2020 Sept 23 ROB 101

Suggestions:

(a) Create a google doc with the Julia commands you have used (update weekly)

(b) Create a google doc or Latex doc with key facts about linear equations, matrices, and vectors.

Chapter 6: Determinants of Products, Matrix Inverse, & Matrix Transpose

Learning Objectives

- Fill in some gaps that we left during our sprint to an effective means for solving large systems of linear equations.

Outcomes

- Whenever two square matrices A and B can be multiplied, it is true that $\det(A \cdot B) = \det(A) \cdot \det(B)$
- You will learn what it means to "invert a matrix," and you will understand that you rarely want to actually compute a matrix inverse!
- If $ad - bc \neq 0$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- Moreover, this may be the only matrix inverse you really want to compute explicitly, unless a matrix has special structure.
- Matrix transpose takes columns of one matrix into the rows of another.

- More on why you should not trust the matrix determinant

Matrix Transpose

$$[v_1 \ \ v_2 \ \cdots \ v_n]^T := \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

and vice-versa

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}^T := [v_1 \ \ v_2 \ \cdots \ v_n]$$

Rows $\rightarrow$ Columns

Columns $\rightarrow$ Rows

$$\boxed{(v^T)^T = v}$$

Same for Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3}$$, then

$$A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}_{3\times 2}$$

$$\boxed{(A^T)^T = A}$$

Read the Chapter for $(A \cdot B)^T = B^T \cdot A^T$

Return of the Matrix Determinant

Suppose A and B are both $n \times n$ matrices, then

$$\boxed{\det(A \cdot B) = \det(A) \cdot \det(B)}$$

Very useful to know

<u>Consequence</u> Suppose A is $n \times n$ and has LU factorization

$$A = L \cdot U.$$

Then L and U are both $n \times n$, and

$$\det(A) = \det(L) \cdot \det(U)$$

<u>Example</u>

$$\underbrace{\begin{bmatrix} -2 & -4 & -6 \\ -2 & 1 & -4 \\ -2 & 11 & -4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} -2 & -4 & -6 \\ 0 & 5 & 2 \\ 0 & 0 & -4 \end{bmatrix}}_{U}$$

$$\det(A) = \det(L) \cdot \det(U)$$
$$= (1)(1)(1) \cdot (-2)(5)(-4) = 40$$

$$\boxed{\det(A) = \det(U)}$$

$\left(\det(A) \neq 0\right) \iff$ (all elements of $\operatorname{diag}(U)$ are non-zero)

Another useful fact

P = permutation matrix

Fact: $|\det(P)| = 1 \quad (\det(P) = \pm 1)$

Example $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\det(P) = (0)(0) - (1)(1) = -1$ ✓

The Matrix Inverse

(this is to matrices what division is to real numbers)

Definition An $n \times n$ matrix A is said to be invertible if there exists another $n \times n$ matrix, denoted A^{-1}, such that

$$\boxed{A \cdot A^{-1} = A^{-1} \cdot A = I_{n \times n}} \quad (*)$$

A^{-1} is called the inverse of A. $\square$

Key Facts

1) You only need to check one side of $(*)$, because

$$(A \cdot A^{-1} = I_{n\times n}) \Longleftrightarrow (A^{-1} \cdot A = I_{n \times n})$$

Remark: This holds when A is made of real numbers or even complex numbers.

2) A^{-1} exists if, and only if, $\det(A) \neq 0$

3) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Does it work?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \left(\frac{1}{ad-bc}\right) \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} =$$

$$= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & \overbrace{(a)(-b)+(b)(a)}^{0} \\ \underbrace{(c)(d)+(d)(-c)}_{0} & (c)(-b)+(d)(a) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Very Useful let $A = n\times n$, $B = n\times n$.

THEN Their product $A\cdot B$ is invertible if, and only if, A and B are both invertible.

Why $\det(A\cdot B) = \det(A)\cdot\det(B)$

Then what is $(A\cdot B)^{-1}$?

$$\boxed{[A\cdot B]^{-1} = B^{-1}\cdot A^{-1}}$$

Notice the flipped order!

<u>Why?</u> $C := B^{-1}\cdot A^{-1}$

$$C\cdot(A\cdot B) = (B^{-1}\cdot A^{-1})(A\cdot B)$$

$$= B^{-1}\cdot \underbrace{A^{-1}\cdot A}_{I}\ B$$

$$= \underbrace{B^{-1} \cdot B}_{I} = I \; !$$

Utility of A^{-1}?

Suppose we have $Ax = b$ and we know $\det(A) \neq 0$.

- Know exists a unique solution
- Know A^{-1} exists.

Can we relate the solution to the matrix inverse?

$$Ax = b$$

multiply both sides by A^{-1}

$$\underbrace{(A^{-1}) \cdot A}\, x = A^{-1} \cdot b$$

$$\boxed{x = A^{-1} b}$$

Beautiful. Every class will want you to use this formula.

But, in ROB 101, we

only do this for small sets of equations.

$\boxed{\text{Theory vs Reality}}$

Major Important Fact

An $n \times n$ matrix A is invertible if, and only if, $\det(A) \neq 0$.

