

Propositional Logic: Hilbert System, \mathcal{H}

CS402, Spring 2017

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Hilbert System, \mathcal{H}

Unlike \mathcal{G} , which deals with sets of formulas, \mathcal{H} is a deductive system for single formulas. In \mathcal{G} , there is one definition of axioms, and multiple rules. In \mathcal{H} , there are many axioms, but only one rule.

Definition 1 (3.9, Ben-Ari)

\mathcal{H} is a deductive system with three axiom schemes and one rule of inference. For any formulas, A, B and C , the following formulas are axioms:

- $\vdash (A \rightarrow (B \rightarrow A))$
- $\vdash ((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))$
- $\vdash (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$

\mathcal{H} uses *Modus Ponens* (MP) as the single inference rule:

$$\frac{\vdash A \quad \vdash A \rightarrow B}{\vdash B} \text{ MP}$$

Theorem 1 (3.10, Ben-Ari)

For any formula ϕ , $\phi \vdash \phi$.

Proof.

Think $A : \phi, B : \phi \rightarrow \phi, C : \phi$ when we refer to axiom schemes.

1. $\vdash (\phi \rightarrow ((\phi \rightarrow \phi) \rightarrow \phi)) \rightarrow ((\phi \rightarrow (\phi \rightarrow \phi) \rightarrow (\phi \rightarrow \phi)))$ Axiom 2
2. $\vdash \phi \rightarrow ((\phi \rightarrow \phi) \rightarrow \phi)$ Axiom 1
3. $\vdash (\phi \rightarrow (\phi \rightarrow \phi)) \rightarrow (\phi \rightarrow \phi)$ MP, 1, 2
4. $\vdash \phi \rightarrow (\phi \rightarrow \phi)$ Axiom 1 ($A, B : \phi$)
5. $\vdash \phi \rightarrow \phi$ MP, 3, 4



Note that $\{\rightarrow, \neg\}$ is an adequate set of operators, i.e. can replace all other binary operators through semantic equivalence.

However, for the sake of expressiveness, we introduce new rules of inference, called *derived rules*, to \mathcal{H} . We then use use derived rules to transform a proof into another (usually longer) proof, which uses just the original axioms and MP.

Consequently, derived rules should be proven to be *sound* with respect to \mathcal{H} . That is, the use of the derived rule does not increase the set of provable theorems in \mathcal{H} . That is, it should be possible to prove a derived rule of interest, without using itself.

Deduction Rule

Definition 2 (3.12)

Let U be a set of formulas, and A a formula. The notation $U \vdash A$ means that the formulas in U are *assumptions* in the proof of A . A *proof* is a sequence of lines $U_i \vdash \phi_i$, such that for each i , $U_i \subseteq U$, and ϕ_i is an axiom, a previously proved theorem, a member of U_i or can be derived by *MP* from previous lines $U'_i \vdash \phi'_i$, $U''_i \vdash \phi''_i$, where $i', i'' < i$.

Definition 3 (3.13)

Deduction rule: suppose you want to prove $A \rightarrow B$. First, we assume A , that is, treat A as if it is an additional axiom, in addition to the given ones, U . Then prove $U \cup \{A\} \vdash B$. This conclusion discharges our initial assumption A . That is, we have

$$\frac{U \cup \{A\} \vdash B}{U \vdash A \rightarrow B}$$

now proved that $A \rightarrow B$. In other words, $U \vdash A \rightarrow B$.

Soundness of the Deduction Rule in \mathcal{H}

Theorem 2 (3.14, Ben-Ari)

*The deduction rule is a **sound** derived rule.*

Proof.

We show, by induction on the length n of the proof of $U \cup A \vdash B$, how to obtain a proof of $U \vdash A \rightarrow B$ that does not use the deduction rule (i.e. show *soundness*).

For $n = 1$, B is proved in a single step. Consequently, B is either an element of $U \cup \{A\}$, an axiom in \mathcal{H} , or a previously proved theorem.

- If B is actually A , then $\vdash A \rightarrow A$ by Theorem 1, so naturally $U \vdash A \rightarrow A$.
- If $B \in U$ (i.e. B is a proven theorem), or B is an axiom, then $U \vdash B$. Then B is proved in a single application of *MP* as follows:

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|----|--|------------------|
| 1. | $U \vdash B$ | Axiom or Theorem |
| 2. | $U \vdash B \rightarrow (A \rightarrow B)$ | Axiom #1 |
| 3. | $U \vdash A \rightarrow B$ | MP, 1, 2 |

Soundness of the Deduction Rule in \mathcal{H}

Proof.

If $n > 1$, the last step in the proof of $U \cup \{A\} \vdash B$ is either a one-step inference of B or an inference of B using *MP*.

In the first case, the result holds by the proof for $n = 1$.

