NR-02

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-1 Numerical Relativity Problems Chapter 2: The 3+1 Deconposition of Einstein's Equations

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https://github.com/zachetienne/nrpytutorial/blob/master/Tutorial-Template_Style_Guide.ipynb

Link to the Style Guide. Not internal in case something breaks.

-1.1.1 NRPy+ Source Code for this module:

None, save the pdf conversion at the bottom of this document.

-1.2 Introduction:

Now we take a look into "so how do we actually go about doing this?" via Numerical Relativity by Baumgarte and Shapiro.

-1.3 Other (Optional):

In order to fascilitate learning, whenever the opportunity arises Sympy will be used.

-1.3.1 Note on Notation:

Any new notation will be brought up in the notebook when it becomes relevant.

-1.3.2 Citations:

[1] https://physics.stackexchange.com/questions/79157/square-bracket-notation-for-anti-symmetric-part-of-a-tensor (3-way and up antisymmetry formula)

[2] https://profoundphysics.com/christoffel-symbols-a-complete-guide-with-examples/ (Christoffel Symbols)

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1 Problem 1 [Back to top]

Demonstrate that the constraint equations 2.2, if satisfied initially, are automatically satisfied at later times when the gravitational field is evolved by using the dynamical equations 2.3. Equivalently, show that the relation $\partial_t(G^{a0} - 8\pi T^{a0}) = 0$ will be satisfied at the initial time $x^0 = t$, hence conclude that 2.2 will be satisfied at $x^0 = t + \delta t$. Hint: use the Bianchi identities together with the equations of energy-momentum conservation to evaluate $\nabla_b(G^{ab} - 8\pi T^{ab})$ at $x^0 = t$

2.2: $G^{a0} = 8\pi T^{a0}$

2.3: $G^{ij} = 8\pi T^{ij}$

1.23 (The Bianchi Identities): $\nabla_e R_{abcd} + \nabla_d R_{abec} + \nabla_c R_{adbe} = 0$

And naturally energy-momentum is conserved. Though, as we know, energy is rather hard to define in relativity...

Note that it at first seems like we are being asked to solve it different ways, but the "equivalently" really is just giving us a hint on where to go and what to do and why. Naturally if the time derivative of a quantity is zero, that quantity is not going to change over an infinitesimal adjustment (δt in our formulation). So let's just start working it out the way the hint suggests.

$$\nabla_b(G^{ab} - 8\pi T^{ab})$$

$$= \nabla_b G^{ab} - \nabla_b 8\pi T^{ab}$$

$$= g_{bf}\nabla_b G^a_f - g_{bf}8\pi\nabla_b T^a_f$$

Now here's the thing, on page 6 we note that as a consequence of the Branchi identities, the covariant divergence of G vanishe AND so does T, via 1.34. Thus we have two terms that are zero. Specifically, zero *always*. Now, granted, we haven't proven WHY the Branchi identities demand this, but it does give us the rather obvious case of

$$= 0$$

That said, this is not what we are looking for, that's the ENTIRE covariant derivative. What WE want is to show that JUST the temporal portions go to zero as well. So let's back up a bit, all the way to the General Relativity book, which will henceforth be refered to as GR. GR6.97 actually gives ust he end result of contracting the Bianchi identities.

$$\nabla_u(2R^u_l - \delta^u_k R) = 0$$

Save for a the placement of hte 2 this is identical to the formulation for the G in terms of R. Which is WHY we can say covariant derivatives of G are always zero.

$$0 = \nabla_j G^{ij} - \nabla_j 8\pi T^{ij} + \nabla_0 G^{a0} - \nabla_0 8\pi T^{a0} + \nabla_b G^{0b} - \nabla_b 8\pi T^{0b} - \nabla_0 G^{00} + \nabla_0 8\pi T^{00}$$

What we have here is spatial components, temporal component column, temporal component row, and then a subtraction of the 00 case which was counted twice. The symmetry of G and T does not allow us to simplify as the derivative on identical components relies on a potentially different variable.

We also note from GR that the Bianchi identities also imply the covariant derivative of T is zero, and this specifically *IS* the local conservation of energy and momentum. From our earlier work we now expand the covariant derivative out, yet we know it must be zero still.

$$0 = g_{bf}\left(\partial_b G^a_f + \Gamma^a_{db}G^d_f - \Gamma^d_{fb}G^a_d\right) - g_{bf}8\pi\left(\partial_b T^a_f + \Gamma^a_{db}T^d_f - \Gamma^d_{fb}T^a_d\right)$$

Since we're looking for a time relation, might as well make the covariant derivative time specifically.

$$\Rightarrow 0 = g_{tf}\left(\partial_t G^a_f + \Gamma^a_{dt}G^d_f - \Gamma^d_{ft}G^a_d\right) - g_{tf}8\pi\left(\partial_t T^a_f + \Gamma^a_{dt}T^d_f - \Gamma^d_{ft}T^a_d\right)$$

