Rob 101 - Computational Linear Algebra
Recitation #3

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1 Matrix Multiplication

1.1 Partitioning Matrices

Let A be an $n \times m$ matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$n \times m$

A partition of A into rows is

$\to m \times 1$

$$\begin{bmatrix} a_1^{\text{row}} \\ a_2^{\text{row}} \\ \vdots \\ a_n^{\text{row}} \end{bmatrix} = \begin{bmatrix} \boxed{a_{11} \quad a_{12} \quad \cdots \quad a_{1m}} \\ \boxed{a_{21} \quad a_{22} \quad \cdots \quad a_{2m}} \\ \vdots \\ \boxed{a_{n1} \quad a_{n2} \quad \cdots \quad a_{nm}} \end{bmatrix}.$$

$1 \times n$

A partition of A into columns is

$$\begin{bmatrix} a_1^{\text{col}} & a_2^{\text{col}} & \cdots & a_m^{\text{col}} \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{matrix}} & \cdots & \boxed{\begin{matrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{matrix}} \end{bmatrix}.$$

Example

$$A = \begin{bmatrix} 5 & 3 & 8 \\ 4 & 2 & 7 \end{bmatrix}$$

1. $a_1^{row} =$ $\begin{bmatrix} 5 & 3 & 8 \end{bmatrix}$
2. $a_2^{col} =$ $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

1.2 Standard Matrix Multiplication

$C = A \cdot B$, Then,

$$C_{ij} := a_i^{\text{row}} \cdot b_j^{\text{col}}.$$

$$C = \begin{bmatrix} -14 & 6 \\ 54 & -11 \end{bmatrix}$$

Example

1. $A = \begin{bmatrix} 2 & -4 & 2 \\ 3 & 4 & 8 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -5 \\ 6 & -3 \\ 3 & 2 \end{bmatrix}$

$$C_{11} = a_1^{row} b_1^{col} = \begin{bmatrix} 2 & -4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix} = -14$$

$$C_{12} = a_1^{row} b_2^{col} = \begin{bmatrix} 2 & -4 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix} = 6$$

$$C_{21} = a_2^{row} b_1^{col} = \begin{bmatrix} 3 & 4 & 8 \end{bmatrix} \begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix} = 54$$

$$C_{22} = a_2^{row} b_2^{col} = \begin{bmatrix} 3 & 4 & 8 \end{bmatrix} \begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix} = -11$$

1.3 Multiplication by Summing over Columns and Rows

$$\underset{m\times n}{C} = \underset{m\times k}{A} \cdot \underset{k\times n}{B} = \sum_{i=1}^{k} a_i^{\text{col}} \cdot b_i^{\text{row}},$$

Example

1. $\underset{2\times 3}{A} = \begin{bmatrix} 2 & -4 & 2 \\ 3 & 4 & 8 \end{bmatrix}$ $\underset{3\times 2}{B} = \begin{bmatrix} 2 & -5 \\ 6 & -3 \\ 3 & 2 \end{bmatrix}$

$\dim C = 2\times 2$, $k = 3$

$$\Rightarrow C = a_1^{col} b_1^{row} + a_2^{col} b_2^{row} + a_3^{col} b_3^{row}$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & -5 \end{bmatrix} + \begin{bmatrix} -4 \\ 4 \end{bmatrix} \begin{bmatrix} 6 & -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 8 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -10 \\ 6 & -15 \end{bmatrix} + \begin{bmatrix} -24 & 12 \\ 24 & -12 \end{bmatrix} + \begin{bmatrix} 6 & 4 \\ 24 & 16 \end{bmatrix}$$

$$C = \begin{bmatrix} -14 & 6 \\ 54 & -11 \end{bmatrix}$$

2 LU Decomposition

Convert the following system of equations into $Ax = b$, and then Decompose the A matrix as $A = LU$. Finally, solve using,Forward Substitution for $Ly = U$ and Back substitution for $Ux = y$

$$x + 2y + z = 13$$
$$x - 3y + 4z = -9$$
$$3x + y - 2z = 9$$

$$X = \begin{bmatrix} e \\ f \\ g \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -3 & 4 \\ 3 & 1 & -2 \end{bmatrix} \quad b = \begin{bmatrix} 13 \\ -9 \\ 9 \end{bmatrix}$$

dim (A) = 3x3,

Initialize, k=1, Temp = A, L=[ ], U=[ ].

Update Temp = Temp - CR s.t. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & \cdot & \cdot \\ 0 & \cdot & \cdot \end{bmatrix}$

alterative notation from classnotes, but comparable!

k=1

$$C = \underbrace{Temp_k^{col}}\frac{1}{pivot} \qquad pivot = \boxed{Temp_k^{col}[k]} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = 1$$

$$= \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} * \frac{1}{1}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$R = Temp_k^{row} = [1 \quad 2 \quad 1]$$

Update Temp = Temp - CR

$$= \begin{bmatrix} 1 & 2 & 1 \\ 1 & -3 & 4 \\ 3 & 1 & -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} [1 \quad 2 \quad 1]$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 1 & -3 & 4 \\ 3 & 1 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 3 & 6 & 3 \end{bmatrix}$$

$\underline{\underline{k=1}}$ $$\text{Temp} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -5 & 3 \\ 0 & -5 & -5 \end{bmatrix}$$

Update L, U.

