GR-07

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-1 General Relativity Problems Chapter 7: Physics in a Curved Spacetime

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https://github.com/zachetienne/nrpytutorial/blob/master/Tutorial-Template_Style_Guide.ipynb

Link to the Style Guide. Not internal in case something breaks.

-1.1.1 NRPy+ Source Code for this module:

None!

-1.2 Introduction:

Now maybe we can apply what we've learned to some actual physical problems. Maybe. One can hope.

-1.3 Other (Optional):

Placeholder.

-1.3.1 Note on Notation:

Any new notation will be brought up in the notebook when it becomes relevant.

-1.3.2 Citations:

[1] (Link) (Placeholder)

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1 Problem 1 [Back to top]

Decide if the following sets are manifolds and say why. If there are any exceptional points at which the sets are not manifolds, give them:

a) Phase space of Hamiltonian mechanics, the space of the canonical coordinates and momenta p_i and q^i

b) Use this to establish 6.64

6.64: $\Gamma^{\alpha}_{\mu\nu,\sigma} = \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\mu,\nu\sigma} + g_{\beta\nu,\mu\sigma} - g_{\mu\nu,\beta\sigma} \right)$

Okay so we have, in general:

$$\nabla_{\beta} g_{\mu\nu} = g_{\mu\nu,\beta} - g_{\alpha\nu} \Gamma^{\alpha}_{\mu\beta} - g_{\mu\alpha} \Gamma^{\alpha}_{\mu\beta} = 0$$

and

$$\frac{1}{2} g^{\alpha\gamma} (g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}) = \Gamma^{\gamma}_{\beta\mu}$$

Adjust indeces and take the derivative.

$$\frac{1}{2} g^{\iota\alpha} (g_{\iota\mu,\nu} + g_{\iota\nu,\mu} - g_{\mu\nu,\iota}) = \Gamma^{\alpha}_{\mu\nu}$$

$$\frac{1}{2} \left[g^{\iota\alpha} (g_{\iota\mu,\nu} + g_{\iota\nu,\mu} - g_{\mu\nu,\iota}) \right]_{,\sigma} = \Gamma^{\alpha}_{\mu\nu,\sigma}$$

So what follows is simpler than it sounds. We use the product rule on the stuff in the middle. Now, if we are evaluating at P, then one might think the three terms in the middle go to zero. They would, if we weren't differentiating. But once we differentiate the g on the outside becomes a single derivative, and IT goes to zero, leaving only the other side of the product rule behind.

$$\frac{1}{2} g^{\iota\alpha} (g_{\iota\mu,\nu\sigma} + g_{\iota\nu,\mu\sigma} - g_{\mu\nu,\iota\sigma}) = \Gamma^{\alpha}_{\mu\nu,\sigma}$$

Which is equivalent to what we sought to show.

c) Fill in the steps needed to establish 6.68.

6.68 is the full tensor:

$$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda} R^{\lambda}_{\beta\mu\nu} = \frac{1}{2} (g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu})$$

From 6.63 we have

$$R^{\lambda}_{\beta\mu\nu} = \Gamma^{\gamma}_{\beta\nu,\mu} - \Gamma^{\gamma}_{\beta\mu,\nu} + \Gamma^{\gamma}_{\sigma\nu} \Gamma^{\sigma}_{\mu\beta} - \Gamma^{\gamma}_{\sigma\beta} \Gamma^{\sigma}_{\mu\nu}$$

$$\Gamma^{\alpha}_{\mu\nu,\sigma} = \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\mu,\nu\sigma} + g_{\beta\nu,\mu\sigma} - g_{\mu\nu,\beta\sigma} \right)$$

$$\frac{1}{2} g^{\alpha\gamma} (g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}) = \Gamma^{\gamma}_{\beta\mu}$$

This is clearly an exercise in algebra. So let's DO THIS!

$$R^{\lambda}_{\beta\mu\nu} = \frac{1}{2} g^{\gamma\iota} \left(g_{\iota\beta,\nu\mu} + g_{\iota\nu,\beta\mu} - g_{\beta\nu,\iota\mu} \right) - \frac{1}{2} g^{\gamma\iota} \left(g_{\iota\beta,\mu\nu} + g_{\iota\mu,\beta\nu} - g_{\beta\mu,\iota\nu} \right) + \frac{1}{2} g^{\alpha\gamma} (g_{\alpha\sigma,\nu} + g_{\alpha\nu,\sigma} - g_{\sigma\nu,\alpha}) \frac{1}{2} g^{\alpha\sigma} (g_{\alpha\mu,\beta} + g_{\alpha\beta,\mu} - g_{\mu\beta,\alpha}) - \frac{1}{2} g^{\alpha\gamma} (g_{\alpha\sigma,\beta} + g_{\alpha\beta,\sigma} - g_{\sigma\beta,\alpha}) \frac{1}{2} g^{\alpha\sigma} (g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,}$$

Remember derivatives can be taken in any order, so flipped bottom indeces will cancel.

$$R^{\lambda}_{\beta\mu\nu} = \frac{1}{2} g^{\gamma\iota} \left(g_{\iota\nu,\beta\mu} - g_{\iota\mu,\beta\nu} + g_{\beta\mu,\iota\nu} - g_{\beta\nu,\iota\mu} \right) + \frac{1}{2} g^{\alpha\gamma} (g_{\alpha\sigma,\nu} + g_{\alpha\nu,\sigma} - g_{\sigma\nu,\alpha}) \frac{1}{2} g^{\alpha\sigma} (g_{\alpha\mu,\beta} + g_{\alpha\beta,\mu} - g_{\mu\beta,\alpha}) - \frac{1}{2} g^{\alpha\gamma} (g_{\alpha\sigma,\beta} + g_{\alpha\beta,\sigma} - g_{\sigma\beta,\alpha}) \frac{1}{2} g^{\alpha\sigma} (g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha})$$

So obviously the leading term is what we want, which means all the other terms better cancel. But do they? Yes. Re-index $\beta\nu$ and they automatically cancel, becoming:

$$R^{\lambda}_{\beta\mu\nu} = \frac{1}{2} g^{\gamma\iota} \left(g_{\iota\nu,\beta\mu} - g_{\iota\mu,\beta\nu} + g_{\beta\mu,\iota\nu} - g_{\beta\nu,\iota\mu} \right)$$

Which if $\iota\alpha$ is adjusted, will cancel with the other multiplied metric, giving us the R we wanted to show at the start. Done.

2 Problem 18 [Back to top]

a) Derive 6.69 and 6.70 from 6.68

Annoying to type out; these are just the symmetries of the Modified Reimann Curvature Tensor.

6.69 points out that swapping the first two and last two indecies makes the result negative, while there is a true symmetry from swapping 13 and 24.

6.70 states that the tensor in form 1234 plus itself in form 1423 and 1342 produces zero. not sure why this is useful but it's true.

b) Show that 6.69 reduces the number of independent components of $R_{\alpha\beta\mu\nu}$ from 4x4x4x4 = 256 to 6x7/2 = 21. Hint: treat pairs of indices. Calculate how many independent choices of pairs there are for the first and second pairs of $R_{\alpha\beta\mu\nu}$

Okay so unlike our previous problems, this one's a 4D cube. Hard to imagine the 4D cube and how its symmetries work. The best thing we can note is that we have four numbers and four indeces: 0123.

Assumign only 6.69 is true, that is, the four-way identity relations, we find numbers that are equal or opposite.

First of all, we need to recognize that R is **antisymmetric** and as such the diagonals are all zero and irrelevant. This means 1111 2222 3333 and 4444 are all just zero. From 6 we also know that having even half of the components the same, such as in 0011, also equals zero. Why?

Well it becomes evident that we don't actually know what antisymmetry in higher dimensions MEANS. However, its clear that the relation 6.69 contains the information: if it transforms something into the negative of itself, it MUST be zero. Let's find all of these. There are a lot of them.

First of all, anything composed of same-pairs has to be zero. 00, 11, 22, 33, etc. This removes a total of sixteen numbers (including the ones where all four are the same).

