Introduction to Logic

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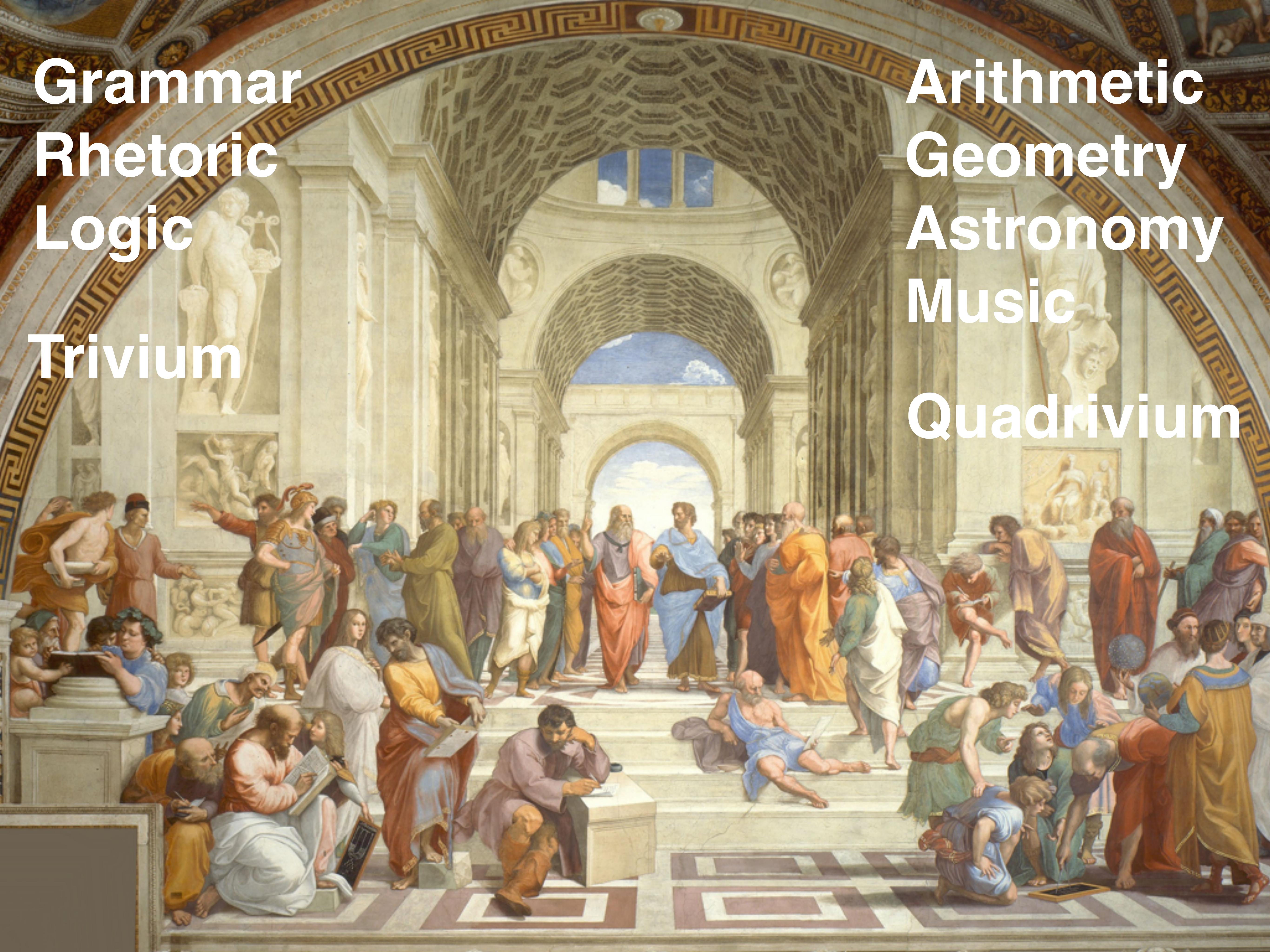
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