Another useful fact about matrix inverses is that if A and B are both $n \times n$ and invertible, then their product is also invertible and

$$\boxed{(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}.}$$

Note that the order is swapped when you compute the inverse. To see why this is true, we note that

$$(A \cdot B) \cdot (B^{-1} \cdot A^{-1}) = A \cdot (B \cdot B^{-1}) \cdot A^{-1} = A \cdot (I) \cdot A^{-1} = A \cdot A^{-1} = I.$$

Hence, $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$ **and NOT** $A^{-1} \cdot B^{-1}$**.**

LU and Matrix Inverses

A consequence of this is that if A is invertible and $A = L \cdot U$ is the LU factorization of A, then

$$A^{-1} = U^{-1} \cdot L^{-1}.$$

While we will not spend time on it in ROB 101, it is relatively simple to compute the inverse of a triangular matrix whose determinant is non-zero.

3 Utility of the Matrix Inverse and its Computation

The primary use of the matrix inverse is that it provides a **closed-form solution** to linear systems of equations. Suppose that A is square and invertible, then

$$\boxed{Ax = b \iff x = A^{-1} \cdot b.} \tag{1}$$

While it is a beautiful thing to write down the closed-form solution given in (1) **as the answer to** $Ax = b$**, one should rarely use it in computations.** It is much better to solve $Ax = b$ by factoring $A = L \cdot U$ and using back and forward substitution, than to first compute A^{-1} and then multiply A^{-1} and b. Later in ROB 101, we'll learn another good method called the QR factorization.

We (your instructors) know the above sounds bizarre to you! You've been told that $Ax = b$ has a unique solution for x if, and only if, $\det(A) \neq 0$, and you know that A^{-1} exists if, and only if, $\det(A) \neq 0$. Hence, logic tells you know that $x = A^{-1}b$ if, and only if, $\det(A) \neq 0$. So what gives?

Theory vs Reality

$x = A^{-1}b$ if, and only if, $\det(A) \neq 0$ is true in the world of perfect arithmetic, but not in the approximate arithmetic done by a computer, or by a hand calculator, for that matter. The problem is that the determinant of a matrix can be "very nice", meaning its absolute value is near 1.0, while, from a numerical point of view, the matrix is "barely invertible".

Example 1 Consider $A = \begin{bmatrix} 1 & 10^{-15} \\ 10^{10} & 1 \end{bmatrix}$. For this small example, we can see that A has a huge number and a tiny number in it, but imagine that A is the result of an intermediate computation in your algorithm and hence you'd never look at it, or the matrix is so large, you would not notice such numbers. If you want to check whether A is invertible or not, you find that $\det(A) = 1 - 10^{-5} = 0.99999$, which is very close to 1.0, and thus the determinant has given us no hint that A has crazy numbers of vastly different sizes in it.

Example 2 Consider a 3×3 matrix

$$A = \begin{bmatrix} 100.0000 & 90.0000 & -49.0000 \\ 90.0000 & 81.0010 & 5.4900 \\ 100.0000 & 90.0010 & 59.0100 \end{bmatrix}. \tag{2}$$

We compute the determinant and check that it is not close to zero. Indeed, $\det(A) = 0.90100$ (correct to more than ten decimal points[1]), and then bravely, we use Julia to compute the inverse, yielding

$$A^{-1} = \begin{bmatrix} -178.9000 & -10,789.0666 & 9,889.0666 \\ 198.7791 & 11,987.7913 & -10,987.981 \\ -110.9878 & -0.1110 & 0.1110 \end{bmatrix}.$$

[1] $\det(A) - 0.9010 = -1.7 \times 10^{-11}$.

As a contrast, we compute $A = L \cdot U$, the LU factorization (without permutation), yielding

$$L = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.9 & 1.0 & 0.0 \\ 1.0 & 1.0 & 1.0 \end{bmatrix} \qquad U = \begin{bmatrix} 100.000 & 90.000 & -49.000 \\ 0.000 & \boxed{-0.001} & 99.000 \\ 0.000 & 0.000 & 9.010 \end{bmatrix}.$$

We see that U has a small number on the diagonal, and hence, if we do back substitution to solve $Ux = y$, for example, as part of solving

$$Ax = b \iff L \cdot Ux = b \iff (Ly = b \text{ and } Ux = y)$$

we know in advance that we'll be dividing by -0.001.

Moreover, we see the diagonal of L is $[1.0, 1.0, 1.0]$, and hence $\det(L) = 1$. The diagonal of U is $[100.0, -0.001, 9.01]$, and hence $\det(U) = 0.901$, and we realize that we ended with a number close to 1.0 in magnitude by multiplying a large number and a small number.

Theory vs Reality

The value of $|\det(A)|$ (magnitude of the determinant of A) is a poor predictor of whether or not A^{-1} has very large and very small elements in it, and hence poses numerical challenges for its computation. For typical HW "drill" problems, you rarely have to worry about this. However, for "real" engineering problems, where a typical dimension (size) of A may be 50 or more, then **please please please avoid the computation of the matrix inverse whenever you can!**

The LU factorization is a more reliable predictor of numerical problems that may be encountered when computing A^{-1}. But once you have the LU factorization, you must ask yourself, do you even need A^{-1}? In the majority of cases, the answer is NO!