2020 Sept 16 ROB 101

Summary

$$A \cdot B := \begin{bmatrix} \boxed{a_{11} \;\; a_{12} \;\; \cdots \;\; a_{1k}} \\ \boxed{a_{21} \;\; a_{22} \;\; \cdots \;\; a_{2k}} \\ \vdots \\ \boxed{a_{n1} \;\; a_{n2} \;\; \cdots \;\; a_{nk}} \end{bmatrix} \cdot \begin{bmatrix} \boxed{\begin{matrix} b_{11} \\ b_{21} \\ \vdots \\ b_{k1} \end{matrix}} & \boxed{\begin{matrix} b_{12} \\ b_{22} \\ \vdots \\ b_{k2} \end{matrix}} & \cdots & \boxed{\begin{matrix} b_{1m} \\ b_{2m} \\ \vdots \\ b_{km} \end{matrix}} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{k} a_{1i}b_{i1} & \sum_{i=1}^{k} a_{1i}b_{i2} & \cdots & \sum_{i=1}^{k} a_{1i}b_{im} \\ \sum_{i=1}^{k} a_{2i}b_{i1} & \sum_{i=1}^{k} a_{2i}b_{i2} & \cdots & \sum_{i=1}^{k} a_{2i}b_{im} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{k} a_{ni}b_{i1} & \sum_{i=1}^{k} a_{ni}b_{i2} & \cdots & \sum_{i=1}^{k} a_{ni}b_{im} \end{bmatrix}. \tag{12}$$

Define $C := A \cdot B$ where $A = [\overset{n}{\cancel{m}} \times k]$
and $B = [k \times \underset{m}{\cancel{n}}]$

Then C is $[m \times n]$ and

$$\boxed{c_{ij} = a_i^{row} \cdot b_j^{col}}$$

$$\underbrace{A}_{[n \times k]} \cdot \underbrace{B}_{[k \times m]} := \underbrace{\begin{bmatrix} a_1^{\text{row}} \cdot b_1^{\text{col}} & a_1^{\text{row}} \cdot b_2^{\text{col}} & \cdots & a_1^{\text{row}} \cdot b_m^{\text{col}} \\ a_2^{\text{row}} \cdot b_1^{\text{col}} & a_2^{\text{row}} \cdot b_2^{\text{col}} & \cdots & a_2^{\text{row}} \cdot b_m^{\text{col}} \\ \vdots & \vdots & \ddots & \vdots \\ a_n^{\text{row}} \cdot b_1^{\text{col}} & a_n^{\text{row}} \cdot b_2^{\text{col}} & \cdots & a_n^{\text{row}} \cdot b_m^{\text{col}} \end{bmatrix}}_{[n \times m]}$$

C = A * B
C[i,j] = A[i:i, :] * B[:, j]
A[i,:]'

End of lecture Monday

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & -2 \\ 6 & 1 \end{bmatrix}$$

we computed (quickly!)

$$A \cdot B = \begin{bmatrix} 17 & 0 \\ 39 & -2 \end{bmatrix} \text{ and } B \cdot A = \begin{bmatrix} -1 & 2 \\ 9 & 16 \end{bmatrix}$$

Today: Very Import that in general $A \cdot B \neq B \cdot A$

ORDER MATTERS

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}_{4\times 4} \qquad B = \begin{bmatrix} 7 & 8 \\ 5 & 6 \\ 3 & 4 \\ 1 & 2 \end{bmatrix}_{4\times 2}$$

$$C := A \cdot B = [4\times 4] \cdot [4\times 2] = [4\times 2]$$

$$C_{31} = a_3^{row} \cdot b_1^{col} = \begin{bmatrix} 9 & 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ 3 \\ 1 \end{bmatrix} = 158$$

$$C_{22} = a_2^{row} \cdot b_2^{col} = \begin{bmatrix} 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 4 \\ 2 \end{bmatrix} = 120 \text{ (?)}$$

