Last in class lecture - Fall 2020

Quadratic program (QP)

Useful Fact about QPs

We consider the QP

$$x^* = \begin{array}{c} \arg\min \\ x \in \mathbb{R}^m \\ A_{in}x \preceq b_{in} \\ A_{eq}x = b_{eq} \\ lb \preceq x \preceq ub \end{array} \frac{1}{2}x^\top Qx + qx \tag{11.55}$$

and assume that Q is symmetric ($Q^\top = Q$) and **positive definite**[a] ($x \neq 0 \implies x^\top Qx > 0$), and that the subset of $\mathbb{R}^m$ defined by the constraints is non empty, that is

$$C := \{x \in \mathbb{R}^m \mid A_{in}x \preceq b_{in},\ A_{eq}x = b_{eq},\ lb \preceq x \preceq ub\} \neq \emptyset. \tag{11.56}$$

Then x^* exists and is unique.

[a] Positive definite matrices are treated in Chapter A.3.

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \underline{x^T Q x} + qx + c \qquad , \; x \in \mathbb{R}^m$$

$$A_{in}\, x \leq b_{in}$$

$$A_{eq}\, x = b_{eq}$$

$$l_b \leq x \leq u_b$$

$Q^T = Q$ symmetric, $\quad x^T Q x > 0$

$x \neq 0$

Example 1:

$$J(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 1)^2$$

$$x^{OPT} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$J(x) = \underline{x_1^2} + 4 - 4x_1 + \underline{x_2^2} + 1 - 2x_2$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underline{5}$$

constant

$$\Rightarrow J(x) = x^T Q x + q x + c$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} -4 & -2 \end{bmatrix}, \quad c = 5$$

Remark:

i) $\max\limits_{x} f(x) = \max f(x) + c$ constant

ii) $\min\limits_{x} f(x) = \min\limits_{x} f(x) - c$

iii) $\max\limits_{x} f(x) = - \min\limits_{x} -f(x)$

iv) $\operatorname{argmax}\limits_{x} f(x) = \operatorname{argmin}\limits_{x} -f(x)$

$$\left.\begin{array}{l} x_1 + 2x_2 \leq 12 \\ 3x_1 + 3x_2 \leq 25 \end{array}\right\} \overset{\text{I}}{\Rightarrow} \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 12 \\ 25 \end{bmatrix}$$

$$A_{in}\, x \leq b_{in}$$

$$\left.\begin{array}{l} x_1 \leq 7 \\ x_2 \leq 5 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{array}\right\} \Rightarrow \begin{array}{l} 0 \leq x_1 \leq 7 \\ 0 \leq x_2 \leq 5 \end{array}$$

