Rob 101 - Computational Linear Algebra
Recitation #8

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1 Subspaces

Suppose that $V \subset \mathbb{R}^n$ is nonempty.

Def. V is a **subspace** of $\mathbb{R}^n$ if any linear combination constructed from elements of V and scalars in $\mathbb{R}$ is once again an element of V. One says that V is **closed under linear combinations.**
In symbols, $V \subset \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if for all real numbers α and β, and all vectors v_1 and v_2 in V

$$\boxed{\underbrace{\alpha v_1 + \beta v_2}_{v_3} \in V.} \tag{1}$$

$v_3 \in V;$

Using this formulation, comment if the following $V \in \mathbb{R}^n$ are Subspaces

1.

$$V := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;\middle|\; x_1^2 + x_2^2 = 2x_1x_2, x_1, x_2 \in \mathbb{R} \right\}.$$

$V \subset \mathbb{R}^2$

$x_1^2 + x_2^2 = 2x_1 x$

$x_1^2 + x_2^2 - 2x_1x_2 = 0$

$(x_1 - x_2)^2 = 0 \Rightarrow x_1 = x_2$

$V := \left\{ \begin{bmatrix} x \\ x \end{bmatrix} ; x \in \mathbb{R} \right\}$

Check 1 $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V$.

Check 2 $v_1 = \begin{bmatrix} a \\ a \end{bmatrix}, v_2 = \begin{bmatrix} b \\ b \end{bmatrix} \in V$

$v_3 = \alpha v_1 + \beta v_2 \in V$

$\alpha \begin{bmatrix} a \\ a \end{bmatrix} + \beta \begin{bmatrix} b \\ b \end{bmatrix} = \begin{bmatrix} \alpha a + \beta b \\ \alpha a + \beta b \end{bmatrix} \in V$

V is a Subspace in $\mathbb{R}^2$

2.

$$V := \left\{ \begin{bmatrix} ax \\ by \\ ax + by + c \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}.$$

Check 1 $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in V$.

$$\begin{bmatrix} x=0 \\ y=0 \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

condition $c = 0$.

$$V := \left\{ \begin{bmatrix} ax \\ by \\ ax+by \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

Check 2 $\alpha v_1 + \beta v_2 \in V$.

$$v_1 = \begin{bmatrix} ax_1 \\ by_1 \\ ax_1 + by_1 \end{bmatrix}; \quad v_2 = \begin{bmatrix} ax_2 \\ by_2 \\ ax_2 + by_2 \end{bmatrix}$$

$$\alpha v_1 + \beta v_2$$

$$= \begin{bmatrix} a\alpha x_1 + \beta a x_2 \\ b\alpha y_1 + \beta b y_2 \\ \alpha a x_1 + \alpha b y_1 + \beta a x_2 + \beta b y_2 \end{bmatrix}$$

$$= \begin{bmatrix} a(\alpha x_1 + \beta x_2) \\ b(\alpha y_1 + \beta y_2) \\ a(\alpha x_1 + \beta x_2) + b(\alpha y_1 + \beta y_2) \end{bmatrix}$$

$x_3 = \alpha x_1 + \beta x_2$; $y_3 = \alpha y_1 + \beta y_2$ $x_3, y_3 \in \mathbb{R}$

$v_3 = \begin{bmatrix} ax_3 \\ by_3 \\ ax_3 + by_3 \end{bmatrix}$; $v_3 \in V$. V is a subspace in $\mathbb{R}^3$

2 Null Space and Range of a Matrix

For any Matrix $A \in \mathbb{R}_{m\times n}$, then the following sets (are actually Subspaces!) can be defined:

Def. $\text{null}(A) := \{x \in \mathbb{R}^m \mid Ax = 0_{n\times 1}\}$ is the **null space** of A.

Def. $\text{range}(A) := \{y \in \mathbb{R}^n \mid y = Ax \text{ for some } x \in \mathbb{R}^m\}$ is the **range** of A.

