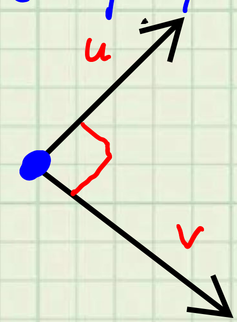
Review Dot Product or Inner Prod.

- $u, v \in \mathbb{R}^n$ $\quad u \bullet v := u^T v = [u_1 \cdots u_n]\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} u_i v_i$
- $u \perp v \iff u \bullet v = 0$ (orthogonal = perpendicular)
- If $u \neq 0$, $v \neq 0$, then $u \perp v \iff$



- An $n \times n$ matrix Q is ORTHOGONAL if
$Q^T \cdot Q = Q \cdot Q^T = I_n$ $\quad \therefore \quad Q^{-1} = Q^T$!

Seems too good to be true, BUT we will show that $A = n \times n$, $\det(A) \neq 0 \iff A = QR$ where Q = orthogonal, R = upper triangular and $\det(R) \neq 0$.

We left off with: What does $Q^T \cdot Q = I$ really mean?

$$Q = [v_1 \;\; v_2 \;\; v_3], \quad Q^T = \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix}$$

$$Q^T \cdot Q = \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix} \cdot \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

Today

$v_1^T v_1 = 1$

$v_2^T v_1 = 0$

$v_3^T v_1 = 0$

$v_1^T v_2 = 0$

$v_2^T v_2 = 1$

$v_3^T v_2 = 0$

$v_1^T v_3 = 0$

$v_2^T v_3 = 0$

$v_3^T v_3 = 1$

$$v_i^T v_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$v_i^T v_i = 1$$

$$\boxed{\begin{array}{c} \|v_i\|^2 \Leftrightarrow \|v_i\| = 1 \\ \\ v_i \perp v_j \quad i \neq j \end{array}}$$

Each column has norm one and the columns are orthogonal to one another!

Def. (a) A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is orthogonal if, for all $i \neq j$,

$v_i \cdot v_j = 0$ (same as $v_i \perp v_j$ $i \neq j$)

(b) $\{v_1, v_2, \ldots, v_n\}$ is orthonormal if

(i) It is orthogonal

(ii) $\|v_i\| = 1 \quad i = 1, 2, \ldots, n.$

Remark: Orthonormal $\iff v_i \cdot v_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$\iff (v_i)^T v_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Length one vectors (normalization)

Suppose $v \neq 0$. $\therefore \|v\| \neq 0$

Claim $\frac{v}{\|v\|}$ has norm one.

$$\beta = \frac{1}{\|v\|} \Rightarrow \frac{v}{\|v\|} = \beta v$$

$$\|\beta v\| = |\beta| \cdot \|v\| = \beta \cdot \|v\| = \frac{\|v\|}{\|v\|} = 1$$

$\blacksquare$

How to construct orthogonal vectors?

Assume $\{u_1, u_2\}$ are linearly independent vectors in $\mathbb{R}^n$. We seek to produce $\{v_1, v_2\}$ orthogonal vectors such that

$\text{span}\{v_1\} = \text{span}\{u_1\}$ dim 1

$\text{span}\{v_1, v_2\} = \text{span}\{u_1, u_2\}$. dim 2

$v_1 \perp v_2$ is our goal!

Step 1 $v_1 := u_1$

Step 2 $v_2 := u_2 - \alpha v_1$ seek α such that $v_1 \perp v_2$.

$$v_2 \perp v_1 \iff v_2 \cdot v_1 = 0$$

$$\iff (u_2 - \alpha v_1) \cdot v_1 = 0$$

$$\iff u_2 \cdot v_1 - \alpha v_1 \cdot v_1 = 0$$

$$\iff \boxed{u_2 \cdot v_1 = \alpha v_1 \cdot v_1}$$

Q? $v_1 \cdot v_1 \neq 0$? Yes, because $v_1 \neq 0$. because $v_1 = u_1$ and $\{u_1, u_2\}$ lin. indep.

$$\therefore \quad \alpha = \frac{u_2 \bullet v_1}{v_1 \bullet v_1}$$

Summarize this

$$v_1 := u_1$$

$$v_2 : u_2 - \frac{u_2 \bullet v_1}{v_1 \bullet v_1} v_1$$

lin. indep of $\{u_1, u_2\} \Rightarrow v_2 \neq 0$, important in the next step of Gram Schmidt Algorithm.

