Review/Summary

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix},$$

n x m rectangular matrix

n = # rows

m = # columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

n x n square matrix

a_{ij} = element of i-th row and j-th column

Linear equations $\longleftrightarrow$ Matrix Form

$$\begin{aligned} x_1 - x_2 &= 1 \\ 2x_1 - 2x_2 &= -1 \end{aligned} \iff \underbrace{\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{b}.$$

Remark

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} -1 \\ -2 \end{bmatrix} x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

How to Handle "Missing" Variables or Coefficients

A zero in row i and column j of a matrix corresponds to the variable x_j being absent from the i-th equation

$$\underbrace{\begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & -1 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 7 \\ 0.5 \\ 7 \end{bmatrix}}_{b} \iff \begin{aligned} 2x_3 &= 7 \\ 2x_1 &= 0.5 \\ -x_2 + 4x_3 &= 7 \end{aligned} \iff \underbrace{\begin{aligned} 0x_1 + 0x_2 + 2x_3 &= 7 \\ 2x_1 + 0x_2 + 0x_3 &= 0.5 \\ 0x_1 - x_2 + 4x_3 &= 7 \end{aligned}}_{\text{we'd never write it like this}} \tag{19}$$

$$\begin{aligned} 2x_3 &= 7 \\ 2x_1 &= 0.5 \\ -x_2 + 4x_3 &= 7 \end{aligned} \longleftrightarrow \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} x_2 + \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} x_3 = \begin{bmatrix} 7 \\ 0.5 \\ 7 \end{bmatrix}$$

7 An Operational View of the Matrix Determinant

What you want and need to know about the determinant function: the first four points are extra important.

Fact 1 The determinant of a square matrix A is a real number, denoted $\det(A)$.

Fact 2 A square system of linear equations (i.e., n equations and n unknowns), $Ax = b$, has a unique solution x for any $n \times 1$ vector b if, and only if, $\det(A) \neq 0$.

Fact 3 When $\det(A) = 0$, the set of linear equations $Ax = b$ may have either no solution or an infinite number of solutions. To determine which case applies (and to be clear, only one case can apply), depends on how "b is related to A", which is another way of saying that we will have to address this later in the course.

Chap. 7

Fact 4 The determinant of the 2×2 square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$A = [a]$
$\det(A) = a$

$$\det(A) := ad - bc.$$

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Fact 5 In written mathematics, but not in programming, you will also encounter the notation $\det(A) = |A|$, so that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc.$$

Examples

$$3x - 3y = 3 \quad (b.1)$$
$$x + 7y = 7 \quad (b.2)$$

Monday

$$\Updownarrow$$

$$\underbrace{\begin{bmatrix} 3 & -3 \\ 1 & 7 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$\det(A) = (3)(7) - (-3)(1) = 24 \neq 0$$

Solution exists and is unique!

If we change "b" $= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we still have existence and uniqueness.

$$-2x + y = 2 \quad (a.1)$$
$$-4x + 2y = -2 \quad (a.2)$$

Monday

$$A = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \Rightarrow \det(A) = (-2)(2) - (1)(-4)$$
$$= -4 + 4 = 0$$

$\therefore$ Could have no solution or an infinite number of solutions; see Chapter 7.

Check for existence and uniqueness of solutions in an example where the unknowns are not in the "correct" order and we have a bunch of "missing coefficients":

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 7 \\ -x_2 + x_3 + 2x_1 &= 0.5 \\ x_1 + 4x_3 &= 7 \\ x_4 + 2x_3 - 5x_5 - 11 &= 0 \\ -4x_2 + 12x_4 &= 0 \end{aligned} \iff \underbrace{\begin{bmatrix} 1 & 1 & 2 & 0 & 0 \\ 2 & -1 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 & -5 \\ 0 & -4 & 0 & 12 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 7 \\ 0.5 \\ 7 \\ 11 \\ 0 \end{bmatrix}}_{b} \tag{25}$$

Solution: Using Julia, one computes $\det(A) = -540 \neq 0$. We therefore conclude that (25) has a unique solution. Grinding through the equations would have been no fun at all! ■

using LinearAlgebra

Diagonal of a Square Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \iff \mathrm{diag}(A) = \begin{bmatrix} a_{11} & a_{22} & \ldots & a_{nn} \end{bmatrix}$$

AKA Main Diagonal

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \mathrm{diag}(A) = [1 \quad 5 \quad 9]$$

Chapter 3 Triangular

Matrices

Learning Objectives

- Equations that have special structure are often much easier to solve
- Some examples to show this.

Outcomes

- Recognize triangular systems of linear equations and distinguish those with a unique answer.
- Learn that the determinant of a square triangular matrix is the product of the terms on its diagonal.
- How to use forward and back substitution.
- Swapping rows of equations and permutation matrices.

Lower Triangular System

$$\begin{aligned} 2x_1 &= 4 \\ 3x_1 - 2x_2 &= 6 \\ x_1 - 3x_2 + 5x_3 &= 7 \end{aligned} \quad \longleftrightarrow \quad \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & -3 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}}_{b}$$

- A is a lower triangular matrix
- all terms ABOVE the diagonal are zero
- $a_{ij} = 0 \quad j > i$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 6 & 0 & 0 & 0 \\ 3 & 7 & 10 & 0 & 0 \\ 4 & 8 & 11 & 13 & 0 \\ 5 & 9 & 12 & 14 & 15 \end{bmatrix}$$

5x5 lower triangular matrix

What if $\tilde{A}$ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ typically not used as a lower triangular matrix.

Key Fact: Determinant of a (square) lower triangular matrix is equal to the product of the terms on the diagonal.

A lower triangular $\Rightarrow \det(A) = a_{11} \cdot a_{22} \cdots a_{nn}$

$$\det\left(\begin{bmatrix} 4 & 0 & 0 & 0 \\ 9 & 2 & 0 & 0 \\ 8 & 3 & -1 & 0 \\ 6 & 8 & 11 & \pi \end{bmatrix}\right) = (4)(2)(-1)(\pi) = -8\pi \neq 0$$

$\det(A) \neq 0 \Leftrightarrow$ all terms on the diagonal are non-zero.

$\det(A) = 0 \Leftrightarrow$ at least one element of the diagonal is zero

Forward Substitution & Lower Triangular Systems

$$\begin{aligned} 2x_1 &= 2 \\ -x_1 - 3x_2 &= -7 \\ 6x_1 + 7x_2 + 8x_3 &= 44 \end{aligned} \longleftrightarrow \begin{bmatrix} 2 & 0 & 0 \\ -1 & -3 & 0 \\ 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ 44 \end{bmatrix}$$

Isolate x_1, x_2, x_3 respectively in rows 1, 2, & 3

$$\begin{aligned} 2x_1 &= 2 \\ -3x_2 &= -7 + x_1 \\ 8x_3 &= 44 - 6x_1 - 7x_2 \end{aligned}$$