution F_i are all continuous, then F has a unique copula C.

The definition of the copula function in Eq. (3) is given in terms of cumulative distribution functions. If we further assume that each F_i and C are differentiable, the joint density $f(x_1, x_2)$ takes the form

$$f(x_1, x_2) = f_1(x_1) \times f_2(x_2) \times c[F_1(x_1), F_2(x_2)]. \tag{4}$$

where $f_i(x_i)$ is the density corresponding to F_i, and where

$$c(u_1, u_2) = \frac{\partial'' C(u_1, u_2)}{\partial u_1 \partial u_2} \tag{5}$$

is the density of the copula. By contrast to the traditional modeling approach that decomposes the joint density as a product of marginal and conditional densities, Eq. (4) states that, under appropriate conditions, the joint density can be written as a product of the marginal densities and the copula density. From Eq. (4), it is clear that the density $C(u_1, u_2)$ encodes information about the dependence structure between the X_i's while the f_i's describe the marginal behaviors. It thus shows that copulas represent a way of trying to extract the dependence and marginal behaviors.

2.1.2. Tail dependence

A useful dependence measure defined by copula is the tail dependence, which measures the probability that two variables are in the lower or upper joint tails. The coefficient of tail dependence is, in this case, a measure of the tendency of markets to crash or boom together.

Table 1
Description of copula functions.

Copula	Copula function	Lower tail dependence (λ_L)	Upper tail dependence (λ_U)
Gaussian	$C_N(u_1, u_2, \rho) = \phi[\phi^{-1}(u_1), \phi^{-1}(u_2)]$	0	0
Frank	$C_{Frank}(u_1, u_2, \theta) = -\frac{1}{\theta}\log = \frac{1}{\theta}\log\left[\frac{\theta(1-e^{-\theta})-(1-e^{-\theta u})(1-e^{-\theta v})}{1-e^{-\theta}}\right]$	0	0
Clayton	$C_C(u_1, u_2, \delta) = [u_1{}^{-\delta} + u_2{}^{-\delta}\}^{-1/\delta}]$	$2^{-1/\delta}$	0
Gumbel	$C_G(u_1, u_2, \delta) = \exp[-\{-\log \ u_1)^{\delta} + (-\log \ u_2)^{\delta}\}^{1/\delta}]$	0	$2-2^{1/\delta}$
BB1	$C(u_1, u_2) = \left(1 + \left[\left(u_1^{-\theta}-1\right)^{\delta} + \left(u_2^{-\theta-1}\right)\right]^{\frac{-1}{\theta}}\right)$	$2^{-1/(\delta\theta)}$	$2-2^{1/\delta}$
BB4	$C(u_1, u_2) = \left((u_1^{-\theta} + u_2^{-\theta} - 1 - \left[\left(u_1^{-\theta}-1\right)^{-\delta} + \left(u_2^{-\theta}-1\right)^{-\delta}\right]^{\frac{-1}{\delta}}\right)^{\frac{-1}{\theta}}$	$\left(2-2^{-1/\delta}\right)^{-1/\theta}$	$2^{-1/\delta}$
BB7	$C(u_1, u_2) = 1 - \left(1 - \left[(1-u_1)^{\theta})^{-\delta} + (1-(1-u_2)^{\theta})^{-\delta} - 1\right]^{\frac{-1}{\delta}}\right)^{\frac{1}{\theta}}$	$2^{-1/\delta}$	$2-2^{1/\theta}$

Note: For information on copulas, see Frank (1979), Gumbel (1960), Joe (1997) and Nelson (1999). This table summarizes the different copula families used in this study.

The coefficients of lower and upper tail dependence (λ_L and λ_U) can be expressed in terms of the copula between X_1 and X_2 as

$$\lambda_L = \lim_{q\to 0} Pr[X_1 \le F_1^{-1}(q)/X_2 \le F_2^{-1}(q)] = \lim_{q\to 0}\frac{C(q, q)}{q} \tag{6}$$

$$\lambda_U = \lim_{q\to 1} Pr[X_1 \ge F_1^{-1}(q)/X_2 \ge F_2^{-1}(q)] = \lim_{q\to 1}\frac{1-2q+C(q, q)}{1-q} \tag{7}$$

where F_1^{-1} and F_2^{-1} are the marginal quantile functions and where λ_L and $\lambda_U \in [0, 1]$. Roughly speaking, $\lambda_L(\lambda_U)$ measures the probability that X_1 is below (above) a low (high) quantile, given that X_2 is below (above) a low (high) quantile. If λ_L or λ_U is positive, then there is lower or upper tail dependence, otherwise there is lower or upper tail independence. Again there is a symmetric tail dependence between two assets when $\lambda_L = \lambda_U$, otherwise it is asymmetric. Different copulas usually represent different dependence structures with the association parameters indicating the strength of dependence. For example, Gaussian copula has zero tail dependence while Clayton copula has left tail dependence and no right tail dependence. On the contrary, Gumbel copula has right tail dependence and no left tail dependence. Further examination of copulas and measures of dependence can be found in Joe (1997) and Nelson (1999).

2.1.3. Copula models

Using copula we study the dependence structure or market co-movement between foreign exchange markets in India. As mentioned in the introduction, the copula family to be applied includes Gaussian copula, Frank copula, Clayton copula, Gumbel copula, and two parameters BB1, BB4, and BB7 copulas.

Li (2000) introduces the Gaussian copula in finance to calibrate default correlation. It is called the Gaussian copula because it communicates dependence in precisely the same way as the Gaussian distribution does, using only pairwise correlations among the variables, but it does so for variables with arbitrary marginal. In univariate analysis, the search for an appropriate distribution often starts with the assessment of the departure from normality of the empirical asset returns. In the same way, the Gaussian copula will serve as a benchmark to test whether the dependence structure in financial returns behaves as the Gaussian framework implies. Table 1 provides a brief description of the Gaussian copula along with other copulas mentioned above.

2.2. Marginal model of AR-t-GARCH-Extreme value theory

2.2.1. Extreme value theory

Extreme value theory relates to the asymptotic behavior of extreme observations of a random variable. It provides the fundamentals for the statistical modeling of rare events, and is used to compute tail related measures. There are two different but related ways of identifying extremes in real data over a certain time horizon. The first approach divides the time horizon into blocks or periods and considers the maximum the variable takes in successive periods, for example months or years. These selected observations constitute the extreme events, also called block maxima.

However, this method is not particularly suited for financial time series because of volatility clustering where extreme events tend to follow one another. As the block maxima method takes into account only the maximum return in each period, a large number of relevant data points are excluded from the analysis. The second approach that utilizes data more efficiently takes into account points above a given threshold. Therefore, the peak over threshold (POT) method has become the method of choice in financial applications.

According to this approach we define a sequence of values that exceed a high threshold u. The distribution of excess values is given by:

$$F_u(y) = \frac{Pr(X-u \le y, X > u)}{Pr(X > u)}, \quad 0 \le y \le x_F - u \tag{8}$$

where $y = y - u$ is the excess over u and x_F is the right endpoint of F. Following Pickands (1975) the limiting distribution of F_u can be approximated by a generalized Parato distribution (GPD) given by

$$\begin{aligned} \mathrm{G}_{\xi\psi}(\mathrm{y}) &= 1 - \left(1 + \frac{\xi y}{\psi}\right)^{-1/\xi}, \quad \text{if } \ \xi \ne 0 \\ &= 1 - \mathrm{e}^{-y/\psi}, \qquad\quad \text{if } \ \xi = 0 \end{aligned} \tag{9}$$

where ξ is the shape parameter, and ψ is the scale parameter. The distribution F can be expressed as function of the conditional excess distribution over the threshold u as follows:

$$F(x) = [1-F(u)]F_u(y) + F(u). \tag{10}$$

The function of $F(u)$ can be estimated non-parametrically by $(n-k)/n$, where n is the total number of observations, and k is the number of observations above the threshold u, using the method of historical simulation (HS). After replacing $F_u(y)$ by $G_{\xi\psi}(y)$, we get the following estimate for F(x):

$$F(x) = 1 - \frac{k}{n}\left[1 + \xi\frac{(x-u)}{\psi}\right]^{-\frac{1}{\xi}} \tag{11}$$

for $X > u$, where ξ and ψ can be estimated by the method of maximum likelihood.

The EVT approach described above focuses directly on the tail but does not acknowledge the fact that financial asset returns are non-iid. Most financial return series exhibit volatility clustering, fat-tailed distributions, and leverage effect. While the fat tails might be modeled directly with EVT, the lack of iid returns is problematic. One approach to this problem is provided by McNeil and Frey (2000). Using a two-stage approach they estimate the conditional volatility using a GARCH model in stage one. The GARCH model serves to filter the return series such that GARCH residuals are closer to iid than the raw return series. Even so, GARCH residuals have been shown to exhibit fat tails. In stage two, they apply EVT to the GARCH residuals. As such, the GARCH-EVT combination accommodates both time-varying volatility and fat-tailed return distributions. In this paper, we follow McNeil and Frey (2000) in combining the EVT approach with appropriate GARCH specification with leverage effect.[1] This approach is denoted as Conditional EVT, the implementation of which is detailed as follow:

We assume that the dynamics of conditional mean returns can be represented by the following AR (s) model

$$r_t = u_t + \varepsilon_t = u_t + \sqrt{h_t} Z_t \tag{12}$$

where $u_t = a_0 + \sum_{i=1}^{s} a_i r_{t-i}$, a_i are parameters, r_{t-i} are lagged returns, ε_t is residual which follows Student-*t* distribution, Z_t is standardized residual which is defined by $\varepsilon_t/\sqrt{h_t}$ and h_t is conditional variance of ε_t. We also assume that the conditional variance h_t follows any one of the asymmetric GARCH (p, q) processes such as EGARCH, TGARCH and PGARCH which are briefly explained below.

2.2.2. EGARCH (p, q) model

Nelson (1991) proposes the following Exponential GARCH (EGARCH) model to allow for leverage effects:

$$\log \quad h_t = \omega + \sum_{i=1}^{p} \alpha_i \frac{|\varepsilon_{t-i}| + \gamma_i \varepsilon_{t-i}}{\sqrt{h_{t-i}}} + \sum_{j=1}^{q} \beta_j \log \quad h_{t-j} \tag{13}$$

Note that the left-hand side is the log of the conditional variance. This implies that the asymmetric effect is exponential, rather than quadratic, and that forecasts of the conditional variance are guaranteed to be nonnegative. Here a positive ε_{t-i} contributes $\alpha_i(1+\gamma_i)|\varepsilon_{t-i}|/\sqrt{h_{t-i}}$ to the log of the conditional volatility, whereas a negative ε_{t-i} contributes $\alpha_i(1-\gamma_i)|\varepsilon_{t-i}|/\sqrt{h_{t-i}}$. The parameter γ_i thus signifies the leverage effect of ε_{t-i}. Again, we expect γ_i to be negative in real applications. The presence of asymmetric volatility can be tested by the null hypothesis H_0: $\gamma_i = 0$. The impact is asymmetric if $\gamma_i \neq 0$.