Otherwise, *MP* was used, so there is a formula C and lines $i, j < n$ in the proof such that line i in the proof is $U \cup \{A\} \vdash C$ and line j is $U \cup \{A\} \vdash C \rightarrow B$. By the inductive hypothesis, $U \vdash A \rightarrow C$ and $U \vdash A \rightarrow (C \rightarrow B)$. Based on these, the proof of $U \vdash A \rightarrow B$ is given by:

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|----|--|----------------------|
| 1. | $U \vdash A \rightarrow C$ | Inductive Hypothesis |
| 2. | $U \vdash A \rightarrow (C \rightarrow B)$ | Inductive Hypothesis |
| 3. | $U \vdash (A \rightarrow (C \rightarrow B)) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow B))$ | Axiom #2 |
| 4. | $U \vdash (A \rightarrow C) \rightarrow (A \rightarrow B)$ | MP, 2, 3 |
| 5. | $U \vdash A \rightarrow B$ | MP, 1, 4 |



Derived Rules in \mathcal{H}

Theorem 3 (3.16, Ben-Ari)

$$\vdash (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$$

Proof.

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|----|--|--------------|
| 1. | $\{A \rightarrow B, B \rightarrow C, A\} \vdash A$ | Assumption |
| 2. | $\{A \rightarrow B, B \rightarrow C, A\} \vdash A \rightarrow B$ | Assumption |
| 3. | $\{A \rightarrow B, B \rightarrow C, A\} \vdash B$ | MP, 1, 2 |
| 4. | $\{A \rightarrow B, B \rightarrow C, A\} \vdash B \rightarrow C$ | Assumption |
| 5. | $\{A \rightarrow B, B \rightarrow C, A\} \vdash C$ | MP, 3, 4 |
| 6. | $\{A \rightarrow B, B \rightarrow C\} \vdash A \rightarrow C$ | Deduction, 5 |
| 7. | $\{A \rightarrow B\} \vdash ((B \rightarrow C) \rightarrow (A \rightarrow C))$ | Deduction, 6 |
| 8. | $\vdash (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$ | Deduction, 7 |



Definition 4 (Rule of Transitivity)

$$\frac{U \vdash A \rightarrow B \quad U \vdash B \rightarrow C}{U \vdash A \rightarrow C}$$

Derived Rules in \mathcal{H}

Theorem 4

$$\vdash (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$$

Proof.

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|----|--|--------------|
| 1. | $\{A \rightarrow (B \rightarrow C), B, A\} \vdash A$ | Assumption |
| 2. | $\{A \rightarrow (B \rightarrow C), B, A\} \vdash A \rightarrow (B \rightarrow C)$ | Assumption |
| 3. | $\{A \rightarrow (B \rightarrow C), B, A\} \vdash B \rightarrow C$ | MP, 1, 2 |
| 4. | $\{A \rightarrow (B \rightarrow C), B, A\} \vdash B$ | Assumption |
| 5. | $\{A \rightarrow (B \rightarrow C), B, A\} \vdash C$ | MP, 4, 3 |
| 6. | $\{A \rightarrow (B \rightarrow C), B\} \vdash A \rightarrow C$ | Deduction, 5 |
| 7. | $\{A \rightarrow (B \rightarrow C)\} \vdash B \rightarrow (A \rightarrow C)$ | Deduction, 6 |
| 8. | $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$ | Deduction, 7 |



Definition 5 (Rule of Exchanged Antecedent)

$$\frac{A \rightarrow (B \rightarrow C)}{U \vdash B \rightarrow A \rightarrow C}$$

Theorems for other operators in \mathcal{H}

Theorem 5

$$\vdash A \rightarrow (B \rightarrow (A \wedge B))$$

Proof.

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|-----|---|---|
| 1. | $\{A, B\} \vdash (A \rightarrow \neg B) \rightarrow (A \rightarrow \neg B)$ | Theorem 1 |
| 2. | $\{A, B\} \vdash A \rightarrow ((A \rightarrow \neg B) \rightarrow \neg B)$ | Exchange of Antecedent |
| 3. | $\{A, B\} \vdash A$ | Assumption |
| 4. | $\{A, B\} \vdash (A \rightarrow \neg B) \rightarrow \neg B$ | MP, 3, 2 |
| 5. | $\{A, B\} \vdash \neg \neg B \rightarrow \neg(A \rightarrow \neg B)$ | Contrapositive |
| 6. | $\{A, B\} \vdash B$ | Assumption |
| 7. | $\{A, B\} \vdash \neg \neg B$ | Double Negation |
| 8. | $\{A, B\} \vdash \neg(A \rightarrow \neg B)$ | MP, 5, 7 |
| 9. | $\{A\} \vdash B \rightarrow \neg(A \rightarrow \neg B)$ | Deduction |
| 10. | $\vdash A \rightarrow (B \rightarrow \neg(A \rightarrow \neg B))$ | Deduction |
| 11. | $\vdash A \rightarrow (B \rightarrow (A \wedge B))$ | $A \wedge B \models \neg(A \rightarrow \neg B)$ |



Prove the following theorems in \mathcal{H} :

- $\vdash (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$ (Contraposition)
- $\vdash \neg\neg A \rightarrow A$ (Double Negation)
- given that $\vdash \text{true}$ and $\vdash \neg \text{false}$, prove $\vdash (\neg A \rightarrow \text{false}) \rightarrow A$ (Reductio Ad Absurdum)