However, what of things of the form xxyz? These and their transformations also all have to be zero, since one of the transforms in 6.69 shuffles only the first pair, so all other combinations therein must be lost.

xxyz, xxzy are automatically zero. So is yzxx but that's true due to another reason so we don't need to count it twice.

So every number which so much as contains a direct pair is toast. Counting only numbers with ONE pair in the first location, gives us six more zeroes for each pair, so six times four is 24. these are completely separate from the 16 before.

We also grab 24 from the second pair.

24+24+16 = 64. There are 64 zeroes in the matrix, which are simply not counted at all. This still leaves 192 potential values.

The cases of indeces with only one number in them are already proven to be zero.

What about cases with only two? Well, anything of xxxy, xyyy, xxyy form is automatically zero, but what of xyxy or xyyx?

xyxy = -yxxy = -xyyx = xyxy

These are perfectly safe, no self-equalizations here. But keep in mind each result of the identity can also be acted upon as well, meaning:

yxxy = -xyxy = -yxyx = xyyx yxyx = -xyyx = -yxxy = yxyx xyyx = -yxyx = -xyxy = yxxy

There are, in this shuffle, four numbers. So every unique combination of two different indeces will consume 4 indeces. There are six of these: 01, 02, 03, 12, 13, 23. So we have (+6) independent values (finally, we're not at zero!) and we reduce hte total number of elements we are considering by 24 again, leaving us 168.

The consideration of three indeces is complicated, four is actually much easier.

(Forgive us for not using zero here, we wrote this part first)

So all posisble combinations of four different numbers will be divided by 4... 4 times 3 times 2 times 1 divided by 4, or just 6 independent values. Or that's what you'd THINK, but no, each of the values here can also twist into even more values with a sort of cascading effect of crazyness! We can apply the cycle again and again and again, oh boy! So instead of four numbers, we get quite a bit more. Start with 1234, 2134, 1243, 3412... and then apply it all to 2134! (and then the others) see how many we get!

1234, 2134, 1243, 3412

2134, **1234**, 2143, 4321

1243, **2143**, **1234**, 4312

3412, **4312**, 3421, **1234**

And the last one only produces things that already exist. But wait, NOW we have to check the new numbers and see what they produce! But that's it, there are only 8. This could even have been reasoned out: we either swap the first two, swap the last two, or swap the locations of the first and last two, of course there are only eight combinations.

For the 4-number combinations, we have 4 times 3 times 2 times 1 divided by 8 is just 3. (+3) to the total, giving us 9 total, and now we only have 144 values left to consider.

Which now... is only the three-index-case.

We already know that all dual-indeces are zero, so we only concern ourselves with non-dual-indeces:, which by nature have to take the forms xyyz, xyzy, yxyz, yxzy. We suspect that each individual one is goign to give us all four options, but let's try it out:

xyyz = -yxyz = -xyzy = yzxy yxyz = -xyyz = -yxzy = yzyx xyzy = -yxzy = -xyyz = zyxy yzxy = -zyxy = -yzyx = xyyz yxzy = -xyzy = -yxyz = zyyx yzyx = -zyyx = -yzxy = yxyz zyxy = -yzxy = -zyyx = xyzy zyyx = -yzyx = -zyxy = yxzy

Each unique set of three will provide *eight* combined indeces. So what are the unique combinations of 3? We have to be careful here, since 1223 and 1332 are distinct, but 1223 and 3221 are not. So let's just write them all out:

0112 0113 2113 0221 0223 1223 0331 0332 1332 1002 1003 2003

So there are (+12). 12 + 9 = 21. Which is exactly the number we wanted!

Let's go ahead and list "representative" values from all 21:

2 combos: 0101 0202 0303 1212 1313 2323

3 combos: 0112 0113 2113 0221 0223 1223 0331 0332 1332 1002 1003 2003

4 combos: 0123 0231 0132

c) Show that 6.70 imposes only one further relation independent of 6.69 on teh components, reducing the total of independent ones to 20.

In the notation we've been using to be fast at typing, 6.70 correlates:

xyzw + xwyz + xzwy = 0

First of all this can't possibly relate numbers together that don't share numbers. All our 2-combos are completely unique, so they can't be related to each other this way. Also even though they can be shuffled into 0011 situations, this is perfectly fine–this does not demand that they be zero because of this, just that one be the reverse of the other!

In the case of our 2-patterns, we get xyyx + xxyy + xyxy = 0 and while the middle one is zero the last is the negative of the first. This will apply to all of them.

3-combos are the same way, but it's not as obvious. However, shuffling indeces around, no matter how it's done, can't change the number of each type of index, keeping them all firmly planetd within their "classes". The three-way sum doesn't change anythign either, as they take the same "shapes" as the 2-combos.

So now we go to the 4 combos, and there *are* no zeroes hanging around the 4 combos, and they certainly *can* be shuffled outside their group. That would mean all 4-combos are related, right, making the independent indeces 19 rather than 20? No! The relation of 6.70 is a + b + c = 0. To find one of them, TWO of the others must be defined. It's like a function f(x,y). Two indepenent parameters are inserted to find the third. So we areduce from 21… to 20.

3 Problem 19 [Back to top]

Prove that $R^{\alpha}_{\beta\mu\nu} = 0$ for polar coordinates in the Euclidean plane. Use 5.45 or equivalent results.

5.45 just gives the Christoffel symbols. Not writing them down again, but 1/r, 1/r, and -r are the only three solid values.

The metric has a diagonal of 1, r^2. The inverse 1, $1/r^2$.

$\Gamma^{\alpha}_{\mu\nu,\sigma} = \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\mu,\nu\sigma} + g_{\beta\nu,\mu\sigma} - g_{\mu\nu,\beta\sigma} \right)$

$$R^{\lambda}_{\beta\mu\nu} = \Gamma^{\gamma}_{\beta\nu,\mu} - \Gamma^{\gamma}_{\beta\mu,\nu} + \Gamma^{\gamma}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\gamma}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}$$

Thus we combine everything together to get the result. (this was originally indexed wrong so above versions may also be indexed wrong.)

R is 2x2x2x2 which means we'll have a grand total of 16 elements here. ALL of them need to be zero. Regardless it should be obvious how to carry out the problem now, it'll just take this thing called TIME.

Let's handle the last terms first since we already know the Christoffel numbers for the non-derivatives.

There are only five resulting terms that exist, and they exist at different "times", never does the sum have more than one component in it. Represent θ by t since we're going to be typing it out a lot. trt = 1/r ttr = 1/r rtt = -r

The combinations require middle term to match the first term of the second.

(rtt)(trt) = -1 (trt)(rtt) = -1 (rtt)(ttr) = -1 (ttr)(trt) = $1/r^2$ (ttr)(ttr) = $1/r^2$

The last term in R's definition is closely related. the only difference is that it swaps the last indeces.

(rtt)(trt) = -1 (trt)(rtt) = -1 (rtr)(ttt) = 0 (!) (ttt)(trr) = 0 (!) (ttr)(ttr) = $1/r^2$

This reveals *two cases* where they simply don't cancel. In terms of R, these cases are:

R(rttr) with -1, and R(trrt) with $1/r^2$. There are also their converses in R(rtrt) with $+1$ and R(trtr) with $-1/r^2$. With luck these will cancel with the derivative terms.

Now while we originally tried to find the derivatives of hte coefficients with respect to the equation, we realized something rather basic. We have the coefficients. Just take the derivatives directly. -r becomes -1, 1/r becomes $-1/r^2$, and only when the derivative is taken with respect to r. This results in three separate places where the derivative coefficients are not zero.

At one, all of the coefficients are ttr,r, and ttr,r - ttr,r is just zero, so they cancel. (uh oh...)

At the others, we have trt,r standing alone in two cases, and then rtt,r standing alone in two cases. R(trtr) has $1/r^2$, R(trrt) has $-1/r^2$. Then we have R(rttr) with 1, and R(rtrt) with -1.

You will note that when the location of R matches, the actual result is opposite for both of what we've calculated. Which means EVERYTHING IS ZERO! WOOHOO YEAH!