2.2.3. TGARCH (p, q) model

Another GARCH model that is capable of modeling leverage effects is the Threshold GARCH (TGARCH) model or also known as the GJR model (Glosten, Jagannathan, & Runkle, 1993), which has the following forms:

$$h_t = \omega + \sum_{i=1}^{p} (\alpha_i + \gamma_i D_{t-i}) \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j} \tag{14}$$

where D_{t-i} is the dummy variable which is equal to 1 if innovation ε_{t-i} is negative and equal to 0 otherwise. That is, depending on whether ε_{t-i} is negative or positive, ε_{t-i}^2 has different effects on the conditional variance; if innovation is negative, the total effects are given by $(\alpha_i + \gamma_i)\varepsilon_{t-i}^2$; when innovation is positive the total effects are given by $\alpha_i \varepsilon_{t-i}^2$.

2.2.4. PARCH (p, q) model

As an alternative to the GARCH process which models conditional variance, independently Schewert (1990) and Taylor (1986) have suggested modeling the conditional standard deviation. Ding, Granger, and Engle (1993) propose the asymmetric power ARCH (PARCH) model where the power parameter δ on the standard deviation is estimated rather than imposed:

$$\sigma_t^\delta = \omega + \sum_{i=1}^{p} \alpha_i (|\varepsilon_{t-i}| - \gamma_i \varepsilon_{t-i})^\delta + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^\delta \tag{15}$$

where $\delta > 0$ and γ_i reflects the so called leverage effect. A positive value of γ_i means that past negative shocks have a deeper impact on current conditional volatility than past positive shocks.

The different asymmetric GARCH models explained above would be fitted here with Student-*t* distribution instead of normal distribution. This is because empirical evidence strongly rejects the idea that financial returns are normally distributed. In fact, it is well established that the financial returns are fat-tailed. Hence we use the fat-tailed density, e.g., Student-*t* with GARCH in order to better account for heavy-tailedness.

With the assumption that the random errors follow Student-*t* distribution, the log-likelihood function of a sample of *m* iid observations is given by:

$$\log(\theta) = \sum_{t=1}^{m} \log \frac{\Gamma[(\upsilon+1)/2]}{\sqrt{\pi(\upsilon-2)}\,\Gamma(\upsilon/2)} - \frac{1}{2}\sum_{t=1}^{m} \left[\log \quad h_t + (\upsilon+1)\log\left(1 + \frac{\varepsilon_t^2}{(\upsilon-2)}\right)\right] \tag{16}$$

where υ is the degree of freedom and $\Gamma(a) = \int_0^{+\infty} x^{a-1} e^x dx$ is the gamma function.

Standardized residuals or innovations (Z_t) can be computed after maximizing Eq. (16) with respect to the unknown parameters:

$$Z_t = \frac{r_t - u_t}{\sqrt{h_t}} \tag{17}$$

The conditional mean (u_t) and variance (h_t) for interval *t* are estimated by using standard 1-step ahead forecasts. The 1-step ahead conditional mean forecast is given by

$$\hat{\mu}_t = \hat{a}_0 + \sum_{i=1}^{s} \hat{a}_i r_{t-i} \tag{18}$$

and the 1-step ahead conditional variance forecast is given by the Eqs. (19)–(21) for EGARCH (p, q), TGARCH (p, q) and PARCH (p, q) models, respectively.

$$\hat{h}_t = \exp\left[\hat{\omega} + \sum_{i=1}^{p} \hat{\alpha}_i \frac{|\varepsilon_{t-i}| + \hat{\gamma}_i \varepsilon_{t-i}}{\sqrt{\hat{h}_{t-i+1}}} + \sum_{j=1}^{q} \hat{\beta}_j \log \quad \hat{h}_{t-j}\right] \tag{19}$$

$$\hat{h}_t = \hat{\omega} + \sum_{i=1}^{p} (\hat{\alpha}_i + \hat{\gamma}_i D_{t-i}) \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \hat{\beta}_j \hat{h}_{t-j} \tag{20}$$

$$\hat{\sigma}_t^\delta = \hat{\omega} + \sum_{i=1}^{p} \hat{\alpha}_i (|\varepsilon_{t-i}| - \hat{\gamma}_i \varepsilon_{t-i})^\delta + \sum_{j=1}^{q} \hat{\beta}_j \hat{\sigma}_{t-j}^\delta \tag{21}$$

[1] Empirical evidences observed in many studies suggest that the conditional variance of financial returns is an asymmetric function of past innovations. More specifically, negative innovations increase volatility more than positive innovations. Since this feature is not addressed by symmetric GARCH model, we have combined the extreme value approach with asymmetric GARCH model.

If the standardized residuals are iid and the fitted model is well-specified, we end stage 1 and in stage 2 we apply EVT to the standardized residuals.

Thus our marginal model is built on AR-t-GARCH-EVT model, in which we use an appropriate AR-t-GARCH model to fit the historical return data, then we model the standardized innovation by two distributions: by the GPD distribution in the lower and upper tails and by the empirical distribution in the remaining part. The marginal distribution of each innovation is given by

$$F(z)=\begin{cases}\dfrac{k^L}{n}\left\{1+\xi^L\dfrac{\left(z-u^L\right)}{\psi^L}\right\}^{-1/\xi^L} & z<u^L\\ \phi(z) & u^L<z<u^R\\ 1-\dfrac{k^R}{n}\left\{1+\xi^R\dfrac{\left(z-u^R\right)}{\psi^R}\right\} & z>u^R\end{cases}\qquad(22)$$

where u^L, u^R are the lower and upper threshold respectively, $\phi(z)$ is the empirical distribution on the interval $[u^L, u^R]$, n is the number of z and k^L is the number of innovation whose value is less than u^L and k^R is the number of innovation whose value is greater than u^R.

2.3. *Portfolio risk analysis*

2.3.1. *VaR and CVaR*

In order to analyze portfolio risk, this paper uses VaR and CVaR, alternatively known as Expected shortfall, to measure the risk. As referred in the introduction a popular measure of market risk is the VaR, which is generally defined as the maximum potential losses in the market value of, say, a financial portfolio with a given level of probability over a specified period. Mathematically, suppose a random variable X with continuous distribution function F models the return distribution of a risky financial portfolio over the specified time horizon. For a given probability $1-\alpha=q$, VaR can be defined as the qth quantile of the distribution F

$$VaR^{\alpha}=F^{-1}(1-\alpha)\qquad(23)$$

where F^{-1} is the so-called quantile function defined as the inverse of the distribution function F.

CVaR is the expected loss in the tail ($CVaR^{\alpha}$), and is defined as the average of the losses that exceeds VaR^{α}. Formally, $CVaR^{\alpha}$ is defined as

$$CVaR^{\alpha}=E(R/R\geq VaR^{\alpha})\qquad(24)$$

2.3.2. *Optimal portfolio with minimum risk*

If we assume the weight of the assets is the same, i.e., $w=[1/2, 1/2]'$ we can compute VaR and CVaR of the equally weighted portfolio. However, the major concern for commercial banks and individual investors is to minimize the risk of the investment portfolio. To address this concern, we compute the optimal weights of each asset that minimize the portfolio risk, using the following algorithm. We assume that the weight in individual asset within a portfolio is $w=[w_1, w_2]'$, where $0\leq w_i\leq 1$ $(i=1, 2)$ and $w_1+w_2=1$. Apparently, there are many weight vectors that satisfy these conditions. For each investment weight vector w, we can get N returns of the portfolio; this means that Returns $=r_{t+1}\times w=(r_{1n}, r_{2n})\times w$. Subsequently, the VaR^{α} of each of these N number of portfolios at a given confidence level $(1-\alpha)$ can be computed. Hence we compute the VaR^{α} of each of these portfolios at a given confidence level. After getting all VaR^{α} values corresponding to all the weight vectors, we can find the minimum VaR^{α}. This enables us to get the weight in each asset that minimizes the VaR^{α}. Following the same procedure we can get the weight in each asset that minimize the $CVaR^{\alpha}$.

2.3.3. *VaR and CVaR estimation procedure*

Estimation of portfolio VaR^{α} and $CVaR^{\alpha}$ requires the knowledge of the assets multivariate distribution, however, it is a strong requirement which can be relaxed using copulas. Since an explicit formula to estimate VaR^{α} and $CVaR^{\alpha}$ for a portfolio is not available even in commonly used cases, alternative approach is the use of Monte Carlo simulation methods. Under dynamic copula models presented earlier, the estimation of VaR^{α} and $CVaR^{\alpha}$ is implemented as follows:

1) Fit an appropriate GARCH model to the return data by quasi-maximum likelihood. That is, maximize the log-likelihood function of the sample assuming Student-t innovations. Obtain parameters estimations and compute standardized residuals to check the adequacy of the GARCH modeling. They are calculated as $Z_{t-n+1}=\frac{r_{t-n+1}-\hat{\mu}_{t-n+1}}{\sqrt{\hat{h}_{t-n+1}}}, \ldots\ldots\ldots\ldots, Z_t=\frac{r_t-\hat{\mu}_t}{\sqrt{\hat{h}_t}}$, where 1-step ahead conditional mean forecast $(\hat{\mu}_t)$ is given by Eq. (18) and conditional variance forecast $(\hat{h}_t)$ is given by Eqs. (19)–(21) for EGARCH (p,q), TGARCH(p,q) and PARCH(p,q), respectively.
2) Fit the GPD to each standardized residuals series after choosing the threshold value u for the upper and lower tail of the distribution. Transform each standardized residuals series into uniform (0,1) using the probability-integral transformation.
3) For each pair of transformed data vectors, fit the most appropriate extreme value copula and estimated their parameters by using a two-step estimation procedure (IFM). If the appropriate copula is not known, several copula are estimated and the appropriate copula is chosen based on model selection criteria.
4) Simulate Q Monte Carlo Scenarios over the time horizon, using the conditional bivariate distribution modeled by the estimated copula model and obtain Q two-dimension vector future returns, $r_{t+1}^{q},\ q=1, \ldots.., Q$ and Q future portfolio returns. The procedure is accomplished in following stages:
   i) Simulate Q uniform variates (0,1) from the copula function estimated in step 3.
   ii) Obtain the (simulated) standardized residuals by using the inverse function of the estimated marginal.
   iii) Get the (simulated) asset returns by using the standardized residuals in step ii, and the forecasted means and variances from step 1.
   iv) Repeat steps i–iii for Q times, and obtain Q two-dimension vector future returns, $r_{t+1}^{q},\ q=1, \ldots.., Q$ and Q future portfolio returns.
5) Given the Q portfolio returns obtained from step iv, an empirical distribution of the loss/profit returns is estimated. VaR_{t+1}^{α} corresponds to the $(1-\alpha)^{\text{th}}$ percentile of this distribution. $CVaR_{t+1}^{\alpha}$ is the average portfolio return of the values that are greater than VaR_{t+1}^{α}.
6) We estimate VaR_{t+1}^{α} and $CVaR_{t+1}^{\alpha}$ of different weighted portfolio to find the efficient set of portfolio.

3. Data and empirical findings

In this section, we first discuss the data set to be used, analyze descriptive statistics and describe empirical copula. Then we present the empirical results to investigate the dependence structure and to analyze portfolio risk.

