14 Sept 2020 ROB 101

Summary

- If A is a square triangular matrix, then $\det(A) =$ product of terms on the diagonal
  $\therefore \det(A) \neq 0 \iff$ all terms on the diagonal are non-zero

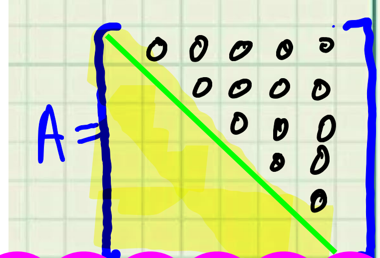
- If A is lower triangular (all terms above the diagonal are zero) and $\det(A) \neq 0$, then $Ax = b$ can be solved by forward substitution

The general form of a lower triangular system with a non-zero determinant is

$$
\begin{aligned}
a_{11}x_1 &= b_1 \quad (a_{11} \neq 0)\\
a_{21}x_1 + a_{22}x_2 &= b_2 \quad (a_{22} \neq 0)\\
\vdots &= \vdots\\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n \quad (a_{nn} \neq 0)
\end{aligned}
$$

and the solution proceeds from top to bottom, like this

$$
\begin{aligned}
x_1 &= \frac{b_1}{a_{11}} \quad (a_{11} \neq 0)\\
x_2 &= \frac{b_2 - a_{21}x_1}{a_{22}} \quad (a_{22} \neq 0)\\
\vdots &= \vdots\\
x_n &= \frac{b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1}}{a_{nn}} \quad (a_{nn} \neq 0)
\end{aligned}
$$



juliahw2 has 100 × 100 system

- Similarly, if A is upper triangular (all terms below the diagonal are zero) and $\det(A) \neq 0$, then

$Ax=b$ can be solved by **back** (aka backward) **substitution.**

The general form of an upper triangular system with a non-zero determinant is

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \quad (a_{11} \neq 0) \\
a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \quad (a_{22} \neq 0) \\
a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \quad (a_{33} \neq 0) \\
\vdots &= \vdots \\
a_{nn}x_n &= b_n \quad (a_{nn} \neq 0),
\end{aligned}
$$

$$
A = \begin{bmatrix}
 & & & & & & \\
0 & & & & & & \\
0 & 0 & & & & & \\
0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 &
\end{bmatrix}
$$

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and the solution proceeds from bottom to top, like this,

$$
\begin{aligned}
x_1 &= \frac{b_1 - a_{12}x_2 - \cdots - a_{1n}x_n}{a_{11}} && (a_{11} \neq 0) \\
x_2 &= \frac{b_2 - a_{23}x_3 - \cdots - a_{2n}x_n}{a_{22}} && (a_{22} \neq 0) \\
\vdots &= \vdots && \vdots \\
x_{n-1} &= \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}} && (a_{n-1,n-1} \neq 0) \\
x_n &= \frac{b_n}{a_{nn}} && (a_{nn} \neq 0),
\end{aligned}
$$

Matrix Multiplication

$$
\underbrace{[a_1 \ a_2 \ \cdots \ a_k]}_{1\times k} \cdot \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}}_{k\times 1} = \underbrace{\sum_{i=1}^{k} a_i b_i}_{1\times 1} = a_1b_1 + a_2b_2 + \cdots + a_kb_k
$$

(Row Vector) · (Column vector)

Must have same number of elements.

Partitioning a Matrix Into Rows and Columns

$$A = \begin{array}{c} \begin{bmatrix} \boxed{1 \quad 2} \\ \boxed{3 \quad 4} \\ \boxed{5 \quad 6} \end{bmatrix} \\ 3\times 2 \end{array} \begin{array}{l} a_1^{row} \\ a_2^{row} \\ a_3^{row} \end{array}$$

Rows of A

$$A = \begin{array}{cc} \left[\boxed{\begin{matrix} 1 \\ 3 \\ 5 \end{matrix}} \right. & \left. \boxed{\begin{matrix} 2 \\ 4 \\ 6 \end{matrix}} \right] \\ a_1^{col} & a_2^{col} \end{array}$$

Columns of A

A partition of A into columns is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} a_1^{\rm col} & a_2^{\rm col} & \cdots & a_m^{\rm col} \end{bmatrix} = \left[\boxed{\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{matrix}} \; \boxed{\begin{matrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{matrix}} \; \cdots \; \boxed{\begin{matrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{matrix}} \right]$$

