**Reminder:** State of Michigan Covid cases are increasing! Take care!

- $S \subset \mathbb{R}^n$ a subset: span generates a subspace

  $\text{span}\{S\} = \{$all linear combinations of elements of $S\}$

  $x \in \text{span}\{S\} \iff x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k,\ k \geq 1$

  $\alpha_i \in \mathbb{R},\ v_i \in S$

- $n \times m$ matrix $A = [a_1^{col}\ a_2^{col}\ \cdots\ a_m^{col}]$
- $\text{col span}(A) := \text{span}\{a_1^{col}, a_2^{col}, \ldots, a_m^{col}\} \subset \mathbb{R}^n$

  subspace of $\mathbb{R}^n$.

- $\text{range}(A) = \{Ax \mid x \in \mathbb{R}^m\} \subset \mathbb{R}^n$
- $\text{null}(A) = \{x \in \mathbb{R}^m \mid Ax = 0\} \subset \mathbb{R}^m$

**Today: How are these all related?**

<u>Facts</u>. $Ax=b$ has a solution if, and only if, $b$ is a linear combination of the columns of $A$.

- $Ax=b$ has a solution $\iff b \in \text{range}(A)$
- $Ax=b$ has a solution $\iff b \in \text{col span}(A)$

Suspect: $\text{range}(A) = \text{col span}(A)$

$$Ax = [a_1^{col} \; a_2^{col} \; \cdots \; a_m^{col}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1 a_1^{col} + x_2 a_2^{col} + \cdots x_m a_m^{col}$$

$$\begin{aligned}
\text{range}(A) &= \{Ax \mid x \in \mathbb{R}^m\} \\
&= \{x_1 a_1^{col} + x_2 a_2^{col} + \cdots + x_m a_m^{col} \mid x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, \ldots, x_m \in \mathbb{R}\} \\
&= \{\alpha_1 a_1^{col} + \alpha_2 a_2^{col} + \cdots + \alpha_m a_m^{col} \mid \alpha_i \in \mathbb{R}\} \\
&= \text{span}\{a_1^{col}, a_2^{col}, \ldots, a_m^{col}\} \\
&= \text{col span}(A)
\end{aligned}$$

✓

Spans help us to generate subspaces!

Basis takes this one step further! "smallest" set of vectors that span a given subspace.

## Basis and Dimension

Def. Suppose that $V$ is a subspace of $\mathbb{R}^n$. Then $\{v_1, v_2, \ldots, v_k\}$ is a basis of $V$ if

a) $\{v_1, v_2, \ldots, v_k\}$ is linearly independent

b) $\mathrm{span}\{v_1, v_2, \ldots, v_k\} = V$

"Goldilocks" small enough to be linearly indep and big enough to generate $V$.

Dimension of $V$ is $k$, the

number of elements in a basis $\square$

<u>Example</u> Let $I_n$ be the $n \times n$ identity matrix and $e_i$ = $i$-th column of $I_n$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

<u>Claim:</u> $\{e_1, e_2, \cdots, e_n\}$ form a basis for $\mathbb{R}^n$.

<u>Why?</u>

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$= x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

$\therefore$ Span $\{e_1, e_2, \ldots, e_n\} = \mathbb{R}^n$

$\therefore$ Check linear indep. But $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} =$

$= \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n \iff \alpha_1 = 0, \ldots, \alpha_n = 0$ $\square$

$\therefore$ Basis ✓ Dimension of $\mathbb{R}^n$ is $n$ $\square$

Claim Let $A$ be an $n\times n$ matrix, with $\det(A)\neq 0$. Then the columns of $A$ form a basis of $\mathbb{R}^n$.

Why?

Write $A=\begin{bmatrix} a_1^{col} & a_2^{col} & \cdots & a_n^{col}\end{bmatrix}$

$\{a_1^{col}, a_2^{col}, \dots, a_n^{col}\}$ basis?

(a) Linear independence?

$$\alpha_1 a_1^{col} + \alpha_2 a_2^{col} + \cdots + \alpha_n a_n^{col} = 0_{n\times 1}$$

$$\Updownarrow$$

$$\underbrace{\begin{bmatrix} a_1^{col} & a_2^{col} & & a_n^{col}\end{bmatrix}}_{A} \underbrace{\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots \\ \alpha_n\end{bmatrix}}_{\alpha} = 0_{n\times 1}$$

$$\Updownarrow$$

$$A\alpha = 0_{n\times 1}$$

$\det(A)\neq 0 \Rightarrow \alpha = 0_{n\times 1}$ is the only solution.

$\therefore$ columns of $A$ are linearly indep.

$$\text{span}\{a_1^{col}, a_2^{col}, \dots, a_n^{col}\} = \mathbb{R}^n \,?$$

Let $b \in \mathbb{R}^n$ be an arbitrary vector.

Seek $\alpha_1, \dots, \alpha_n$ such that

$$b = \alpha_1 a_1^{col} + \alpha_2 a_2^{col} + \cdots + \alpha_n a_n^{col}$$

$$\Updownarrow$$

$$\underbrace{\begin{bmatrix} a_1^{col} & a_2^{col} & & a_n^{col} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\alpha} = b$$

$$\Updownarrow$$

$$A\alpha = b$$

$\det(A) \neq 0 \Rightarrow$ exists a solution to $A\alpha = b \Rightarrow$ span ✓

# Rank and Nullity of a Matrix

Let $A = n \times m$

**Def.**

a) $\operatorname{rank}(A) = \dim \operatorname{range}(A) = *$

b) $\operatorname{nullity}(A) = \dim \operatorname{null}(A)$

* Number of linearly indep columns of A

**Useful Properties of Rank and Nullity**

For an $n \times m$ matrix $A$, key results on its rank and nullity are given below:

**Fact** $\operatorname{rank}(A) + \operatorname{nullity}(A) = m$, the number of columns in $A$. This is known as the **Rank-Nullity Theorem**.

**Fact** $\operatorname{rank}(A^\top A) = \operatorname{rank}(A)$.

**Fact** $\operatorname{rank}(A^\top) = \operatorname{rank}(A)$.

**Fact** $\operatorname{nullity}(A^\top \cdot A) = \operatorname{nullity}(A)$.

**Fact** $\operatorname{nullity}(A^\top) + m = \operatorname{nullity}(A) + n$.

**Fact** For any $m \times k$ matrix $B$, $\operatorname{rank}(A \cdot B) \leq \operatorname{rank}(A)$.

**Fact** For any $p \times n$ matrix $C$, $\operatorname{rank}(C \cdot A) \leq \operatorname{rank}(A)$.