4 Problem 20 [Back to top]

Fill in the algebra necessary to establish 6.73

6.73: $\nabla_\alpha \nabla_\beta V^\mu = V^\mu_{,\beta\alpha} + \Gamma^\mu_{\nu\beta,\alpha} V^\nu$

okay so this isn't as simple as it looks. The previous step, 6.72, provides:

$$\nabla_\alpha \nabla_\beta V^\mu = \nabla_\alpha (V^\mu_{;\beta}) = (V^\mu_{;\beta})_{,\alpha} + \Gamma^\mu_{\sigma\alpha} V^\sigma_{;\beta} + \Gamma^\sigma_{\beta\alpha} V^\mu_{;\sigma}$$

But we're evaluating this at the point P so any actual Γ are zero. Which means we just have...

$$= (V^\mu_{;\beta})_{,\alpha}$$

Which we can expand into:

$$= V^{\mu}_{,\beta\alpha} + \Gamma^{m}u_{\nu\beta}V^{\mu}_{,\alpha} + \Gamma^{m}u_{\nu\beta,\alpha}V^{\mu}$$
$$= V^{\mu}_{,\beta\alpha} + \Gamma^{m}u_{\nu\beta,\alpha}V^{\mu}$$

And hey look that's what we wanted.

5 Problem 21 [Back to top]

Consider the sentences following 6.78. Why does the argument in parenthesis not aply to the signs in:

$$V^{\alpha}_{;\beta} = V^{\alpha}_{,\beta} + \Gamma^{\alpha}_{\mu\beta}V^{\mu};; V_{\alpha;\beta} = V_{\alpha,\beta} - \Gamma^{\mu}_{\alpha\beta}V_{\mu}$$

6.78 it essentially pointing out that the double covariant derivative acting on a $\binom{1}{1}$ tensor produces components of that tensor acted upon by the R-tensor, but always with positive sign. We note above that the relation there shows that, no, it does not have to be positive in the case of a total derivative acting on a vector (or one-form). the one-form version produces a minus sign.

The argument in parentehsis is "They must all have the same sign because rasiing and lowering indices with g is unaffected by ∇_{α} since $\nabla g = 0$."

It seems as though the reason this doesn't work is because we're not applying a coefficient to every term, like we are in 6.78. The partial derivative still remains, and its not multiplied by any tensor or tensor-like component.

In fact we recall from earlier that the Christoffel coefficietns aren't even a tensor at *all*, so the entire thing falls apart as the term with the sign is not invariant (although the total is.)

6 Problem 22 [Back to top]

Fill in the algebra necessary to establish 6.84, 6.85, and 6.86

Right, rather than trying to funble our way through this, since this problem is basically asking us to work out the details of "non-paralellism", we're going to start from teh beginning, which is 6.79, and derive *every* part of it since as of starting this problem we are not sure what exactly we did to get here.

We start by considering two geodesics, let htem be V and V' so their tangents are $\vec{V}$ and $\vec{V}'$. They start paralell to each other, at points A and A'. Let the affine parameter be λ.

Let there be another vector $\vec{\xi}$ which connects the two geodesics at a single value of the parameter λ. Think of it as a line straddling the distance between the two geodesics. At the 'start' it connects A and A', and will move on to connect B and B', etcetera.

To make this a simple version, we create a local inertial coordiante system at A itself, and also say that the coordinate x^0 points along geodesic V and that the coordinate is scaled to match the progression λ. One case of this would be traveling in the x direction on the cartesian plane at one unit distance per unit time. We have $V^{\alpha} = dx^{\alpha}/d\lambda$ which is just the definition of the tangent vector, but because of the way we've defined everything we have made at A the vector have components $V^{\alpha} = \delta^{\alpha}_{0}$ which is to say we are pointing directly in the (1,0,0,…) direction.

Then we note the equation of the geodesic at A is \$ $\frac{\mathrm{d}^2x^\alpha}{d\lambda^2|\_A=0\text{\$.Butbeforewemoveon,let'sseewherethiscomesfrom.Theactualdefinitionofthegeodesicis}}$

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\beta}\frac{x^\mu}{d\lambda}\frac{x^\beta}{d\lambda} = 0$$

.

However as we're evaluating this at a specific point A, the Christoffel symbols vanish as they are all zero, thus our geodesic is correct. AT A.

At A' we have a different story, the symbols most certainly do not vanish.

Fortunately for us it does simplify quite a bit, as we defined the vector V to be (1,0,0,...) and since the geodesics start OUT tangent, this is $\vec{V}'$ as well. This means that only the x^0 components evaluate, leaving a single Christoffel symbol.

$$\frac{d^2x^\alpha}{d\lambda^2}|_{A'} + \Gamma^\alpha_{00}(A') = 0$$

.

Where the "of A"' portion reminds us that the Christoffel symbols change based on what point they're at.

The next step is a minor approximation. We don't know what Γ is at A', but we do know A' is separated from A by our connecting vector $\vec{\xi}$. The approximation we make is that we assume the derivative is mostly constnat over the path between A and A' which, if the separation is small, is a pretty good approximation. This is, in math:

$$\Gamma^\alpha_{00}(A') \approx \Gamma^\alpha_{00,\beta}\xi^\beta$$

Note that while the Christoff symbol at A is zero, its derivative is not (necessarily), which is why we can make the above statement. Also note that there is a sum here over all directions that the connecting vector points. Ideally we start the situaiton in such a way that there's only one term, though.

We now have at A'...

$$\frac{d^2x^\alpha}{d\lambda^2}|_{A'} = -\Gamma^\alpha_{00,\beta}\xi^\beta$$

.

Now we can combine our A and A' equations to say:

$$\frac{d^2x^\alpha}{d\lambda^2}|_{A'} - \frac{d^2x^\alpha}{d\lambda^2}|_{A} = -\Gamma^\alpha_{00,\beta}\xi^\beta$$

.

Curiuosly, since A and A' are separated by $\vec{\xi}$ we can also say

$$\frac{d^2x^\alpha}{d\lambda^2}|_{A'} - \frac{d^2x^\alpha}{d\lambda^2}|_{A} = \frac{d^2\xi^\alpha}{d\lambda^2}$$

.

Which combines to

$$-\Gamma^{\alpha}_{00,\beta}\xi^{\beta} = \frac{d^2\xi^{\alpha}}{d\lambda^2}$$

.

Which gives us a more explicit description of how $\vec{\xi}$ changes.

Now we go grab another equation: 6.48. Which states:

$$U^{\beta}V^{\alpha}_{;\beta} = 0 \Leftrightarrow \frac{d}{d\lambda}\vec{V} = \nabla_{\vec{U}}\vec{V} = 0$$

Our ultimate goal here is the full second covariant derivative of ξ. Using the above, we can actually write out

$$\nabla_V \nabla_V \xi^{\alpha} = \nabla_V(\nabla_V \xi^{\alpha}) = \frac{d}{d\lambda}(\nabla_V \xi^{\alpha})$$

But this really doesn't seem right, if we extract it as we know we should using $V^{\alpha}_{;\beta} = V^{\alpha}_{,\beta} + \Gamma^{\alpha}_{\mu\beta}V^{\mu}$, we get:

$$= \frac{d}{d\lambda}(\nabla_V \xi^{\alpha}) + \Gamma^{\alpha}_{\beta 0}(\nabla_V \xi^{\beta})$$

IF this turns out well, this means 6.84 has a typo, an $=$ instead of a $+$. Now we can expand each internal term as well...

$$= \frac{d}{d\lambda}(\frac{d}{d\lambda}\xi^{\alpha} + \Gamma^{\alpha}_{\beta 0}\xi^{\beta}) + \Gamma^{\alpha}_{\beta 0}(\frac{d}{d\lambda}\xi^{\alpha} + \Gamma^{\alpha}_{\beta 0}\xi^{\beta})$$

That term on the right is 0 since it's a Christoffel symbol without a derivative being taken, and we are working in the A frame.

$$= \frac{d}{d\lambda}(\frac{d}{d\lambda}\xi^{\alpha} + \Gamma^{\alpha}_{\beta 0}\xi^{\beta})$$

Note that since we set the parameter to be in the same direction as x^0 we can adjust notation for it as "0"

$$= \frac{d^2}{d\lambda^2}\xi^{\alpha} + \Gamma^{\alpha}_{\beta 0,0}\xi^{\beta} + \Gamma^{\alpha}_{\beta 0}\xi^{\beta}_{,0}$$

$$= \frac{d^2}{d\lambda^2}\xi^{\alpha} + \Gamma^{\alpha}_{\beta 0,0}\xi^{\beta}$$

This is 6.85. Which means that yes there was a typo up there. Hooo boy.

