Rob 101 - Computational Linear Algebra
Recitation #5

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1 Set of Linearly Independent Vectors

By Definition of Linear Independence, we know that,

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0 \iff \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Using this Definition, determine if the following vectors are linearly independent.

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}, v_3 = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$$

Pro-tip for checking Linear Independence

The set of vectors $\{v_1, v_2, \ldots, v_m\}$ is linearly independent if the any of the following are true:
Define A to be matrix whose columns are the given set of vectors. $A = \begin{bmatrix} v_1 & v_2 & \ldots & v_m \end{bmatrix}$

- $\det(A^\top \cdot A) \neq 0$ or A is invertible.
- For any LU Factorization $P \cdot (A^\top \cdot A) = L \cdot U$ of $A^\top A$, the $m \times m$ upper triangular matrix U has no zeros on its diagonal.

Solve for linear independence for the following set of vectors, written in the A matrix, using the Pro-tip in Julia:

1.
$$A = \begin{bmatrix} -4.0 & 4.5 & 3.3 \\ -1.3 & -3.5 & 1.0 \\ 2.4 & 4.9 & -4.2 \\ -3.8 & 1.6 & -2.0 \end{bmatrix} \tag{1}$$

2.
$$A = \begin{bmatrix} 2.4 & 4.1 & 0.9 & -0.5 & -0.9 & -2.2 & -4.6 & 3.0 & 0.1 & -4.0 \\ -0.6 & 4.0 & -4.6 & -4.5 & -3.0 & 1.1 & 4.7 & -3.2 & 4.1 & -3.6 \\ -2.1 & 1.7 & -5.0 & 2.4 & 1.7 & 0.8 & 4.1 & -4.3 & -3.7 & -4.7 \\ 1.9 & -0.8 & -4.9 & 4.3 & -5.0 & 4.3 & 1.1 & 0.1 & 1.0 & 1.7 \\ 0.6 & 4.0 & -3.8 & -0.2 & -3.5 & -2.6 & 3.9 & 5.0 & 2.7 & 1.0 \\ 3.2 & 4.6 & 1.5 & 4.6 & 4.3 & 1.0 & -3.2 & -0.3 & -2.8 & -0.3 \end{bmatrix} \tag{2}$$

2 Solutions of Ax=b and Linear Independence

Existence

The system of equation $Ax = b$ has a solution if:

b is a linear combination of the columns of A

Using this definition, lets find if the following system of equations have a solution:

$$\begin{aligned} -a + 3b + 5c &= 20 \\ -2a - 2c &= -8 \\ -3a + 3b + 4c &= 10 \end{aligned}$$

Semi-Pro-tip for checking existence of solution

We know that the system of equation $Ax = b$ has a solution if:

b is a linear combination of the columns of A

Thus, b appended as a column of A should have the number of Linearly Independent Columns.
Mathematically written as:
Define $A_{\mathrm{e}} := [A \;\; b]$ by appending b to the columns of A. Then we do the corresponding LU Factorizations

- $P \cdot \left(A^\top A\right) = L \cdot U$
- $P_{\mathrm{e}} \cdot \left(A_{\mathrm{e}}^\top A_{\mathrm{e}}\right) = L_{\mathrm{e}} \cdot U_{\mathrm{e}}$.

Fact: $Ax = b$ has a solution if, and only if, U and U_{e} have the same number of linearly independent columns.
We check the number of independent columns in U and U_{e} using our Semi-Pro-tip (Appendix)!
Using this definition, lets find if the following system of equations have a solution in Julia:

1.

$$A = \begin{bmatrix} -2.4 & -2.1 & -2.6 \\ -0.8 & 2.9 & -1.2 \\ 1.5 & 1.0 & 1.5 \\ -2.3 & 2.0 & -0.4 \\ 2.7 & 0.6 & -1.3 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 7 \\ 6 \\ 9 \\ 9 \\ 9 \end{bmatrix}$$

2.

$$A = \begin{bmatrix} -4.0 & 1.6 & -2.1 & -0.8 & -3.8 & 4.6 & -2.2 \\ -1.3 & 3.3 & 1.9 & 4.0 & 1.5 & -0.9 & 1.1 \\ 2.4 & 1.0 & 0.6 & 4.6 & -0.5 & -3.0 & 0.8 \\ -3.8 & -4.2 & 3.2 & 0.9 & -4.5 & 1.7 & 4.3 \\ 4.5 & -2.0 & 4.1 & -4.6 & 2.4 & -5.0 & -2.6 \\ -3.5 & 2.4 & 4.0 & -5.0 & 4.3 & -3.5 & 1.0 \\ 4.9 & -0.6 & 1.7 & -4.9 & -0.2 & 4.3 & -4.6 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 10 \\ 10 \\ 7 \\ 9 \\ 2 \\ 8 \\ 2 \end{bmatrix}$$

Uniqueness

The system of equation Ax = b has a unique solution if:

b is a linear combination of the columns of A, and, the columns of A are linearly independent.

Using this definition, find if the above Julia example has a unique solution:

$$A = \begin{bmatrix} -4.0 & 1.6 & -2.1 & -0.8 & -3.8 & 4.6 & -2.2 \\ -1.3 & 3.3 & 1.9 & 4.0 & 1.5 & -0.9 & 1.1 \\ 2.4 & 1.0 & 0.6 & 4.6 & -0.5 & -3.0 & 0.8 \\ -3.8 & -4.2 & 3.2 & 0.9 & -4.5 & 1.7 & 4.3 \\ 4.5 & -2.0 & 4.1 & -4.6 & 2.4 & -5.0 & -2.6 \\ -3.5 & 2.4 & 4.0 & -5.0 & 4.3 & -3.5 & 1.0 \\ 4.9 & -0.6 & 1.7 & -4.9 & -0.2 & 4.3 & -4.6 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 10 \\ 10 \\ 7 \\ 9 \\ 2 \\ 8 \\ 2 \end{bmatrix}$$

3 Appendix

3.1 Semi-Pro Tip: When Is it Enough to Look at the Diagonal of U?

Let $P \cdot \left(A^\top A\right) = L \cdot U$ be an LU Factorization of $A^\top A$, and define k as above to be the number of linearly independent columns of U. When is k equal to the number of non-zero elements on the diagonal of U?

- If the diagonal of U has no zero elements, then $k = m$ and all columns of A are linearly independent.
- If the diagonal of U has one zero element, then $k = m - 1$ and one can select $k = m - 1$ linearly independent columns from A.
- If the diagonal of U has two or more zero elements, then additional computations are necessary. For example, if the rows of U that have zeros on their diagonal element are identically zero, then k equals the number of non-zero elements on the diagonal of U.