We can arrange terms together and divide out the metric…

$$\Rightarrow 0 = \partial_t G^a_f - 8\pi \partial_t T^a_f + \Gamma^a_{dt} G^d_f - \Gamma^d_{ft} G^a_d - 8\pi \Gamma^a_{dt} T^d_f + 8\pi \Gamma^d_{ft} T^a_d$$

Here's the curious thing about this–the Christoffel symbols on both sides match with flipped signs, and the signs for G and T are in the same place. We have NOT applied any restrictions to G or T, their indeces are still free, so they hit everything. Remember that in general

$$G^{ab} = 8\pi T^{ab}$$

. Which means that since the Christoffel symbols match, the four trailing terms will CANCEL!

$$\Rightarrow 0 = \partial_t (G^a_f - 8\pi T^a_f)$$

Which is what we sought. Which is far more trivial than we made it by overthinking. This kind of HAS to be true, however the subtraction used on the Christoffel symbols would not apply here, as the derivatives are being taken and the derivative of 0 need not actually be 0.

We know that the conditions of Einstein's equations are satisfied initially, and we have just seen that with said conditions the temporal derivative is zero, thus so long as Einstein's equations are satisfied the evolution is valid.

Which is good, but remember that such assumptions vanish at cosmological scales due to the Cosmological Constant.

2 Problem 2 [Back to top]

Show that the evolution equations 2.11 and 2.12 preserve the constraint 2.5. i.e., show that

$$\frac{\partial}{\partial t} \mathcal{C}_E = 0$$

2.5: $C_E = D_i E^i - 4\pi\rho = 0$

2.11: $\partial_t A_i = -E_i - D_i \Phi$

2.12: $\partial_t E_i = D_i D^j A_j - D^j D_j A_i - 4\pi j_i$

OKAY so there are pages upon pages of scratch work here that have gone absolutely nowhere, time to move on.

3 Problem 3 [Back to top]

Show that a transformation to a new "tilded" gauge according to

$$\tilde{\Phi} = \Phi - \frac{\partial \Lambda}{\partial t}$$

$$\tilde{A}_i = A_i + D_i\Lambda$$

leaves the physical fields E^i and B^i unchanged

We do note that to be a gauge transformation the shifts have to be sufficiently small, so we know that the additional terms are small in comparison to the original ones. Thus we consider Λ and its derivative to be small.

Let's go with a relation that has E in it:

$$\partial_t A_i = -E_i - D_i\Phi$$

$$\Rightarrow E_i = -\partial_t A_i - D_i\Phi$$

So if we replace our adjustments with our substitutions this BETTER be the same.

$$\Rightarrow E_i = -\partial_t A_i - \partial_t D_i\Lambda - D_i\Phi + D_i\partial_t\Lambda$$

$$\Rightarrow E_i = -\partial_t A_i - D_i\Phi$$

So yes E has to be the same.

But what of B? Well, we can kind of think of this logically. E and B are intrisically linked, if E changes, B changes, if E stays the same, B stays the same. Thus, if one is unchanged, both are unchanged. 2.8 allows one to be calculated from the other.

4 Problem 4 [Back to top]

In the so-called radiation, Coulomb, or transverse gauge, the divergence (or longitudinal) part of A is chosen to vanish. $D_iA^i = 0$. So that A_i is purely transverse. Show that in this gague Φ plays the role of a Coulomb potential, $D^iD_i\Phi = -4\pi\rho$ and that the vector potential A_i satisfies a simple inhomogeneous wave equation

$$\Box A_i = -\partial_t^2 A_i + D^jD_jA_i = -4\pi j_i + D_i(\partial_t\Phi)$$

This appears to be a problem in two parts, first we have to show that the potential is as always, knowing only that the physical covariant derivative for A always goes to zero. This knowledge does not immediately remove most restrictions, however–after all, derivatives of zero can still be something other than zero. The bane of our existence.

NOTE: there was a section here about j=0. It does not.

What DOES help us solve it is 2.5, which, when reworded, states that $D_iE^i = 4\pi\rho$. With this knowledge, we can apply a covariant derivative on 2.11

$$D^i\partial_t A_i = -D^iE_i - D^iD_i\Phi$$

The last term is the one we want. The left-hand side looks unevaluatable, but since the divergence vanishes, it better vanish equally at all times, thus the derivative of zero is zero in this case. This leaves us with:

$$\Rightarrow D^i D_i \Phi = -D^i E_i$$

$$\Rightarrow D^i D_i \Phi = -4\pi\rho$$

Which is what it should be.

And now the second part. We need to show:

$$\Box A_i = -\partial_t^2 A_i + D^j D_j A_i = -4\pi j_i + D_i(\partial_t \Phi)$$

Which is a relation of the square laplacian. The left hand side andt he middle follow automatically. However, the last step does not. Also it seems to imply that the current, j, is not zero, like we determined before.