Anyway we can finally substitute in our second derivative to get:

$$\begin{aligned} &= -\Gamma^{\alpha}_{00,\beta}\xi^{\beta} + \Gamma^{\alpha}_{\beta 0,0}\xi^{\beta} \\ &= (\Gamma^{\alpha}_{\beta 0,0} - \Gamma^{\alpha}_{00,\beta})\xi^{\beta} \end{aligned}$$

now the problem claims that this equals $R^{\alpha}_{00\beta}$. While it does you have to take the first two indeces as fungible to make it work. The derivative coefficients remain, the non-derivative ones cancel each other out as they are symmetric across our little point here.

$$= R^{\alpha}_{00\beta}\xi^{\beta}$$
$$= R^{\alpha}_{\mu\nu\beta}V^{\mu}V^{\nu}\xi^{\beta}$$

Okay this last step needs some jusitifcation. Why does acting on our vectors reduce their indeces to zero? Well, after all, we've excessively defined V in terms of the point A, going in (1,0,0,0,...) so there WERE no other indeces. So yeah going that way checks out.

The question now is why we can go from 00 to genral indeces and the equation isn't messed up... but think of it this way. We could ALWAYS define V to be in a singular direction, so naturally it has to hold in general as well.

And now... we are done.

Ooogh. What does this even SAY? Well, since R is zero in a flat space, paralell lines remain parallel. However, R on a non-flat space have their separation values change, as R scales it somehow.

And that's all this needed to show.

7 Problem 23 [Back to top]

Prove 6.88 (oh no.) *Be careful: one cannot simply differentiate 6.67 since it is valid only at P, not in the neighborhood of P.*

6.88: $R_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2}(g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda})$

Oh BOY.

Anyway, we are proving that the partial derivative of R is is the derivative of all the g components that make it up. However, if we back way way up, we note that all the equations we've been using so far are only valid at P, since we've been removing the extra coefficients. So we need to back up to:

$$R^{\lambda}_{\beta\mu\nu} = \Gamma^{\gamma}_{\beta\nu,\mu} - \Gamma^{\gamma}_{\beta\mu,\nu} + \Gamma^{\gamma}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\gamma}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}$$

We can see obviously from the mentioned 6.67 and 6.68 that the already-differentiated terms will become exactly the relation we want. It's the terms we ignored at P that we need to prove cancel, that is:

$$\Gamma^{\gamma}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\gamma}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}$$

These guys.

Which we then apply the product rule to to actually take their derivatives:

$$\Gamma^{\gamma}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu,\lambda} + \Gamma^{\gamma}_{\sigma\mu,\lambda}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\gamma}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu,\lambda} - \Gamma^{\gamma}_{\sigma\nu,\lambda}\Gamma^{\sigma}_{\beta\mu}$$

Unfortunately it's not obvioust that they cancel, which means we'll have to *expand* them.

$$\Gamma^{\alpha}_{\mu\nu,\sigma} = \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\mu,\nu\sigma} + g_{\beta\nu,\mu\sigma} - g_{\mu\nu,\beta\sigma} \right)$$

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma} (g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha})$$

At which point we went "we're not tpying that up" and wrote down the substitutions in a notebook. Index salad, index salad...

First we noted the leading terms are all $1/4$ inverse-g. The inverse-g terms combined into two distinct types, meaning that the first and third could add, as could the second and fourth. Meaning that if one of them canceled, both of them would cancel. Furthermore, the only swap between the pairs was a $\mu\nu$ swap, so all we needed to find was the result for ONE of the terms. This still wasn't exactly trivial.

Work implies that nothing cancels. Then we realize that the two "different" leading coefficients we found really are not, since the metric is by definition SYMMETRIC. So really all of them can add together as they wish.

We showed (on the paper) that the pairs 13 and 24 did not cancel. HOWEVER, it is clear from the coefficietns $\mu\nu$ swap AND the term the derivative acts on swaps, then everything is in fact equal and cancels.

Here's an example term: $g_{\alpha\sigma,\mu} g_{\iota\beta,\nu\lambda}$. If we perform the swap we end up with $g_{\alpha\sigma,\nu\lambda} g_{\iota\beta,\nu}$ which actually IS the same because by index salading we can set all the dummy indeces to anything.

Note: wouldn't this change R? It seems like for an arbitrary R these wouldn't line up perfectly. Ah, but let's look at which indeces are being adjusted: $\alpha\iota$ are summation indeces, they have no relevance on R. Which leaves σ and β. The thing is, by swapping it on one term, we have to swap it on all terms, which makes them fly past each other except when $\sigma = \beta$, which can't always be true as β only takes one value on any given sum.

And so we are at a loss but think that maybe spending more time isn't worth it.

HOLD ON A SECOND.

WE'RE STUPID.

We're still evaluating at point P. Which means, while we needed to open up the Christoffel symbols to take their derivatives, WE ARE STILL IN P. EVERY ONE OF THOSE TERMS IS ZERO. EVERY ONE OF THEM.

AGH.

8 Problem 24 [Back to top]

Establish 6.89 from 6.88

6.88 is what we were yelling at in the previous problem.

$R_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2}(g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda})$

Now we wish to show that $R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0$ Which, since we are really tired of typing out all these greek letters, we shall just represent as R(1234,5) + R(1253,4) + R(1245,3) = 0.

Expanding we don't care about the fractional coefficient. We end up with:

g(14,235) - g(13,245) + g(23,145) - g(24,135)

+g(13,254) - g(15,234) + g(25,134) - g(23,154)

+g(15,243) - g(14,253) + g(24,153) - g(25,143)

Now we can re-arrange on eitehr side of the comma. Ascending order seems reasonable.

g(14,235) - g(13,245) + g(23,145) - g(24,135)

+g(13,245) - g(15,234) + g(25,134) - g(23,145)

+g(15,234) - g(14,235) + g(24,135) - g(25,134)

Now, does EVERYTHING cancel?

YES! And we're done.

9 Problem 25 [Back to top]

a) Prove that the Ricci tensor is the only independent contraction of $R^{\alpha}_{\beta\mu\nu}$, all others are multiples of it.

The Ricci tensor is $R_{\alpha\beta}$ derived from $R^{\mu}_{\alpha\mu\beta}$, which is to say a contraction along two indeces. (We'll just assume we're always contracting along two indeces and not worry about other options, such as three.)

We wish to show that all other possible contractions are either null, or some multiple of the Ricci tensor. For now we shall *assume* the tensor is symmetric so 12 and 21 would be the same tensor, but notably in the next part we have to prove that property.

Regardless, let's label our indeces xyzw since we're lazy. The 2-combinations on offer are:

xy xz xw yz yw (the definition) zw

So there are five other options. Using "a" as the "singular" element, these become:

xyaa xaza xaaw ayza ayaw (definition) aazw

So first of all from **Problem 18** we know that anything with "aa" in on either the left or right side just flat out reduces to zero. So that gets rid of two of our options right away.

xaza xaaw ayza ayaw (definition)

So now the trick is to show that these are equivalent to each other, give or take a sign. This can actually be shown simply by the "shuffling" of combinations we once did. Like so:

a1a2 = -1aa2 = -a12a = a2a1

And as we know from **Problem 18** the three-combinations will cycle through EVERY possible option. Thus, every other combination has to be plus or minus the others.

We don't even have to assume symmetry to know this, actually, as the inverted versions will also be hit!

b) Show that the Ricci tensor is symmetric

So, basically, $R^{\mu}_{\alpha\mu\beta} = R^{\mu}_{\beta\mu\alpha}$

So basically, can we say a1a2 = a2a1? Well... let's see.

a1a2 = -1aa2 = -a12a = a2a1.