That is, the j-th column is the $n \times 1$ column vector

$$a_j^{\rm col} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix},$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} a_1^{\text{row}} \\ a_2^{\text{row}} \\ \vdots \\ a_n^{\text{row}} \end{bmatrix} = \begin{bmatrix} \boxed{a_{11} \; a_{12} \; \cdots \; a_{1m}} \\ \boxed{a_{21} \; a_{22} \; \cdots \; a_{2m}} \\ \vdots \\ \boxed{a_{n1} \; a_{n2} \; \cdots \; a_{nm}} \end{bmatrix}.$$

That is, the i-th row is the $1 \times m$ row vector

$$a_i^{\text{row}} = [a_{i1} \;\; a_{i2} \;\; \cdots \;\; a_{im}],$$

Standard Matrix Multiplications : Rows x Columns

Let A and B be matrices

suppose that A is $n \times k$ (n rows & k columns) and that B is $k \times m$ (k rows and m columns). Then the product $A \cdot B$ makes sense and results in an $n \times m$ matrix.

n, m, and k are arbitrary integers

greater than or equal to 1.

- [3x3 matrix] $\cdot$ [3x7 matrix] = [3x7]
- [1xn] $\cdot$ [nx1] = [1x1]
- [5x5] $\cdot$ [5x5] = [5x5]
- [3x4] $\cdot$ [4x2] = [3x2]
- [4x5] $\cdot$ [4x4] = undefined

Examples

$A \cdot B =: C$

$(C := A \cdot B)$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}_{2\times 2} = \begin{bmatrix} a_1^{row} \\ a_2^{row} \end{bmatrix} = \begin{bmatrix} \boxed{1 \quad 3} \\ \boxed{2 \quad 4} \end{bmatrix}$$

$$B = \begin{bmatrix} 5 \\ 6 \end{bmatrix}_{2\times 1} = \begin{bmatrix} b_1^{col} \end{bmatrix}$$

$$\underset{2\times 2}{A} \cdot \underset{2\times 1}{B} = \begin{bmatrix} a_1^{row} \\ a_2^{row} \end{bmatrix} \cdot \begin{bmatrix} b_1^{col} \end{bmatrix} = \underbrace{\begin{bmatrix} a_1^{row} \cdot b_1^{col} \\ a_2^{row} \cdot b_1^{col} \end{bmatrix}_{2\times 1}}_{C}$$

$$a_1^{row} \cdot b_1^{col} = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = (1)(5) + (3)(6)$$

$$= 23$$

$$a_2^{row} b_1^{col} = \begin{bmatrix} 2 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = (2)(5) + (4)(6)$$

$$= 34$$

$$A \cdot B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 23 \\ 34 \end{bmatrix}$$

The above is how a computer does matrix multiplication.

Next, a more human view!

$$A \cdot B = \begin{bmatrix} \boxed{1 \quad 3} \\ \boxed{2 \quad 4} \end{bmatrix} \cdot \begin{bmatrix} \boxed{\begin{matrix} 5 \\ 6 \end{matrix}} \end{bmatrix} = \begin{bmatrix} (1)(5) + (3)(6) \\ (2)(5) + (4)(6) \end{bmatrix}$$

$$= \begin{bmatrix} 23 \\ 34 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow 3 \times 1$$

3×2 $\qquad$ 2×1

$$A \cdot B = \begin{bmatrix} x_1 + 4x_2 \\ 2x_1 + 5x_2 \\ 3x_1 + 6x_2 \end{bmatrix}$$

General Definition

$$A = n \times k = \begin{bmatrix} a_1^{row} \\ \vdots \\ a_i^{row} \\ \vdots \\ a_n^{row} \end{bmatrix} \qquad a_i^{row} = 1 \times k$$

$$B = k \times m = \begin{bmatrix} b_1^{col} \cdots b_j^{col} \cdots b_m^{col} \end{bmatrix}$$

$$b_j^{col} = k \times 1$$

$C := A \cdot B$ where

$c_{ij} := a_i^{row} \cdot b_j^{col}$ ij element of C

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & c_{22} & \cdots & c_{2m} \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nm} \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2} = \begin{bmatrix} a_1^{row} \\ a_2^{row} \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & -2 \\ 6 & 1 \end{bmatrix}_{2\times2} = \begin{bmatrix} b_1^{col} & b_2^{col} \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & -2 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} (1)(5)+(2)(6) & (1)(-2)+(2)(1) \\ (3)(5)+(4)(6) & (3)(-2)+(4)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 0 \\ 39 & -2 \end{bmatrix}$$

$$B \cdot A = \begin{bmatrix} 5 & -2 \\ 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 9 & 16 \end{bmatrix}$$