Introducing bankruptcy choice

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In this note we introduce a consumption-savings model where we give agents the option to file for bankruptcy, which is characterized as an optimal stopping problem. This is a simple version of the households' problem found in the consumer bankruptcy economy described in Mellior and Shibayama (2018). As shown in Moll (2016) we can solve real options by formulating them as a linear complementarity problem (LCP) or with the use of the so-called splitting method. The LCP approach is much faster and will be adopted here. The codes for running this example are `cTsolver.m`, `mainfile.m` and `LCP.m`. Let's first describe the problem, then move on to a discussion of the appropriate boundary condition and finally look at the implementation of the algorithm.

The problem

As in Achdou et al. (2017) agents maximize utility subject to a flow budget constraint. The only idiosyncratic shock affects income z, which is a two point jump process, where λ_L and λ_H are the Poisson rates of jumps from low to high and high to low income, respectively. Agents can save by accumulating wealth a. Negative values of a mean agents are in debt. Additionally, the agent can now choose a time T where it files for bankruptcy[1]. Upon filing for bankruptcy the agent obtains the value of default V^D and loses the value of not being in default V^N. In order to keep things simple we do not explain where V^D comes from, we just take it as given. Moreover, V^D may or may not depend on[2] a. Hence, our problem is shown next.

$$V_i^N(a_t) = \max_{c,T} E_t \left[\int_t^T e^{-\rho(s-t)} u(c) \mathrm{d}s + e^{-\rho(T-t)} V^D(a_T) \right] \tag{1}$$

$$\text{s.t.} \quad \frac{\mathrm{d}a}{\mathrm{d}t} = z_i + ra - c \tag{2}$$

Furthermore, we impose the following:

- There is an exogenous debt limit $a \geq \underline{a}$ where $-\infty < \underline{a} < 0$.
- z is a jump process where $z_H > z_L$ and $i = L, H$.
- CRRA utility function - $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$
- For the sake of simplicity assume that only the low income type can file for bankruptcy.

Following Moll (2016) we can show that the HJB equation[3] is

$$\rho V_i^N = \max_c \; u(c) + \frac{\partial V_i^N}{\partial a} S_i + \lambda_i [V_j^N - V_i^N] \qquad i = L, H \quad i \neq j \tag{3}$$

with the constraint that

$$V_i^N(a) \geq V^D(a). \tag{4}$$

We can express it as a variational inequality

[1] Assume that filing is immediately followed by a discharge of all debts. Also, note that in this note default and bankruptcy are used interchangeably.

[2] More on this further below.

[3] For notational convenience I will be denoting the drift as S instead of $\frac{\mathrm{d}a}{\mathrm{d}t}$.

$$\min \left\{ \rho V_i^N - u(c) - \frac{\partial V_i^N}{\partial a} S_i - \lambda_i [V_j^N - V_i^N], V_i^N - V^D \right\} = 0. \tag{5}$$

Equation (5) can be conveniently solved as a LCP. In reality, instead of looking for the optimal stopping time T, we will be solving for the threshold value a^* where the agent optimally chooses to default (the area where $a < a^*$ is the default region). At the default threshold a^*, the value function V^N satisfies the following optimality conditions[4]:

- Value matching

$$V_L^N(a^*) = V^D(a^*) \tag{6}$$

- Smooth pasting

$$V_L^{N\prime}(a^*) = V^{D\prime}(a^*) \tag{7}$$

Finally, in the no default region we have the standard first order condition in consumption given by

$$u'(c_i) = \frac{\partial V_i^N}{\partial a}. \tag{8}$$

Introducing bankruptcy choice requires 2 modifications of Ben Moll's Huggett model; these are discussed next in two steps.

Step 1: Boundary condition at $\underline{a}$

The approach taken in this note will require us to think about the boundary condition at a^* *and consumption* at $\underline{a}$. The reason is that, unlike other methods, we will fix the asset grid and keep it the same throughout our computations. If we find that the optimal default threshold is $a^* > \underline{a}$ we will still have to compute how agents behave in the space $a < a^*$. We illustrate this by first considering the case of a value function of default V^D that is flat (it does not depend on a). We then show what happens when the slope becomes positive.

