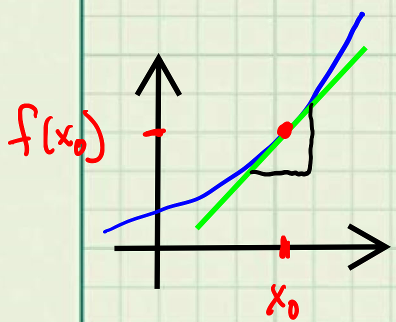
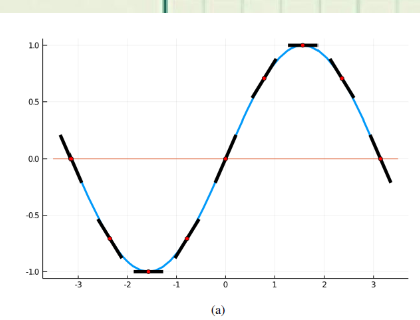
Review: Goal: find roots of functions $f: \mathbb{R}^n \to \mathbb{R}^n$, that is $f(x^*) = 0_{n\times 1}$.

Bisection does not geralize well to vector-valued functions, that is, $n>1$.



$$\frac{df}{dx}(x_0) = \text{slope at } x_0 \approx \frac{\text{rise}}{\text{run}}$$

$$\frac{df}{dx}(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h} \quad \text{forward difference approx}$$

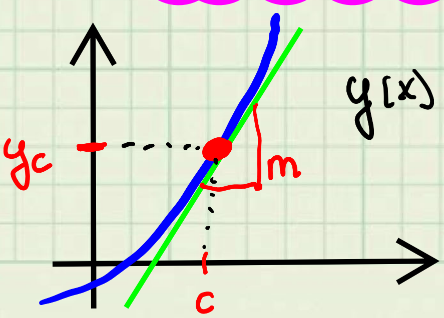


$$\frac{df}{dx}(x_0) \approx \frac{f(x_0) - f(x_0-h)}{h} \quad \text{backward difference approx}$$

$$\frac{df}{dx}(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h} \quad \text{symmetric difference approx.}$$

Linear Approximation of a function at a point x_0

$$\boxed{f(x) = f(x_0) + \frac{df}{dx}(x_0)(x - x_0)}$$



$$y(x) = y_c + m(x - c)$$

$$\boxed{\begin{aligned} c &= x_0 \\ m &= \frac{df}{dx}(x_0) \\ y_c &= f(x_0) \end{aligned}}$$

Linear Approx $\Rightarrow$ Newton's Algorithm

x_k = current approx of a root

Find x_{k+1} by solving linear approx:

$$0 = f(x_{k+1}) \approx f(x_k) + \frac{df}{dx}(x_k)\,(x_{k+1} - x_k)$$

$$\therefore \quad \frac{df}{dx}(x_k)\, x_{k+1} \approx \frac{df}{dx}(x_k)\, x_k - f(x_k)$$

IF $\frac{df}{dx}(x_k) \neq 0 \Rightarrow x_{k+1} = x_k - \left[\frac{df}{dx}(x_k)\right]^{-1} f(x_k)$

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x = x0; h = 0.01; tol = 1e-6
for k = 1:N
  x = x - [ (f(x+h) - f(x-h)) / 2h ]^-1 . f(x)