From the middle step, we can expand the time derivative of A_i via 2.11

$$-\partial_t^2 A_i + D^j D_j A_i$$

$$= -\partial_t(-E_i - D_i\Phi) + D^j D_j A_i$$

$$= \partial_t E_i + D_i \partial_t \Phi + D^j D_j A_i$$

With this we ave one of the terms we want. We can use 2.12 for the time derivative of E to get...

$$= D_i D^j A_j - D^j D_j A_i - 4\pi j_i + D_i \partial_t \Phi + D^j D_j A_i$$

The divergence vanishes, and two terms cancel, leaving us with:

$$= -4\pi j_i + D_i \partial_t \Phi$$

Which is what we sought.

5 Problem 5 [Back to top]

Show that the normalized 1-form $\omega_a = \alpha\Omega_a$ is rotation-free $\omega_{[a}\nabla_b\omega_{c]} = 0$

So first of all we need to be careful with the notation– the brackets refer to the antisymmetric portion of the tensor. in this case it would be the tensor composed of all these things smashed together, which is a three-way antisymmetric tensor. We use 1 to grab the three-index antisymmetry formula.

$$\omega_{[a}\nabla_b\omega_{c]}$$

$$= \alpha^2\Omega_{[a}\nabla_b\Omega_{c]}$$

$$= \alpha^2\nabla_{[a}t\nabla_b\nabla_{c]}t$$

2.20 lets us know that $\nabla_{[a}\nabla_{b]}t = 0$. Perhaps we can make use of this. This fact may be rather obvious since derivative order can be changed at will, but the extra t we have in the middle here complicates things now. The antisymmetric expansion gives us:

$$= \alpha^2\frac{1}{3!}\left[\nabla_a t\nabla_b\nabla_c t + \nabla_c t\nabla_a\nabla_b t + \nabla_b t\nabla_c\nabla_a t - \nabla_a t\nabla_c\nabla_b t - \nabla_b t\nabla_a\nabla_c t - \nabla_c t\nabla_b\nabla_a t\right]$$

We note that every term cancels because the last two ∇s can be shuffled at will, meaning a + term will always cancel with a - term. Thus, it goes to zero.

Now as for what exactly this means physically... what DOES rotation-free mean? Presumably it means it's immune to rotation or lacks the capacity to rotate, but the physical intuition is not coming today.

6 Problem 6 [Back to top]

Show that $\gamma^a_b v^b$, where v^b is an arbitrary spacetime vector, is purely spatial.

Just so we're clear, $v^b = (t, x, y, z)$. The final result of what we get here better not have any time in it.

2.30 gives us the projection operator, $\gamma^a_b = \delta^a_b + n^a n_b$

All together, we get the result

$$\delta^a_b v^b + n^a n_b v^b$$

Notably we only sum over b, a is what our final vector is goign to be.

$$= v^a + n^a n_b v^b$$

This initial projection just makes use of the dirac delta, only the term that matches the index will survive. As for the other term, however, the terms of the sum don't just automatically vanish.

The way to prove something is spatial in this case is to prove that nothing about it lies along the normal vector to time, which is to say, we apply n^a to it again and see where we end up:

$$n^a v^a + n^a n^a n_b v^b$$

In essence, this better equal zero!

Since we are trying to equal zero here, we can pull out a metric from every term to lower the index of a. We actually pull the dirac delta back out since it'll be useful now.

$$g^{aa} n_a \delta^a_b v^b + g^{aa} n_a n^a n_b v^b$$

$$= g^{aa} n_b v^b - g^{aa} n_b v^b = 0$$

Where the last step is $n^a n_a = -1$. And we're done!

7 Problem 7 [Back to top]

Show that for the second rank tensor T_{ab} we have

$$T_{ab} = \perp T_{ab} - n_a n^c \perp T_{cb} - n_b n^c \perp T_{ac} + n_a n_b n^c n^d T_{cd}$$

.

Curiously, this seems to be a case of working backward. We have projections of the tensor, we need to reclaim the original tensor with them. (Or think of it as putting the tensor in terms of its projections… sorta. The motivation is a bit unclear). We are warned in teh book to use the projection symbolw ith some care since it only applies to the free indices of the tensor that it operates on.

First step, let's expand all the projections via 2.31.

$$T_{ab} = \gamma^c_a \gamma^d_b T_{cd} - n_a n^c \gamma^a_c \gamma^d_b T_{ad} - n_b n^c \gamma^d_a \gamma^b_c T_{bd} + n_a n_b n^c n^d T_{cd}$$

.

Now we recall from 2.24 that every normal vector *contains* a metric. The best part is we can choose exactly what form that metric takes for maximum simplification. More specifically, we choose indeces such that we end up with forms akin to $g^{ab} g_{bc} = g^a_c = \delta^a_c$ and deltas are really easy to make vanish. Furthermore, we played with indeces and found:

$$n^a = -g^{ab}\omega_b \Rightarrow n_c = -g_{ac} g^{ab} \omega_b = -\delta^b_c \omega_b = -\omega_c$$

Which makes this quite a bit simpler. Watch the negative sign! (Which doesn't show up here since we always end up with two of them)

This turns out to not be all that helpful but it IS useful to knnow. Instead of doing this, let's just shuffle the index salad to make all the T terms match.