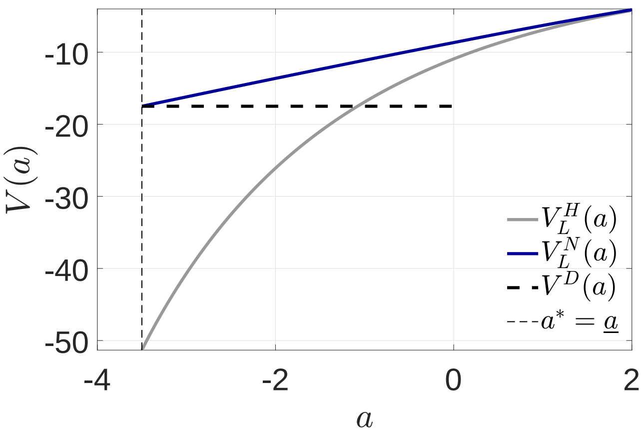

Figure 1: Corner solution

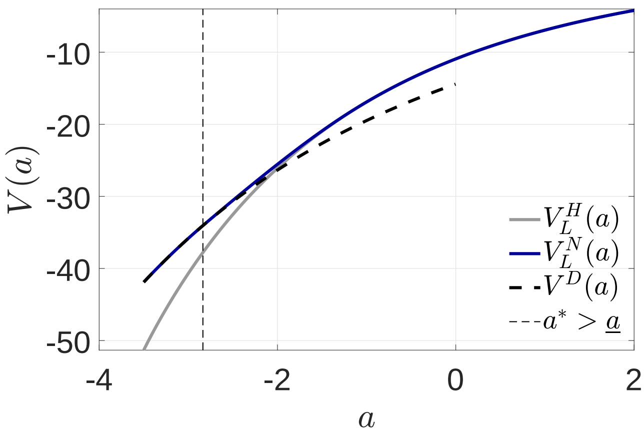

Figure 2: Smooth pasting

[4]See Dixit and Pindyck (1994). If the default threshold is at $\underline{a}$, smooth pasting may not be satisfied (more on this below).

Bankruptcy when the value of default is flat

In Figures (1) and (2) V_L^H depicts the value function of a low-income-type agent that is not allowed to file for bankruptcy (a Huggett value function). V_L^N and V^D represent the values of the low income type when bankruptcy is allowed and that of default, respectively. Figure (1) illustrates how a flat value of default gives rise to a corner solution. A flat value function implies that the marginal penalty of more debt in the default regime is zero. Using the smooth pasting condition would imply infinite consumption[5] at $\underline{a}$. When bankruptcy occurs at the debt limit and the slope of $V^D(a)$ is too small, we will not be able to rely on smooth pasting. However, we can still use value matching, equation (6). In order to do so, define $F(c(\underline{a}))$ as the discrepancy in value matching at $\underline{a}$. Let $\underline{c} = c(\underline{a}) = c(a^*)$.

$$F(\underline{c}) = V_L^N(\underline{a}) - V^D \tag{9}$$

Plug the HJB equation for V_L^N, equation (3), along with the FOC[6] to obtain the next expression.

$$F(\underline{c}) = \frac{u(\underline{c}) + u'(\underline{c})S_L + \lambda_L V_H^N(\underline{a})}{(\rho + \lambda_L)} - V^D \tag{10}$$

Let c^* represent the level of consumption at $\underline{a}$ that yields $F(\underline{c}) = 0$, or, if $F(\underline{c})$ does not have a solution, the value of $\underline{c}$ such that $F(\underline{c})$ is minimized. If default is optimal then $F(c^*) = 0$ and value matching is satisfied. Otherwise, $F(\underline{c}) > 0$ and we revert back to a Huggett model. To stress the link, consider the slope of (10) with respect to $\underline{c}$ and Figure (3).

$$\frac{\partial F(\underline{c})}{\partial \underline{c}} = \frac{\partial^2 u(\underline{c})}{\partial \underline{c}^2}(z_L + r\underline{a} - \underline{c})$$

Because $\partial^2 u(\underline{c})/\partial \underline{c}^2 < 0$, it is obvious that $F(\underline{c})$ is U-shaped and that it takes its minimum at $\underline{c} = z_L + r\underline{a}$ (i.e., at $S_L = 0$). The minimum of $F(\underline{c})$ is precisely the boundary condition of a Huggett model where default is not allowed/not optimal. Therefore, one can think of the standard Huggett model as having $F(c^*) > 0$, as depicted in Figure (3)[7]. When the agent files for bankruptcy the drift will be negative and $c(\underline{a}) > z + r\underline{a}$.