3.1. *Data*

In this study we choose daily data on five foreign exchange rates including USD/INR, SF/INR, JPY/INR, GBP/INR and EURO/INR to demonstrate the application of GARCH-EVT-Copula model. The data series to be used here are from February 1, 1996 to June 30,

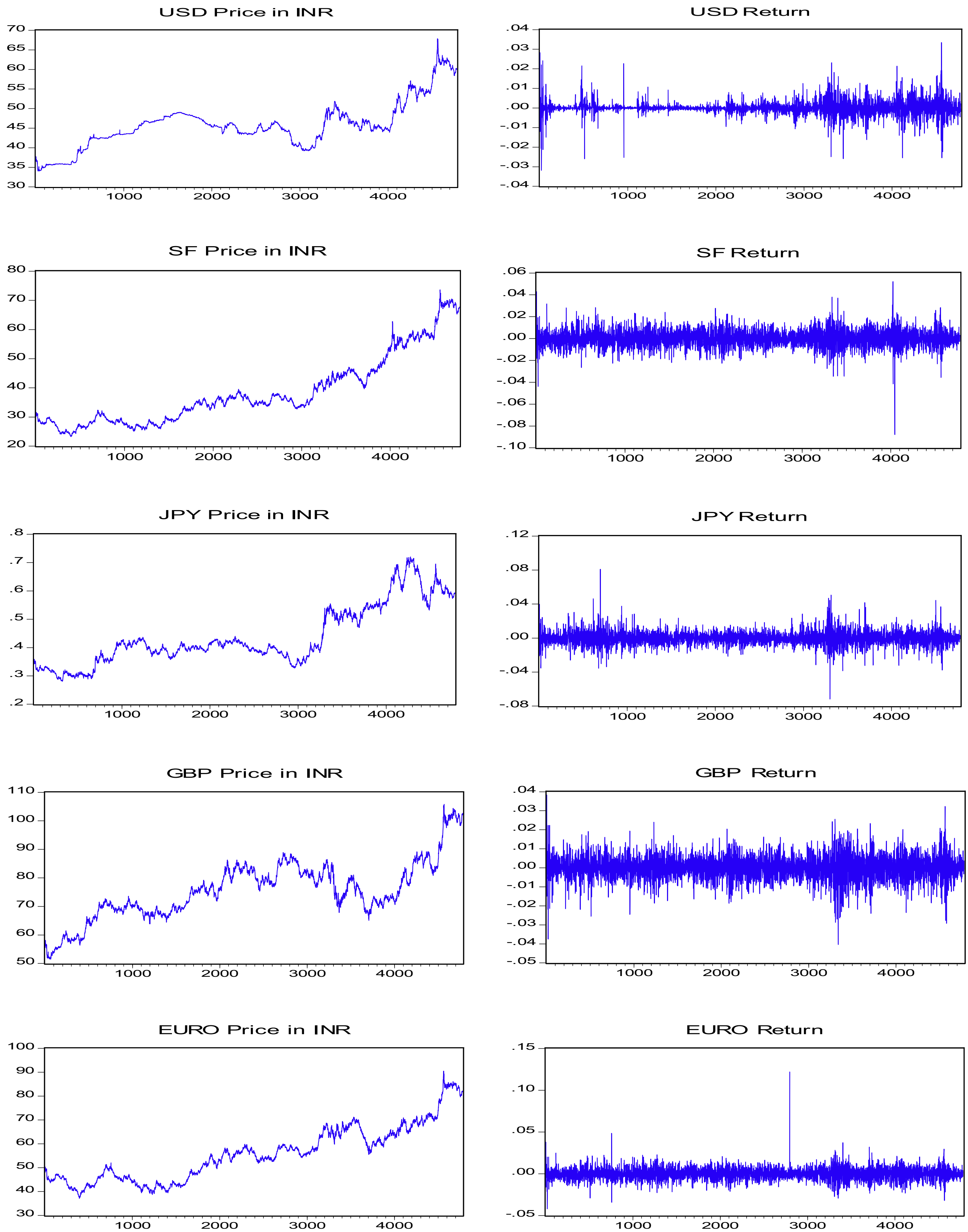

Fig. 1. Price and return plots from five foreign exchange: USD/INR, SF/INR, JPY/INR, GBP/INR and EURO/INR (from February 1, 1996 to June 30, 2014).

Table 2
Descriptive statistics and correlation matrix of returns.

Statistics	USD	SF	JPY	GBP	EURO
Panel A: Descriptive statistics of five forex return series					
Mean	0.000104	0.00017	0.000115	0.000129	0.000135
Median	0	9.60E − 05	0	0.000212	0.000189
Std. deviation	0.003635	0.007559	0.007931	0.006185	0.007152
Skewness	0.036699	−0.235676	0.368922	−0.225803	1.041679
Kurtosis	14.99374	8.831317	9.707798	5.480225	22.06387
Jarque–Bera	28645.16	6815.338	9067.946	1265.532	73232.54
Max	0.033493	0.052064	0.080945	0.038256	0.12191
Min	−0.031837	−0.087863	−0.071686	−0.040303	−0.041983
Q(16)	133.82 (0.000)	41.279 (0.001)	30.579 (0.015)	18.325 (0.305)	45.025 (0.000)
$Q^2(16)$	2307.9 (0.000)	228.31 (0.000)	733.46 (0.000)	1415.6 (0.000)	13.638 (0.626)
Number of observations (T)	4779	4779	4779	4779	4779
Panel B: Correlation matrix of forex return series					
USD	1	0.4172	0.4709	0.4695	0.4053
SF	0.4172	1	0.4944	0.6014	0.8183
JPY	0.4709	0.4944	1	0.3354	0.3825
GBP	0.4695	0.6014	0.3354	1	0.6553
EURO	0.4053	0.8183	0.3825	0.6553	1

Note: The table presents descriptive statistics of the five foreign exchange return series over the period from February 1, 1996 to June 30, 2014. All the series exhibit both significant skewness and kurtosis. In the same way as the skewness and kurtosis reveal important distributional difference between the return series, the pairwise correlation suggests diversity in the dependence structure of bivariate series.

2014 (the data source is http://www.Kshitis). We use the daily logarithmic returns defined as $r_t = l_n P_t - l_n P_{t-1}$, where P_t is the daily closing price of exchange rate at time *t*. The data series give us 4779 observations for each exchange rate. The usual descriptive statistics of the five series are displayed at Panel A of Table 2. Panel B of the table gives the Pearson's correlation matrix.

To assess the distributional characteristics and stochastic properties of the return data, we must first examine some descriptive statistics reported at panel A of Table 2. The mean daily return of all exchange rates is positive justifying an upward movement of daily price of exchange rate shown at left panel of Fig. 1. The sample skewness is negative for three exchange rates i.e., USD, SF and GBD which suggests that the negative shocks are more frequent than the positive ones for these three rates. For the rest two rates the skewness is positive suggesting more frequent positive shocks. The kurtosis estimate is very high for all the rates which means that return distribution is leptokurtic, with heavier tails than the normal distribution. The non-zero skewness and high kurtosis clearly indicate the non-normality of the distribution which is confirmed by the high Jarque–Bera statistics. On the basis of Ljung–Box Q statistic, the hypothesis that all correlation coefficients up to 16 are jointly zero is rejected for all series except GBP. Therefore, we can conclude that the return series mostly present some linear dependence in returns. In addition, the statistics of serial correlations in squared returns [Q^2 (16)] are significant for all series except EURO which suggest that there is non-linear dependence in those return series. This indicates volatility clustering which is clearly observed in the return series shown at the right panel of Fig. 1. It is also observed in some cases that volatility appears to be more when price declines than that when price increases. Hence we may need an asymmetric GARCH model, namely EGARCH or TGARCH or APARCH model, whichever is appropriate, to filter the return series of each market separately. Together with Panel A of Table 2, Fig. 1 demonstrates the defining characteristics of the foreign exchange markets: high volatility, occasional extreme movements, volatility clustering and fat tailed distributions. The evidences further motivate the exploration of conditional EVT to get the marginal distribution.

Panel B of Table 2 reports the Pearson's linear correlation matrix which may provide a preliminary idea of linear relationship between foreign exchange returns in India. The positive and high correlation values indicate that foreign currency markets move together in the same direction.

As it is mentioned earlier, we have taken five foreign exchange rates i.e., USD/INR, SF/INR, JPY/INR, GBP/INR and EURO/INR. Out of these five exchange rates if we consider one pair of rates at a time to apply bivariate copula models, we would have following ten pairs of rates: USD-SF, USD-JPY, USD-GBP, USD-EURO, SF-JPY, SF-GBP, SF-EURO, JPY-GBP, JPY-EURO, and GBP-EURO. We would describe below empirical copula of these ten pairs.

To examine the preliminary dependence structure of the data, we have first obtained the empirical copula table. To that end we have ranked each pair of foreign exchange return series in ascending order and divided each series evenly into 10 bins, such that bin 1 includes the observations with the lowest values and bin 10 the observations with the highest values. As we are interested in knowing the values of one series associated with the values of other series, we count the number of observations in each (i, j) cell. If the two series were perfectly positively or negatively correlated we would see most of observations lying on the diagonal connecting the upper-left with the lower-right corner and the lower-left with the upper-right corner, respectively; and if they are independent then we would expect the numbers in each cell to be about the same. Also, if there is lower tail dependence between the two series we would expect more observations in cell (1,1), and if there is upper tail dependence we would observe a larger number of observations in cell (10,10).

Table 3 reports the empirical copula for the ten pairs of foreign exchange returns. For all the pairs, cells (1,1) and (10,10) contain a greater number of observations than any other cell. So for example consider the USD-SF pair, whose cell (1,1) and (10,10) contains 148 and 140 observations respectively, which means out of 4779 observations there 148 and 140 occurrences when both USD and SF returns lie in their lowest 10th percentile and highest 10th percentiles, respectively. Numbers in other cells of this pair are much smaller than those in these two cells, which is the evidence of both upper and lower tail dependence. Comparing cell (1, 1) and cell (10, 10) of all other pairs, we note even stronger evidences of both upper and lower tail dependence.

The evidence suggests to use parametric copula possessing upper and lower tails dependence. We experiment three copulas, namely, BB1, BB4 and BB7 which possess both tails dependence. For a matter of comparison we also use Gaussian copula, Frank copula, Clayton copula, and Gumbel copula.

Table 3
Empirical copula for returns.