$$\begin{aligned}T_{ab} &= \gamma_a^c\gamma_b^d T_{cd} - n_c n^a \gamma_a^c\gamma_b^d T_{cd} - n_c n^b \gamma_a^d\gamma_b^c T_{cd} + n_a n_b n^c n^d T_{cd} \\ &= \left(\gamma_a^c\gamma_b^d - n_c n^a \gamma_a^c\gamma_b^d - n_c n^b \gamma_a^d\gamma_b^c + n_a n_b n^c n^d\right) T_{cd}\end{aligned}$$

Index salad: a and b are entirely interchangeable and arbitrary.

$$= \left(\gamma_a^c\gamma_b^d - n_c n^a \gamma_a^c\gamma_b^d - n_c n^b \gamma_b^d\gamma_a^c + n_a n_b n^c n^d\right) T_{cd}$$

Somehow, we need to show that the term in the parentheses is equivalent to $\delta_a^c\delta_b^d$. It is ALMOST factorable itno a binomial that can be outright split up, but nto quite. If we could somehow lower the cd and raise the ab without actually *changing* the final term, we would get exactly what we want.

However, it sure seems like that is NOT true. In fact in our notes we ended up proving that, in any specific case, $n_a n^b \neq n_b n^a$. Same goes for the last term, doing it twice does not "undo" the problem, unless we were in the Euclidean metric, which we most definitely are not.

Unless we were in the Euclidean metric.

~~PREVIOUS WORK WE'RE PRETTY SURE IS WRONG~~

Each term in the sum is independent and ends with a particular tensor T. We can re-arrange the index to put it in ab terms, giving us:

$$= (\gamma_c^a\gamma_d^b - n_a n^c \gamma_c^a\gamma_d^b - n_b n^c \gamma_d^a\gamma_c^b + n_c n_d n^a n^b) T_{ab}$$

Now we engage in the practice of Index Salad trying to get the term in parentheses to reduce to 1. Curious, this almost looks like a binomial expansion. Only a and b matter for interacting with the Tensor, so we can shuffle c and d as we wish, getting some nice like terms:

$$= (\gamma_c^a\gamma_d^b - n_a n^c \gamma_c^a\gamma_d^b - n_b n^d \gamma_c^a\gamma_d^b + n_c n_d n^a n^b) T_{ab}$$

The indeces on the last term aren't neat, we want up to be down and down to be up. Unfortunately it does not seem possible to just "choose metrics" that will make everything cancel when we adjust the indeces.

Now we recall from 2.24 that every normal vector *contains* a metric. The best part is we can choose exactly what form that metric takes for maximum simplification. More specifically, we choose indeces such that we end up with forms akin to $g^{ab}g_{bc} = g_c^a = \delta_c^a$ and deltas are really easy to make vanish. Furthermore, we played with indeces and found:

$$n^a = -g^{ab}\omega_b \Rightarrow n_c = -g_{ac}g^{ab}\omega_b = -\delta_c^b\omega_b = -\omega_c$$

Which makes this quite a bit simpler. Watch the negative sign! (Which doesn't show up here since we always end up with two of them)

$$= (\gamma_c^a\gamma_d^b - n_a n^c \gamma_c^a\gamma_d^b - n_b n^d \gamma_c^a\gamma_d^b + n_c n_d n^a n^b) T_{ab}$$

. . . Wait hold on while the relation above appears potentially useful, we still cannot decomposet his into a square binomial like it looks like we SHOULD. If we COULD we would end up with something along the lines of:

$$= (\gamma^a_c - n_a n^c)(\gamma^b_d - n_b n^a)T_{ab}$$

2.30 informs us that the binomials in parentheses are delta functions! Specifically:

$$= (\delta^a_c)(\delta^b_d)T_{ab}$$

$$= T_{cd} = T_{ab}$$

So yes this LOOKS about right. This would all be automatic if we could prove $n_a n_b n^c n^d = n_c n_d n^a n^b$

Wait, hold on, agh, there's the obvious answer right there. We CAN shuffle the indeces around, even a and b, because while they WOULD interact with T, they actually in the end DO NOT. Sure, the two diracs change the indeces from ab to cd... but then we change them right bac. In effect, there is no change at all to the tensor. Which means that even if we shuffle indeces around of the summing terms, the result is still 1. Hah!

...Yes this is a little shaky but it has to be true, though as for why we can't do it without this step we are not sure.

If we split each n up into its component parts, we *can* get each to behave, ending up with the "equality"

$$\omega_a \omega_b \omega_c \omega_d (-g^{cd})^2 = \omega_a \omega_b \omega_c \omega_d (-g^{ab})^2$$

Which, if we are allowed to shuffle the indeces, yes is true. IF we are allowed to shuffle the indeces. It seems like we should due to the order of operations we have set up, but we are hesitant.

8 Problem 8 [Back to top]

Show that the 3-dimensional covariant derivative is compatible with the spatial metric γ_{ab}*, that is, show that* $D_a \gamma_{bc} = 0$

Let's write this out carefully. When acting on a scalar, the covariant derivative follows 2.40: $D_a f = \gamma^b_a \nabla_b f$ For our purposes replace b with d since we have other indeces around.