Using this definition, Find the Null Space and Range of the following:

1.
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\text{null}(A) = \{x \in \mathbb{R}^3 \mid Ax = 0_{3\times 1}\}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_3 = 0;\ x_2 = 0;\ x_3 = 0$

$\text{null}(A) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\} \iff$ Columns of A are linearly independent

$\text{Range}(A) = \{y \in \mathbb{R}^3 \mid y = Ax\ ;\ x \in \mathbb{R}^3\}$ $\quad x = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$

$\text{Range}(A) = \left\{y = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix};\ \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \text{ or } x \in \mathbb{R}^3\right\}$

2.
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \end{bmatrix}.$$

$m > n$

$\text{null}(A) = \left\{x \in \mathbb{R}^3;\ Ax = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$ $\quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$x_1 + 2x_2 + 3x_3 = 0 \quad -(1)$

$2x_2 + 2x_3 = 0 \quad -(2)$

$(1) - (2)$

$x_1 + x_3 = 0$

$x_1 = -x_3 = \alpha$

Replace in (2)

$x_2 = -x_3$

$x = \alpha \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $\quad \text{null } A = \left\{\alpha \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \middle| \alpha \in \mathbb{R}\right\}$

$\text{Range}(A) = \left\{\alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} \middle| \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\right\}$

we can eliminate α_3. $\quad \text{Range}(A) = \left\{\alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right\}$

3 Column Span

Let A be an $n \times m$ matrix.

$$A := \text{span}\left\{a_1^{\text{col}}, \ldots, a_m^{\text{col}}\right\}.$$

We can also discuss, rank and nullity of A here as:

Def. $\text{rank}(A) := \dim \text{col span}\{A\}$.

Def. $\text{nullity}(A) := \dim \text{null}(A)$.

Using these definitions comment on the Results of the Rank-Nullity theorem for

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \end{bmatrix}.$$

$\text{null}(A) = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\}$

$\text{nullity}(A) = 1$

$\text{Range}(A) = \left\{ y = \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} \middle| \alpha_1, \alpha_2 \in \mathbb{R} \right\}$

$= \text{col span}\{A\} = \alpha_1 a_1^{col} + \alpha_2 a_2^{col} + \alpha_3 a_3^{col}.$

$a_3^{col} = a_1^{col} + a_2^{col}.$

we eliminate α_3.

$\text{Rank}(A) = 2$

Rank-Nullity Theorem.

$m = 3 = \text{nullity}(A) + \text{rank}(A)$

[Hence Proved]

4 Basis

Suppose that V is a subspace of $\mathbb{R}^n$. Then $\{v_1, v_2, \ldots, v_k\}$ is a **basis for V** if

1. the set $\{v_1, v_2, \ldots, v_k\}$ is linearly independent, and
2. span $\{v_1, v_2, \ldots, v_k\} = V$. $\rightarrow v \in V = \alpha_1 v_1 + \alpha_2 v_2 \ldots \alpha_k v_k$

The **dimension of V is k**, the number of basis vectors
Find the basis and Dimension for the following Subspaces:

1.
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \end{bmatrix}, S = \text{col span}\{A\}$$

$S = \{\alpha_1 a_1^{col} + \alpha_2 a_2^{col} + \alpha_3 a_3^{col};\ \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}$

$v = \{v_1, v_2, v_3 \ldots\} = \text{span}\{v\} = S$

$v = \{a_1^{col}, a_2^{col}, a_3^{col}\} = \text{span}\{v\} = \text{colspan}\{A\}$

$a_3^{col} = a_1^{col} + a_2^{col}$; we can eliminate a_3^{col}

$v = \{a_1^{col}, a_2^{col}\}$; basis of S.

dimension of $S = 2$

Step 1
$S = \{ \quad \}$
$= \{\alpha_1 v_1 + \alpha_2 v_2 \ldots\}$

Step 2
$V = \{v_1, v_2 \ldots v_k\}$

Step 3
Remove all v_m that are linearly dependent

2.
$$V := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;\middle|\; x_1^2 + x_2^2 = 2x_1x_2, x_1, x_2 \in \mathbb{R} \right\}.$$

$V = \left\{ \begin{bmatrix} x \\ x \end{bmatrix} \middle| x \in \mathbb{R} \right\}$

$= \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\}$

$\downarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$v = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ $\text{span}\{v\} = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = V$

v is a basis for V, dimension of $V = 1$.