Pair											Pair										
USD-SF	148	79	63	54	26	22	28	26	20	12	USD-JPY	155	87	65	56	21	23	25	16	16	14
	52	69	73	66	51	36	29	37	33	32		62	55	71	64	59	42	39	41	25	20
	44	57	50	49	61	59	44	44	41	29		42	64	54	46	71	65	43	28	34	31
	49	43	51	55	63	57	43	44	38	35		44	42	50	65	50	58	47	47	47	28
	28	38	41	35	36	50	39	45	35	25		35	38	38	34	33	54	35	43	34	28
	49	68	59	48	69	68	64	57	55	47		44	68	70	55	54	62	65	57	58	51
	45	46	40	56	59	49	39	45	44	55		35	46	42	57	52	62	57	47	40	40
	31	28	51	62	49	55	60	52	47	43		28	39	42	46	57	60	68	47	55	36
	21	33	32	33	31	54	76	64	74	60		25	25	29	35	36	45	64	84	77	58
	10	17	18	20	33	28	56	64	91	140		7	14	17	20	21	31	35	68	92	172
USD-GBP	164	83	70	37	25	23	19	18	25	14	USD-EURO	142	90	57	48	31	22	22	26	19	21
	57	72	72	54	46	44	42	31	35	25		47	65	63	63	60	48	40	33	29	30
	40	59	52	60	58	53	38	48	43	27		40	47	58	52	70	45	48	41	40	31
	37	43	59	52	43	64	56	53	38	33		50	47	46	60	51	54	45	45	46	34
	24	33	35	46	52	55	37	33	32	25		28	34	41	46	38	46	38	43	30	28
	37	53	70	75	72	61	73	57	51	35		44	65	71	54	69	67	68	49	53	44
	41	46	46	46	63	46	54	46	48	42		42	40	46	52	52	48	51	53	41	53
	30	41	34	50	62	61	58	48	52	42		33	38	49	48	39	64	58	55	58	36
	30	28	26	31	31	38	72	82	73	67		29	31	31	31	36	54	68	69	73	56
	17	20	14	27	25	34	29	62	81	168		15	22	16	24	30	32	40	64	89	145
SF-JPY	178	85	56	43	30	24	25	14	14	12	SF-GBP	214	86	54	45	19	19	16	8	10	6
	80	92	79	54	48	32	32	24	16	21		87	106	81	76	41	28	18	20	12	9
	57	70	70	74	50	49	32	31	22	23		46	78	88	80	60	49	33	21	17	6
	27	58	67	75	66	55	49	31	27	23		32	71	81	70	65	57	44	28	21	9
	35	45	57	58	74	76	49	34	33	17		22	45	52	62	84	63	59	48	26	17
	22	35	48	55	49	73	53	63	51	29		18	36	34	45	53	79	85	60	43	25
	27	31	44	47	49	57	65	70	43	45		27	20	32	38	54	72	69	75	53	38
	23	23	27	31	35	71	66	69	69	64		13	13	25	29	45	51	80	89	87	46
	16	28	21	26	33	36	74	79	88	77		6	16	20	18	36	47	50	67	115	103
	8	11	9	15	20	29	33	63	115	171		12	7	11	15	20	14	24	62	94	219
SF-EURO	335	85	27	7	9	5	3	3	1	2	JPY-GBP	141	71	65	43	37	28	26	23	16	27
	82	214	117	26	16	9	6	5	1	2		75	77	60	57	53	46	30	30	27	23
	24	103	158	116	39	20	9	3	4	2		48	76	70	70	58	37	39	36	26	18
	12	34	94	158	98	46	23	6	5	2		46	53	58	62	65	58	49	42	26	19
	4	17	44	107	140	101	39	16	6	4		28	42	61	55	52	66	54	41	35	20
	5	9	20	35	99	146	109	36	16	3		34	38	40	54	55	82	61	52	50	36
	4	5	10	18	37	89	149	122	29	15		28	34	41	40	48	56	62	57	62	50
	5	4	3	9	26	48	93	171	100	19		27	31	23	31	44	48	67	79	76	52
	3	5	3	1	8	12	38	98	223	87		19	27	26	41	29	33	57	65	85	96
	2	3	2	1	4	4	9	18	93	342		31	29	34	25	36	25	33	53	75	137
JPY-EURO	159	70	62	34	31	28	25	27	20	21	GBP-EURO	222	110	43	26	24	22	14	9	2	5
	74	78	72	58	52	43	26	27	32	16		96	102	87	67	44	31	26	11	9	5
	49	73	79	72	59	41	41	31	22	11		54	76	98	76	55	40	33	24	17	5
	38	64	63	66	55	55	54	40	25	18		40	71	84	77	63	43	42	33	16	9
	30	45	46	69	71	63	40	32	33	25		16	44	55	67	82	69	60	33	38	13
	29	36	49	57	70	61	56	65	46	33		18	25	40	67	62	81	67	65	36	18
	24	33	41	38	47	53	81	59	59	43		11	27	27	36	63	75	79	75	52	33
	25	31	33	34	37	52	55	85	70	56		8	11	18	31	51	59	82	80	86	52
	19	21	24	27	28	45	57	47	98	112		6	10	20	19	18	41	46	101	118	99
	29	28	9	23	26	39	43	65	73	143		5	3	6	12	14	19	29	47	104	239

Note: Observations for each series totalled 4779. In each table, along the horizontal axis and in ascending order (from left to right) are the ranks for foreign exchange returns for the first foreign exchange returns named in the first column; and along the vertical axis and in ascending order (from top to bottom) are the ranks for the second foreign exchange returns named in the first column.

3.2. *Empirical findings*

We apply AR-t-GARCH-EVT-Copula model to study the dependence structure and to analyze portfolio risk. Firstly, we fit AR-t-GARCH-EVT model to each of the five exchange rate series and get the marginal distribution of residuals. Then, we use bivariate copulas to describe dependence structure between any pair of foreign exchange return series, separately for each of ten pairs. Finally, we do portfolio risk analysis using VaR and CVaR and fitting their efficient frontiers.

There are usually two approaches to estimate a parametric copula model, namely one stage full maximum likelihood (ML) and inference for the margins (IFM). The ML approach jointly estimates the parameters in the marginal models and the parameters of the copula model simultaneously. The IFM method proposed by Joe and Xu (1996) breaks the estimation into two steps: at the first step, one can estimate the parameters in the marginal distributions; at the second step, given the estimated marginal parameters, the copula parameters are estimated.

Compared with the ML, the IFM method is less computationally intensive. Moreover, the large number of parameters in the simultaneous ML estimation could make numerical maximization of the likelihood function difficult. Since our models involve a large number of parameters, we adopt the IFM method for our estimation as well.

3.2.1. *Estimates of marginal models*

Since the actual foreign exchange return series have fat tail and exhibit volatility clustering, EVT appears to be an appropriate approach for modelling the fat tail behaviour. But as referred ear-

Table 4
Parameter estimates for the marginal distribution models.

Statistics	USD	SF	JPY	GBP	EURO
Mean equation					
a_0	−3.35E − 06 (0.7943)	0.000124 (0.1656)	−4.82E − 05 (0.6020)	0.000188	0.000118 (0.1474)
a_1	0.121072	−0.080855		−0.051153	−0.087030
a_2	−0.049241				
Variance equation					
ω	2.84E − 06	−0.253534	1.38E − 05 (0.3737)	1.39E − 05 (0.4447)	0.000231 (0.2816)
α_1	0.375119	0.084192	0.060590	0.064475	0.051436
γ_1	−0.129128	0.033616	−0.224758	−0.124916	−0.134420 (0.1024)
β_1	0.754235	0.980917	0.930109	0.928511	0.943231
δ	1.325446		1.472645	1.400154	0.863861
Log-likelihood	23043.03	16893.42	16881.89	17894.44	17236.25

Note: The table reports ML estimates of the fitted AR-GARCH models with Student-t governing the error terms. The fitted models are AR(2)-PARCH (1,1) for USD; AR(1)-EGARCH(1,1) for SF; AR(0)-PARCH(1,1) for JPY; AR(1)-PARCH (1,1) for GBP and AR(1)-PARCH (1,1) for EURO. For each data series, parameter estimates are based on the sample period from February 01, 1996 to June 30, 2014. The majority of parameter estimates are statistically significant at better than 1% level and their *p* value is not shown. *p* Values are shown in parentheses only when not significant at the 1% level.

Table 5
Diagnostic statistics of standardized residuals.

Statistics	USD	SF	JPY	GBP	EURO
Skewness	21.40052	0.015537	0.260631	−0.059616	3.55531
Kurtosis	906.0971	4.42563	4.966682	3.724168	92.47792
Jarque–Bera	163E + 08 (0.000)	404.8981 (0.000)	824.2881 (0.000)	107.2557 (0.000)	1604322 (0.000)
Q(16)	18.651 (0.287)	13.716 (0.620)	16.701 (0.405)	8.8327 (0.920)	5.7759 (0.990)
$Q^2(16)$	0.0466 (1.000)	11.659 (0.767)	18.455 (0.298)	11.934 (0.749)	0.2424 (1.000)
Number of observations	4777	4778	4779	4778	4778

Note: The table reports diagnostic statistics of standardised residuals from the fitted AR(s)-GARCH (p, q) model with Student-*t* distribution governing the error terms. The statistics are the basis of the EVT estimation. The *p* values are given in the parentheses.

lier applying EVT to the return series (r_t) is inappropriate because random variable (r_t) is not iid. Therefore, we first use the suitable AR-GARCH model to the return series to generate the standardized residual series which is iid, if the GARCH model is tenable. Then we model the standardized innovations with EVT.

The appropriate AR-GARCH model for each series is selected by adopting the following procedure. First, various asymmetric GARCH models i.e., AR(0)-GARCH (1, 1) models with leverage effect are estimated and compared using the usual information criteria such as AIC, BIC and log-likelihood statistics. Once the asymmetric model is selected, the AR-GARCH specifications are augmented with additional AR, and ARCH, GARCH lagged terms when necessary to eliminate autocorrelation in the standardized and squared standardized residuals, respectively. Based on this procedure, we have selected AR(2)-PARCH (1,1) for USD; AR(1)-EGARCH(1,1) for SF; AR(0)-PARCH(1,1) for JPY; AR(1)-PARCH (1,1) for GBP and finally AR(1)-PARCH (1,1) for EURO.

Table 4 presents the estimated parameters of the mean and variance equations of the selected model applied to daily return series in each market. The constant term and AR (s) coefficients in the mean equation are mostly found to be significant. Similarly, the parameters in the variance equation: the constant, the ARCH (p) coefficients, and the GARCH (q) coefficients are found to be significant in majority of the cases. The value of γ_1 is also significant in each series except EURO, indicating asymmetric volatility in those return series. Estimates of the conditional mean and standard deviation series ($\hat{\mu}_{t-n+1},\ldots,\hat{\mu}_t$) and ($\sqrt{\hat{h}_{t-n+1}},\ldots,\sqrt{\hat{h}_t}$) can be calculated recursively from Eqs. (18) and (19) or (20) or (21), whichever is appropriate, respectively after substitution of sensible starting values.

Standardized residuals are calculated both to check adequacy of the selected models and to use in stage 2 of the method. They are calculated as

$$Z_{t-n+1} = \frac{r_{t-n+1} - \hat{\mu}_{t-n+1}}{\sqrt{\hat{h}_{t-n+1}}}, \ldots\ldots\ldots\ldots, \; Z_t = \frac{r_t - \hat{\mu}_t}{\sqrt{\hat{h}_t}} \tag{25}$$

and should be iid if the fitted model is tenable.

Table 5 presents diagnostic statistics of standardized residuals. We have earlier noted the significant value of Ljung–Box Q(16) statistics in Table 2 which indicate that raw returns are mostly serially correlated and hence are not iid as required by EVT. In contrast, Ljung–Box Q(16) statistics reported in Table 5 suggest that the standardized residuals are iid. Thus the filtering procedure advocated by McNeil and Frey (2000) has been effective in producing iid residuals on which EVT can be implemented. Q^2 (16) statistic of standardized residuals in all series failed to detect serial correlations in squared standardized residuals, suggesting that the selected asymmetric GARCH models are well specified. However, it appears from the table that skewness and excess kurtosis remain in the standardized residuals. It is also noted that the standardized residual series are not normally distributed as suggested by Jarque–Bera statistics. All these findings motivate the second stage of McNeil and Frey's EVT implementation, where fat tails of the standardized residuals are explicitly modeled.