Example

$A = 2\times 1$ and $B = 1\times 3$

Then $A \cdot B = [2\times 1]\cdot[1\times 3] = [2\times 3]$

Let's see what this looks like

$$A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}_{2\times 1} \qquad B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}_{1\times 3}$$

$$C := A\cdot B = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \end{bmatrix}_{2\times 3}$$

Starting Down a Road Less Traveled

$$\begin{bmatrix} \boxed{1 \quad 3} \\ \boxed{2 \quad 4} \end{bmatrix} \cdot \begin{bmatrix} \boxed{\begin{matrix}5\\3\end{matrix}} & \boxed{\begin{matrix}6\\4\end{matrix}} \end{bmatrix} = \begin{bmatrix} 14 & 18 \\ 22 & 28 \end{bmatrix}$$

$$\begin{array}{cc} \begin{bmatrix} \boxed{\begin{matrix}1\\2\end{matrix}} & \boxed{\begin{matrix}3\\4\end{matrix}} \end{bmatrix} & \cdot \begin{bmatrix} \boxed{5 \quad 6} \\ \boxed{3 \quad 4} \end{bmatrix} \begin{matrix} b_1^{row} \\ b_2^{row} \end{matrix} \\ a_1^{col} \quad a_2^{col} & \end{array}$$

$a_1^{col} \cdot b_1^{row} + a_2^{col} \, b_2^{row}$

$$\begin{bmatrix} 5 & 6 \\ 10 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = \begin{bmatrix} 14 & 18 \\ 22 & 28 \end{bmatrix}$$

Alternative Method for Matrix Multiplication

{Column times row method}

{standard way: row times column method}

Partition A into columns

$$A = \begin{bmatrix} a_1^{col} \cdots a_i^{col} \cdots a_k^{col} \end{bmatrix}_{n \times k} \qquad a_i^{col} = [n \times 1]$$

Partition B into rows

$$B = \begin{bmatrix} b_1^{row} \\ \vdots \\ b_i^{row} \\ \vdots \\ b_k^{row} \end{bmatrix}_{k \times m} \qquad b_i^{row} = 1 \times m$$

Then define $C := A \cdot B$

$$C = \sum_{i=1}^{k} \underbrace{a_i^{col} \cdot b_i^{row}}_{n \times m}$$

$$a_i^{col} \cdot b_i^{row} = [n \times 1] \cdot [1 \times m] = [n \times m]$$

Example

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 0 \end{bmatrix}_{2\times 3} \qquad B = \begin{bmatrix} 0 & 5 \\ 1 & 2 \\ -2 & 1 \end{bmatrix}_{3\times 2}$$

$$A \cdot B = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 0 & 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 5 \\ 0 & 20 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 12 \\ 2 & 24 \end{bmatrix} =: C$$

Let's do C_{21} by standard multiplication

$$C_{21} = a_2^{row} \cdot b_1^{col} = \begin{bmatrix} 4 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = 2$$

LU Factorization without Permutations

[redacted]

Notes for Computational Linear Algebra by Jessy Grizzle, Director of Michigan Robotics

https://umich.instructure.com/courses/403066/files/folder/Booklet%3A%20Notes%20for%20Computational%20Linear%20Algebra

Material added by JWG on 16 Sept 2020

Suppose we have

$$A = \begin{bmatrix} 2.0 & 3.0 & 4.0 \\ 6.0 & 14.0 & 15.0 \\ -4.0 & -11.0 & -7.0 \end{bmatrix} \tag{1}$$

and we wish to solve $Ax = b$ for

$$b = \begin{bmatrix} 20.0 \\ 79.0 \\ -47.0 \end{bmatrix} \tag{2}$$

We would go sigh, what a pain!

