# Propositional Logic: Semantics (2/3)

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# Overview

- Logical Equivalence and Substitution
- Satisfiability, Validity, and Consequence

# Logical Equivalence

**Definition 1 (2.26)**

Let $A_1, A_2 \in \mathscr{F}$. If $\nu_{\mathscr{I}}(A_1) = \nu_{\mathscr{I}}(A_2)$ for *all* interpretations $\mathscr{I}$, then $A_1$ is *logically equivalent* to $A_2$, denoted $A_1 \equiv A_2$.

| $\mathscr{I}(p)$ | $\mathscr{I}(q)$ | $\nu_{\mathscr{I}}(p \vee q)$ | $\nu_{\mathscr{I}}(q \vee p)$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $F$ |

# Logical Equivalence

We can extend the result of the previous example from atomic propositions to general formulas.

**Theorem 1 (2.28)**

*Let $A_1$ and $A_2$ be any formulas. Then $A_1 \vee A_2 \equiv A_2 \vee A_1$.*

**Proof.**

1. Let $\mathscr{I}$ be an arbitrary interpretation for $A_1 \vee A_2$. Then, $\mathscr{I}$ is also an interpretation for $A_2 \vee A_1$, because $\mathscr{P}_{A_1} \cup \mathscr{P}_{A_2} = \mathscr{P}_{A_2} \cup \mathscr{P}_{A_1}$.
2. Similarly, $\mathscr{I}$ is an interpretation for $A_1$ and $A_2$.
3. Therefore, $\nu_{\mathscr{I}}(A_1 \vee A_2) = T \leftrightarrow (\nu_{\mathscr{I}}(A_1) = T \vee \nu_{\mathscr{I}}(A_2) = T) \leftrightarrow \nu(A_2 \vee A_1) = T$.

$\square$

# Logical Equivalence

**Theorem 2 (2.29)**

*$A_1 \equiv A_2$ if and only if $A_1 \leftrightarrow A_2$ is true in every interpretation.*

- **Object Language:** the language we set out to study, i.e. propositional logic in our current case.
- **Metalanguage:** the language that is used to discuss an object language.

What is the difference between $\leftrightarrow$ and $\equiv$?

- **Material Equivalence** ($\leftrightarrow$): just another statement in the object language; truth value depends on the specific interpretation.
- **Logical Equivalance** ($\equiv$): semantic statement, i.e. if $p$ is logically equivalent to $q$, it means that under every possible interpretation, $p$ and $q$ logically means the same thing. This is a statement in the metalanguage.

# Logical Substitution

Logical equivalance justifies *substitution* of one formula for another that is equivalent.

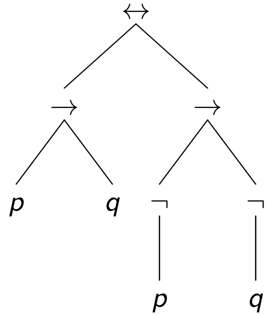
Let us present the intermediate steps first.

**Definition 2 (2.30)**

$A$ is subformula of $B$ if the formation tree for $A$ occurs as a subtree of the formation tree for $B$. $A$ is *proper* subformula of $B$ if $A$ is a subformula of $B$, but A is not identical to $B$.

## Example 1 (2.31)

The formula $(p \to q) \leftrightarrow (\neg p \to \neg q)$ contains the following proper subformulas: $p \to q, \neg p \to \neg q, \neg p, \neg q, p$ and $q$



**Definition 3 (2.32)**

If $A$ is a subformula of $B$, and $A'$ is an arbitrary formula, then $B'$, the *substitution* of $A'$ for $A$ in $B$, denoted $B\{A \leftarrow A'\}$, is the formula obtained by replacing all occurrences of the subtree for $A$ in $B$ by the tree for $A'$.

**Theorem 3 (2.34)**

*Let $A$ be a subformula of $B$ and let $A'$ be a formula such that $A \equiv A'$. Then $B \equiv B\{A \leftarrow A'\}$.*

Substitution can be naturally used to *simplify* formulas.

$$p \wedge (\neg p \vee q) \equiv (p \wedge \neg p) \vee (p \wedge q) \equiv \mathit{false} \vee (p \wedge q) \equiv p \wedge q$$

# Logical Equivalence

## Definition 4 (2.35)

A binary operator, $o$, is *defined from* a set of operators, $O = \{o_1, \ldots, o_n\}$ iff there is a logical equivalence $A_1 \; o \; A_2 \equiv A$ where $A$ is a formula constructed from occurrences of $A_1$, $A_2$, and operators in $O$.