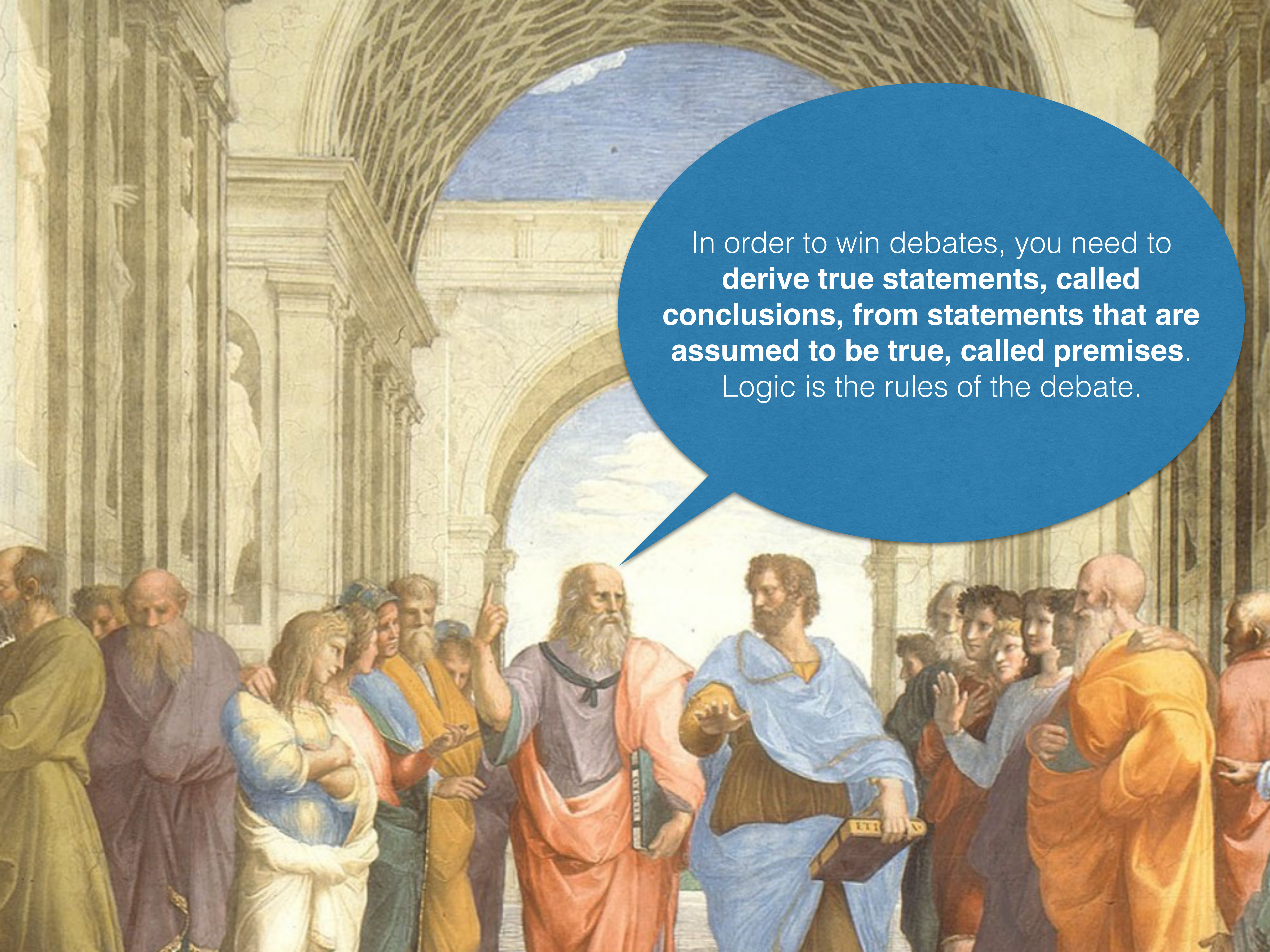
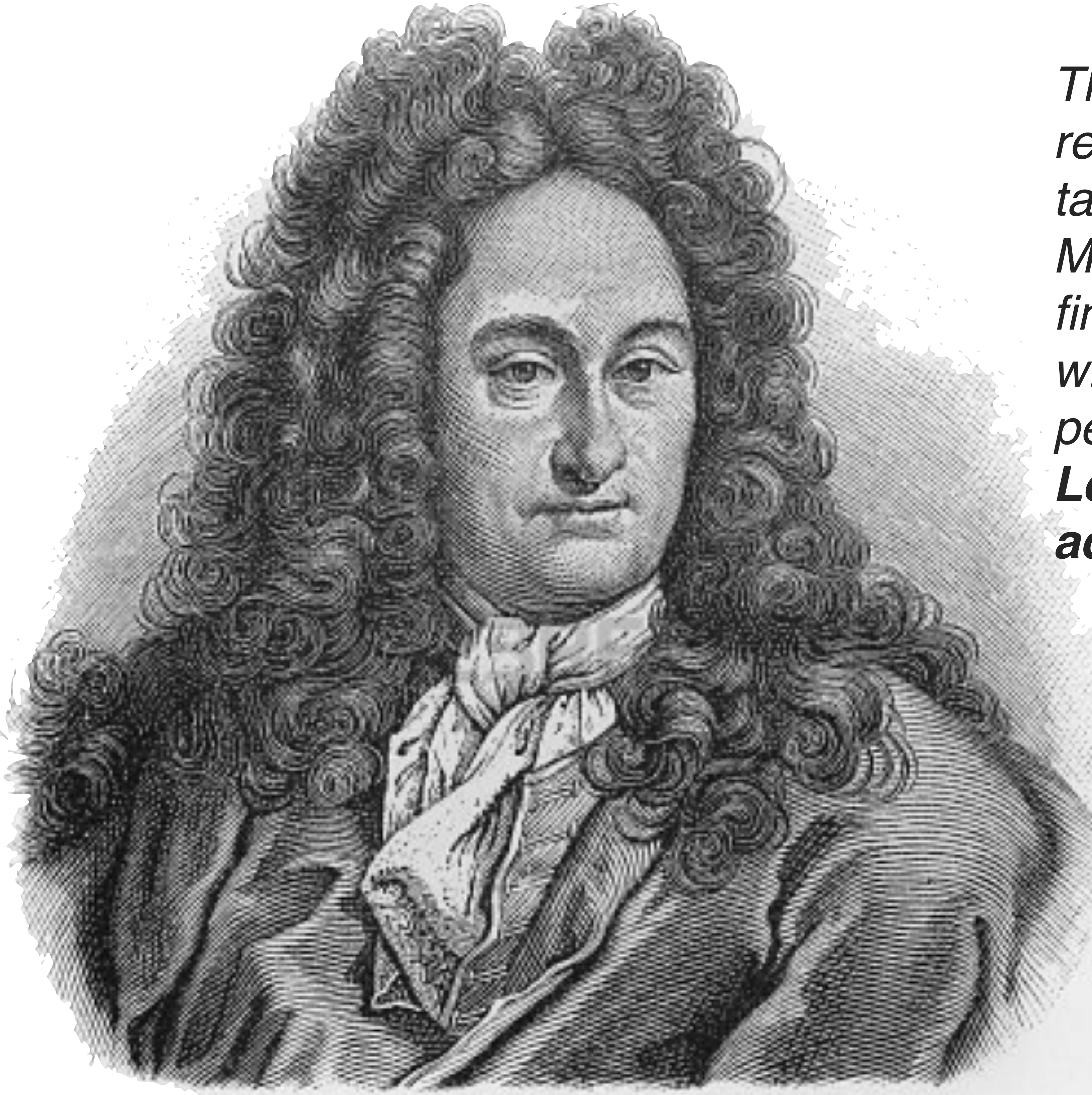