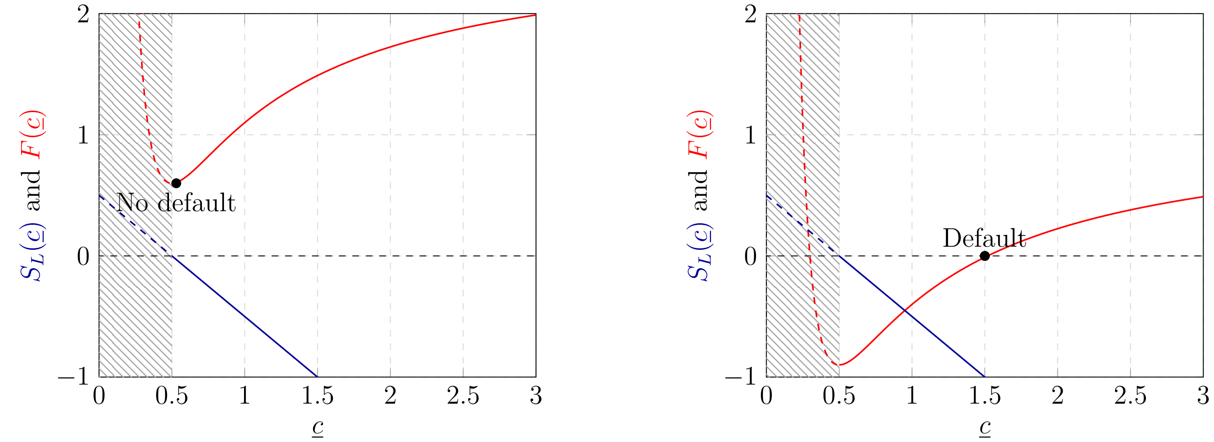

Figure 3

[5]This consumption spike never takes place when agents optimize; infinite consumption at a very short span of time violates consumption smoothing. Hence, if $\frac{\partial V^D(a)}{\partial a}$ is too flat, agents will increase borrowing (i.e., pushing a^* to the left). The agent will increase its consumption level for all time periods from t_0 until the moment of bankruptcy filing, thus spreading out the consumption spike. As a result, the level of $V_L^N(a)$ shifts up for the entire state space. For an exposition on why smooth pasting is not satisfied when the value function of default is flat see the appendix in Mellior and Shibayama (2018).

[6]Note that when bankruptcy is optimal, the FOC is still satisfied at $\underline{a}$. For a discussion beyond this note, see Mellior and Shibayama (2018).

[7]We make explicit the dependence of the drift on $\underline{c}$ to highlight the connection between the sign of savings and bankruptcy choice.

We can see in Figure (3) that when bankruptcy is optimal, $F(c^*)$ will have two roots[8]. We can immediately discard the smaller root since it yields positive savings at the debt limit. Positive savings at $\underline{a}$ imply we are moving away from the default boundary and runs into a contradiction with exercising the option to default. That is why the drift and $F(\underline{c})$ function are dashed and greyed out for consumption below $z_L + r\underline{a}$. The matlab file `cTsolver.m` will take care of finding this root at $\underline{a}$.

Animating the boundary condition

Anything that relatively increases the option value of bankruptcy raises consumption at the moment of default c^*. Hence, increasing V^D and ρ and decreasing λ_L make consumption at the boundary go up. When the value of bankruptcy goes up there are downwards vertical shifts in the value matching equation - see Figure (3) and the animation[9] in Figure (4).