Acting on other tensors involves tacking on more γ functions based on the number of indeces. For instance, in our case:

$$D_a \gamma_{bc} = \gamma^b_e \gamma^c_f \gamma^d_a \nabla_d \gamma_{ef}$$

We just need to show that this is in fact zero.

...Yeah haven't the foggiest idea how to do this.

9 Problem 9 [Back to top]

Show that for a scalar product $v^a w_a$ the Leibnitz rule

$$D_a(v^b w_b) = v^b D_a w_b + w_b D_a v^b$$

Only holds if v^a and v_b are purely spatial

Oho, trying to tell me the product rule is wrong are you? Well... yeah that makes sense.

Anyway a scalar product is a SCALAR so the 3D derivative becomes

$$\gamma_a^c \nabla_c(v^b w_b)$$

Now we know the product rule applies to the normal covariant derivative, so we can use it here.

$$\gamma_a^c \left[\nabla_c(v^b)w_b + v^b \nabla_c(w_b)\right]$$

Now, it sure seems like no matter what γ can just be pulled in and show that the product rule still applies... but let's think about what γ IS. The form we have is 2.30, which is the projection of a 4-dimensional tensor into a spatial slice. Well, if v and w are spatial, then the result is obvious: projection does nothing if it's already projected, so the rule must still hold.

But that's only half of the proof. We need to show that it is NOT true when v or w or both has a temporal component. This is actually easy to see: let there be v = (1,x,y,z) and w = (0,x,y,z). When they have their fancy dot product they just end up with "$x^2 + y^2 + z^2$" the temporal components completley cancel, then everything goes throgh and acts just fine.

For the side of the product rule where D acts on v alone will remove the temporal component entirely. However, in the case where D acts on w, the temporal component in v is still there. Thus when γ goes through it WILL act on that vector and project it. Which is to say *a situation may be constructed where γ changes the left side of the rule without the right, necessarily breaking the equality.* Essentially what we've done is proof by counterexample, ableit somewhat generally.

Put another way, if either w or v has a temporal component, when γ goes through the vector by necessity must be altered to remove it, it doesn't matter how exactly. However, on the opposite side, there is no alteration occuring, that is to say, "no change".

The end result of all this would be something akin to:

$$D_a(v^b w_b) = v^{b'} D_a w_b + w_b D_a v^b$$

Where the prime vector has been altered in some fashion. This is not a general rule, this is just our specific case.

How exactly γ acts can be left vague. Good for us!

10 Problem 10 [Back to top]

*Show that the twist ω_{ab} has to vanish as a consequence of n^a being rotation-free. See **Problem 5**.*

What **Problem 5** actually shows is that ω_a is rotation free. That said, since it is rotation free, and n^a is constructed from it and the metric, obviously the same holds for it.

So the question is why does this make the twist vanish? (Such an excellent name, the Twist...) The twist is given by 2.48.

$$\omega_{ab} = \gamma_a^c \gamma_b^d \nabla_{[c} n_{d]}$$

And the rotation-free requirement is 2.23.

$$\omega_{[a} \nabla_b \omega_{c]} = 0 = n_{[a} \nabla_b n_{c]}$$

Problem 7 can confirm that this is actually a direct substitution, just with a negative sign. Notably since there are two of them the signs cancel.

Note that all ω are actually $\alpha\Omega$ which means we actuallky have:

$$\omega_{ab} = \gamma_a^c \gamma_b^d \alpha \nabla_{[c} \Omega_{d]}$$

And from 2.20 we know that that antisymmetric portion goes to zero. Thus everything goes to zero and the twist vanishes.

...Seems too simple...

11 Problem 11 [Back to top]

Show that the extrinsic curvature of t=constant hypersurfaces of the Schwarzchild metric 2.35 vanishes.

The curvature is given by 2.49

$$K_{ab} = -\gamma_a^c \gamma_b^d \nabla_c n_d$$

.

We just need to evaluate this for Schwarzchild geometry. Which we actually have outlined in other places. The spatial metric is 2.39

$$\gamma_{ab} = \left(1 + \frac{M}{2r}\right)^4 diag(0, 1, r^2, r^2 sin^2\theta)$$

And the normal vector is

$$n^a = -g^{ab}\omega_b = \frac{1+M/2r}{1-M/2r}(1,0,0,0)$$

Now, these are not exactly in the right forms. But we can alter the curvature equation to *make* it the right forms!

$$K_{ab} = -g^{cf}g^{dg}g_{dh}\gamma_{fa}\gamma_{gb}\nabla_c n^h$$

.

Now, in OUR metric, these terms only exist in certain locations. The easiest result is to remove all non-diagonals.

$$K_{ab} = -g^{ca}g^{db}g_{dh}\gamma_{aa}\gamma_{bb}\nabla_c n^h$$

.

$$K_{ab} = -g^{ca}\delta^b_h\gamma_{aa}\gamma_{bb}\nabla_c n^h$$

.

$$K_{ab} = -g^{ca}\gamma_{aa}\gamma_{bb}\nabla_c n^b$$

.