In order to win debates, you need to
derive true statements, called
conclusions, from statements that are
assumed to be true, called premises.
Logic is the rules of the debate.

Syllogism

- The most famous one:
    - All men must die.
    - Socrates is a man.
    - Therefore, Socrates must die (O).
- The Usual Suspects (1995, directed by Bryan Singer)
    - You walk with a slight limp.
    - We know that the criminal walks with a slight limp.
    - Therefore, you must be the criminal (X).

Leibniz: Symbolic and Formal Reasoning



Gottfried Wilhelm Leibniz.

The only way to rectify our reasonings is to make them as tangible as those of the Mathematicians, so that we can find our error at a glance, and when there are disputes among persons, we can simply say: ***Let us calculate, without further ado, to see who is right.***

Logic in 19th Century

- Logic was still in the realm of philosophy, not in mathematics and science, mainly because natural language was too ambiguous to deal with logic.

- With Boole and De Morgan, symbolic logic was born (1847); Frege and Pierce pushed the limit further near the end of 19th Century

  - Use of formal symbols (δ, ϕ, $\wedge$, $\vee$, $\neg$, etc)

  - Separation of syntactic representation and semantic interpretation

  - Construction of formal rules purely based on syntactic representations

Logic in 19th Century

- Near the end of the 19th century, there were strange signs in the world of mathematics, in the form of **paradoxes**.

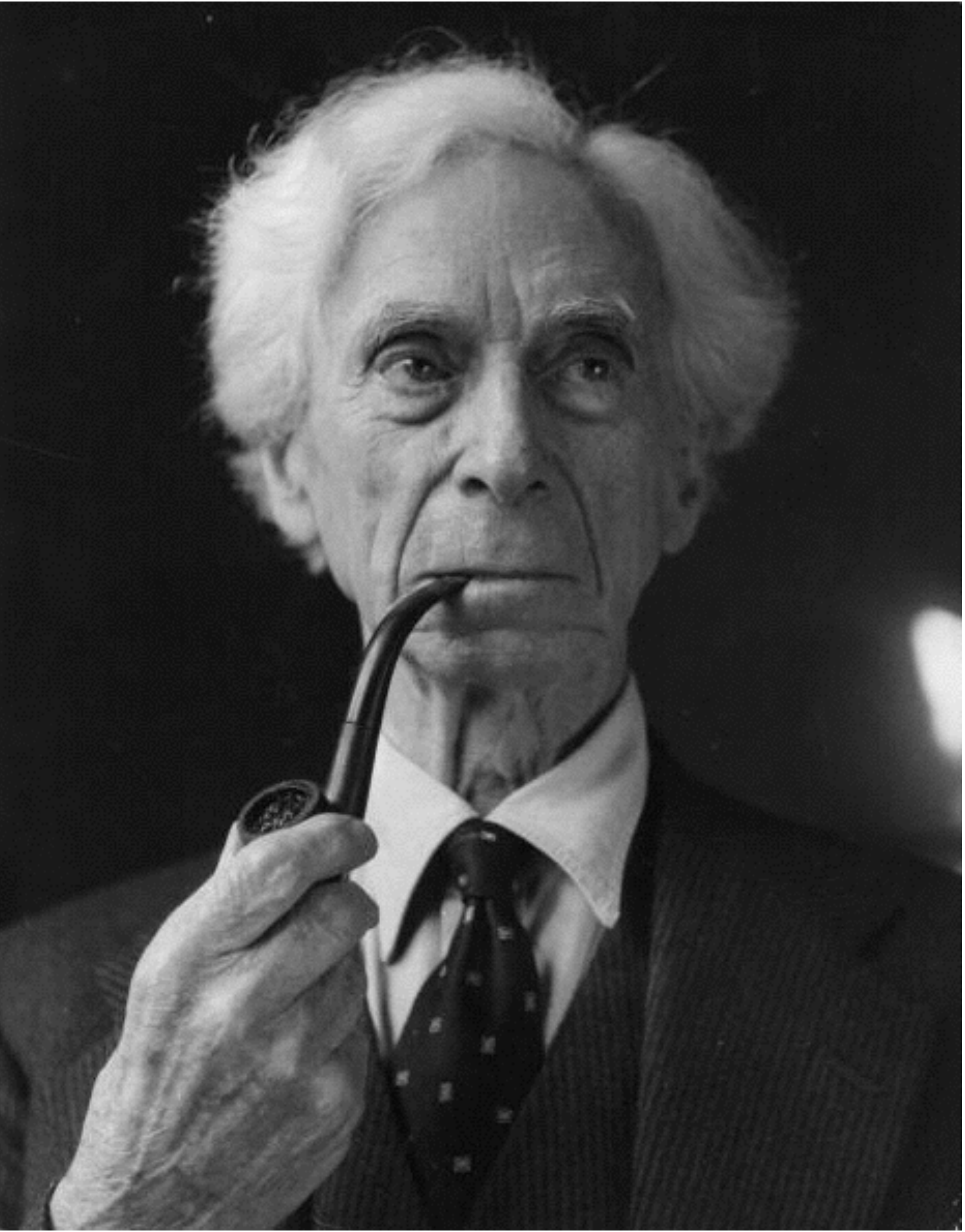
- Many of these have something to do with the meta-nature of sets, or recursive self-reference.