  if |f(x)| < tol
    xstar = x
    break
  end
end
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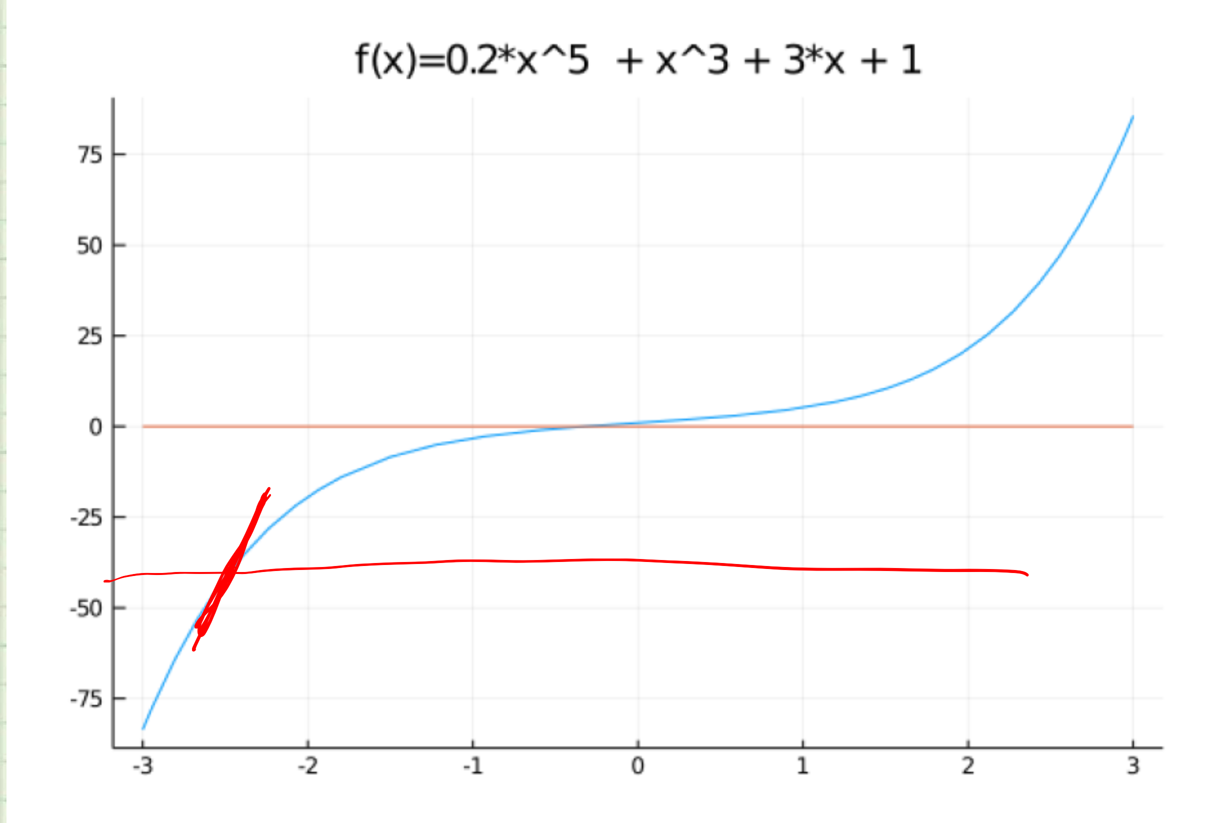
N = max # loops

Newton's Algorithm

Example $f(x) = 0.2x^5 + x^3 + 3x + 1$

find a root using Bisection & Newton



$h = 10^{-2}$, $x_0 = 2$ Newton, $tol = 10^{-4}$

Bisection : bracketing points $a = -2$, $b = 1$

Symmetric Diff. Approx.

Bisection

a	c	b	f(c)	k
−2.0000	−0.5000	1.0000	−0.6313	0.0000
−0.5000	0.2500	1.0000	1.7658	0.0000
−0.5000	−0.1250	0.2500	0.6230	1.0000
−0.5000	−0.3125	−0.1250	0.0314	2.0000
−0.5000	−0.4063	−0.3125	−0.2880	3.0000
−0.4063	−0.3594	−0.3125	−0.1257	4.0000
−0.3594	−0.3359	−0.3125	−0.0466	5.0000
−0.3359	−0.3242	−0.3125	−0.0075	6.0000
−0.3242	−0.3184	−0.3125	0.0120	7.0000
−0.3242	−0.3213	−0.3184	0.0023	8.0000
−0.3242	−0.3228	−0.3213	−0.0026	9.0000
−0.3228	−0.3220	−0.3213	−0.0001	10.0000
−0.3220	−0.3217	−0.3213	0.0011	11.0000
−0.3220	−0.3218	−0.3217	0.0005	12.0000
−0.3220	−0.3219	−0.3218	0.0002	13.0000
−0.3220	−0.3220	−0.3219	0.0000	14.0000

Newton: 5 iterations!

x_k	$f(x_k)$	$\frac{df(x_k)}{dx}$	k
2.0000	21.4000	31.2209	0.0000
1.3146	8.0005	11.1709	1.0000
0.5984	3.0247	4.2025	2.0000
−0.1214	0.6341	3.0445	3.0000
−0.3296	−0.0255	3.3379	4.0000
−0.3220	−0.0001	3.3219	5.0000

Will Newton's Algorithm Always Converge?

NO!

fun example: $f(x) = (x-1)^{1/3}$

$= \text{sgn}(x-1)\sqrt[3]{|x-1|}$

$\text{sign} = \begin{cases} +1 & x-1>0 \\ -1 & x-1<0 \end{cases}$

Newton diverges because $\frac{df}{dx}(x_0)$ does not exist for $x_0 = 1$. $\square$

Goal: Vector valued functions

$f: \mathbb{R}^n \to \mathbb{R}^n$, seek $f(x^*) = 0_{n\times 1}$

Method: To develop linear approx.'s for $f: \mathbb{R}^m \to \mathbb{R}^n$, and then for $m=n$, derive Newton-Raphson algorithm: Matrix version of Newton's Algorithm

For general $m, n \geq 1$, we seek

$$f(x) \approx f(x_0) + A(x - x_0)$$

$x_0 \in \mathbb{R}^m$, $f(x_0) \in \mathbb{R}^n$, $x, x_0 \in \mathbb{R}^m$

and hence $A = n \times m$ matrix

What is A? In other words, how to compute A from $f(x)$???