Hey look at that, we've shown it trivially. Wow. Glad we did **Problem 18** to completion, it made thinking about this easy.

10 Problem 26 [Back to top]

*Use **Problem 17** to prove 6.94*

6.94 states, simply, that $g^{\alpha\beta}_{;\mu} = 0$

Which. This is extremely trivial, it seems, as **Problem 17** has $g^{\alpha\beta}_{,\mu}(P) = 0$

The trivial adjustment is that while the partial derivative at P is zero, that means the Chrsitoffel symbol is zero since we are at P, so the partial derivative is equal to the total derivative.

So. Uh. Yeah. That's it. Go home, it's over.

11 Problem 27 [Back to top]

Fill in the algebra necessary to establish 6.95, 6.97, 6.99.

So since this is an extended derivation, we're going to start from the beginning of the section, that is, by applying the Ricci contraction to the Bianchi identities. The Bianchi identities are $R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0$

The actual Ricci concentraiton is given by $R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta}$.

We specifically accomplish this applicaiton by applying a raising metric to the Bianchi identities. We can apply it independently of the derivative since the full derivative of the metric is zero and can thus be treated as a cosntant. All the contraction really does is adjust indeces.

$$g^{\alpha\mu}(R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu}) = 0$$

$$\Rightarrow (R^{\mu}_{\beta\mu\nu;\lambda} + R^{\mu}_{\beta\lambda\mu;\nu} + R^{\mu}_{\beta\nu\lambda;\mu}) = 0$$

$$\Rightarrow R\beta\nu;\lambda - R_{\beta\lambda;\nu} + R^{\mu}_{\beta\nu\lambda;\mu} = 0$$

The subtraction is there because we had to shuffle some indeces to get the right form for conversion. Which apparently is 6.95, but that's trivially true, at least it seems so. Shuffle index 3 and 4, get a minus sign.

We now have the contracted Bianchi identities.

Apparently this isn't sueful, so we contract AGAIN, this time to the Ricci Scalar, which is $R = g^{\mu\nu} R_{\mu\nu}$. If we apply another raising metric...

$$g^{\beta\nu}(R\beta\nu;\lambda - R_{\beta\lambda;\nu} + R^{\mu}_{\beta\nu\lambda;\mu}) = 0$$

$$\Rightarrow R_{;\lambda} - R^{\nu}_{\lambda;\nu} + g^{\beta\nu} R^{\mu}_{\beta\nu\lambda;\mu} = 0$$

$$\Rightarrow R_{;\lambda} - R^{\nu}_{\lambda;\nu} - g^{\beta\nu} R^{\mu}_{\beta\lambda\nu;\mu} = 0$$

$$\Rightarrow R_{;\lambda} - R^{\nu}_{\lambda;\nu} - R^{\mu}_{\lambda;\mu} = 0$$

That last step was us essentially defining a new contraction. (Mildly confused on validity... need to do it in the same order as the metric perhaps? Addendum: what apears to be happening is an ignoring of the first term, and using the last four terms to do the collapse. This involves swapping $\nu\lambda$ positions which are the 34 index, and there you have it) To match the 9.96 given in the book, we adjust indeces.

$$\Rightarrow R_{;\lambda} - R^{\mu}_{\lambda;\mu} - R^{\mu}_{\lambda;\mu} = 0$$

Which we can simplify to

$$(2R^{\mu}_{\lambda} - \delta^{\mu}_{\lambda} R)_{;\mu} = 0$$

the 2R term is obvious, thos parts add, and the signs can be inverted by carrying everything across the equals sign. But where did the delta come from? Well, do note that we are SUMMING over μ, so it hits every value. The point is to say that the R term only exists when $\mu = \lambda$. And then we have our result. This is 6.97.

To get to 6.99 we have to define a symmetric tensor, specifically G.

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = G^{\beta\alpha}$$

Our goal is to find G hidden within 6.97. Let's work it out.

$$(2R^{\mu}_{\lambda} - \delta^{\mu}_{\lambda} R)_{;\mu} = 0$$
$$\Rightarrow 2(R^{\mu}_{\lambda} - \frac{1}{2}\delta^{\mu}_{\lambda} R)_{;\mu} = 0$$
$$\Rightarrow 2g^{\lambda\beta}(R^{\mu}_{\lambda} - \frac{1}{2} g^{\mu}_{\lambda} R)_{;\mu} = 0$$

Here we remembered that the dirac is the metric with but up and down indeces.

$$\Rightarrow 2(R^{\mu\beta} - \frac{1}{2} g^{\mu\beta} R)_{;\mu} = 0$$

Change index.

$$\Rightarrow 2(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R)_{;\mu} = 0$$

Hey look, that's G.

$$\Rightarrow 2(G^{\alpha\beta})_{;\mu} = 0$$
$$\Rightarrow 2G^{\alpha\beta}_{;\mu} = 0$$
$$\Rightarrow G^{\alpha\beta}_{;\mu} = 0$$

And we're done here, that is 6.99.

12 Problem 28 [Back to top]

a) Derive 6.19 by using the usual coordiante transformation from Cartesian to spherical polars.

6.19 is the spherical metric, that is, the metric with the diagonal $1, r^2, r^2 sin\theta$. However, we need to actually find it from scratch to truly solve the problem. So basically, we just know the answer: now we need to derive it.

The "simple" way to find the metric is 6.11, $(g) = (\Lambda)(\eta)(\Lambda)^T$, where η is the standard metric, (1,1,1) diagonal in this case. In tensor notaiton, this is $g_{\alpha\beta} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\eta_{\mu\nu}$

This requires knowing the translation matrix from cartesian to spherical, which is defined by the Jacobian. 5.13 gives the exact format for 2x2, which we extend into 3x3. We do admit to having to look up the exact relations for xyz to spherical. 7 reveals...

$$x = rsin\theta cos\phi; y = rsin\theta sin\phi; z = rcos\theta$$

Which we now put into a Jacobian matrix. Rows are by cartesian xyz, rows are by r$\theta\phi$.

$$\begin{bmatrix} sin\theta cos\phi & rcos\theta cos\phi & -rsin\theta sin\phi \\ sin\theta sin\phi & rcos\theta sin\phi & rsin\theta cos\phi \\ cos\theta & -rsin\theta & 0 \end{bmatrix}$$

Since the cartesian metric is the identity, our operation is the same as multiplying this matrix by its transpose.

$$\begin{bmatrix} sin\theta cos\phi & sin\theta sin\phi & cos\theta \\ rcos\theta cos\phi & rcos\theta sin\phi & -rsin\theta \\ -rsin\theta sin\phi & rsin\theta cos\phi & 0 \end{bmatrix}$$

This took some notebook work to fully calculate, as there were nine separate terms.

After reviewing the definitions we realized we had the transpose backwards. Whoops. This rather quickly gave us our diagonal, though, just reverse the transposes and out pop $1, r^2, r^2 sin\theta$, tah-dah!

We did not test every single off-diagonal to see if they were zero because it should be automatic, but we did test one just to see. And yes it was zero.

b) Deduce from 6.19 that the metric of the surface of a sphere of radius r has components \$ $g\_\{\theta\theta\} = r2, g\_\{\phi\phi\} = r2sin2\theta, g\_\{\theta\phi\} = 0$\$$intheusualsphericalcoordinates.$

Obviously the transform the metric provides must still hold even for points restricted to a spherical shell of constant radius, so the diagonal becomes $(r^2, r^2 sin^2\theta)$ rather trivially.

c) Find the componetnst $g^{\alpha\beta}$ for the sphere.

Finding the inverse is simple: invert everything, the diagonal is now $1/r^2, 1/(r^2 sin^2\theta)$. Tah-dah!

13 Problem 29 [Back to top]

In polar coordinates, calculate the Reimann curvature tensor of the spehre of unit radius, whose metric is given in **Problem 28**. Note that in two dimensions there is onely one independent component, by the same argument as **Problem 18b**. So calculate $R_{\theta\phi\theta\phi}$ and obtain all other components in terms of it.

