

# ROB 101 - Computational Linear Algebra

## Recitation #2

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## 1 Solution of Linear System of Equations

In solving  $Ax = b$ , we exploit the form of  $A$  for some finding solutions using Forward and Backward Substitution. However, for us to do that  $A$  must be a Triangular Matrix.

### 1.1 Triangular matrices

**Fact:** For any Triangular Matrix  $A$ ,  $\det(A)$  is the product of all elements on its diagonal.

#### 1.1.1 Upper Triangular Matrices

A matrix,  $A$  is upper Triangular, if and only if, for every element  $a_{ij}$  in the matrix, i.e. the element at  $i^{th}$  row and  $j^{th}$  column, the following condition holds true:

$$a_{ij} = 0 \text{ if } i < j$$

Example:

$$\begin{bmatrix} 2 & 3 & 7 \\ 0 & 2 & 8 \\ 0 & 0 & 9 \end{bmatrix}$$

Method of Solution:

#### 1.1.2 Lower Triangular Matrices

A matrix,  $A$  is Lower Triangular, if and only if, for every element  $a_{ij}$  in the matrix, i.e. the element at  $i^{th}$  row and  $j^{th}$  column, the following condition holds true:

$$a_{ij} = 0 \text{ if } j > i$$

Example:

$$\begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 7 & 8 & 9 \end{bmatrix}$$

Method of Solution:

Build a matrix of coefficients, A, and check if the Matrix is Triangular or not from the following system of equations:

1.

$$2x_1 + 2x_2 = 24$$

$$2x_2 = 14$$

$$A = \begin{bmatrix} & \\ & \end{bmatrix} \quad \det(A) =$$

Remark:

2.

$$2x_1 + 3x_2 + 4x_3 = 23$$

$$6x_1 + 7x_2 = 22$$

$$9x_1 = 18$$

$$A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \quad \det(A) =$$

Remark:

3.

$$2x_1 + 2x_2 + 3x_3 + 4x_4 = 20$$

$$5x_2 + 6x_3 + 7x_4 = 34$$

$$8x_3 + 9x_4 = 25$$

$$10x_4 = 10$$

$$A = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \quad \det(A) =$$

Remark:

4.

$$2x_1 = 4$$

$$8x_1 + 3x_2 = 28$$

$$2x_1 + 3x_2 + x_3 = 22$$

$$2x_1 + 2x_2 + 3x_3 + 2x_4 = 46$$

$$A = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \quad \det(A) =$$

Remark:

## 1.2 Forward Substitution

$$2x_1 = 4$$

$$8x_1 + 3x_2 = 28$$

$$2x_1 + 3x_2 + x_3 = 22$$

$$2x_1 + 2x_2 + 3x_3 + 2x_4 = 46$$

### 1.3 Backward Substitution

$$2x_1 + 2x_2 + 3x_3 + 4x_4 = 20$$

$$5x_2 + 6x_3 + 7x_4 = 34$$

$$8x_3 + 9x_4 = 25$$

$$10x_4 = 10$$

## 2 Appendix

**Review:** The general form of a lower triangular system with a non-zero determinant is

$$\begin{aligned}
 a_{11}x_1 &= b_1 & (a_{11} \neq 0) \\
 a_{21}x_1 + a_{22}x_2 &= b_2 & (a_{22} \neq 0) \\
 &\vdots = \vdots \\
 a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n & (a_{nn} \neq 0)
 \end{aligned} \tag{1}$$

and the solution proceeds from top to bottom, like this

$$\begin{aligned}
 x_1 &= \frac{b_1}{a_{11}} & (a_{11} \neq 0) \\
 x_2 &= \frac{b_2 - a_{21}x_1}{a_{22}} & (a_{22} \neq 0) \\
 &\vdots = \vdots \\
 x_n &= \frac{b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1}}{a_{nn}} & (a_{nn} \neq 0).
 \end{aligned} \tag{2}$$

Similarly, The general form of an upper triangular system with a non-zero determinant is

$$\begin{aligned}
 a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 & (a_{11} \neq 0) \\
 a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 & (a_{22} \neq 0) \\
 a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 & (a_{33} \neq 0) \\
 &\vdots = \vdots \\
 a_{nn}x_n &= b_n & (a_{nn} \neq 0),
 \end{aligned} \tag{3}$$

and the solution proceeds from bottom to top, like this,

$$\begin{aligned}
 x_1 &= \frac{b_1 - a_{12}x_2 - \cdots - a_{1n}x_n}{a_{11}} & (a_{11} \neq 0) \\
 x_2 &= \frac{b_2 - a_{23}x_3 - \cdots - a_{2n}x_n}{a_{22}} & (a_{22} \neq 0) \\
 &\vdots = \vdots & \vdots \\
 x_{n-1} &= \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}} & (a_{n-1,n-1} \neq 0) \\
 x_n &= \frac{b_n}{a_{nn}} & (a_{nn} \neq 0),
 \end{aligned} \tag{4}$$