A Cretan philosopher says that all Cretan people only tell lies.

A Sophist is sued for his tuition by the school that educated him.

He argues that he must win, since, if he loses, the school didn't educate him well enough, and doesn't deserve the money.

The school argues that he must lose, since, if he wins, he was educated well enough and therefore should pay for it.



Russell's Paradox

Let A be the set of all sets X such that X is not a member of itself.

By definition, A is a member of A if and only if A is not a member of A.

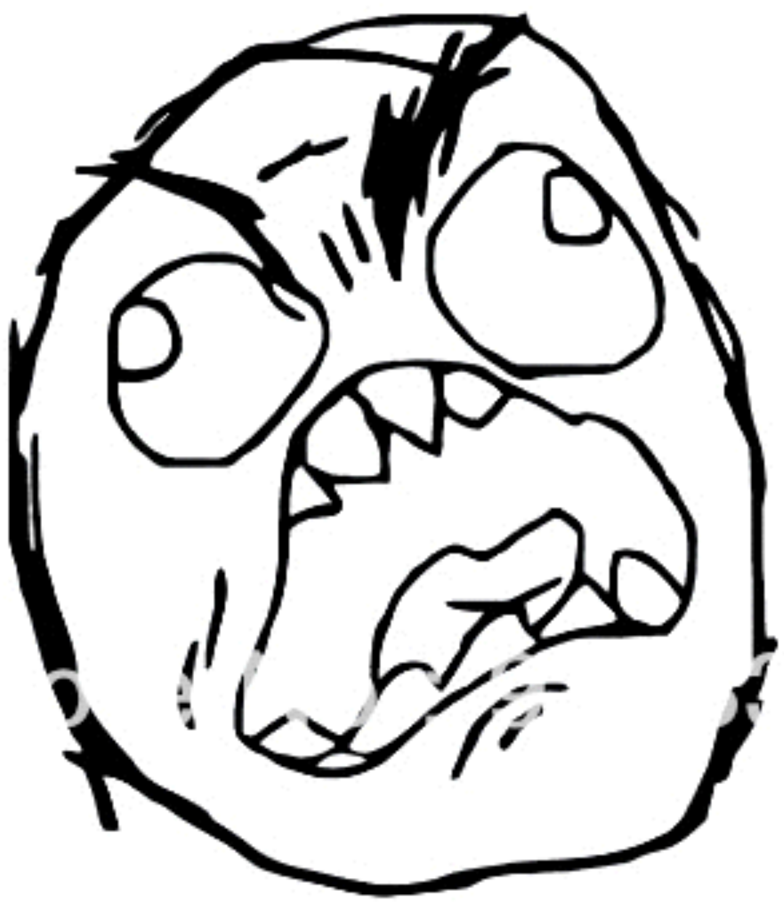
More formally,

$$A = \{X | X \notin X\}$$

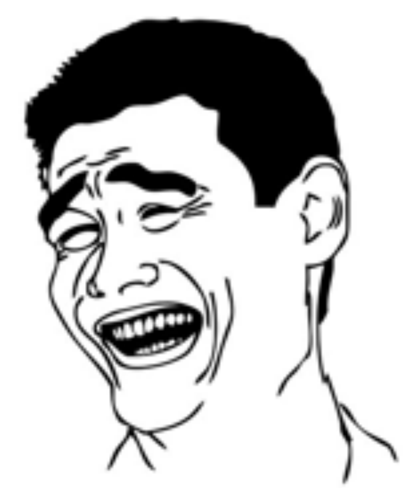
$$A \in A \leftrightarrow A \notin A$$

What the…?

Why is it even possible to write these things down using perfectly normal-looking maths?!?!!!



There must be something broken in maths - we need to fix it.



The Grand Aim: justify mathematical deduction by formalising a system of logic, <u>so that the set of derivable/provable statements is the same as the set of true statements</u>.

Hilbert Program

- David Hilbert, 1862-1943
- “Wir müssen wissen. Wir werden wissen.”