Right, so, R has 16 components in a two-dimensional case. We can actually write them all out in BINARY!

0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111

Naturally anything with any doubled anything is zero, so we ignore those.

0101, 0110, 1001, 1010

*only four components are potentially independent, and we know from **Problem 18** shuffling that they are all related to each other. By the suggestion in the problem, we have:*

0101 = -1001 = -0110 = 0101

-1001 = 0101 = 1010 = -0110

Which correlates all four of the values. Which means... now we have to figure out how to CALCULATE one of the terms. Shocked and terrified gasp,

From 6.62 we know the tensor is defined by the christoffel coefficients and their derivatives. Which means we're gonna have to grab everything and do some tedious calculaitons... but since we've never found an actual value for R before, we're gonna actually do it.

$$R^{\lambda}_{\beta\mu\nu} = \Gamma^{\gamma}_{\beta\nu,\mu} - \Gamma^{\gamma}_{\beta\mu,\nu} + \Gamma^{\gamma}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\gamma}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}$$

$$\Gamma^{\alpha}_{\mu\nu,\sigma} = \frac{1}{2} g^{\alpha\beta}\left(g_{\beta\mu,\nu\sigma} + g_{\beta\nu,\mu\sigma} - g_{\mu\nu,\beta\sigma}\right)$$

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha})$$

So, just calculate ONE.

R(0101) is our starting point. This becomes

$$R^0_{101} = \Gamma^0_{11,0} - \Gamma^0_{10,1} + \Gamma^0_{\sigma 0}\Gamma^\sigma_{11} - \Gamma^0_{\sigma 1}\Gamma^\sigma_{10}$$

$$= \Gamma^0_{11,0} - \Gamma^0_{10,1} + \Gamma^0_{00}\Gamma^0_{11} - \Gamma^0_{01}\Gamma^0_{10} + \Gamma^0_{10}\Gamma^1_{11} - \Gamma^0_{11}\Gamma^1_{10}$$

Now all these coefficients are given by the various metrics within them. Rather than writing it all down, since that would be qutie tedious, we talk it out. The only metrics that exist are on the diagonals. Lower metric is $r^2, r^2 sin^2\theta$ and the upper metric is $1/r^2, 1/(r^2 sin^2\theta)$. All off-diagonal terms are zero, which will reduce quite a lot to actually zero. In fact, the "sum" in every one of the coefficients reduces to a single number, since the upper metric in each one only exists on one of the terms.

After all of it, only one of the derivative coefficients and one of the non-derivative terms remains, combining to be

$$cos^2\theta - sin^2\theta + \frac{1}{tan^2\theta}$$

Which, with a sign adjustment, are all the terms in the R tensor.

Most of this was just double and triple checking indeces. It takes time.

14 Problem 30 [Back to top]

Calculate the Reimann curvature tensor of the cylinder. Since the cylidner is flat, this should vanish. Use whatever coordinates you likem and make sure you write down the metric properly!

Tempting to move on past this one, but more metric calculation is worthwhile.

The nice thing about cylindrical is that it's basically just fancy polar.

$x = rcos\theta$

$y = rsin\theta$

$z = z$ *(nice!)*

The polar metric is just a condensed version of this one. The only change is the addition of z=z, which makes a rather interesting addition of just... 1. Thus the full metric has diagonal (1, r^2, 1), in terms of $r\theta z$.

Notably, every portion of the Reimann curvature tensor is composed of derivatives of these metrics with respect to the coordinates. Only the middle term has a derivative that survives: 2r, and then a second derivative of 2. *Virtually everything goes to zero. We only need to concern ourseveles with $g_{\theta\theta,r}$ and $g_{\theta\theta,rr}$*

Thus, we might be able to LOOK BACKWARD. Our equations:

$$R^\gamma_{\beta\mu\nu} = \Gamma^\gamma_{\beta\nu,\mu} - \Gamma^\gamma_{\beta\mu,\nu} + \Gamma^\gamma_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\gamma_{\sigma\nu}\Gamma^\sigma_{\beta\mu}$$

$$\Gamma^\alpha_{\mu\nu,\sigma} = \frac{1}{2}g^{\alpha\beta}\left(g_{\beta\mu,\nu\sigma} + g_{\beta\nu,\mu\sigma} - g_{\mu\nu,\beta\sigma}\right)$$

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha})$$

So rather than working out each part, we work out all possible coefficients that contain the only terms that exist and see if they cancel or not.

Of the standard coefficients there are only 1/r, 1/r, -r. Which makes a lot of sense, really.

The signs seem a slight bit off, but the derivatives have $1/r^2, 1/r^2, -1$

There is significant annoyance in showing that all these things cancel.

The singular coefficients pair up to form three -1 terms, and one $1/r^2$ *term. Two of those -1 terms are entirely self-canceling.*

The rest better cancel with the derivatives. The -1 term in fact cancels with the -1 term from the derivatives in both cases.

One of the derivatives' $1/r^2$ *case cancels with itself* $\Gamma^{\theta}_{\theta r,r}$.

However, the last two seem to add. This problem woudl be fixed if the derivative coefficietns were NEGATIVE. Which ostentiably they should be, as they are the derivatives of hte other coefficients, which should just be able to be taken without issue. The derivative of 1/r is flat out $-1/r^2$*. But the result we get from the formula up above is positive, no sign change. Is it typed down wrong...? No...*

Well clearly we've shown that it does cancel, but the issue of why the formula above doesn't put in the right sign is greatly concerning.

This would be fixed if the inverse metric was $-1/r^2$*. But it's not. Because that doesn't reduce to the 111 diagonal.*

Annoyingly looking things up doesn't help here since everyone else just uses symmetry. Faster, to be sure...

No multiplying by the lowering metric at the end to get R into a different form doesn't change anything.

So, in conclusion, yes the cylidner is flat. We are not sure why the derivative formula doesn't do the proper negative stuff. Concerning.

Man even using it in a different metric didn't work... the sign error still occurs...

So basically let's take derivatives directly and not use that equation. There may be some validity criteria we have forgotten.

15 Problem 31 [Back to top]

Prove that covariant differnetiation obeys the usual product rule.

Skipped. Would be tedious. Is kind of obvious. Also already proven in other math courses.

16 Problem 32 [Back to top]

A four-dimensional manifold has coordinsates (u,v,w,p) in which the metric has components $g_{uv} = g_{ww} = g_{pp} = 1$, all other indepdendent componetns vanishing.

a) Show that the manifold is flat and the signature is $+2$

The signature is usually the sum of the diagonals, and that is the same as taking the determinant... except apparently not, as the determinant is shown to be zero. So let's go back and figure out what exactly signature MEANS.

ALSO metrics are symmetric by definition so g_{vu} better also be 1.

The signature is just the sum of the diagonal elements. So +2. That was much easier than we thought it was.

b) The result in a) implies the manifold must be Minkowski spacetime. Find a coordinate transformation to the usual coordinates (t,x,y,z). Hint: you may find it useful to calculate $\vec{e}_v \cdot \vec{e}_v$ and $\vec{e}_u \cdot \vec{e}_u$

well w=y p=z, so we can just ignore them.

So what matrix transforms (0,1)(1,0) to (-1,0)(0,1)?

That matrix is (0,1)(-1,0)

Which basically states t = -v and x = u.

17 Problem 33 [Back to top]

A 'three-sphere' is the three-dimensional surface in four-dimensional Euclidean space (coordinates x y z w) given by the equation $x^2 + y^2 + z^2 + w^2 = r^2$, where r is the radius of the sphere.

a) Define new coordinates (r, θ, ϕ, χ) by the equations $w = rcos\chi, z = rsin\chi cos\theta, x = rsin\chi sin\theta cos\phi, y = rsin\chi sin\theta sin\phi$ Show that (θ, ϕ, χ) are coordinates for the sphere. These genrealize to the familiar polar coordinates.

Tempting to do a by-words argument, but this is 4D, so let's try to figure out how to show it mathematically.