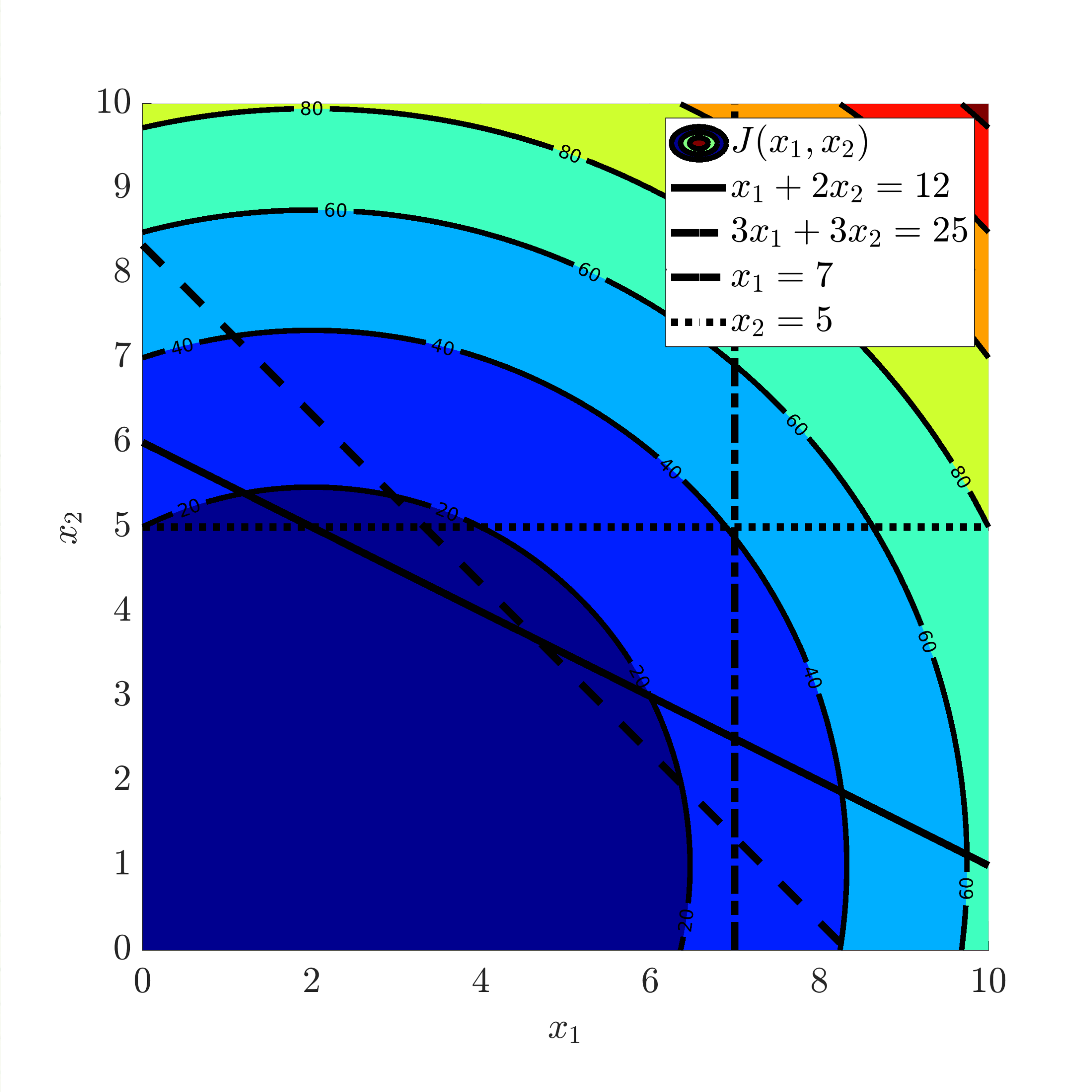
$$\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

