# Propositional Logic: Deductive Proof & Natural Deduction Part 2

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Shin Yoo

|  | Introduction | Elimination |
|---|---|---|
| $\wedge$ | $\dfrac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge i$ | $\dfrac{\phi \wedge \psi}{\phi}\ \wedge e_1 \qquad \dfrac{\phi \wedge \psi}{\psi}\ \wedge e_2$ |
| $\vee$ | $\dfrac{\phi}{\phi \vee \psi}\ \vee i_1$ | $\dfrac{\psi}{\phi \vee \psi}\ \vee i_2 \qquad \dfrac{\phi \vee \psi \quad \overset{\phi}{\underset{\chi}{\vdots}} \quad \overset{\psi}{\underset{\chi}{\vdots}}}{\chi}\ \wedge e$ |
| $\rightarrow$ | $\dfrac{\overset{\phi}{\underset{\psi}{\vdots}}}{\phi \rightarrow \psi}\ \rightarrow_i$ | $\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}\ \rightarrow_e$ |

How do we know the validity of these rules?

Truth tables. In other words, $\bigwedge$ *premise* $\models$ *consequent*

# Basis for Inference Rules

How about the following?

|  | Introduction | Elimination |
|---|---|---|
| $\neg$ | $\phi$ <br> $\vdots$ <br> $\dfrac{\bot}{\neg\phi}$ $\neg i$ | $\dfrac{\neg\phi \quad \begin{matrix}\neg\phi\\\vdots\\\psi\end{matrix} \quad \begin{matrix}\neg\phi\\\vdots\\\neg\psi\end{matrix}}{\phi}$ $\neg e$ |

These cannot be justified by truth tables. Rather, these are justified by the Reductio Principle of propositional logic.

### Theorem 1 (Reductio Principle)

*Let $\Gamma$ be a set of formulas, $\phi$ and $\psi$ a formula. If $\Gamma \cup \{\varphi\} \vDash \psi$ and $\Gamma \cup \{\varphi\} \vDash \neg\psi$, then $\Gamma \vDash \neg\varphi$. If $\Gamma \cup \{\neg\varphi\} \vDash \psi$ and $\Gamma \cup \{\neg\varphi\} \vDash \neg\psi$, then $\Gamma \vDash \varphi$.*

A related question: prove that $\land$ and $\lor$ cannot define $\lnot$.

**Repetition:** since $\models P \rightarrow P$, we can derive an inference rule based on it.

$$\frac{\phi}{\phi} \ \text{Repetition}$$

For example, this rule can be used to prove $Q \rightarrow (P \rightarrow Q)$:

| 1. | $Q$ | Assumption |
|---|---|---|
| 2. | $P$ | Assumption |
| 3. | $Q$ | Repetition, 1 |
| 4. | $P \rightarrow Q$ | $\rightarrow_i$, 2-3 |
| 5. | $Q \rightarrow (P \rightarrow Q)$ | $\rightarrow_i$, 1-4 |

Now, introduction of double negation:

$$\frac{\phi}{\neg\neg\phi}$$

| 1. | $\phi$ | Assumption |
| 2. | $\neg\phi$ | Assumption |
| 3. | $\phi$ | Repetition, 1 |
| 4. | $\neg\neg\phi$ | $\neg_i$, 2-3 (i.e. $\neg\phi \rightarrow \neg\phi \wedge \neg\phi \rightarrow \phi$) |
| 5. | $\phi \rightarrow \neg\neg\phi$ | $\rightarrow_i$, 1-4 |

## On Derived Rules

Elimination of double negation:

$$\frac{\neg\neg\phi}{\phi}$$

| | | |
|---|---|---|
| 1. | $\neg\neg\phi$ | Assumption |
| 2. | $\neg\phi$ | Assumption |
| 3. | $\neg\neg\phi$ | Repetition |
| 4. | $\phi$ | $\neg_e$, 2-3 |
| 5. | $\neg\neg\phi \rightarrow \phi$ | $\rightarrow_i$, 1-4 |

Prove the validity of the following sequents.

- $(s \to p) \lor (t \to q) \vdash (s \to q) \lor (t \to p)$
- $\vdash (p \to q) \lor (q \to r)$