Perhaps the best way is to take the definition of r and show it's independent from the values of the angles. Which is actually pretty easy! insert the definitions of x, y, z, and w into the r definition, the radius completely cancels out. However, we will need to prove that the sum of all the squares equals 1 to confirm this.

$$\begin{aligned}
cos^2\chi + sin^2\chi cos^2\theta + sin^2\chi sin^2\theta cos^2\phi + sin^2\chi sin^2\theta sin^2\phi \\
= cos^2\chi + sin^2\chi cos^2\theta + sin^2\chi sin^2\theta \\
= cos^2\chi + sin^2\chi \\
= 1
\end{aligned}$$

Well would you look at that, the equality is satisfied and adjusting any of the three angles does absolutely nothing to the radius. Therefore, the three angles are all on *the three-sphere.*

There is a minor point that remains though: how can we show that the angles span the entire three-sphere, and that we haven't just gotten part of it? Well, we can reason this one out. Since r is always positive, it can't provide the sign to the various x, y, z, w components, to the sign must come from the angles. So every possible x w y z can be made with them, given an r, there's no bounds they can't reach.

Arguably this points to a much simpler two-line proof: the sphere is defined by constant r, thus the other three coordinates have to characterize it. Since they span Cartesian space, they must span the entire sphere.

b) Show that the metric of the three-sphere of radius r has components in these coordinates $g_{\chi\chi} = r^2, g_{\theta\theta} = r^2 sin^2\chi, g_{\phi\phi} = r^2 sin^2\chi sin^2\theta$, all other components vanishing. (Use the same method as **Problem 28**.)

Basically, find the conversion from cartesian metric. Except we aren't exactly converting from cartesian, but rather the matrix with the (1,1,1,1) diagonal.

This actually will be a lot of steps and not particuarly illuminating. The method to be outlined is simple: take the Jacobian. Since the 4-cartesian metric is just the identity, to find the terms all one has to do is multiply the transformation matrix (Jacobian) with its own transpose. This will result in 16 rather long evaluations that should result in the 4-metric. Whatever portion of the metric is on the radius section can be ignored. (It'll probably be 1 anyway).

Actually let's go ahead and calculate the radius portion of the metric, since it's not given. All we need to do is find the dot product of everything with a derivative taken with respect to r...

...

...we arleady calculated this in part a). The metric $g_{rr} = 1$. Nice.

Anyway this means we can actually find rather easily the diagonals by taking dot products of the obvious derivatives. The $g_{\phi\phi}$ result is actually self evident, since the first two become zero and the last two have the same terms affixed to the usual trig addition to 1 rule.

But still, we're not going to show the diagonals are zero. Let someone else do that.

18 Problem 34 [Back to top]

Establish the following identities for a general metric tensor in a general coordinate system. You may find 6.39 and 6.40 useful.

6.39: $g_{,\mu} = gg^{\alpha\beta}g_{\beta\alpha,\mu}$

6.40: $\Gamma^{\alpha}_{\mu\alpha} = \frac{\sqrt{-g}_{,\mu}}{\sqrt{-g}}$

a) $\Gamma^{\mu}_{\mu\nu} = \frac{1}{2}(ln|g|)_{,\nu}$

Okay so first of all Christoff Symbols are symmetric, $\TeX{a}

Compute 20 independent ocmponetns of $R_{\alpha\beta\mu\nu}$...

I'm going to put this nicely.

Heck no. It was hard enough and annoying enough to compute ONE *component in* ***Problem 30****.*

The method is simple enough. Calculate metric and inverse metric (which, for this problem, are provided via the length of the line element. The integration factor is everyone's friend!) From that calculate the Christoffel symbols that we have done MULTIPLE TIMES already. And then the derivatives. Since we're actually calculating we can take the derivatives directly. Do this twenty times.

My goodness that would be beyond tedious...

19 Problem 36 [Back to top]

A four-dimensional manifold has coordinates (t,x,y,z) and line element $ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2, dy^2, dz^2)$, where $|\phi(t,x,y,z)|$ « 1 everywhere. At any point P with coordinates (t_0, x_0, y_0, z_0), find a coordinate transformation to a locally inertial coordinate system, to first order in ϕ. At what rate does such a frame accelerate with respect to the original coordinates, agian to first order in ϕ?

The metric's diagonal can be taken directly from the line element, $(-1-2\phi, 1-2\phi, 1-2\phi, 1-2\phi)$ *Note that this is close to -1,1,1,1 but not quite, given the sign on* ϕ*.*

The determinant of this is $-1-2\phi+(1-2\phi)^3$ *and we can start to see where the first-order approximation comes in. If we expand out the cubed term, we'll arrive at* $-1-2\phi+1-6\phi+12\phi^2-8\phi^3$*. Combining and ignoring second order terms, the determinant is simply* -8ϕ

At any point, ϕ *has a value betwewn 1 and -1, so we can just treat it like a constant, or perhaps more accurately a parameter. Provided a* ϕ*, we will be able to find a coordinate transform to the locally inertial coordinate system. In terms of matrices, this is rather trivial:* $(-1/(-1-2\phi), 1/(1-2\phi), 1/(1-2\phi), 1/(1-2\phi))$*. The matrix with this diagonal will turn the metric into (-1,1,1,1).*

This is sufficient to define a coordinate transformation. What we've essentially done is apply a metric to the metric. We can then translate this backward to get a new line element:

$$ds^2 = \frac{1}{(1+2\phi)}dt^2 + \frac{1}{(1-2\phi)}(dx^2, dy^2, dz^2)$$

Keep in mind this is not the actual line element of our transformed matrix, just the metric we used. Our ultimate goal is to find an acceleration rate of the NEW frame in relation to the old one. Though, naturally, the line element of the new frame is constant (whih is what we wanted for it to be locally inertial). Hmm... how DO we find an acceleration... or a speed *for that matter?*

Perhaps we shouldn't use the alternate metric to accomplish the transformation but actually find the result directly. For any inertial metric, there is always a transformation that transforms to every other metric, and vice-versa. This transformation is determined by the Jacobian. Which we usually calculate by knowing *the coordinate transformation (x=whateverx') first, not the other way around.*

Let our "inertial" forms be t', x', y', and z'.

If we think of transforming FROM the primed ineces, we get a much easier problem, for then the metric we're transforming from is "nearly" the identity, and the matrix that equals the new metric follows a discernable pattern. Specifically, we can derive the diagonals rather easily:

$$\frac{\partial}{\partial t}(-t' + x' + y' + z') = -(1+2\phi)$$
$$\frac{\partial}{\partial x}(-t' + x' + y' + z') = 1-2\phi$$
$$\frac{\partial}{\partial y}(-t' + x' + y' + z') = 1-2\phi$$
$$\frac{\partial}{\partial z}(-t' + x' + y' + z') = 1-2\phi$$

And naturally all of the diagonals are zero, but those relations are harder to write. However, they are necessary conditions, otherwise there are far too many solutions above up there. (making the four variables translate directly along a linear path is a solution, for instance, but that doesn't keep zero diagonals.)

Diagonals, by the way, look like this:

$$\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial t} - \frac{\partial x'}{\partial t}\frac{\partial t'}{\partial x} - \frac{\partial y'}{\partial t}\frac{\partial t'}{\partial y} - \frac{\partial z'}{\partial t}\frac{\partial t'}{\partial z} = 0$$

Remember that $\partial t'$ *sections are always negative.*

So while these may not help us they do give us helpful bounds by which to test our possible solutions.

Examining determinants, we have our determinant of -8ϕ*. we know this is equal to the negative square of the transformation matrix's determinant, via 6.15, so we find it's determinant to be* $\pm\sqrt{8\phi}$*. In terms of volume element, this becomes unambiguously* $\sqrt{8\phi}$*.*

Put more characteristically, $\sqrt{8\phi}dtdxdydz$ = *dt'dx'dy'dz'.*

20 Problem 37 [Back to top]

a) "Proper Volume" of a two-dimensional manifold is usually called "proper area". Using the metric in **Problem 28**, integrate 6.18 to find the proper area of a sphere of radius r.

Ah, this is a matter of the integration factor!