What if we knew that

$$\underbrace{\begin{bmatrix} 2.0 & 3.0 & 4.0 \\ 6.0 & 14.0 & 15.0 \\ -4.0 & -11.0 & -7.0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 3.0 & 1.0 & 0.0 \\ -2.0 & -1.0 & 1.0 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} 2.0 & 3.0 & 4.0 \\ 0.0 & 5.0 & 3.0 \\ 0.0 & 0.0 & 4.0 \end{bmatrix}}_{U} \tag{3}$$

Could we use this to our advantage?

Let's write out $Ax = b$ using $A = L \cdot U$

$$\underbrace{\begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 3.0 & 1.0 & 0.0 \\ -2.0 & -1.0 & 1.0 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} 2.0 & 3.0 & 4.0 \\ 0.0 & 5.0 & 3.0 \\ 0.0 & 0.0 & 4.0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 20.0 \\ 79.0 \\ -47.0 \end{bmatrix}}_{b} \tag{4}$$

Our unknown is x. Let's define an intermediate unknown y=U x

$$\underbrace{\begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 3.0 & 1.0 & 0.0 \\ -2.0 & -1.0 & 1.0 \end{bmatrix}}_{L} \cdot \underbrace{\underbrace{\begin{bmatrix} 2.0 & 3.0 & 4.0 \\ 0.0 & 5.0 & 3.0 \\ 0.0 & 0.0 & 4.0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x}}_{y} = \underbrace{\begin{bmatrix} 20.0 \\ 79.0 \\ -47.0 \end{bmatrix}}_{b} \tag{5}$$

Which gives

$$\underbrace{\begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 3.0 & 1.0 & 0.0 \\ -2.0 & -1.0 & 1.0 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 20.0 \\ 79.0 \\ -47.0 \end{bmatrix}}_{b} \tag{6}$$

and hence

$$\begin{aligned} y_1 &= 20.0 \\ y_2 &= 79.0 - 3y_1 = 79.0 - 60.0 = 19.0 \\ y_3 &= -47.0 + 2.0y_1 + y_2 = -47.0 + 00.0 + 19.0 = 12.0 \end{aligned} \tag{7}$$

Now we can use U x = y to solve for x

$$\underbrace{\begin{bmatrix} 2.0 & 3.0 & 4.0 \\ 0.0 & 5.0 & 3.0 \\ 0.0 & 0.0 & 4.0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{y} = \begin{bmatrix} 20.0 \\ 19.0 \\ 12.0 \end{bmatrix} \tag{8}$$

which gives

$$\begin{aligned} &2.0x_1 = 20.0 - 3.0x_2 - 4.0x_3 \implies x_1 = \frac{20.0 - (3.0)(2.0) - (4.0)(3.0)}{2.0} = 1.0 \\ &5.0x_2 = 19.0 - 3.0x_3 \implies x_2 = \frac{19.0 - (3.0)(3.0)}{5} = 2.0 \\ &4.0x_3 = 12.0 \implies x_3 = 3.0 \end{aligned} \tag{9}$$

Reminder of why this works: Ax = b and A = L $\cdot$ U gives

$$\underbrace{\begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 3.0 & 1.0 & 0.0 \\ -2.0 & -1.0 & 1.0 \end{bmatrix}}_{L} \cdot \underbrace{\underbrace{\begin{bmatrix} 2.0 & 3.0 & 4.0 \\ 0.0 & 5.0 & 3.0 \\ 0.0 & 0.0 & 4.0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x}}_{y} = \underbrace{\begin{bmatrix} 20.0 \\ 79.0 \\ -47.0 \end{bmatrix}}_{b} \tag{10}$$

Learning Objectives

- How to reduce a hard problem to two much easier problems
- The concept of "factoring" a matrix into a product of two simpler matrices that are in turn useful for solving systems of linear equations.

Outcomes

- Our first encounter in lecture with an explicit algorithm
- Learn how to do a *special case* of the LU decomposition, where L is a lower triangular matrix and U is an upper triangular matrix.
- Use the LU decomposition to solve linear equations
- More advanced: what we missed in our first pass at LU factorization: a (row) permutation matrix.