Book shows $\text{span}\{v_1, v_2\} = \text{span}\{u_1, u_2\}$

Gram-Schmidt Process

Suppose that that the set of vectors $\{u_1, u_2, \ldots, u_m\}$ is linearly independent and you generate a new set of vectors by

$$\begin{aligned}
v_1 &= u_1 \\
v_2 &= u_2 - \left(\frac{u_2 \bullet v_1}{v_1 \bullet v_1}\right) v_1 \\
v_3 &= u_3 - \left(\frac{u_3 \bullet v_1}{v_1 \bullet v_1}\right) v_1 - \left(\frac{u_3 \bullet v_2}{v_2 \bullet v_2}\right) v_2 \\
&\vdots \\
v_k &= u_k - \sum_{i=1}^{k-1} \left(\frac{u_k \bullet v_i}{v_i \bullet v_i}\right) v_i \quad \text{(General Step)}
\end{aligned} \tag{9.20}$$

Then the set of vectors $\{v_1, v_2, \ldots, v_m\}$ is

- orthogonal, meaning, $i \neq j \implies v_i \bullet v_j = 0$
- for all $1 \leq k \leq m$, $\text{span}\{v_1, v_2, \ldots, v_k\} = \text{span}\{u_1, u_2, \ldots, u_k\}$, and
- linearly independent.

The last item is redundant, but it worth stating so as to emphasize if you start with a set of **basis vectors**, then you will end with a **basis made of orthogonal vectors**. If you then normalize the **orthogonal basis**, you will produce an **orthonormal basis**!

Example 9.19 *You are given that the set below is a basis for $\mathbb{R}^3$. Produce from it an orthonormal basis.*

$$\{u_1, u_2, u_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution:

Step 1 is to apply Gram-Schmidt to produce an orthogonal basis.

$$v_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$v_1 \bullet v_1 = (v_1)^\top v_1 = 2;$$

$$v_2 = u_2 - \frac{u_2 \bullet v_1}{v_1 \bullet v_1} v_1$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \underbrace{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{3} \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 3 \end{bmatrix}$$

$$v_2 \bullet v_2 = v_2^\top v_2 = \frac{19}{2}$$

$$v_3 = u_3 - \frac{u_3 \bullet v_1}{v_1 \bullet v_1} v_1 - \frac{u_3 \bullet v_2}{v_2 \bullet v_2} v_2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \underbrace{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{1} \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \underbrace{\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{3\frac{1}{2}} \frac{1}{\frac{19}{2}} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{7}{38} \\ \frac{7}{38} \\ \frac{21}{19} \end{bmatrix} = \begin{bmatrix} -\frac{6}{19} \\ \frac{6}{19} \\ -\frac{2}{19} \end{bmatrix}.$$

Collecting the answers, we have

$$\{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 3 \end{bmatrix}, \begin{bmatrix} -\frac{6}{19} \\ \frac{6}{19} \\ -\frac{2}{19} \end{bmatrix} \right\}$$

Step 2: Normalize to obtain an orthonormal basis (often useful to do this, but not always required).

$$\tilde{v}_1 = \frac{v_1}{\|v_1\|} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\tilde{v}_2 = \frac{v_2}{\|v_2\|} = \frac{\sqrt{38}}{38} \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}$$

$$\tilde{v}_3 = \frac{v_3}{\|v_3\|} = \frac{\sqrt{19}}{19} \begin{bmatrix} -3 \\ 3 \\ -1 \end{bmatrix}$$

All of this is quite tedious by hand, while being super fast and fun in Julia!

Orthonormal and Orthognal Matrices

An $n \times m$ rectangular matrix Q is **orthonormal**:

- if $n > m$ (tall matrices), its columns are orthonormal vectors, which is equivalent to $Q^\top \cdot Q = I_m$; and
- if $n < m$ (wide matrices), its rows are orthonormal vectors, which is equivalent to $Q \cdot Q^\top = I_n$.

A square $n \times n$ matrix is **orthogonal** if $Q^\top \cdot Q = I_n$ and $Q \cdot Q^\top = I_n$, and hence, $Q^{-1} = Q^\top$.

Remarks:

- For a **square matrix**, $n = m$, $(Q^\top \cdot Q = I_n) \iff (Q \cdot Q^\top = I_n) \iff (Q^{-1} = Q^\top)$.
- For a tall matrix, $n > m$, $(Q^\top \cdot Q = I_m) \not\Rightarrow (Q \cdot Q^\top = I_n)$.
- For a wide matrix, $m > n$, $(Q \cdot Q^\top = I_n) \not\Rightarrow (Q^\top \cdot Q = I_m)$.

Careful