We have already extracted the iid filtered residuals from each returns series. Now we construct the marginal of each series using the empirical CDF for the interior and the GPD estimates for the upper and lower tails of the filtered residuals Z_t. Fitting the GPD to the filtered returns requires selection of the lower and upper thresholds. However, this selection is subject to a trade-off between variance and bias. By increasing the number of observations for the series of maxima (a lower threshold), some observations from the centre of the distribution are introduced in the series, and the tail index is more precise but biased (i.e., there is less variance). On the other hand, choosing a high threshold reduces the bias but makes the estimator more volatile (i.e., there are fewer observations). Thus the threshold estimation becomes more of an art than a science in balancing this trade-off between bias and variance. Following De Melo Mendes (2005), we set the threshold values such that 10% of the residuals are reserved for the lower left and the upper right quadrants (i.e., the empirical 10% and 90% quantiles in each margin). In what follows, we refer the tail dependence coefficient which

Table 6
Parameter estimates for GPD model.

Statistics	USD	SF	JPY	GBP	EURO
Total observations					
Panel A: Left tail					
EVT threshold u	−1.02178	−1.20046	−1.17971	−1.27218	−1.23634
No. of exceedences k					
GPD shape parameter (ξ)	0.1923* (3.4391)	−0.0011 (−0.0341)	0.0219 (0.4418)	−0.0193 (−0.4537)	−0.0847* (−2.2911)
GPD scale parameter (β)	0.4726** (13.9465)	0.5602** (17.6799)	0.5233** (14.8024)	0.5434** (16.0086)	0.5806** (17.0328)
Log-likelihood	−211.7	−200.5	−178.5	−177.2	−177.2
Panel B: Right tail					
EVT threshold u	1.123452	1.254712	1.206194	1.23917	1.236797
No. of exceedences k	478	478	478	478	478
GPD shape parameter (ξ)	0.3562** (6.6573)	−0.0430 (−1.0852)	0.1016 (1.9560)	−0.0815 (−1.8308)	0.1512* (3.9589)
GPD scale parameter (β)	0.5002** (14.5961)	0.5886** (16.5289)	0.5750** (14.4812)	0.5660** (15.6845)	0.4780** (16.8982)
Log-likelihood	−316.5	−204.1	−262	−166.6	−197.5
Total observations	4779	4779	4779	4779	4779

Note: The table reports ML estimates of the GPD for the AR-GARCH-EVT model. The t statistics are given in the parenthesis, while asterisks demonstrate the level of statistical significance.
* Denotes significance at 5% level.
** Denotes significance at 1% level.

describes how large (or small) returns of a foreign currency occur with large (or small) returns of another to as bull and bear markets (Cherubini, Luciano, & Vecchiato, 2004; Longin & Solnik, 2001; Nelson, 1991).

To find the optimal estimates of shape (ξ) and scale () parameters, the maximum likelihood method is applied on $G_{\xi\psi}(y)$ in Eq. (9) using the log-likelihood function:

$$\log \quad L(\xi, \psi, y_1, \ldots, y_k) = \sum_{j=1}^{k} \log \ G_{\xi\psi}(y_j) =$$

$$-k\log \ \psi - \left(1 + \frac{1}{\xi}\right) \sum_{j=1}^{k} \log \left(1 + \xi \frac{y_j}{\psi}\right) \tag{26}$$

where $y_j = Z_j - u$ and Zj defines the standardized residuals exceeding the identified threshold value of u. Maximizing the log likelihood of Eq. (26) s. t. $\psi > 0$, $1 + \xi y_j/\psi > 0$, the most likely values of ξ and ψ are estimated and reported for left tail at panel A and for right tail at panel B of Table 6.

Note that USD series has significant positive value of shape parameter (ξ) at both panels A and B of the table, which indicate its heavy tailedness for both tails. However, EURO has short tailedness for left tail and heavy tailedness for right tail since its shape parameter (ξ) value is significantly negative at panel A and significantly positive at panel B. It is also observed that SF and GBP have insignificant short tailedness, while JPY has insignificant heavy tailedness for both tails. To evaluate the GPD fit in the tails of the distribution, in Fig. 2, we show the qq plots of the lower and upper tail exceedances against the quantiles obtained from the GPD fit. The approximate linearity of these plots indicates the GPD model seems to be a good choice.

3.2.2. Estimates of the copula models and dependence structure

In this subsection, first we explain how the estimation of a parametric copula function can be carried out independently from the estimation of the marginal distributions and then investigate the dependence structure between currency markets. We use this approach to calibrate the copula family on ten pairs of foreign exchange returns. Suppose we assume a parametric form for the marginal for the marginal distributions of the X_i's and the copula C and denote θ the vector of parameters of the margins and the copula. The calibration of the bivariate model could be obtained by classically maximizing the log-likelihood:

$$l(\theta) = \sum_{t=1}^{T} \log \ f(x'_1, x'_2; \theta). \tag{27}$$

Recall the bivariate joint density function in Eq. (4). Using Eq. (4), the log-likelihood in Eq. (27) can also be written

$$l(\theta) = \sum_{t=1}^{T} \log \ c[F_1(x'_1), F_2(x'_2); \theta] + \sum_{t=1}^{T}\sum_{i-1}^{2} \log \ f_i(x'_i; \theta). \tag{28}$$

Although possible in theory, this estimation procedure may prove difficult in practice when the dimensionality of the problem increases.

Joe and Xu (1996) note, however, that the copula representation splits the parameters into specific parameters for the marginal distributions and common parameters for the dependence structure. Let us denote θ_i the vector of parameters of the marginal distribution of the random variable X_i. Further recall that the distribution function in Eq. (2) takes the form $F_i(\theta_i; \theta_i)$, while $c(u_1, u_2; \delta)$ denotes the parametric counterpart of the copula density in Eq. (5). The maximum likelihood approach reduces to the maximization of the following quantity:

$$l(\theta) = \sum_{t=1}^{T} \log \quad c[F_1(x'_1; \ \theta_1), F_2(x'_2; \theta_2); \delta] + \sum_{t=1}^{T}\sum_{i=1}^{2} \log \ f_1(x'_i; \theta_i). \tag{29}$$

To this point, using the kernel density estimates[2] $\hat{f}_i$ and a Gaussian quadrature to compute the empirical cumulative distribution functions $\hat{F}_i$, we transform the realizations of the X_i's representing foreign exchange returns into realizations of uniform random variables U_i. We thus obtain series

$$\left[\hat{u}'_i = \hat{F}_i(x'_i)\right]_{t=1}^{T} \quad \text{for} \quad i = 1, 2. \tag{30}$$

The parametric estimation of the seven copula functions is then obtained as follows:

[2] The kernel density estimator of the random variable X_i is defined by Silverman (1986): $\hat{f}_i(x) = \frac{1}{nh}\sum_{t=1}^{T} K\left(\frac{x - x_i^t}{h}\right)$, where $\left(x_i^t\right)_t^T$ is the ith observed return series.

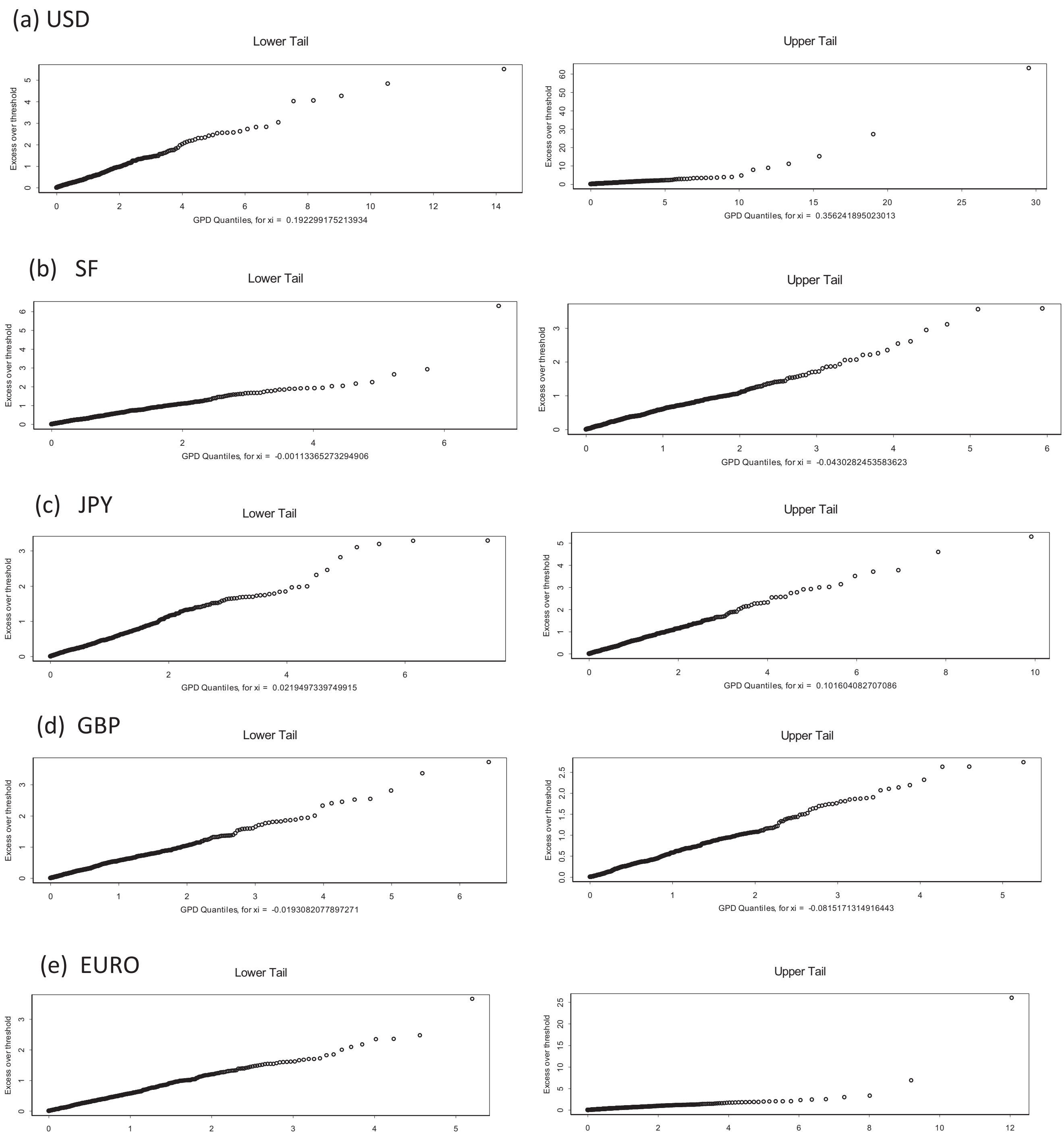

Fig. 2. Estimated tails from GPD models fit to lower and upper tail exceedances.

$$\hat{\delta} = \arg\max(\delta) = \arg\max \sum_{t=1}^{T} \log \quad c(\hat{u}_1,\ \hat{u}_2;\ \delta). \tag{31}$$

It corresponds to step 2 in the inference functions for the margins method introduced in Joe and Xu (1996). Table 7 gives the parameters obtained for each pair of returns for each copula function together with the t-statistic of the estimates in the parentheses.