Hilbert Program

- **Formalism**: represent all mathematical statements in precise formal language.
- **Completeness**: all true mathematical statements can be proved using the formalism.
- **Consistency**: the formalism of the mathematics does not include any contradiction.
- **Conservation**: a proof that any result about “real objects” obtained by reasoning about “ideal objects (such as uncountable sets)” can be proved without using “ideal objects”
- **Decidability**: there should be an algorithm for deciding the truth of any mathematical statement.

Gödel’s Incompleteness Theorem

- On any computable axiomatic system that is powerful enough to describe the arithmetic of the natural numbers:
  - If the system is consistent, it cannot be complete.
  - The consistency of the axioms cannot be proven within the system.

Gödel's Incompleteness Theorem

- On any computable axiomatic system that is powerful enough to describe the arithmetic of the natural numbers:

  - A statement *G* can be formulated in the language of *S*, of which, by reasoning outside S, we can establish that it is true, whereas G is not provable in S.

  - The statement G of the first theorem is, in S, equivalent to the statement which expresses that S is consistent.

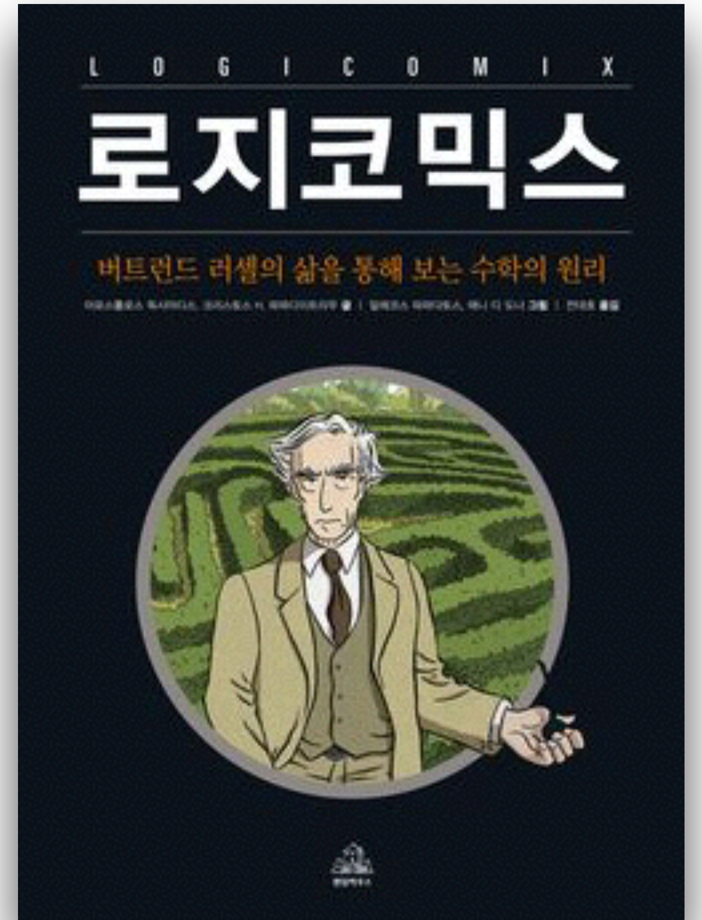
Was the Hilbert Program an wasted effort?

- Although it is not possible to formalise ALL mathematics, it is possible to do so for a large and practical subset of it.

- Although it is not possible to prove completeness for Peano Arithmetics, it is possible to prove for many interesting systems.

- We now know the boundaries of finitary proofs better.