Similarly, an unary operator $o$ is *defined from* a set of operators, $O = \{o_1, \ldots, o_n\}$ iff there is a logical equivalence $o \; A_1 \equiv A$ where $A$ is a formula constructed from occurrences of $A_1$, and operator $o$.

## Example 2

- $\leftrightarrow$ is defined from $\{\rightarrow, \wedge\}$ because $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$.
- $\rightarrow$ is defined from $\{\neg, \vee\}$ because $A \rightarrow B \equiv \neg A \vee B$.
- $\wedge$ is defined from $\{\neg, \vee\}$ because $A \wedge B \equiv \neg(\neg A \vee \neg B)$.

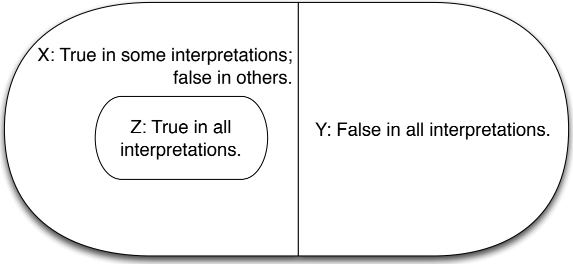
# Satisfiability, Validity, and Consequences

## Definition 5 (2.38)

- A propositional formula $A$ is *satisfiable* iff $\nu_{\mathscr{I}}(A) = T$ for *some* interpretation $\mathscr{I}$.
- A satisfying interpretation is called a *model* for $A$.
- $A$ is *valid*, denoted $\models A$, iff $\nu_{\mathscr{I}}(A) = T$ for *all* interpretation $\mathscr{I}$.
- A valid propositional formula is also called a *tautology*.
- $A$ is *unsatisfiable* if and only if it is not satisfiable, that is, if $\nu_{\mathscr{I}}(A) = F$ for *all* interpretations $\mathscr{I}$.
- $A$ is falsifiable, denoted $\not\models A$, if and only if it is not valid, that is, if $\nu_{\mathscr{I}}(A) = F$ for *some* interpretaion $\mathscr{I}$.

## Theorem 4 (2.39)

*A is valid iff $\neg A$ is unsatisfiable. A is satisfiable iff $\neg A$ is falsifiable.*




- $X$ (and, therefore, $Z$): Satisfiable.
- $Y$: Unsatisfiable.
- $Z$: Valid.
- $(X - Z) \bigcup Y$: Falsifiable (i.e. can be shown to be false).

**Definition 6 (2.40)**

Let $\mathscr{U} \subseteq \mathscr{F}$ be a set of formulas. An algorithm is a *decision procedure* for $\mathscr{U}$ if given an arbitrary formula $A \in \mathcal{F}$, it terminates and return the answer 'yes' if $A \in \mathscr{U}$ and the answer 'no' if $A \notin \mathscr{U}$.

By Theorem 2.39, a decision procedure for satisfiability can be used as a decision procedure for validity. Let $\mathcal{V}$ be the set of all satisfiable formulas. To decide the validity of $A$, we can apply the decision procesure for satisfiability of $\neg A$. This decision procedure is called a *refutation procedure*.

**Example 3**

Is $(p \to q) \leftrightarrow (\neg p \to \neg q)$ valid?

**Example 4**

$p \vee q$ is satisfiable but not valid.

# Logical Consequence

**Definition 7 (2.42)**

Extension of satisfiability from a single formula to a set of formulas: a set of formulas $U = A_1, \ldots, A_n$ is (*simultaneously*) *satisfiable* iff there exists an interpretation $\mathscr{I}$ such that $\nu_{\mathscr{I}}(A_1) = \ldots = \nu_{\mathscr{I}}(A_n) = T$. The satisfying interpretation is called a *model* of $U$. $U$ is *unsatisfiable* iff for every interpretation $\mathscr{I}$, there exists an $i$ such that $\nu_{\mathscr{I}}(A_i) = F$.

# Logical Consequence

**Definition 8 (2.48)**

Let $U$ be a set of formulas and $A$ a formula. $A$ is a *logical consequence* of $U$, denoted $U \models A$, iff every model of $U$ is a model of $A$.

**Theorem 5 (2.50)**

*$U \models A$ iff $A_1 \wedge A_2 \ldots \wedge A_n \rightarrow A$, where $U = \{A_1, \ldots, A_n\}$.*

If $U = \emptyset$, the logical consequence is the same as the validity.