II

$$\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

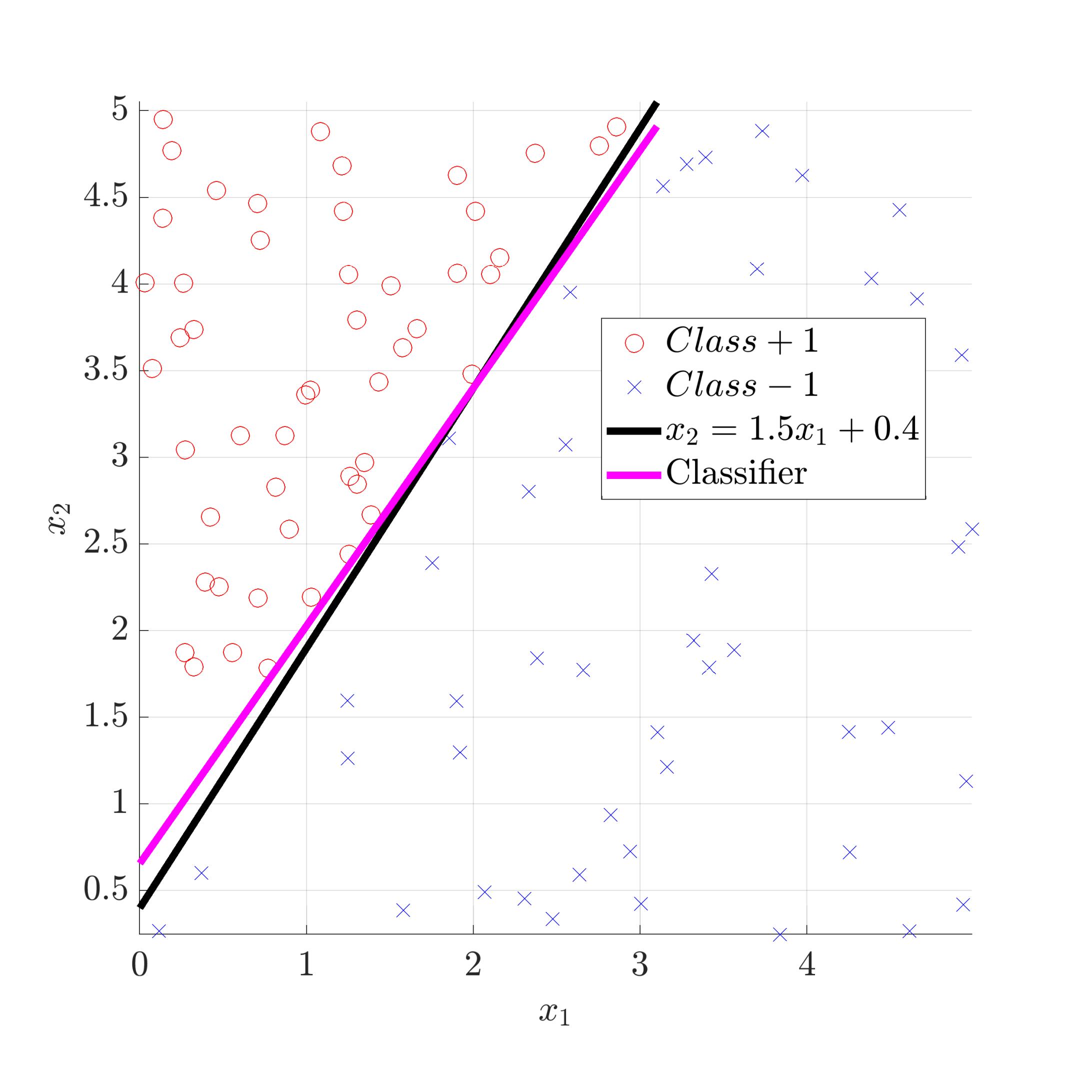
Combine I & II

$$\begin{bmatrix} -Inf \\ -Inf \\ 0 \\ 0 \end{bmatrix} \leq \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{4\times 2} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{2\times 1} \leq \begin{bmatrix} 12 \\ 25 \\ 7 \\ 5 \end{bmatrix}$$



Contour plot

Example 2 : Max-margin Classifier



$a^T x + b = 0$

each × or ○ is a data point x_i

$x_i \in \mathbb{R}^2$ (inputs), labels are ± 1,

$y_i \in \{-1, +1\}$ (target or output).

Data set $D = \{(x_i, y_i)\}_{i=1}^{n}$

$$a^T x + b = 0 \rightarrow w^T \bar{x} = 0$$

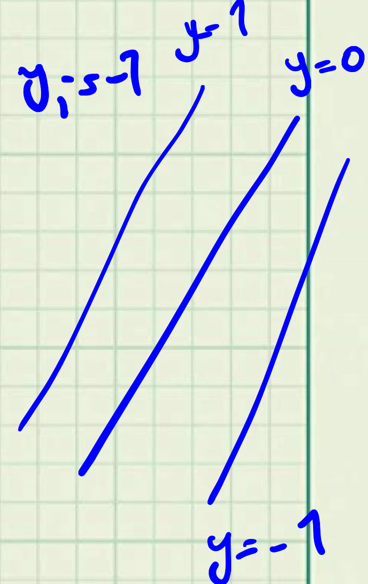
$$w := \begin{bmatrix} a \\ b \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}$$

b: bias

$$\min \frac{1}{2} \|w\|^2$$

$$\text{s.t.} \quad w^T \bar{x}_i \geq 1 \quad \text{if } y_i = +1$$

$$w^T \bar{x}_i \leq -1 \quad \text{if } y_i = -1$$



$$\Rightarrow \quad \min_w \frac{1}{2} w^T w$$

$$\text{s.t.} \quad y_i (w^T \bar{x}_i) \geq 1$$