$L = [L \quad c]$

$U = [U \, ; \, R]$

$$\Rightarrow L = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} ; \quad U = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

$\underline{\underline{k=2}}$ $$\text{Temp} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -5 & 3 \\ 0 & -5 & -5 \end{bmatrix}$$

$$\text{pivot} = \text{Temp}_k^{col}[k] = -5$$

$$c = \text{Temp}_k^{col} \frac{1}{\text{pivot}}$$

$$= \begin{bmatrix} 0 \\ -5 \\ -5 \end{bmatrix} \frac{1}{-5}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$R = \text{Temp}_k^{row} = \begin{bmatrix} 0 & -5 & 3 \end{bmatrix}$$

Update $Temp = Temp - CR$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -5 & 3 \\ 0 & -5 & -5 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & -5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -5 & 3 \\ 0 & -5 & -5 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & -5 & 3 \\ 0 & -5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -8 \end{bmatrix}$$

Update L, U.

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 3 \end{bmatrix}$$

$\underline{\underline{k=3}}$

$$Temp = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -8 \end{bmatrix} \qquad pivot = Temp_k^{col}[k] = -8$$

$$c = Temp_k^{col}\ {}^{1}/_{pivot}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -8 \end{bmatrix} \frac{1}{-8} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$R = Temp_k^{row} = \begin{bmatrix} 0 & 0 & -8 \end{bmatrix}$$

Update Temp = Temp − CR

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -8 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -8 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} L & C \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} U ; R \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 3 \\ 0 & 0 & -8 \end{bmatrix}$$

Confirm that $A = LU$.

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -3 & 4 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 3 \\ 0 & 0 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 1 & -3 & 4 \\ 3 & 1 & -2 \end{bmatrix}$$

$Ax=b \Rightarrow \overbrace{L\underbrace{Ux}_{y}=b} \Rightarrow Ly=b$

Forward Substitution

$b=\begin{bmatrix}13\\-9\\9\end{bmatrix}$; $y=\begin{bmatrix}h\\i\\j\end{bmatrix}$ $L=\begin{bmatrix}1&0&0\\1&1&0\\3&1&1\end{bmatrix}$

For $n=3$

$x_1=\frac{b_1}{a_{11}}$

$x_2=\frac{b_2-a_{21}x_1}{a_{22}}$

$x_3=\frac{b_3-a_{31}x_1-a_{32}x_2}{a_{33}}$

$h=\frac{13}{1}$

$i=\frac{-9-1(h)}{1}=-9-13=-22$

$j=\frac{9-3(h)-1(i)}{1}=9-3(13)-(-22)$

$=-8$ $\quad y=\begin{bmatrix}13\\-22\\-8\end{bmatrix}$

$Ux=y$; $U=\begin{bmatrix}1&2&1\\0&-5&3\\0&0&-8\end{bmatrix}$; $x=\begin{bmatrix}e\\f\\g\end{bmatrix}$

Back Substitution

For $n=3$

$x_1=\frac{b_1-a_{12}x_2-a_{13}x_3}{a_{11}}$

$x_2=\frac{b_2-a_{23}x_3}{a_{22}}$

$x_3=\frac{b_3}{a_3}$

$e+2f+g=13 \quad -(1)$

$-5f+3g=-22 \quad -(2)$

$-8g=-8 \quad -(3)$

(3) $g=\frac{-8}{-8}=1$

(2) $f=\frac{-22-3g}{-5}=\frac{-22-3}{-5}=5$

(1) $e=13-2f-g=13-2(5)-1$

$=2$

$x=\begin{bmatrix}2\\5\\1\end{bmatrix}$

Check that $Ax=b$!

3 Appendix

Review: The general form of a lower triangular system with a non-zero determinant is

$$\begin{aligned} a_{11}x_1 &= b_1 \quad (a_{11} \neq 0) \\ a_{21}x_1 + a_{22}x_2 &= b_2 \quad (a_{22} \neq 0) \\ \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n \quad (a_{nn} \neq 0) \end{aligned} \tag{1}$$

Frwo ard. Substitution.

and the solution proceeds from top to bottom, like this

n=3

$$\begin{aligned} x_1 &= \frac{b_1}{a_{11}} \quad (a_{11} \neq 0) \\ x_2 &= \frac{b_2 - a_{21}x_1}{a_{22}} \quad (a_{22} \neq 0) \\ \vdots &= \vdots \\ x_n &= \frac{b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1}}{a_{nn}} \quad (a_{nn} \neq 0). \end{aligned} \tag{2}$$

Similarly, The general form of an upper triangular system with a non-zero determinant is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \quad (a_{11} \neq 0) \\ a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \quad (a_{22} \neq 0) \\ a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \quad (a_{33} \neq 0) \\ \vdots &= \vdots \\ a_{nn}x_n &= b_n \quad (a_{nn} \neq 0), \end{aligned} \tag{3}$$

and the solution proceeds from bottom to top, like this,

$$\begin{aligned} x_1 &= \frac{b_1 - a_{12}x_2 - \cdots - a_{1n}x_n}{a_{11}} && (a_{11} \neq 0) \\ x_2 &= \frac{b_2 - a_{23}x_3 - \cdots - a_{2n}x_n}{a_{22}} && (a_{22} \neq 0) \\ \vdots &= \vdots && \vdots \\ x_{n-1} &= \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}} && (a_{n-1,n-1} \neq 0) \\ x_n &= \frac{b_n}{a_{nn}} && (a_{nn} \neq 0), \end{aligned} \tag{4}$$