# Theories

*Logical consequence* is the central concept in the foundations of mathematics; validity is often trivial and not very interesting. For example, Euclidean geometry is an extensive set of logical consequences, all deduced from the five axioms.

**Definition 9 (2.55)**

Let $\mathscr{T}$ be a set of formulas. $\mathscr{T}$ is *closed under logical consequence* iff for all formulas $A$, if $\mathscr{T} \models A$ then $A \in \mathscr{T}$. A set of formulas that is closed under logical consequence is a *theory*. The elements of $\mathscr{T}$ are theorems.

**Definition 10**

Let $\mathscr{T}$ be a theory. $\mathscr{T}$ is said to be *axiomatizable* iff there exists a set of formulas $U$ such that $\mathscr{T} = \{A | U \models A\}$. The set of formulas $U$ are the axioms of $\mathscr{T}$. If $U$ is finite, $\mathscr{T}$ is said to be *finitely axiomatizable*.

## Examples of Theory

| | $p$ | $q$ | $r$ | $p \vee q \vee r$ | $q \rightarrow r$ | $r \rightarrow p$ |
|---|---|---|---|---|---|---|
| $\mathscr{I}_1$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $\mathscr{I}_2$ | $T$ | $T$ | $F$ | $T$ | $F$ | $T$ |
| $\mathscr{I}_3$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $\mathscr{I}_4$ | $T$ | $F$ | $F$ | $T$ | $T$ | $T$ |
| $\mathscr{I}_5$ | $F$ | $T$ | $T$ | $T$ | $T$ | $F$ |
| $\mathscr{I}_6$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ |
| $\mathscr{I}_7$ | $F$ | $F$ | $T$ | $T$ | $T$ | $F$ |
| $\mathscr{I}_8$ | $T$ | $F$ | $F$ | $F$ | $T$ | $T$ |

- $U = \{p \vee q \vee r, q \rightarrow r, r \rightarrow p\}$
- Interpretation $\nu_1, \nu_3, \nu_4$ are models of $U$ (i.e. interpretations that make all formulas in $U$ true).
- Which of the following are true?
  1. $U \models p$
  2. $U \models q \rightarrow r$
  3. $U \models r \vee \neg q$
  4. $U \models p \wedge \neg q$

## Examples of Theory

| | $p$ | $q$ | $r$ | $p \vee q \vee r$ | $q \rightarrow r$ | $r \rightarrow p$ |
|---|---|---|---|---|---|---|
| $\nu_1$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $\nu_2$ | $T$ | $T$ | $F$ | $T$ | $F$ | $T$ |
| $\nu_3$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $\nu_4$ | $T$ | $F$ | $F$ | $T$ | $T$ | $T$ |
| $\nu_5$ | $F$ | $T$ | $T$ | $T$ | $T$ | $F$ |
| $\nu_6$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ |
| $\nu_7$ | $F$ | $F$ | $T$ | $T$ | $T$ | $F$ |
| $\nu_8$ | $T$ | $F$ | $F$ | $F$ | $T$ | $T$ |

Theory of $U = \{p \vee q \vee r, q \rightarrow r, r \rightarrow p\}$, i.e. $\mathcal{T}$ (U):

- $U \subseteq \mathcal{T}(U)$ because for all formula $A \in \mathcal{F}$, $A \models A$.
- $p \in \mathcal{T}(U)$ because $U \models p$.
- $(q \rightarrow r) \in \mathcal{T}(U)$ because $U \models (q \rightarrow r)$.
- $p \wedge (q \rightarrow r) \in \mathcal{T}(U)$ because $U \models p \wedge (q \rightarrow r)$.

**Theory of Euclidean Geometry** is based on the set of 5 axioms, $U = A_1, A_2, A_3, A_4, A_5$ such that:

- $A_1$: Any two points can be joined by a unique straight line.
- $A_2$: Any straight line segment can be extended indefinitely in a straight line.
- $A_3$: Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
- $A_4$: All right angles are congruent.
- $A_5$: For every line $l$ and for every point $P$ that does not lie on $l$, there exists a unique line $m$ through $P$ that is parallel to $l$.

The ancient Greeks suspected whether $A_5$ is a logical consequence of the other four. For about 2,000 years, various mathematicians tried to show $\{A_1, \ldots, A_4\} \models A_5$. Only in 1868, Beltrami showed that $A_5$ is independent from the rest. In other words, we accept $A_5$ as an axiom.

Beltrami also showed that non-Euclidean geometry (i.e. $U$ with $A_5$ replaced with alternatives) is *consistent*.