And that seals it! The only time n exists is when b=t, but when b=t, the γ does not exist! and the other way around is true as well meaning that, no matter what, the extrinsic curvature will vanish. Now at first we wonder why we need to think about why t=const here, what if t wasn't const? Well. Remember that the hypersurfaces we've been modeling this entire time have t=const as the assumption the *entire time*. It's baked in to what we've done above.

12 Problem 12 [Back to top]

Show that the acceleration a_a is purely spatial, $n^a a_a = 0$

Have worked on this one for QUITE some time, ended up with:

$$n^a a_a = -\alpha^3 g^{ac}g^{bd}\nabla_c t\nabla_d t\nabla_b\nabla_a t$$

Since t is a scalar function 5.53 from General Relativity can give us:

$$= -\alpha^3 g^{ac}g^{bd}\partial_c t\partial_d t\partial_b\partial_a t$$

Which does not seem to have a way to evaluate to zero, at least not obviously. Taking two derivatives of a scalar function is not guaranteed to be zero by any means. So what's going on here?

Moving on, too much time spent.

13 Problem 13 [Back to top]

Show that the acceleration a_a is related to the lapse α according to

$$a_a = D_a ln\alpha$$

Given how little luck we had with the previous problem, perhaps it is unsurpising that this one is unsolved.

14 Problem 14 [Back to top]

Find the acceleration a_a for the normal observer 2.38 in Schwarzchild spacetime.

The definition of acceleration is $a_a = n^b \nabla_b n_a$. For the Schwarzchild metric we would need to grab Christoffels to deal with this... and for the alternative method we would ALSO need them so egh let's just go grab them from 2. Now that we have them, we can use 2.38:

$$n^a = \frac{1 + M/2r}{1 - M/2r}(1, 0, 0, 0)$$

This means, rather obviously, that only n^t actually exists.

The normal observer is the one moving along the normal vector, which essentially means that the acceleration itself is only happening in the time component as well. (We can see this since the xyz terms in our expression all vanish.)

Curiously the sum over the b index also goes to nothing except for t, which leaves us with

$$a_t = n^t \nabla_t n_t$$

.

Of course, the issue is that our vector n isn't in the right index form, so we have to:

$$= n^t \nabla_t g_{tt} n^t$$

.

Now we happen to just know what the Schwarzchild metric is, so this isn't an issue. The tt component is $-\frac{1-M/2r}{1+M/2r}$ which... well would you look at that it reduces the part in the derivative to -1. And that... makes the entire thing zero.

But wait, didn't we say there should be temporal acceleration?

Yes. But remember this is the physical acceleration, there's no temporal componenent. So... Yeah the normal observer has no acceleration.

15 A.1 An Aside on Lie Derivatives

The book introduced Lie Derivatives and said to refer to the appendix and guess what, there are problems back there, so we're going to DO them.

16 Problem A1 [Back to top]

Show that the expression

$$\mathcal{L}_{\mathbf{X}} T^a_b = X^c \nabla_c T^a_b - T^c_b \nabla_c X^a + T^a_c \nabla_b X^c$$

wehre ∇_a denotes a covariant derivative with a symmetric connection, is equivalent to A.8

$A.8: L_{**X**} T^a_b = X^c ∂_cTΘa_b − TΘ_b∂_cXΘa + TΘa_c∂_bXΘc$

The symmetric connection merely means theat the Christoffels are symmetric on their lower indeces. Which... well we kind of usually assume they are but in the GENERAL Lie Derivative case they may not be. Regardless, what we ultimately need here is to show that every Christoffel term vanishes, leaving only the partial derivative terms. For that, we turn to our book on General Relativity. Equations 6.33 through 6.35 give us all the information we need about covariant derivatives acting on tensors. We already know the partial derivative terms are prsent in A.8, so we actually want to show that:

$$0 = X^c(\Gamma^a_{uc} T^u_b - \Gamma^u_{cb} T^a_u) - T^c_b(\Gamma^a_{uc} X^u) + T^a_c(\Gamma^c_{ub} X^u)$$

Note that every term is summed over both c and u. WIth some clever index shuffling, we can arrive at:

$$\Rightarrow 0 = X^u \Gamma^a_{cu} T^c_b - X^u \Gamma^c_{ub} T^a_c - T^c_b \Gamma^a_{uc} X^u + T^a_c \Gamma^c_{ub} X^u$$

Which only cancels if the Christoffels in the first and third terms can shuffle their lower indeces. Which is, in fact, a thing we were given.

17 Problem A2 [Back to top]

*Show that $L_{**X**}(rΘaω

*Show that for a p-form $\tilde{\mathbf{\Omega}}$, $ L{**X**} = L{**X**} $*

Now that's a mess of bolding.

Okay so annoyingly the trick to how to deal with this comes from A.18, which is in the NEXT SECTION and we didn't even look at it, harumph. Anyway, the expression above is annoying, let's write it out a little differently.

$$\mathcal{L}_{\boldsymbol{X}} \partial_a \Omega_{bcdef...}$$

Now we know the one-form is just the "gradient" (or what we often think of as the gradient), and all it does is take partial derivatives of every single component in the p-form.