- The decision problem led to recursion theory, metamathematics, and mathematical logic, which in turn led to Church-Turing thesis, which in turn led to the foundation of computer science.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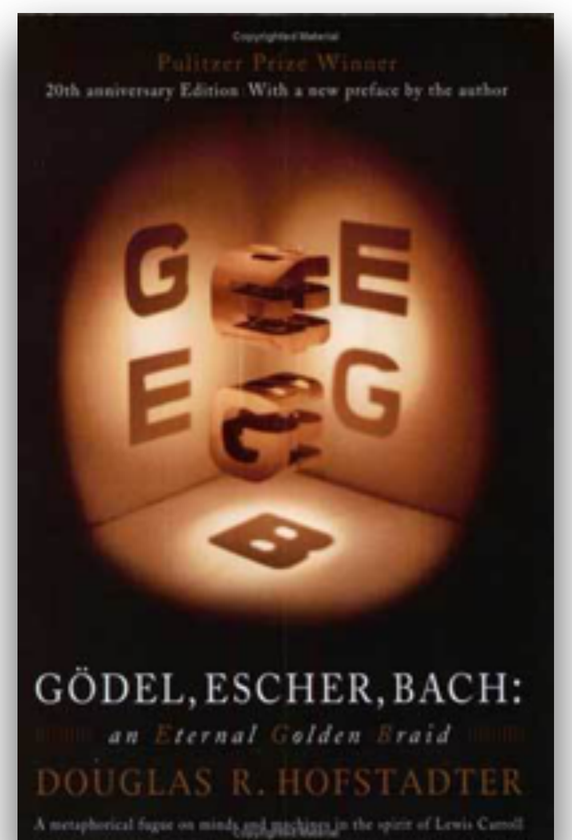
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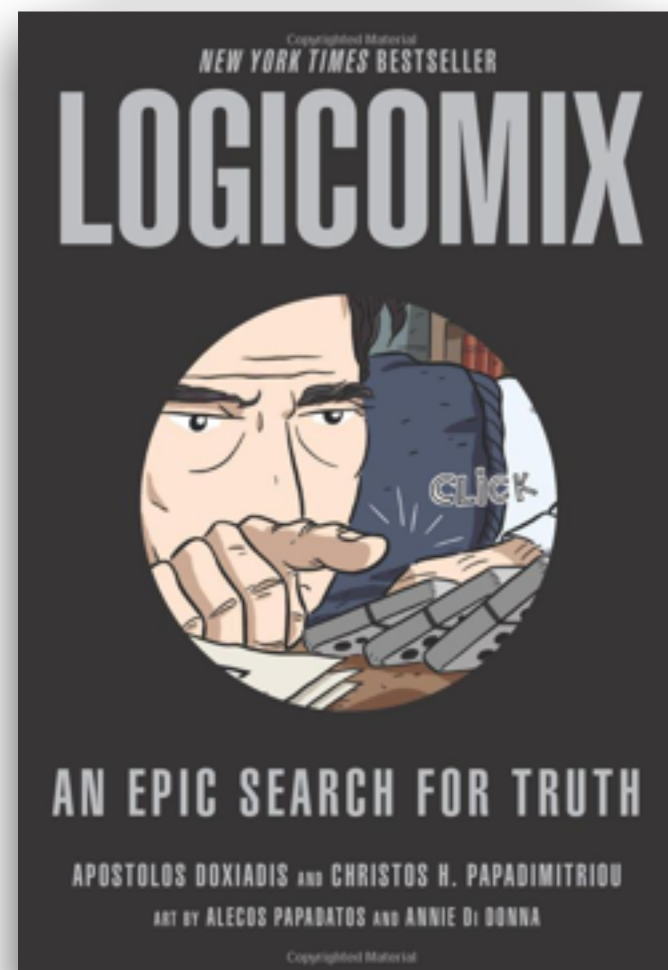
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“OK, but what’s the use of logic in computer science?”

Software Fault:

2002 NIST Estimated Economic Loss: $59.5B

One view about S/W faults

- Software is an abstract, logical system.
- Requirements for software can be specified clearly and precisely using formal logic.
- Then, we can prove that the given implementation satisfies the formal specification, i.e. we can **prove the absence of faults**.