Since Gaussian copula is considered as the copula of multivariate normal distribution, we can compare Gaussian copula's parameters to the Pearson linear correlations of the ten pairs reported in Table 2. The results in Table 7 show that the estimates of the Gaussian copula are similar to those of the Pearson linear correlations for certain pairs, such as SF-JPY, SF-GBP, and GBP-EURO. The differences between the Gaussian copula and the linear correlation are trivial for returns that do not have significant fat tails. On the other hand, for pairs evincing major divergence from normality in terms of tails, i.e., USD-EURO, the correlations estimated by Gaussian copula vary from those estimated by the Pearson correlation.

The findings also suggest that the dependence between returns vary from pair to pair. Among the ten pairs, the relationship is strongest between SF and EURO, and weakest between USD and EURO. It is also observed that in general the correlations of Frank copula are higher than those of Gumbel copula.

Table 7
Estimation of copula parameters.

Pairs	Parameters	Gaussian	Frank	Clayton	Gumbel	BB1	BB4	BB7
USD-SF	δ	0.2859 (22.3874)	1.8054 (19.9926)	0.3476 (16.16)	1.2064 (94.5887)	1.1335 (73.1356)	0.3544 (17.9747)	0.2478 (10.5961)
	θ					0.1617 (6.3506)	0.1725 (6.6731)	1.1579 (59.3189)
USD-JPY	δ	0.3230 (26.2023)	2.0951 (22.9993)	0.39595 (17.93)	1.2485 (92.9720)	1.1695 (70.8594)	0.3958 (20.0087)	0.2772 (11.4176)
	θ					0.1674 (6.4514)	0.1806 (6.8668)	1.2011 (57.9711)
USD-GBP	δ	0.3037 (24.1730)	1.8864 (20.7842)	0.36344 (16.7)	1.2265 (94.0509)	1.1646 (72.9630)	0.3970 (20.9465)	0.2324 (9.9453)
	θ					0.1332 (5.3979)	0.1399 (5.6467)	1.1998 (59.7068)
USD-EURO	δ	0.2683 (20.6884)	1.6823 (18.6407)	0.31786 (15.02)	1.1942 (95.3185)	1.1316 (75.2752)	0.3523 (17.3230)	0.2203 (9.7414)
	θ					0.1405 (5.7882)	0.1508 (5.4456)	1.1591 (61.0339)
SF-JPY	δ	0.5015 (51.8090)	3.5627 (36.6223)	0.70876 (27.91)	1.4736 (88.3583)	1.3315 (63.2906)	0.5829 (26.2149)	0.4957 (16.8812)
	θ					0.2579 (8.7783)	0.2652 (9.0609)	1.3971 (52.8040)
SF-GBP	δ	0.6047 (77.0149)	4.7343 (45.2505)	0.96313 (34.01)	1.6877 (85.9498)	1.5179 (60.4494)	0.7801 (30.6897)	0.6369 (19.1509)
	θ					0.2670 (8.8408)	0.2713 (9.0860)	1.6303 (52.4241)
SF-EURO	δ	0.8692 (325.7874)	11.9989 (67.9414)	2.88989 (54.75)	3.2241	2.5847	1.8303	2.0864
	θ					0.5506	0.5862	3.0153
JPY-GBP	δ	0.3627 (30.7140)	2.5268 (27.0351)	0.47006 (20.34)	1.3067 (90.5238)	1.2190 (67.1720)	0.4447 (21.4099)	0.3201 (12.3074)
	θ					0.1757 (6.4751)	0.1944 (7.0144)	1.2557 (55.0782)
JPY-EURO	δ	0.4201 (38.2492)	2.9253 (30.8928)	0.56062 (23.37)	1.3710 (89.9069)	1.2649 (67.8568)	0.5057 (25.2643)	0.3819 (13.9572)
	θ					0.2049 (7.8573)	0.2153 (8.6208)	1.3167 (54.2984)
GBP-EURO	δ	0.6554 (94.9713)	5.3349 (49.2153)		1.8055 (86.0001)	1.6123 (63.5538)	0.8842 (28.4255)	0.7229 (20.5777)
	θ					0.2829 (10.6252)	0.2845 (7.4600)	1.7512 (52.3934)

Note: We use the empirical distribution functions F_i to transform the return series into uniform variables $\left[\hat{u}'_i = \hat{F}_i(x'_i)\right]_{t=1}^{T}$ for $i = 1, 2$. Through this transformation, the parameters can be estimated by classical maximum likelihood method as explained in Eq. (30). The estimated parameters for each copula function and each pair of returns are summarized in the table. In parentheses we provide the t statistics of the parameters, which show that all parameters are significant at 1% level.

Table 8
Comparison of copula models.

Pairs	Parameters	Gaussian	Frank	Clayton	Gumbel	BB1	BB4	BB7
USD-SF	L-L	203.7673	200.4620	172.3000	191.5917	**215.8979**	206.1580	209.7015
	AIC	−405.5346	−398.9240	−342.6361	−381.1834	−**427.7959**	−408.3160	−415.4029
	BIC	−399.0627	−392.4520	−336.1641	−374.7114	−**414.8519**	−395.3720	−402.4590
USD-JPY	L-L	263.3421	265.9468	212.0000	259.0750	**284.1636**	270.3224	275.5568
	AIC	−524.6841	−529.8935	−422.067	−516.1500	−**564.3273**	−536.6448	−547.1137
	BIC	−518.2122	−523.4216	−415.595	−509.6780	−**551.3833**	−523.7008	−534.1697
USD-GBP	L-L	231.2442	216.7400	185.1000	236.6600	**253.9440**	247.4111	248.5480
	AIC	−460.4885	−431.4800	−368.1727	−471.3199	−**503.8880**	−490.8222	−493.0961
	BIC	−454.0165	−425.0080	−361.7007	−464.8479	−**490.9440**	−477.8783	−480.1521
USD-EURO	L-L	178.4214	174.1325	149.7000	178.1629	**198.4685**	186.3845	194.2149
	AIC	−354.8427	−346.2650	−297.4461	−354.3258	−**392.9370**	−368.7691	−384.4298
	BIC	−348.3707	−339.7930	−290.9742	−347.8538	−**379.9930**	−355.8251	−371.4858
SF-JPY	L-L	692.0608	692.7308	532.4000	675.8952	**722.1582**	706.6203	695.9246
	AIC	−1382.122	−1383.462	−1062.859	−1349.790	−**1440.316**	−1409.241	−1387.849
	BIC	−1375.650	−1376.990	−1056.387	−1343.318	−**1427.372**	−1396.297	−1374.905
SF-GBP	L-L	1087.583	1102.064	823.6000	1105.768	**1151.607**	1124.297	1107.658
	AIC	−2173.166	−2202.128	−1645.110	−2209.536	−**2299.213**	−2244.593	−2211.315
	BIC	−2166.694	−2195.656	−1638.638	−2203.064	−**2286.269**	−2231.649	-2198.372
SF-EURO	L-L	3366.039	3533.449	2810.000	3559.222	**3709.347**	3645.794	3519.058
	AIC	−6730.078	−7064.898	−5617.655	−7116.443	−**7414.693**	−7287.587	−7034.116
	BIC	−6723.606	−7058.426	−5611.183	−7109.972	−**7401.749**	−7274.643	−7021.172
JPY-GBP	L-L	337.0584	369.9878	273.8000	351.7847	**376.7295**	352.5257	360.7647
	AIC	−672.1169	−737.9756	−545.5374	−701.5695	−**749.4589**	−701.0515	−717.5294
	BIC	−665.6449	−731.5036	−539.0654	−695.0975	−**736.5149**	−688.1075	−704.5854
JPY-EURO	L-L	463.8833	485.8952	365.6000	471.0417	**503.4455**	481.6554	484.2986
	AIC	−925.7665	−969.7904	−729.6741	−940.0835	−**1002.891**	−959.3108	−964.5971
	BIC	−919.2945	−963.3184	−722.6741	−933.6115	−**989.9470**	−946.3668	−951.6532
GP-EURO	L-L	1341.100	1338.105	998.6	1349.243	**1400.813**	1377.625	1345.988
	AIC	−2680.199	−2674.210	−1995.208	−2696.486	−**2797.626**	−2751.251	−2687.977
	BIC	−2673.727	−2667.738	−1988.736	−2690.015	−**2784.682**	−2738.307	−2675.033

Note: The bold values in the table refer to the values of LL, AIC, and BIC for the best fitted copula model.

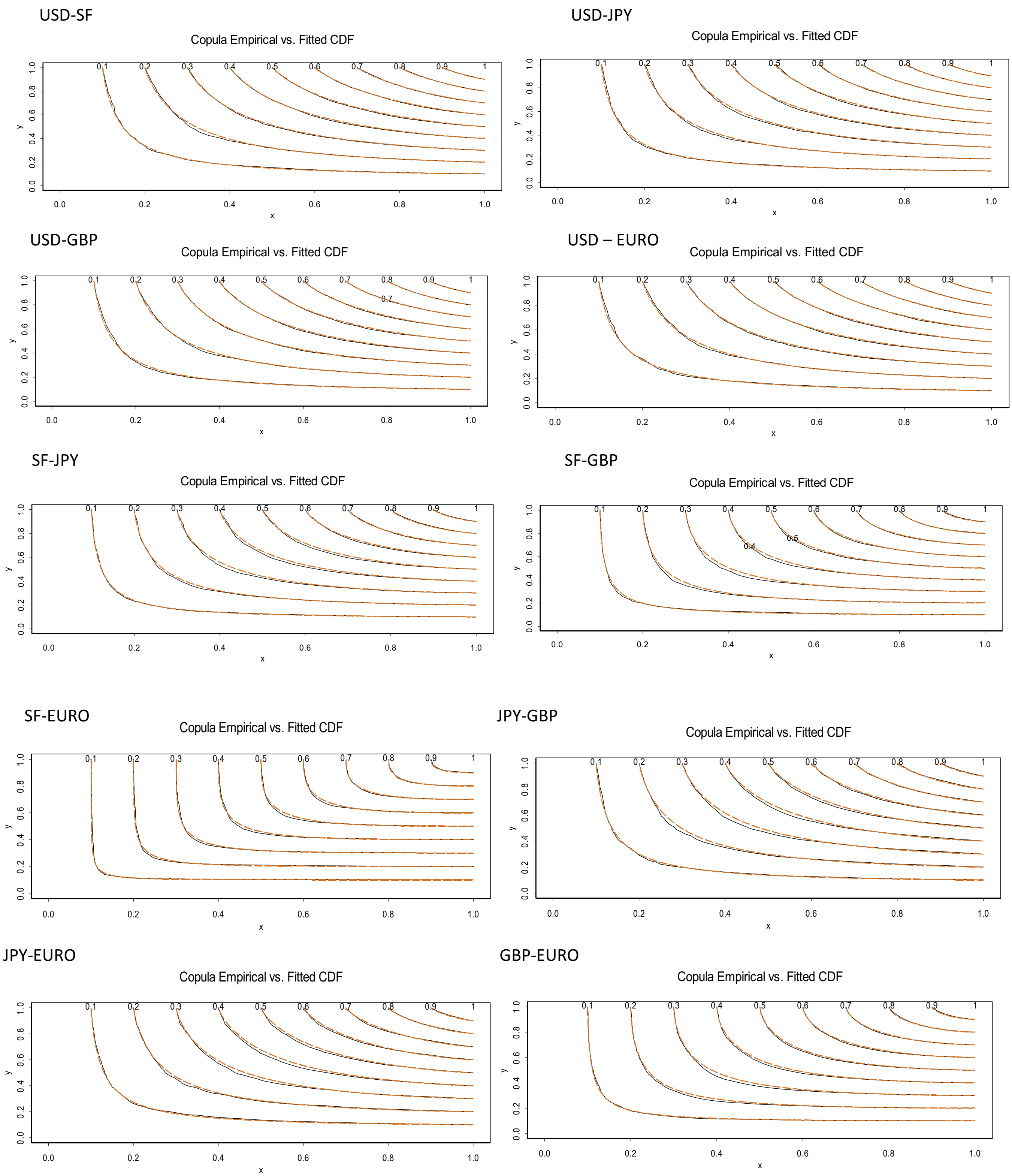

Fig. 3. The Contours of the empirical copula of the best fitted BB1 for all the pairs. The irregular lines in each plots correspond to the empirical fits, and the solid lines corresponds to the BB1 copula.