The diagonal was $r^2, r^2 sin^2\theta$*. Multiply it all together and then square root to get the actual integration factor of* $r^2 sin\theta$ *which we've used before.*

For our bounds, we have two angles: θ *goes from 0 to* π *and* ϕ *goes twice that. The integral over* $sin\theta$ *gives 2, so the total is* $4\pi r^2$*, the well known surface area of a sphere.*

b) Do the same for the three-sphere of **Problem 33**.

See the problem here is that we don't know our bounds.

The integration factor itself isn't that difficult: $r^3 sin^2\chi sin\theta$*. But what are the limits on our angles? We can DEDUCE it from the original coordinate transforms. Parameters that are only inside a cosine function only need to vary from 0 to* π *to span all possible values, but sine functions have to go further. So we posit that only* ϕ *goes to* 2π*. Let's see if this is reasonable.*

We do need the integral of $sin^2\chi$ *from 0 to* π*. Geogebra gives* $\pi/2$ *This gets rid of the 2 from the* $sin\theta$*, suggesting that the "surface area" of a 3-sphere is...* $2\pi^2 r^3$*.*

And Wikipedia agrees YES! Boo-yeah.

21 Problem 38 [Back to top]

Integrate 6.8 to find the length of a circle of constant coordinate θ on a sphere of radius r.

6.8: \$l = $\int$ _{λ_0}Θ{λ_1}|$\vec{V}\cdot\vec{V}$|Θ{1/2}$d\lambda$\$

The dot product of the tangent vector of a circle to itself would be $(0,1)\cdot(0,1)$ *The metric is (1,*r^2*) in the case of a circle. And yes, we are aware the problem says constant* θ *but we don't have to listen to that, we can turn this into a purely polar problem with no loss of generality.*

Thus, the dot product is r^2. We take the root of this and are just left with r.

So the full integral is $r(\lambda_1 - \lambda_0)$

Which means the REAL question is how much λ do we have on this integral? Well... 2π. We're going all the way around once. Though we should probably find the mathematical reason for this rather than just because "we know the trig functions."

To do that we leave polar coordinates behind for a moment. We know that a circle in parametric cartesian is some form of $(sin\lambda, cos\lambda)$ for (x,y). It is easy to show that this cycles every 2π, and in cartesian coordinates the integration is direct 1 to 1. And if we increased the "speed", we would need to increase the vectors we used at the start! So that's why it's this way.

Just to prove it to ourselves, let the circle be given by $(sin2\lambda, cos2\lambda)$. The tengent vector given by the derivative will be $(2sin2\lambda, 2cos2\lambda)$. In pola, this is (0,2). Which would make our dot product $4r^2$ which is square rooted to 2r. Then we integrate and since we're going twice as fast, we should in the same "time" lambda get twice as much distance. And we do, 8π. Tah-dah!

22 Problem 39 [Back to top]

a) For any two vector fields $\vec{U}$ and $\vec{V}$, their **Lie Bracket** is defined to be the vector field $[\vec{U}, \vec{V}]$ with components

$$[\vec{U}, \vec{V}]^\alpha = U^\beta \nabla_\beta V^\alpha - V^\beta \nabla_\beta U^\alpha$$

Show that

$$[\vec{U}, \vec{V}] = -[\vec{V}, \vec{U}]$$

$$[\vec{U}, \vec{V}]^\alpha = U^\beta \frac{\partial V^\alpha}{\partial x^\beta} - V^\beta \frac{\partial U^\alpha}{\partial x^\beta}$$

This is one tensor field in which partial derivatives need not be accompanied by Christoffel symbols!

This is... trivially true. Just flip U and V, note that it results in a sign change. Huh.

b) Show that $[\vec{U}, \vec{V}]$ is a derivative operator on $\vec{V}$ along $\vec{U}$, i.e., show that for any scalar f

$$[\vec{U}, f\vec{V}] = f[\vec{U}, \vec{V}] + \vec{V}(\vec{U} \cdot \nabla f).$$

This is sometimes called the **Lie Derivative** with respect to $\vec{U}$ and is denoted by

$$[\vec{U}, \vec{V}] = \$_{\vec{U}}\vec{V}, \vec{U} \cdot \nabla f = \$_{\vec{U}} f$$

Then eq 6.101 (the first one in this part) would be written in the more conventional form of the Leibnitz rule for the derivative operator $ $_{} $ (agh, this is supposed to be the pound sign, but that can't be typed!)

$$\$_{\vec{U}}(f\vec{V}) = f\$_{\vec{U}}\vec{V} + \vec{V}\$_{\vec{U}}f$$

The result of a) shows that this derivative operator may be defined without a connection or metric, and is therefore very fundamental.

This actually isn't bad at all. Just physically evaluate it, though do so in tensor notaiton.

$$fU^\beta\nabla_\beta V^\alpha - fV^\beta\nabla_\beta U^\alpha + V^\alpha U^\beta\nabla_\beta ff$$

First and last terms can be adjusted via product rule.

$$= U^\beta\nabla_\beta fV^\alpha - fV^\beta\nabla_\beta U^\alpha$$

$$= [\vec{U}, f\vec{V}]^\alpha$$

c)Calculate the components of the Lie Derivative of a one-form field $\tilde{\omega}$ from the knowledge that, for any vector field $\vec{V}$, $\tilde{\omega}(\vec{V})$ is a scalar like f above, and from the definition that \$ \$_ {} \$ is a one-form field:

$$\$_{\vec{U}}[\tilde{\omega}(\vec{V})] = (\$_{\vec{U}}\tilde{\omega})(\vec{V}) + (\tilde{\omega})(\$_{\vec{U}}\vec{V})$$

This is the analog of 6.103.

So we're going to solve for the Lie Derivative. This is going to be a little funky. Treat the part we're going for as G, which means the middle term is $G(V^\alpha)$. With this, we can start solving.

$$U^\beta\nabla_\beta\omega_\alpha(V^\alpha) - \omega_\beta(V^\beta)\nabla_\beta U^\alpha = G_\alpha(V^\alpha) + \omega_\alpha\left[U^\beta\nabla_\beta V^\alpha - V^\beta\nabla_\beta U^\alpha\right]$$

The simple and ugly answer is:

$$\Rightarrow \left(U^\beta\nabla_\beta\omega_\alpha(V^\alpha) - \omega_\beta(V^\beta)\nabla_\beta U^\alpha - \omega_\alpha\left[U^\beta\nabla_\beta V^\alpha - V^\beta\nabla_\beta U^\alpha\right]\right)/V^\alpha = G_\alpha$$

Let's at least try to simplify this.

$$\Rightarrow \left(U^\beta\nabla_\beta\omega_\alpha V^\alpha - \omega_\beta V^\beta\nabla_\beta U^\alpha - \omega_\alpha U^\beta\nabla_\beta V^\alpha + \omega_\alpha V^\beta\nabla_\beta U^\alpha\right)/V^\alpha = G_\alpha$$

$$\Rightarrow \left(U^\beta\nabla_\beta\omega_\alpha V^\alpha - \omega_\alpha U^\beta\nabla_\beta V^\alpha\right)/V^\alpha = G_\alpha$$

Good enough.

23 Addendum: Output this notebook to $\LaTeX$-formatted PDF file [Back to top]

The following code cell converts this Jupyter notebook into a proper, clickable $\LaTeX$-formatted PDF file. After the cell is successfully run, the generated PDF may be found in the root NRPy+ tutorial directory, with filename `GR-06.pdf` *(Note that clicking on this link may not work; you may need to open the PDF file through another means.)*

***Important Note**: Make sure that the file name is right in all six locations, two here in the Markdown, four in the code below.*

- *GR-06.pdf*
- *GR-06.ipynb*
- *GR-06.tex*

[1]:
```
import cmdline_helper as cmd    # NRPy+: Multi-platform Python command-line interface
cmd.output_Jupyter_notebook_to_LaTeXed_PDF("GR-06")
```

```
Created GR-06.tex, and compiled LaTeX file to PDF file GR-06.pdf
```

[ ]:
```
```

[ ]:
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```