Remark: $m = n = 1$, $\underset{1\times 1}{A} = a = \frac{df}{dx}(x_0)$

$f: \mathbb{R}^m \longrightarrow \mathbb{R}$ (n=1 for simplicity)

$$f(x) = f(x_0) + \underset{1\times m}{A} \cdot \underbrace{(x - x_0)}_{m\times 1}$$

$$A = [a_1 \ a_2 \ \cdots \ a_m]_{1\times m}$$

$$f(x) - f(x_0) = [a_1 \ a_2 \ \cdots \ a_m](x - x_0)$$

Let $\{e_1, e_2, \ldots, e_m\}$ be the natural basis for $\mathbb{R}^m$, hence e_i = i-th column of I_m

Let $x = x_0 + he_i$ $\Longrightarrow$ $x - x_0 = he_i$ $\quad h > 0$

$$f(x_0 + he_i) - f(x_0) = [a_1 \ a_2 \ \cdots \ a_m]\, he_i$$
$$= a_i h$$

Note $[a_1 \ a_2 \ \cdots \ a_m]\, h \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = h a_1$

$$[a_1 \; a_2 \; \cdots \; a_m] \, h \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = h a_2$$

$$\boxed{a_i = \frac{f(x_0 + he_i) - f(x_0)}{h}}$$

~~i-th~~ partial derivative of f with respect to x_i

Def. $f : \mathbb{R}^m \to \mathbb{R}$, $x_0 \in \mathbb{R}^m$, $h > 0$ small.

$$\frac{\partial f}{\partial x_i}(x_0) = \frac{f(x_0 + \underbrace{he_i}) - f(x_0)}{h}$$

(forward diff approx)

$$= \Big(f(x_{01}, x_{02}, \ldots, \underbrace{x_{0i} + h}, x_{0\,i+1}, \ldots, x_{0m}) - f(x_{01}, x_{02}, \ldots, x_{0m})\Big) / h$$

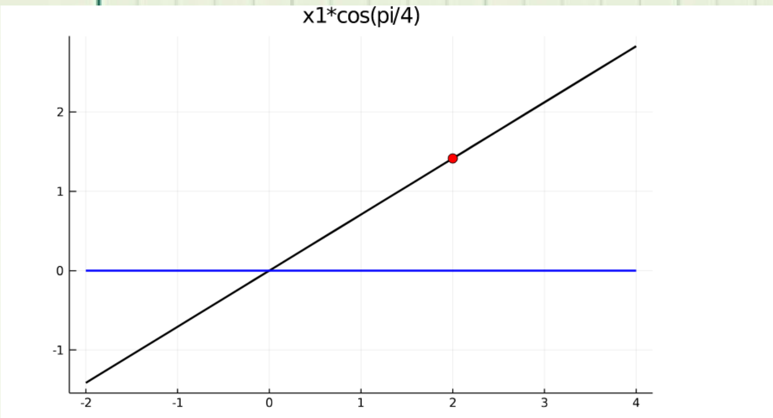
Partial derivative of f w.r.t. x_i.

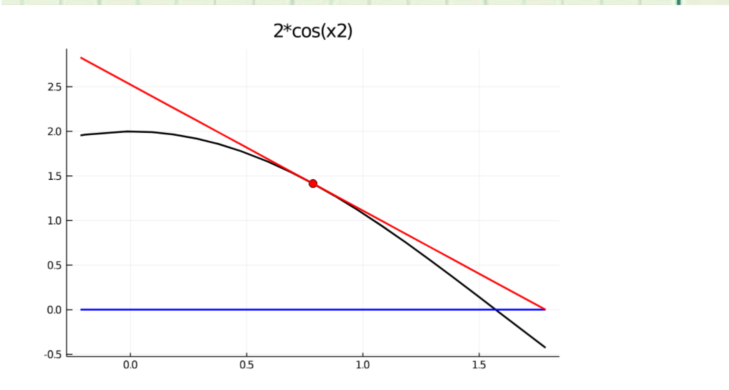
$$\frac{\partial f}{\partial x_i}(x_0) = \frac{f(x_0 + he_i) - f(x_0 - he_i)}{2h}$$

Example:

$$f(x_1, x_2) = x_1 \cos(x_2)$$

$$x_0 = \begin{bmatrix} 2 \\ \pi/4 \end{bmatrix}$$





Slope computed with all variables but one set to a constant.

$$x_0 = \begin{bmatrix} 2 \\ \pi/4 \end{bmatrix}$$

$$\frac{\partial f}{\partial x_1}(x_0) = \frac{f(2+h, \pi/4) - f(2-h, \pi/4)}{2h}$$

$$= \frac{(2+h)\cos(\pi/4) - (2-h)\cos(\pi/4)}{2h}$$

$$= \cos(\pi/4) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\frac{\partial f}{\partial x_2}(x_0) = \frac{f(2, \pi/4 + h) - f(2, \pi/4 - h)}{2h}$$

$$= \frac{2\cos(\pi/4 + h) - 2\cos(\pi/4 - h)}{2h}$$

$h = 10^{-3}$

≈ -1.4142

Analytical answer $= -\sqrt{2}$