Lotka Voltera : prey-predator model

1 Context

In Trieste (Italy), at the end of the First World War, the fishing catches had decreased. The fishery office had noted that the proportion of shark type fishes, not very interesting for consumption, had increased considerably compared to the interesting sardine type fishes. The fishery office asked for help from Volterra who modeled the shark-sardine system.

2 Model

The system model is based on two differential equations where $x(t)$ and $y(t)$ represent respectively the quantity of sardines and the quantity of sharks :

$$\begin{cases} \dot{x}(t) &= a.x(t) - b.x(t).y(t) \\ \dot{y}(t) &= c.x(t).y(t) - d.y(t) \end{cases}$$

with $a, b, c, d > 0$

This model, also called the Lokta-Volterra system, means that in the absence of sharks, sardines proliferate $\dot{x} = ax$, in the absence of sardines, sharks disappear $\dot{y} = -dy$. The term with $x(t).y(t)$ represents the meeting of sharks and sardines which increases the number of sharks and decreases the number of sardines.

The impact of the sardine fishery on the evolution of both populations can also be modeled. The model of the continuous population dynamics of sharks remains the same, while the model for the evolutionary dynamics of sardines becomes :

$$\begin{cases} \dot{x}(t) &= a.x(t) - b.x(t).y(t) - f.x(t) \qquad \text{if fishing authorized} \\ \dot{x}(t) &= a.x(t) - b.y(t).y(t) \qquad \text{if fishing not authorized} \end{cases}$$

with $a, b > 0$ and $f \geqslant 0$

The evolution of sardine fishery is as follows:

- initially the fishery is prohibited
- if fishing is prohibited, it becomes authorized as soon as the sardine population exceeds a threshold x_max
- if fishing is allowed, it becomes prohibited as soon as the sardine population falls below a threshold x_min