Propositional Logic: Soundness and Completeness of $\mathcal{H}$

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Shin Yoo

Goals of Logic

- To check whether a given formula ϕ is valid (semantic)
- To prove a given formula ϕ (syntactic)

$\models \phi$ Semantic Methods (Truth table)	$\equiv$	Semantic Tableau	$\equiv$	$\vdash \phi$ Syntactic Methods ($\mathcal{G}$, $\mathcal{H}$, etc)

We can use what we have proved so far.

$\vdash \phi$ Hilbert System, $\mathcal{H}$	$\equiv$	$\mathcal{G}$	$\equiv$	Semantic Tableau	$\equiv$	$\models \phi$ Semantic Methods (Truth table)

Soundness of $\mathcal{H}$

Theorem 1 (3.34, Ben-Ari)

$\mathcal{H}$ is sound, that is, if $\vdash A$ in $\mathcal{H}$ then $\models A$.

How do we prove this? Structural Induction, that is:

1. Show that three axioms of $\mathcal{H}$ are all valid, and
2. Show that if the premises of Modus Ponens rule is valid, then so is the conclusion.

Show that three axioms of $\mathcal{H}$ are all valid. To show that A is valid, we can show $\neg A$ is not satisfiable, i.e., that the semantic tableau of $\neg A$ is *closed*.

Axiom 1

$$\neg(A \rightarrow (B \rightarrow A))$$

$$A, \neg(B \rightarrow A)$$

$$A, B, \neg A$$

Axiom 3

$$\neg((\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B))$$

$$\neg B \rightarrow \neg A, \neg(A \rightarrow B)$$

$$\neg B \rightarrow \neg A, A, \neg B$$

$$\neg\neg B, A, B \qquad \neg A, A, \neg B$$

$$B, A, \neg B$$

Tableau for Axiom 2:
$\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$?

Soundness of $\mathcal{H}$

Show that if MP is sound. We use Reductio Ad Absurdum.

Proof.

Suppose that MP is not sound. There would be a set of formulas $\{A, A \rightarrow B, B\}$ such that A and $A \rightarrow B$ are valid but B is not.If B is not valid, there is an interpretation ν such that $\nu(B) = F$. Since A and $A \rightarrow B$ are valid, for **any** interpretation, **including** ν, resulting in $\nu(A) = \nu(A \rightarrow B) = T$. The truth table then states $\nu(B) = T$, which results in a contradiction in terms of our choice of ν. □

$\mathcal{H}$ is sound.

Completeness of $\mathcal{H}$

Theorem 2 (3.35, Ben-Ari)

$\mathcal{H}$ is complete, that is, if $\models A$ then $\vdash A$ in $\mathcal{H}$.

- Any valid formula can be proved in $\mathcal{G}$ (Thm 3.8). We will show how a proof in $\mathcal{G}$ can be mechanically transformed into a proof in $\mathcal{H}$.
- A set of formulas U is provable in $\mathcal{G} \equiv$ a single formula $\bigvee U$ is provable in $\mathcal{H}$.
- Certain axioms of $\mathcal{G}$ are trivial: $\{p, \neg p\}$ is an axiom in $\mathcal{G}$, then $\vdash (p \vee \neg p)$ in $\mathcal{H}$, using Thm 3.10 ($\vdash A \rightarrow A$). (Note that $A \vee B \leftrightarrow \neg A \vee B$)
- How about $\{q, \neg p, r, p, s\}$? Not trivial.

Lemma 1 (3.36, Ben-Ari)

If $U' \subseteq U$ and $\vdash \bigvee U'$, then $\vdash \bigvee U$ in $\mathcal{H}$.

Proof.

Suppose we have a proof of $\bigvee U'$. By repeated application of Thm 3.31, we can transform this into a proof of $\bigvee U''$, where U'' is a permutation of the elements of U (Thm 3.31 Weakening: $\vdash A \rightarrow A \vee B$). Now by repeated applications of the commutativity and associativity of disjunction, we can move the elements of U'' to their proper places (Thm 3.32 is the Commutativity Rule, Thm 3.33 is the Associativity Rule). □

Completeness of $\mathcal{H}$

Now we use the induction on the structure of proofs in $\mathcal{G}$ in order to prove the completeness of $\mathcal{H}$. That is, we show that for any proof in $\mathcal{G}$, there exists a mechanically corresponding proof in $\mathcal{H}$.

Proof.

Case 1: If U is an **axiom**, it contains a pair of complementary literals, and $\vdash p \vee \neg p$ is provable in $\mathcal{H}$. By Lemma 1, this can be transformed into a proof of $\bigvee U$.

Lem 3.36: If $U' \subseteq U$ and $\vdash \bigvee U'$, then $\vdash \bigvee U$ in $\mathcal{H}$.

Proof. Cont.

Case 2: If U is not an axiom in $\mathcal{G}$, the last step in the proof of G is the application of either $\alpha-$ or $\beta-$ rule.

- $\alpha-$rule: $\dfrac{\vdash U_1 \cup \{A_1, A_2\}}{\vdash U_1 \cup \{A_1 \vee A_2\}}$. By the inductive hypothesis, $\vdash (\bigvee U_1 \vee A_1) \vee A_2$ in $\mathcal{H}$, from which we get $\vdash \bigvee U_1 \vee (A_1 \vee A_2)$ by associativity.

Completeness of $\mathcal{H}$

Proof.

- $\beta-$rule: $\dfrac{\vdash U_1 \cup \{A_1\} \quad \vdash U_2 \cup \{A_2\}}{\vdash U_1 \cup U_2 \cup \{A_1 \wedge A_2\}}$. By the inductive hypothesis, $\vdash \bigvee U_1 \vee A_1$ and $\vdash \bigvee U_2 \vee A_2$ in $\mathcal{H}$. From these, we need to find a proof of $\vdash \bigvee U_1 \vee \bigvee U_2 \vee (A_1 \wedge A_2)$.