Comparing the copula-based model is an important task after fitting copula with marginal. The most widely used methods to measure goodness of fit test which allow to rank the copula according to the fit are LL, AIC, and BIC. The respective values of LL, AIC, and BIC are reported in Table 8. The model with the highest LL but with lowest AIC and BIC should be considered as the best one. Based on the above criterion, we find BB1 is the best fit for all pairs. Fig. 3 shows the Contours of the empirical copula of the best fitted BB1 for all the pairs. The irregular lines in each plots correspond to the empirical fits, and the solid lines corresponds to the BB1 copula.

Table 9
Tail dependence coefficient of best Copula BB1 model.

Pair	Lower tail dependence (λ_L)	Upper tail dependence (λ_U)
USD-SF	0.022782568	0.156786482
USD-JPY	0.028997599	0.191158161
USD-GBP	0.011466542	0.18664182
USD-EURO	0.012782274	0.154892991
SF-JPY	0.132852008	0.317002924
SF-GBP	0.180812436	0.421225713
SF-EURO	0.614432477	0.692426453
JPY-GBP	0.039308712	0.234172342
JPY-EURO	0.068947783	0.270234666
GBP-EURO	0.218786088	0.462877689

Note: The table reports the estimated values of the lower and upper tail dependence for each pair. The lower tail dependence parameters are estimated as $\lambda_L = 2^{-1/(\delta\theta)}$, and upper tail dependence parameters are estimated as $\lambda_U = 2 - 2^{1/\delta}$. The estimated values of the lower and upper tail dependence parameters are obtained from the above equations, respectively, using the values of δ and θ of the respective pairs reported in Table 7.

The visual inspection from the Contours of all models (not shown here) also indicates that the BB1 is the best for all the pairs.

Following the best copula BB1 model we estimate both lower and upper tails dependence for all the pairs. The lower tail dependence parameters are estimated as

$$\lambda_L = 2^{-1/(\delta\theta)} \tag{32}$$

and upper tail dependence parameters are estimated as

$$\lambda_U = 2 - 2^{1/\delta} \tag{33}$$

For each pair, the estimated values of the lower and upper tail dependence parameters are obtained from Eqs. (32) and (33), respectively, using the values of δ and θ of the respective pairs reported in Table 7. The estimated values are reported in Table 9. In the financial data context upper tail dependence means dependence in boom market, while lower tail dependence means dependence in bear market. It is observed that dependences in boom market are different than the dependences in bear market. Since upper tail dependence parameters are higher than lower tail dependence parameter for all pairs, the foreign exchange rates in India are more likely to rise together than to fall together. We note the pair SF-EURO is the strongest tail dependent pair for both positive and negative co-exceedances. The second and the third position are occupied by GBP-EURO and SF-GBP pairs. The USD-EURO pair shows the smallest degree of upper tail dependence, while USD-JPY pair reports the smallest degree of lower tail dependence.

To interpret the tail dependence results, take the upper tail and the lower tail dependence parameters λ_U and λ_L for SF-EURO pair. For this pair, λ_U is estimated to be 0.6924, meaning that given SF having a price jump above a certain value, the probability of EURO having a price jump above a corresponding value is about 69.24%. The lower tail dependence parameter λ_L for the pair is estimated to be 0.6144, meaning that given SF having a price drop below a certain value, the probability of EURO having a price drop below a corresponding value is about 61.44%.

3.2.3. Portfolio risk analysis

3.2.3.1. Estimating VaR and CVaR. To examine and measure portfolio risk we have already introduced the most commonly used techniques such as VaR and CVaR. So far we have followed first three steps for their estimation as described in the methodology part and identified BB1 as the most appropriate copula. Using BB1 copula we would now quantify the portfolio risk of ten separate portfolios, each of which comprising of different pairs of currencies as referred earlier. As mentioned in the estimation procedure, in order to compute risk measures, we have to use Monte Carlo simulation because analytical methods exist only for a multivariate normal distribution (i.e., a Gaussian copula). When copula functions are used, it is relatively easy to construct and simulate random scenarios from the joint distribution of $Z_{1,t}$ and $Z_{2,t}$ based on any choice of marginal and any type of dependence structure. Hence, we now follow step 4 in which our strategy consists of first simulating $Q = 10{,}000$ dependent uniform variates from the estimated parameters of the selected BB1 copula model and transforming them into standardized residuals by inverting the semi-parametric marginal CDF of each currency. We then consider the simulated standardized residuals and calculate the returns by reintroducing the GARCH volatility and the conditional mean term observed in the original return series. Finally, given the simulated return series and a particular weight vector $w = [w_1, w_2]'$ for each pair of currency, we compute the portfolio return R. Then following step 5 we estimate VaR^{α}_{t+1} and $CVaR^{\alpha}_{t+1}$ for a given level of confidence $(1-\alpha)$.

3.2.3.2. Fitting efficient frontiers. We estimate VaR^{α}_{t+1} and $CVaR^{\alpha}_{t+1}$ of N number different weighted portfolios to find the efficient frontier of portfolio. Changing the weight vector $w = [w_1, w_2]'$ from $w = [0, 1]'$ to $w = [1, 0]'$ with an increment of 0.001 (i.e., changing proportion of first asset of 0–100% in increment 0.1%), we get $N = 1001$ different combinations of portfolio expected return and risk for each pair of currency. If we plot the N number of return and risk combinations on a graph, we get the efficient frontier of portfolios of a pair of currencies. At the confidence level $1-\alpha = 95\%$, we apply both the measures of risk: VaR^{α}_{t+1} and $CVaR^{\alpha}_{t+1}$ to optimize the portfolio and obtain VaR^{α}_{t+1} and $CVaR^{\alpha}_{t+1}$ efficient frontiers of the portfolio under different expected returns of each pair, as shown in Fig. 4. The figure shows that $CVaR^{\alpha}_{t+1}$ is always larger than VaR^{α}_{t+1}. Both VaR^{α}_{t+1} and $CVaR^{\alpha}_{t+1}$ efficient frontiers are not convex for the pairs in which USD currency is one of two assets in the portfolio. But the frontiers are convex in case of portfolios which comprise any two currencies other than USD.

The VaR^{α}_{t+1} and $CVaR^{\alpha}_{t+1}$ efficient frontiers are similar to a mean-variance frontier except for the definition of risk: VaR^{α}_{t+1} and $CVaR^{\alpha}_{t+1}$ relative to the benchmark return instead of standard deviation (σ). The efficient frontiers exhibit that for a given level of expected return, the portfolio manager can foresee risk i.e., the average loss both in terms of VaR^{α}_{t+1} and $CVaR^{\alpha}_{t+1}$ in 95% cases. We compute the optimum weights of each currency that minimize the portfolio risk and identify the global minimum risk portfolio with a 95% confidence level. The optimal portfolio weights under the minimum portfolio risk (VaR^{α}_{t+1} and $CVaR^{\alpha}_{t+1}$) and the corresponding VaR^{α}_{t+1} and $CVaR^{\alpha}_{t+1}$ for 95% confidence level are presented in Table 10. From the table it appears that if the objective is to minimize portfolio risk, currency portfolio manager should concentrate fully in USD in case of a two assets portfolio in which the USD is one of the two assets (first four portfolios). However, if the manager wants to manage portfolio which comprises any two assets other than the USD (next six portfolios), he should invest in both the assets proportionately as reported in the table to minimize the portfolio risk.

4. Implications and policy lessons

In the preceding section, we have investigated the dependence between currency returns and analyze the portfolio risk. In this section we would try to explain the implications and policy lessons of the study for investors, risk players and policy makers in foreign exchange markets in India. While explaining the implications and policy lessons, we have emphasized and compared the particular results of India.

The dependence structure exhibits the asymmetric dependence between any two exchange rates. The degree of asymmetry in dependence in the upper and lower tails reported in Table 9 shows