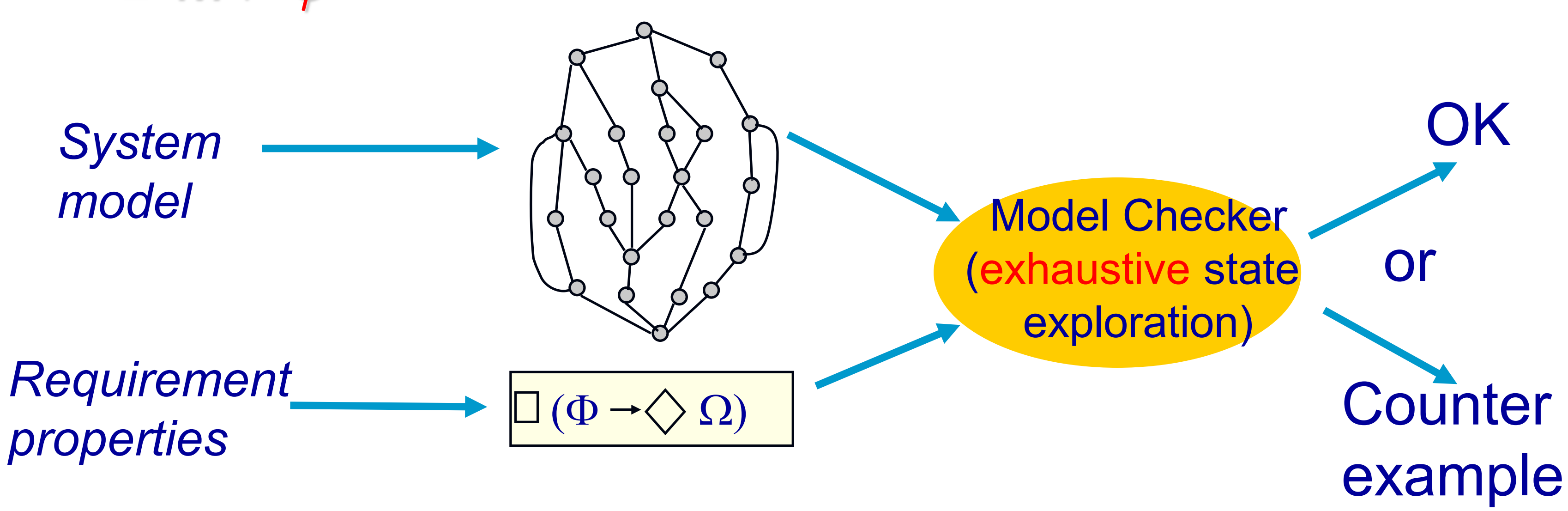
Case in Favour of Logic

- Natural language is often **ambiguous**, **inconsistent**, and **incomplete**.
  - A programmer was told: “Buy six bottles of milk. Oh, and, if there are eggs, buy 12”. The programmer returned home with 12 bottles of milk.
  - John eats an apple **and** Mary eats an orange.
  - John drove on **and** hit Mary.
  - The sun is shining **and** John feels happy.
  - John is a Korean **or** a computer scientist.
  - John is working **or** he is resting at home.

Case in Favour of Logic

Design Analysis

- We definitely want to guarantee/prove that a system design model M satisfies a requirement property ϕ
  - $M \models \phi$



Case against Logic

- Some requirements are harder to capture with formal logic. Games, for example?
- Some properties are impossible to capture with formal logic. Side-channel attacks, energy consumption, etc.
- Formal verification had a hard time scaling up.
- So far, it seems like people **do not like** to write specifications in formal logic.

Why not just use programming languages?

- It is **not impossible** to write requirement specifications in conventional PL. In fact, TDD says we should do so.
- But PL is about the low level details of the steps required to achieve a goal, not the goal itself. Carrying all the details makes it cumbersome to reason about the requirements using code.
- On the other hand, logic is designed for the job.

Hierarchy of Logical System

- **Propositional Logic**: also sentential logic. Atomic symbols are sentences (propositions). It studies whether propositions, that are formed by other propositions and logical connectives, are true or false.

- **Predicate Logic**: first, second, and higher order logic (informally refers to the first order logic). It introduces quantifier (all… some…) to variables (first order) or relations (second order).

Hierarchy of Logic

- **Modal Logic**: extends propositional/predicate logic by introducing modality to sentences. Instead of "John is happy", it can state "John is **usually** happy" and reason about "usually" (which is the modality here). The modality may concern aspects of sentences that are temporal (A is true **until** B), or deontic (It is **obligatory** that A is true), etc.

Terminology

- **Proposition**: simply means a sentence, formally stated. Can be true or false.
- **Axiom**: in classical logic, this is a statement that is so evident or well-established that we accept without controversy or question. In modern logic, the meaning of axiom being “true” becomes tricky.
- **Theorem**: a statement that has been proven on the basis of other theorems or axioms. Theorems are logical consequences of axioms.
- **Definition**: a statement of the meaning of a term. It can be intensional (try to give the essence of a new term) or extensional (try to enumerate every single object that a term describes).