Rob 101 - Computational Linear Algebra
Recitation #8

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1 Subspaces

Suppose that $V \subset \mathbb{R}^n$ is nonempty.

Def. V is a **subspace** of $\mathbb{R}^n$ if any linear combination constructed from elements of V and scalars in $\mathbb{R}$ is once again an element of V. One says that V is **closed under linear combinations.**
In symbols, $V \subset \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if for all real numbers α and β, and all vectors v_1 and v_2 in V

$$\boxed{\alpha v_1 + \beta v_2 \in V.} \tag{1}$$

Using this formulation, comment if the following $V \in \mathbb{R}^n$ are Subspaces

1. $$V := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;\middle|\; x_1^2 + x_2^2 = 2x_1x_2, x_1, x_2 \in \mathbb{R} \right\}.$$

2.

$$V := \left\{ \begin{bmatrix} ax \\ by \\ ax + by + c \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}.$$

2 Null Space and Range of a Matrix

For any Matrix $A \in \mathbb{R}_{m \times n}$, then the following sets (are actually Subspaces!) can be defined:

Def. $\text{null}(A) := \{x \in \mathbb{R}^m \mid Ax = 0_{n \times 1}\}$ is the **null space** of A.

Def. $\text{range}(A) := \{y \in \mathbb{R}^n \mid y = Ax \ \text{ for some } \ x \in \mathbb{R}^m\}$ is the **range** of A.

Using this definition, Find the Null Space and Range of the following:

1.
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

2.
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \end{bmatrix}.$$

3 Column Span

Let A be an $n \times m$ matrix.

$$A := \text{span}\left\{a_1^{\text{col}}, \ldots, a_m^{\text{col}}\right\}.$$

We can also discuss, rank and nullity of A here as:

Def. $\text{rank}(A) := \dim \text{col span}\{A\}$.

Def. $\text{nullity}(A) := \dim \text{null}(A)$.

Using these definitions comment on the Results of the Rank-Nullity theorem for

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \end{bmatrix}.$$

4 Basis

Suppose that V is a subspace of $\mathbb{R}^n$. Then $\{v_1, v_2, \ldots, v_k\}$ is a **basis for V** if

1. the set $\{v_1, v_2, \ldots, v_k\}$ is linearly independent, and
2. span $\{v_1, v_2, \ldots, v_k\} = V$.

The **dimension of V is k**, the number of basis vectors
Find the basis and Dimension for the following Subspaces:

1.
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \end{bmatrix}, S = \text{col span}\{A\}$$

2.
$$V := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;\middle|\; x_1^2 + x_2^2 = 2x_1x_2, x_1, x_2 \in \mathbb{R} \right\}.$$

3.
$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \; S = \text{span}\{a\}$$

5 Gram-schmidt

Suppose that that the set of vectors $\{u_1, u_2, \ldots, u_m\}$ is linearly independent then you can generate a new set of orthogonal vectors $\{v_1, v_2, \ldots, v_m\}$ as:

$$
\begin{aligned}
v_1 &= u_1 \\
v_2 &= u_2 - \left(\frac{u_2 \bullet v_1}{v_1 \bullet v_1}\right) v_1 \\
v_3 &= u_3 - \left(\frac{u_3 \bullet v_1}{v_1 \bullet v_1}\right) v_1 - \left(\frac{u_3 \bullet v_2}{v_2 \bullet v_2}\right) v_2 \\
&\vdots \\
v_k &= u_k - \sum_{i=1}^{k-1} \left(\frac{u_k \bullet v_i}{v_i \bullet v_i}\right) v_i \qquad \text{(General Step)}
\end{aligned}
\tag{2}
$$

You are given that the set below is a basis for $\mathbb{R}^3$. Produce from it an orthonormal basis.

$$
\{u_1, u_2, u_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}
$$

6 QR Factorization

Suppose that A is an $n \times m$ matrix with linearly independent columns. Then there exists an $n \times m$ matrix Q with orthonormal columns and an upper triangular, $m \times m$, invertible matrix R such that $A = Q \cdot R$. Moreover, Q and R are constructed as follows:

- Let $\{u_1, \ldots, u_m\}$ be the columns of A with their order preserved so that

$$A = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}$$

- Q is constructed by applying the Gram-Schmidt Process to the columns of A and normalizing their lengths to one,

$$\{u_1, u_2, \ldots, u_m\} \xrightarrow[\text{Process}]{\text{Gram-Schmidt}} \{v_1, v_2, \ldots, v_m\}$$

$$Q := \begin{bmatrix} \frac{v_1}{||v_1||} & \frac{v_2}{||v_2||} & \cdots & \frac{v_m}{||v_m||} \end{bmatrix}$$

- Because $Q^\top Q = I_m$, it follows that $A = Q \cdot R \iff R := Q^\top \cdot A$.

Find the QR Factorization of

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$