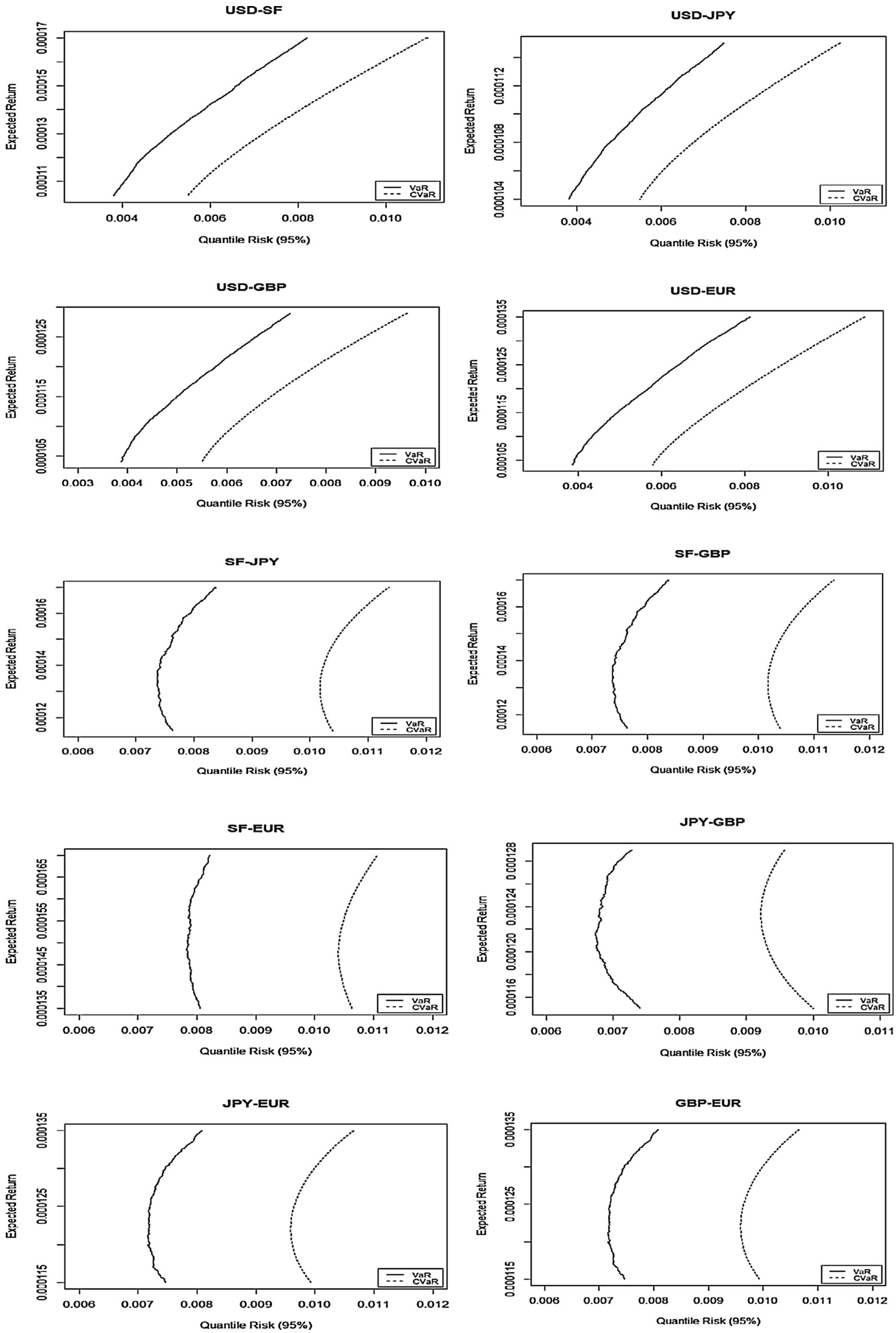

Fig. 4. Efficient VaR^{α}_{t+1} and $CVaR^{\alpha}_{t+1}$ frontiers at 95% confidence level.

that two exchange rates exhibit greater correlation during market upturns than market downturns. In our model, upper (lower) tail dependence measures the dependence between the exchange rates on days INR is depreciating (appreciating) against both the foreign currencies. The grater values of upper tails than the lower tails suggest that foreign currencies in India are more dependent during INR depreciations against foreign currencies than during INR appreciations. Several studies have found asymmetric tail depen-

Table 10
The optimum investment proportion of portfolio with minimum risk.

Pair	Min VaR^{α}_{t+1} with optimal weights		Min $CVaR^{\alpha}_{t+1}$ with optimal weights	
	Min VaR^{α}_{t+1}	Weights	Min $CVaR^{\alpha}_{t+1}$	Weights
SD-SF	0.003792	USD = 0.9981 SF = 0.0019	0.005480845	USD = 1 SF = 0
USD-JPY	0.003812001	USD = 1 JPY = 0	0.005488812	USD = 1 JPY = 0
USD-GBP	0.003867677	USD = 1 GBP = 0	0.008202016	USD = 1 GBP = 0
USD-EURO	0.003859056	USD = 1 EURO = 0	0.008202016	USD = 1 EURO = 0
SF-JPY	0.007359141	SF = 0.3237 JPY = 6763	0.010167512	SF = 0.3054 JPY = 0.69.46
SF-GBP	0.007110309	SF = 0.4383 GBP = 0.5617	0.009794136	SF = 0.1572 GBP = 0.8428
SF-EURO	0.007822133	SF = 0.3833 EURO = 0.6167	0.010398574	SF = 0.3627 EURO = 0.6373
JPY-GBP	0.006736713	JPY = 0.5264 GBP = 0.4736	0.009215307	JPY = 0.4012 GBP = 0.5988
JPY-EURO	0.007156882	JPY = 0.754 EURO = 0.246	0.009587377	JPY = 0.6537 EURO = 0.3463
GBP-EURO	0.007151465	GBP = 0.5941 EURO = 0.4059	0.009806652	GBP = 0.71 EURO = 0.29

Note: We estimate VaR^{α}_{t+1} and $CVaR^{\alpha}_{t+1}$ of N number different weighted portfolios at the confidence level $1-\alpha = 95\%$. Changing the weight vector $w = [w_1, w_2]'$ from $w = [0, 1]'$ to $w = [1, 0]'$ with an increment of 0.001 (i.e., changing proportion of first asset of 0 to 100% in increment 0.1%), we get $N = 1001$ different combinations of portfolio expected return and risk for each pair of currency. The table shows the optimum investment proportion of portfolio with minimum portfolio risk (VaR^{α}_{t+1} and $CVaR^{\alpha}_{t+1}$).

dence between foreign currency returns. Patton (2001, 2006) and Tursunalieva and Silvapulle (2007) find evidence that the mark-dollar and yen-doller exchange rates are more dependent when they are depreciating against the dollar than when they are appreciating. That is dependence was greater during appreciation of the USD (depreciations of both the Yen and DM) than during depreciations of the USD (appreciations of both the Yen and DM). The empirical findings of asymmetric dependence in our study are similar with their findings. The asymmetric dependence structure in Indian foreign currency markets, has not been previously reported in the empirical exchange rate literature. Thus the study enrich findings in existing literature.

These results, which have not been documented in the existing literature, have direct implications for investors and risk managers during extreme currency market movements in India. Since upper tail dependence parameters are much higher than lower tail dependence parameter for all pairs, it is suggested that investors should avoid holding the respective pairs of currency during up market. It is also evident that the pair SF-EURO is the strongest tail dependent pair for both positive and negative co-exceedances. The USD-EURO pair shows the smallest degree of upper tail dependence while USD-GBP pair reports the smallest degree of lower tail dependence. Thus, portfolio manager would not get much diversification benefit if she combines SF and Euro both during boom and bear markets. However, she can diversify the portfolio risk by including USD and EURO in boom market and USD and GBP during bear market. The finding of asymmetric tail dependence also implies that the Gaussian dependence hypothesis that underlies modern portfolio theories may be inadequate. Picking up the tail dependence could lead to a more realistic assessment of the linkage between foreign exchange markets and possibly more accurate risk management. The findings have potentially important implications for portfolio decisions and hedging problems involving their exchange rates, as it implies that linear correlation is not sufficient to describe their dependence structure. Thus, for example, a hedge constructed using linear correlation may not offer the degree of protection it would under a multivariate normal distribution.

Turning to the results reported in Table 10 which presents the optimum portfolio weights under minimum portfolio risk (portfolio VaR and CVaR), we find that, if the manager wants to manage portfolio which comprises any two foreign currencies other than the USD, she should take long positions in both the currencies to minimize portfolio risk. However, if the manager wants to manages two assets portfolio (USD along with any other currency), she has to go for short sales of other currency and invest the total amount to USD to minimize the portfolio risk. Almost full concentration of investment to USD is not surprise given the situation that USD remains to be the most important currency in Indian national foreign exchange reserves, and will continue to be the major currency of government and private investment. This helps explain why USD accounts for the largest weight in the foreign exchange portfolio with minimum risk, during our entire sample period.

The empirical findings regarding optimum investment proportion with minimum risk observed in Wang et al. (2010) are similar with our findings. They have used Gaussian Copula, t Copula and Clayton Copula to measure foreign exchange portfolio VaR and CVaR. In their study it is observed that when minimizing portfolio risk, optimal portfolio weights are similar across different copula and different confidence levels. The optimum investment tends to concentrate in the investment of USD. It is consistent with the fact that USD remains to be the most important currency even in Chinese foreign exchange markets too. Their study shows t copula best describes the dependency structure between foreign exchanges in their portfolio. It also captures portfolio risk better than Clayton and Gaussian Copulas. In our study BB1 copula can best measure the dependency structure. Hence we have used BB1 copula to measure foreign exchange portfolio risk.

In order to obtain the best optimal portfolio, one has to maximize risk premium per unit of risk. More particularly, an investor who wants to be, say, 95% confident that his or her maximum average loss will not be more than VaR/CVaR limit, however, attains the possible return therefore selects the point on the efficient VaR frontier, where excess premium per unit of risk is maximized.

Using the annual risk free rate which is the 91-day T-Bill rates, we can maximize the risk premium per unit of risk and can attain optimum allocation between two currencies at different confidence levels. Focussing on downside risk as an alternative measure for risk in financial markets can enable us to develop a framework for portfolio selection that moves away from the standard mean-variance approach. Introducing CVaR into the measure for risk has the benefit of allowing the risk–return trade-off to be analyzed for various associated confidence levels. Since the riskiness of an asset increases with the choice of the confidence level associated with the downside risk measure, risk become a function of the individual's risk aversion level. The portfolio selection problem is still to maximize expected return, however whilst minimizing the downside risk as captured by CVaR. This allows us to develop a very generalized framework for portfolio selection. The portfolio managers can use a mean-CVaR model for portfolio selection instead of the traditional mean-variance mode. Though CVaR has not so far been fully explored for portfolio performance evaluation, this seems a natural application, given its common use to manage risk. Thus, the study can be extended to evaluate portfolio performance by measuring reward-to-CVaR ratio which gives the same kind of ranking for portfolio performance as the Sharpe ratio.

Potential users of our approach would be traders/brokers on currency markets who are interested to compute best optimal currency portfolio in a mean-CVaR framework. Banks, too, may use the proposed model for the purpose of internal risk control.

5. Conclusion

In this paper we endeavor to investigate the dependence structure and estimate portfolio risk using data from foreign exchange markets in India. To study the different foreign exchange rates dependence, we specify both the marginal models for the currency returns and a joint model for the dependence. We employ the AR-t-GARCH-EVT models for the marginal distribution of each of five currency returns series. For the joint model, we choose seven copulas with different dependence structure. The empirical copula in Table 2 suggests both upper and lower tail dependences for almost all pairs of exchange rates. Hence we use BB1, BB2 and BB7 copulas which possess both tails dependence and for a matter of comparison we also use Gaussian copula, Frank copula, Clayton copula and Gumbel copula. Using LL, AIC, and BIC values we find BB1 as the best fitted copula for all the pairs. As a whole the evidences of tail dependence coefficients of the best copula model suggest that currency markets are more likely to boom together than to crash together.

We have measured portfolio risk in terms of VaR_{t+1}^{α} and $CVaR_{t+1}^{\alpha}$. We apply both the measures of risk and obtain VaR_{t+1}^{α} and$CVaR_{t+1}^{\alpha}$ efficient frontiers of the portfolio at 95% confidence level. The efficient frontiers allow us to develop a very generalized framework for portfolio selection. One can select a portfolio to maximize return for a given level of risk as captured by VaR_{t+1}^{α} and $CVaR_{t+1}^{\alpha}$. Similarly one can minimize the risk for a given level of return. We have identified the global minimum risk portfolio with 95% confidence level for all pair of currencies. The optimum investment for a minimum risky portfolio tends to concentrate fully to USD, which is consistent with the fact that USD remains to be the most important foreign currency in Indian economy.

The findings of the study could lead to a more realistic assessment of the linkage between foreign exchange markets and possibly accurate risk management. The findings can help the policy makers better design their policy for foreign exchange management and investors better manage the foreign exchange risk.

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