Rob 101 - Computational Linear Algebra
Recitation #4

Tribhi Kathuria

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1 Vectors in $\mathbb{R}^n$

We identify $\mathbb{R}^n$ with the set of column vectors on length n

So for us, saying $x \in \mathbb{R}^n$ is the same as saying $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, where, $x_i \in \mathbb{R}$

Thus any matrix A of n rows and m columns is a set of m vectors in $\mathbb{R}^n$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} a_1^{\text{col}} & a_2^{\text{col}} & \cdots & a_m^{\text{col}} \end{bmatrix} \iff a_j^{\text{col}} := \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} \in \mathbb{R}^n, 1 \le j \le m \tag{1}$$

Example:

Given matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 5 & 6 & 8 \end{bmatrix}$ is a set of 3 vectors in $\mathbb{R}^3$

Write down this set of vectors:

$a_1^{col} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ $\quad a_2^{col} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ $\quad a_3^{col} = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix} \in \mathbb{R}^3$

Review

Vector addition; $x \in \mathbb{R}^n, y \in \mathbb{R}^n$

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$

Scalar Multiplication with vectors, $\alpha_1 \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 x_1 \\ \alpha_1 x_2 \\ \vdots \\ \alpha_1 x_n \end{bmatrix}$

2 Linear Combination

A vector, $v \in \mathbb{R}^n$ is said to be a Linear Combination of vectors $v_1, v_2 \cdots v_m \in \mathbb{R}^n$ if there exits real numbers $\alpha_1, \alpha_2 \cdots \alpha_m$ such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m$$

Using this formulation, find if the given vector v, is a linear combination of the vectors $v_1, v_2 \cdots v_m$ in the question, if true, also find the vector of coefficients, α

1. $v = \begin{bmatrix} 8 \\ 1 \\ -8 \end{bmatrix}$ $v, v_1, v_2, v_3 \in \mathbb{R}^3$

   $v_1 = \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix}$, $v_2 = \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

   Let's assume there is $\alpha_1, \alpha_2, \alpha_3$ such that

   $V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$ —(1)

   $$\begin{bmatrix} 8 \\ 1 \\ -8 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix} + \alpha_2 \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 8 \\ 1 \\ -8 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 5\alpha_2 + 0\alpha_3 \\ 2\alpha_1 + \alpha_2 + 0\alpha_3 \\ -6\alpha_1 + 6\alpha_2 + 2\alpha_3 \end{bmatrix}$$

   $$= \begin{bmatrix} -1/3 \\ -2/3 \\ 2 \end{bmatrix} + \begin{bmatrix} 25/3 \\ 5/3 \\ 10 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -20 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ -8 \end{bmatrix}$$ with $\alpha_1 = -1/3$, $\alpha_2 = 5/3$, $\alpha_3 = -10$

   $\therefore$ v is a linear combination of v_1, v_2, v_3.

2. $v = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$

   $v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   $\alpha_1 = 2, \alpha_2 = 1, \alpha_3 = 3$

   $\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$

   $2\begin{bmatrix} 3 \\ 4 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   $= \begin{bmatrix} 6+2+3 \\ 8+2-3 \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \end{bmatrix} = V$

   v is a linear combination of v_1, v_2, v_3

3. $v = \begin{bmatrix} 7 \\ 5 \\ 4 \end{bmatrix}$

$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$V = \alpha_1 v_1 + \alpha_2 V_2$

$$\begin{bmatrix} 7 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ 2\alpha_1 \\ 3\alpha_1 \end{bmatrix} + \begin{bmatrix} 2\alpha_2 \\ \alpha_2 \\ 0 \end{bmatrix}$$

$\Rightarrow \alpha_1 + 2\alpha_2 = 7$ (A)

$2\alpha_1 + \alpha_2 = 5$ (B)

$3\alpha_1 = 4$ (C)

From (C) $\alpha_1 = 4/3$

Replacing in (B), $2(4/3) + \alpha_2 = 5$

$\alpha_2 = 5 - \frac{8}{3} = \frac{7}{3}$

This should satify eq. A.

$\frac{4}{3} + 2\left(\frac{7}{3}\right) = 7$

$\Rightarrow \frac{18}{3} \neq 7$

We could not find a set of solution α_1, α_2 that would satify the condition (A, B, C).

Thus, v is not a linear combination of v_1, v_2, V_3.

3 Linear Independence

the vectors $\{v_1, v_2, ..., v_m\}$ are **linearly independent** if the **only** real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ yielding a linear combination of vectors that adds up to the zero vector,

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0, \tag{2}$$

are $\boldsymbol{\alpha}_1 = 0, \boldsymbol{\alpha}_2 = 0, \ldots, \boldsymbol{\alpha}_m = 0$.
Concise definition of Linear Independence:

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0 \iff \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Using this Definition, determine if the following vectors are linearly independent.

1. $v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\alpha_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4\alpha_1 + 2\alpha_2 \\ \alpha_1 + 3\alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$4\alpha_1 + 2\alpha_2 = 0$
$\Rightarrow \alpha_2 = -2\alpha_1 \xrightarrow{\text{Replace}}$ $\alpha_1 + 3\alpha_2 = 0$
$\alpha_1 + 3(-2\alpha_1) = 0$
$\Rightarrow \alpha_1 - 6\alpha_1 = 0 \Rightarrow \boxed{\alpha_1 = 0}$

Replace

$\alpha_2 = 0$

$(\alpha_1, \alpha_2) = (0, 0)$
Thus, v_1 & v_2 are linearly independent

2. $v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $v_4 = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$

$v_4 = 2v_1 + 1v_2 + 3v_3$ (From prev. problem)

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0.$

$3\alpha_1 + 2\alpha_2 + \alpha_3 + 11\alpha_4 = 0$ —(1)
$4\alpha_1 + 2\alpha_2 - \alpha_3 + 7\alpha_4 = 0$ —(2)

v_1, v_2, v_3, v_4 are not linearly independent.

Put, $\{\alpha_1 = 2, \alpha_2 = 1, \alpha_3 = 3, \alpha_1 = -1.\}$

$3(2) + 2(1) + 3 + 11(-1)$
$6 + 2 + 3 - 11 = 0$ ✓

$4(2) + 2(1) - 3 + 7(-1) = 0$
$10 - 10 = 0$ ✓