3.
$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, S = \text{span}\{a\}$$

$S = \left\{ \alpha \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\}$

$\rightarrow v = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$; $\text{span}\{v\} = \left\{ \alpha \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\} = S$

$\{v\}$ are linearly independent

$v = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is a basis for S.

dimension of S is 1.

$v_1 \perp v_2 \in \mathbb{R}^m$
$dot(v_1, v_2) = 0;\ v_1 \bullet v_2 = 0 \Leftrightarrow v_1^T v_2 = 0$

5 Gram-schmidt

Suppose that that the set of vectors $\{u_1, u_2, \ldots, u_m\}$ is linearly independent then you can generate a new set of orthogonal vectors $\{v_1, v_2, \ldots, v_m\}$ as:

$$\begin{aligned}
v_1 &= u_1 \\
v_2 &= u_2 - \left(\frac{u_2 \bullet v_1}{v_1 \bullet v_1}\right) v_1 \\
v_3 &= u_3 - \left(\frac{u_3 \bullet v_1}{v_1 \bullet v_1}\right) v_1 - \left(\frac{u_3 \bullet v_2}{v_2 \bullet v_2}\right) v_2 \\
&\vdots \\
v_k &= u_k - \sum_{i=1}^{k-1} \left(\frac{u_k \bullet v_i}{v_i \bullet v_i}\right) v_i \qquad \text{(General Step)}
\end{aligned} \tag{2}$$

You are given that the set below is a basis for $\mathbb{R}^3$. Produce from it an orthonormal basis.

$$\{u_1, u_2, u_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Step 1

$$v_1 = u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Step 2

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) (1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - (0)(1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = u_2$$

Step 3

$$v_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} (1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \left(\frac{1}{2}\right) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$v_1 \perp v_2;\ v_2 \perp v_3;\ v_1 \perp v_3.$

6 QR Factorization

$Q^\top Q = I_m$

Suppose that A is an $n \times m$ matrix with linearly independent columns. Then there exists an $n \times m$ matrix Q with orthonormal columns and an upper triangular, $m \times m$, invertible matrix R such that $A = Q \cdot R$. Moreover, Q and R are constructed as follows:

- Let $\{u_1, \ldots, u_m\}$ be the columns of A with their order preserved so that

$$A = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}$$

$u_1 = a_1^{col},\ u_2 = a_2^{col} \ldots$

- Q is constructed by applying the Gram-Schmidt Process to the columns of A and normalizing their lengths to one,

$$\{u_1, u_2, \ldots, u_m\} \xrightarrow[\text{Process}]{\text{Gram-Schmidt}} \{v_1, v_2, \ldots, v_m\}$$

orthogonal

$\Downarrow$ normalize.

$$Q := \begin{bmatrix} \frac{v_1}{||v_1||} & \frac{v_2}{||v_2||} & \cdots & \frac{v_m}{||v_m||} \end{bmatrix}$$

- Because $Q^\top Q = I_m$, it follows that $A = Q \cdot R \iff R := Q^\top \cdot A$.

Find the QR Factorization of

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

prev. example

$$\{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$\Downarrow$ normalize.

$$Q = \left[\frac{1}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ \frac{1}{\sqrt{1^2+1^2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \ \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$R = \overset{Q^\top}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2/\sqrt{2} & \end{bmatrix}$$