Well guess what, derivatives commute and can be taken in any order. However, from a cursory inspection of A.18 it appears constants are thrown into the middle of everything, so it's not eactly a trivial result... The first term is the problem child–every subsequent term has the Ω out front where the derivative can easily access it no problem.

But the first term...

$$\mathcal{L}_{\boldsymbol{X}}\Omega_{bcdef\ldots} = X^o\nabla_o\Omega_{bcdef\ldots} + \text{nice terms}$$

Now if we add the one-form back in,

$$\partial_a\mathcal{L}_{\boldsymbol{X}}\Omega_{bcdef\ldots} = \partial_a X^o\nabla_o\Omega_{bcdef\ldots} + \text{nice terms}$$

Now for the other relation, we actually get two terms out since if the Lie Derivative acts on the entire "one form" at once, it produces a term for every single index, including the index on the exterior derivative.

$$\mathcal{L}_{\boldsymbol{X}}\partial_a\Omega_{bcdef\ldots} = X^o\nabla_o\partial_a\Omega_{bcdef\ldots} + \partial_o\Omega_{bcdef\ldots}\nabla_a X^o + \text{nice terms}$$

Because we didn't have a term that changed the differential index before. However, all the "nice terms" are identical and just go poof, making the equality:

$$\partial_a X^o\nabla_o\Omega_{bcdef\ldots} = X^o\nabla_o\partial_a\Omega_{bcdef\ldots} + \partial_o\Omega_{bcdef\ldots}\nabla_a X^o$$

Which we might be able to work with, let's see.

Expand the left side by the product rule.

$$\Rightarrow \nabla_o\Omega_{bcdef\ldots}\partial_a X^o + X^o\partial_a\nabla_o\Omega_{bcdef\ldots} = X^o\nabla_o\partial_a\Omega_{bcdef\ldots} + \partial_o\Omega_{bcdef\ldots}\nabla_a X^o$$

Since derivatives commute we have the two terms closes to the equals sign cancel.

$$\Rightarrow \nabla_o\Omega_{bcdef\ldots}\partial_a X^o = \partial_o\Omega_{bcdef\ldots}\nabla_a X^o$$

Now this is quite promising. In fact, covariant and partial derivatives are interchangeable... IF we have a symmetric affine connection, that is, that the Christoffels commute. Notably, this assumption has been *implicit* in all our work so far since we relied on A.11, A.13, A.14, all of which use the covariant derivative version.

We are not sure this assumption is valid, to be sure... HOWEVER! The *general* form of the Lie Derivative uses partial derivaties, so if we doubt this step, we can revert to A.8 and use the partial derivative version. Which means that, even in the GENERAL case...

$$\Rightarrow \partial_o\Omega_{bcdef\ldots}\partial_a X^o = \partial_o\Omega_{bcdef\ldots}\partial_a X^o$$

Which IS clearly true and what we sought to prove.

18 Problem A4 [Back to top]

*Let $x^a(\lambda)$ be the integral curves of a vector field x^a, and let Y^a be a second vector field. Show that if Y^a is **Lie dragged** along X^a, $\mathcal{L}_{\boldsymbol{X}} Y^a = 0$, then it will connect points of equal λ along the congruence $x^a(\lambda)$*

Okay let's just try to evaluate this, as a vector field IS a vector, in a way.

$$\mathcal{L}_{\boldsymbol{X}} Y^a$$

$$= X^b \nabla_b Y^a - Y^b \nabla_b X^a$$

$$= [X, Y]^a$$

So we are actually *given* that this equals zero. We need to take this zero and, from it, show that Y connects poitns of equal λ along the given congruence. What we're doing is looking for a situation where this commutator equals zero. This is obviously only true when the fields are equal to each other, that is, X=Y. (Or one of them is zero but that's trivial.)

This means that Y has the same integral curves as X. Thus we have shown what we sought.

(Mild confusion… we'll see if this leads to any problems later.)

19 Problem 15 [Back to top]

*Following the example of **Problem 7**, show that the 4-dimensional Reimann tensor ${}^{(4)}R_{abcd}$ can be written as:*

$${}^{(4)}R_{abcd} = \gamma^p_a \gamma^q_b \gamma^r_c \gamma^s_d {}^{(4)}R_{pqrs} - 2\gamma^p_a \gamma^q_b \gamma^r_{[c} n_{d]} n^s {}^{(4)}R_{pqrs} - 2\gamma^p_c \gamma^q_d \gamma^r_{[a} n_{b]} n^s {}^{(4)}R_{pqrs} + 2\gamma^p_a \gamma^r_{[c} n_{d]} n_b n^q n^s {}^{(4)}R_{pqrs} - 2\gamma^p_b \gamma^r_{[c} n_{d]} n_a n^q n^s {}^{(4)}R_{pqrs}$$

Now as we recall **Problem 7** was annoying and involved some "questionable" index shuffling, but we'll see if we can do it agian. The antisymmetric parts are a bit problematic.. let's examine what they actually mean.