1.	$\vdash \bigvee U_1 \vee A_1$	Induction Hypothesis
2.	$\vdash \neg \bigvee U_1 \rightarrow A_1$	$A \vee B \models \neg A \rightarrow B$
3.	$\vdash A_1 \rightarrow (A_2 \rightarrow (A_1 \wedge A_2))$	Derived rule on $\wedge$
4.	$\vdash \neg \bigvee U_1 \rightarrow (A_2 \rightarrow (A_1 \wedge A_2))$	MP, 2, 3
5.	$\vdash A_2 \rightarrow (\neg \bigvee U_1 \rightarrow (A_1 \wedge A_2))$	Exchanged Antecedents
6.	$\vdash \bigvee U_2 \vee A_2$	Induction Hypothesis
7.	$\vdash \neg \bigvee U_2 \rightarrow A_2$	$A \vee B \models \neg A \rightarrow B$
8.	$\vdash \neg \bigvee U_2 \rightarrow (\neg \bigvee U_1 \rightarrow (A_1 \wedge A_2))$	MP, 7, 5
9.	$\vdash \neg \bigvee U_2 \rightarrow (\bigvee U_1 \vee (A_1 \wedge A_2))$	$A \vee B \models \neg A \rightarrow B$
10.	$\vdash \bigvee U_2 \vee (\bigvee U_1 \vee (A_1 \wedge A_2))$	$A \vee B \models \neg A \rightarrow B$
11.	$\vdash \bigvee U_1 \vee \bigvee U_2 \vee (A_1 \wedge A_2))$	Associativity

$\square$

Consistency of $\mathcal{H}$

Definition 1 (3.38, Ben-Ari)

A set of formulas U is inconsistent iff for some formula A, $U \vdash A$ and $U \vdash \neg A$. U is consistent iff U is not inconsistent.

Theorem 3 (3.39, Ben-Ari)

U is inconsistent iff for all A, $U \vdash A$.

Proof.

Let A be an arbitrary formula. Since U is inconsistent, for some formula B, $U \vdash B$ and $U \vdash \neg B$. By Thm 3.21, $\vdash B \rightarrow (\neg B \rightarrow A)$. Using MP twice, $U \vdash A$. The converse is trivial. □

Consistency of $\mathcal{H}$

Corollary 1 (3.40)

U is consistent iff for some A, $U \nvdash A$.

Theorem 4 (3.41)

$U \vdash A$ iff $U \cup \{\neg A\}$ is inconsistent.

Variants of $\mathcal{H}$

Variant Hilbert Systems almost always have MP as the single rule, while having different choice of primitive operators and axioms. For example, a variant $\mathcal{H}$' replaces the third axiom with:

$$\text{Axiom 3'}: \vdash (\neg B \to \neg A) \to ((\neg B \to A) \to B)$$

Theorem 5

$\mathcal{H}$ and $\mathcal{H}$' are equivalent.

Variants of $\mathcal{H}$

Proof of Axiom 3' in $\mathcal{H}$:

1.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash \neg B$	Assumption
2.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash \neg B \rightarrow A$	Assumption
3.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash A$	MP, 1, 2
4.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash \neg B \rightarrow \neg A$	Assumption
5.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash A \rightarrow B$	Contrapositive, 4
6.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash B$	MP, 3, 5
7.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash \neg B \rightarrow B$	Deduction, 7
8.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash (\neg B \rightarrow B) \rightarrow B$	Theorem 3.29
9.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash B$	MP, 8, 9
10.	$\{\neg B \rightarrow \neg A\} \vdash (\neg B \rightarrow A) \rightarrow B$	Deduction, 9
11.	$\vdash (\neg B \rightarrow \neg A) \rightarrow (\neg B \rightarrow A) \rightarrow B$	Deduction, 10

$\mathcal{H}$" has the same MP rule, but a different set of axioms:

1. $\vdash A \vee A \rightarrow A$
2. $\vdash A \rightarrow A \vee B$
3. $\vdash A \vee B \rightarrow B \vee A$
4. $\vdash B \rightarrow C \rightarrow (A \vee B \rightarrow A \vee C)$

$\mathcal{H}$"' has only one axiom called Meredith's Axiom:

$$(\{[(A \rightarrow B) \rightarrow (\neg C \rightarrow \neg D)] \rightarrow C\} \rightarrow E) \rightarrow [(E \rightarrow A) \rightarrow (D \rightarrow A)]$$

Arrrrgh.

Subformula Property

Definition 2 (3.48, Ben-Ari)

A deductive system has the **subformula property** if any formula appearing in a proof of A is either a subformula of A or the negation of a subformula of A.

$\mathcal{G}$ has the subformula property; $\mathcal{H}$ does not. Why?

If a deductive system has the subformula property, then mechanical proof may become possible, since

- there exist only a finite number of subformulas for a finite formula ϕ
- there exist only a finite number of inference rules

The rest is the machine's work.

Automated Proof

One desirable property of a deductive system is to generate an automated/mechanical proof.

- We have decision procedures to check validity of a propositional formula automatically (i.e., truth table and semantic tableau).
- Note that decision procedures require knowledge on all interpretations (i.e., infinite number of interpretations, in general) which is not feasible except for propositional logic.

A deductive proof requires only a finite set of sets of formulas, because a deductive proof system analyses the target formula only, not its interpretations.

- Many research works to develop (semi)automated theorem prover.
- No obvious connection between the formula and its proof in $\mathcal{H}$; makes a proof in $\mathcal{H}$ difficult (no mechanical proof).
- One has to rely on one's brain to select proper axioms.