For bubbles paths reconstructions it has been used: Laplacian of Gaussian filter (LoG) (Torre V., Poggio T., 1986, Kong et al., 2013), algorithm of detection of local extremes of image brightness and Kalman filter (Kalman R.E., 1960; Maybeck, 1979; Negenborn, 2003; Welch and Bischop, 2006) with Hungarian algorithm (Kuhn, H. W., 1955). The path of the bubble is created frame by frame. Laplacian of Gaussian filter produces monochromatic frame pictures where local extremes of image brightness are estimating the bubble centre. The algorithm of detection of local extremes of image brightness calculates the coordinates of all local extremes on each bubble. The initial position of the bubble is calculated using LoG and is corrected by the Kalman filter. For Kalman's filter motion model 2D space (x, y) was used. State matrix and state vector of the motion model in 2D space were applied in the form [Gustafsson F. et al., 2002]:

$$\begin{bmatrix} x_t \\ y_t \\ \dot{x}_t \\ \dot{y}_t \\ \ddot{x}_t \\ \ddot{y}_t \end{bmatrix} = \begin{bmatrix} I & It & I\frac{1}{2}t^2 \\ 0 & I & It \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{y-1} \\ \dot{x}_{t-1} \\ \dot{y}_{t-1} \\ \ddot{x}_{t-1} \\ \ddot{y}_{t-1} \end{bmatrix} \tag{Y}$$

The resulting point is assigned by the Hungarian algorithm to the point estimated in the previous frame of the video (using LoG algorithm). The implementation of above operation sequence for all frames of the video creates the bubble path.

The values from motion model are compared with values from extrema detection algorithm in the correction phase of the Kalman filter, as equation Z describes:

$$\hat{x}_k = \bar{x}_k + K_k(z_k - H\bar{x}_k) \,. \tag{Z}$$

Final estimate in the k frame ($\hat{x}_k$) is equal to the sum of expected position of the bubble (value from model $\bar{x}_k$) and a difference between extrema positions form LoG (z_k) and bubble positions from model ($H\bar{x}_k$). Impact of this difference ($z_k - H\bar{x}_k$) on the estimate value ($\hat{x}_k$) is regulated by the gain K_k. The Hungarian Methods associates position points form model with extrema position points form LoG so ($z_k - H\bar{x}_k$) difference is counted properly.

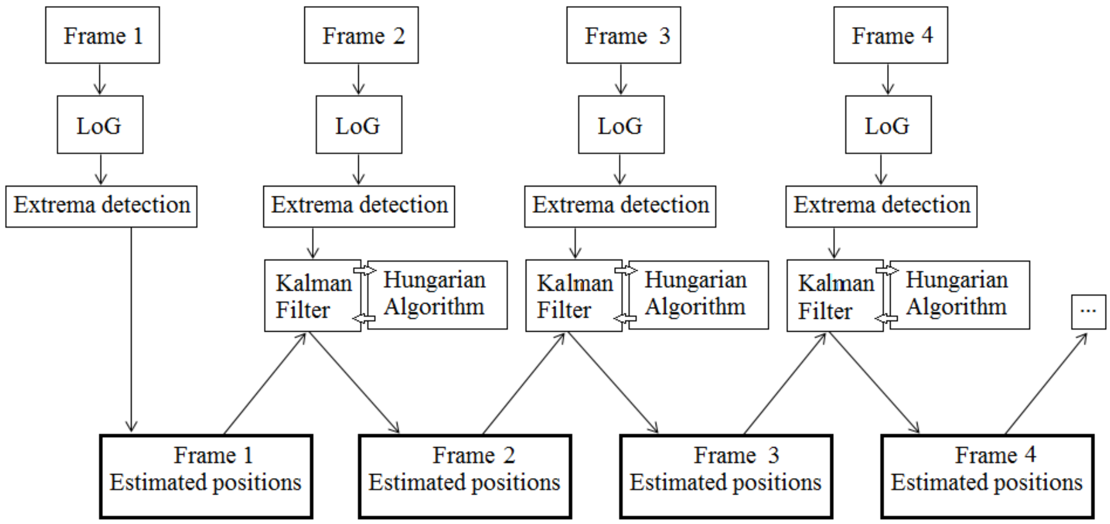

Fig. X Bubble path reconstruction algorithm scheme

The schema of bubble path reconstruction algorithm is shown in Fig. X. Algorithm steps for first 4 frames are presented. Every frame of the analysed video is filtered by the LoG filter. After LoG filtration, extrema detection algorithm finds local extrema in the frame. For the first frame, these local extrema are treated as estimated bubble positions. Extrema positions for the second frame are treated as input for the Kalman filter. Estimated bubble positions from first frame are also used by the Kalman filter. Basing on estimated positions from the first frame and motion model implemented in its construction, Kalman filter predicts bubble positions. At this stage of the algorithm, the Hungarian Algorithm is used. Main task of the Hungarian Algorithm is to find corresponding bubble positions between predicted positions from motion model and positions of image extrema. Corresponding pairs (one point from prediction and one local extrama from detection) are used for correcting positions predicted by the Kalman filter. After correction phase estimated bubble positions for frame 2 are ready. For the third frame, Kalman filter uses estimated positions of the bubble from second frame and local extrema of the image detected in the third frame. With the use of the Hungarian Algorithm, estimated bubble positions for third frame are created. In the fourth frame, estimated bubble positions from the third frame and local extrema from the forth frame are used. Every next frame is analysed in such manner.

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