$$\gamma^r_{[c} n_{d]} = \frac{1}{2}(\gamma^r_c n_d - \gamma^r_d n_c)$$

notably this is actually helpful if we EXPAND, then it'll get rid of all the 2-coefficients and give us more terms, but all temrs that are more similar to each other.

$${}^{(4)}R_{abcd} = \gamma^p_a \gamma^q_b \gamma^r_c \gamma^s_d {}^{(4)}R_{pqrs} - \gamma^p_a \gamma^q_b \gamma^r_c n_d n^s {}^{(4)}R_{pqrs} + \gamma^p_a \gamma^q_b \gamma^r_d n_c n^s {}^{(4)}R_{pqrs} - \gamma^p_c \gamma^q_d \gamma^r_a n_b n^s {}^{(4)}R_{pqrs} + \gamma^p_c \gamma^q_d \gamma^r_b n_a n^s {}^{(4)}R_{pqrs} + \gamma^p_a \gamma^r_c n_d n_b n^q n^s {}^{(4)}R_{pqrs} - \gamma^p_a \gamma^r_d n_c n_b n^q n^s {}^{(4)}R_{pqrs} - \gamma^p_b \gamma^r_c n_d n_a n$$

$$\Rightarrow {}^{(4)}R_{abcd} = (\gamma^p_a\gamma^q_b\gamma^r_c\gamma^s_d - \gamma^p_a\gamma^q_b\gamma^r_c n_d n^s + \gamma^p_a\gamma^q_b\gamma^r_d n_c n^s - \gamma^p_c\gamma^q_d\gamma^r_a n_b n^s + \gamma^p_c\gamma^q_d\gamma^r_b n_a n^s + \gamma^p_a\gamma^r_c n_d n_b n^q n^s - \gamma^p_a\gamma^r_d n_c n_b n^q n^s - \gamma^p_b\gamma^r_c n_d n_a n^q n^s + \gamma^p_b\gamma^r_d n_c n_a n^q n^s)\, {}^{(4)}R_{pqrs}$$

So basically we need to show that the things in the parentheses reduce to nothing more than a "change the index of R" situation. Which pretty clearly means it becomes $\delta^p_a\delta^q_b\delta^r_c\delta^s_d$

$${}^{(4)}R_{abcd} = (\gamma^p_a\gamma^q_b\gamma^r_c\gamma^s_d - \gamma^p_a\gamma^q_b\gamma^r_c n_d n^s + \gamma^p_a\gamma^q_b\gamma^r_d n_c n^s - \gamma^p_c\gamma^q_d\gamma^r_a n_b n^s + \gamma^p_c\gamma^q_d\gamma^r_b n_a n^s + \gamma^p_a\gamma^r_c n_d n_b n^q n^s - \gamma^p_a\gamma^r_d n_c n_b n^q n^s - \gamma^p_b\gamma^r_c n_d n_a n^q n^s + \gamma^p_b\gamma^r_d n_c n_a n^q n^s)\, {}^{(4)}R_{pqrs}$$

Much like **Problem 7** even after much monkeying, the relations just don't seem to be equal.

20 Problem 16 [Back to top]

Show that

$$\nabla_a V^a = \frac{1}{\alpha} D_a(\alpha V^a)$$

For any spatial vector V^a. Hint: One possible derivation uses 2.51 and 2.62; a more elegant aproach starts with the identity A.44

2.51 is from **Problem 13** and is $a_a = D_a ln\alpha$

2.62 is the long

$$D_a V^b = \gamma^p_a\gamma^b_q\nabla_p V^q = \gamma^p_a(g^b_q + n_q n^b)\nabla_p V^q = \gamma^p_a\nabla_p V^b - \gamma^p_a n^b V^q \nabla_p n_q = \gamma^p_a\nabla_p V^b - n^b V^e \gamma^p_a\gamma^q_e\nabla_p n_q = \gamma^p_a\nabla_p V^b + n^b V^e K_{ae}$$

Meanwhile A.44 gives us

$$\nabla_a X^a = \frac{1}{\sqrt{|g|}}\partial_a(\sqrt{|g|}X^a)$$

21 Addendum: Output this notebook to LaTeX-formatted PDF file [Back to top]

The following code cell converts this Jupyter notebook into a proper, clickable LaTeX-formatted PDF file. After the cell is successfully run, the generated PDF may be found in the root NRPy+ tutorial directory, with filename `NR-02.pdf` (Note that clicking on this link may not work; you may need to open the PDF file through another means.)

Important Note: Make sure that the file name is right in all six locations, two here in the Markdown, four in the code below.

- NR-02.pdf
- NR-02.ipynb
- NR-02.tex

[1]:
```
import cmdline_helper as cmd    # NRPy+: Multi-platform Python command-line interface
cmd.output_Jupyter_notebook_to_LaTeXed_PDF("NR-02")
```

Created NR-02.tex, and compiled LaTeX file to PDF file NR-02.pdf

[ ]:
```
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[ ]:
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