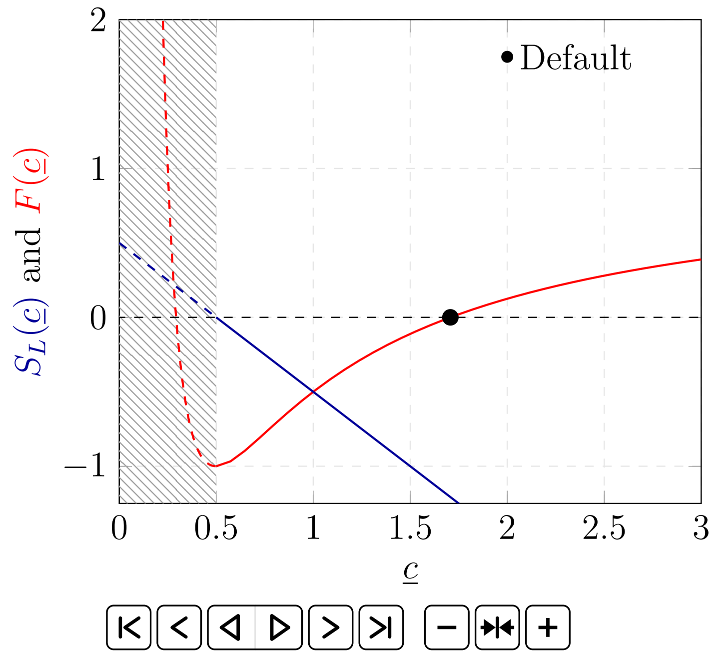

Figure 4

As we can see in the figures, the unique minimum of (10) is situated where the drift $S_L(\underline{c})$ is equal to zero. This is the standard boundary condition for $c(\underline{a})$ in the Huggett model without bankruptcy. In that case the agent can no longer issue more debt when it reaches the exogenous debt limit. Our method will capture this boundary condition when $F(\underline{c})$ does not cross the zero line. Furthermore, when bankruptcy becomes attractive equation (10) has two roots. One implies a positive drift - located in the greyed out area. As mentioned above we discard this case. Thus we keep the other root, which yields a negative drift (consumption larger than $z_L + r\underline{a}$) and is consistent with exercising the option to default.

Non-flat value of default

If the marginal penalty of more debt in the default region is positive and small, we should not expect the results to change. The agent will push a^* towards $\underline{a}$ for the same reasons outlined above. This can be seen in Figure (5). The relative difference in curvature of $V^D(a)$ and $V_L^N(a)$ will determine whether default takes place in the interior of the state space. Figure (2) shows a case where both $V_L^N(a)$ and

[8]This reveals a connection with what Klaus Wälde described as the so-called twin solutions of HJB equations, developed in further detail in Wälde (2010). Essentially, the HJB equation for a standard problem has two roots in consumption, but one of them imply a value function that is not concave; for standard problems we should focus only on the root that gives us a concave value function. In our bankruptcy model we pick the root that gives us a negative drift at a^*.

[9]You need to open this note with a good PDF reader (Adobe Acrobat reader works fine) in order to play the animation.

$V^D(a)$ display more curvature and where $a^* > \underline{a}$. The relative curvature of the two value functions can be affected by many factors not considered on this note for the sake of brevity. These could be bankruptcy laws and risk premia, for instance.

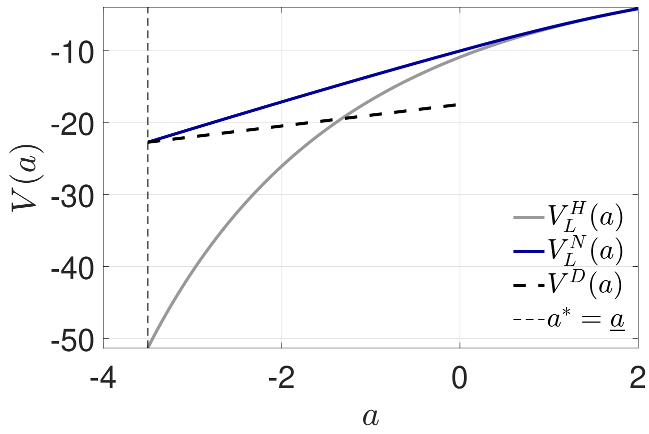

Figure 5

The message is that if $V^D(a)$ is high enough to meet value matching and has a positive slope, then a^* may be in the interior of the state space or at $\underline{a}$. Smooth pasting will be satisfied in the former but generally not in the latter[10], as shown in Figures (2) and (5). Regardless of which case takes place, we still have to compute consumption at $\underline{a}$. Our solver takes care of finding consumption at this point, following the approach outlined above. Checking value matching and smooth pasting at a^* will not require a solver when default occurs in the interior; the LCP algorithm takes care of this[11]. The approach taken in this note is general enough to catch both cases.

Putting everything together

The discussion above is summarized in Table (1).

Table 1: Boundary Conditions

Default	Boundary	Value matching	Smooth pasting	Drift at $\underline{a}$
Yes, at $a^* > \underline{a}$	interior solution	yes	yes	-
Yes, at $a^* = \underline{a}$	corner solution	yes	no, $\frac{\partial V_L^N(\underline{a})}{\partial a} > \frac{\partial V^D(\underline{a})}{\partial a}$	-
No	Huggett-Achdou	no, $V_L^N(\underline{a}) > V^D(a)$	no	0

The final step is to fix the entries of the $\boldsymbol{A}$ matrix, consistent with the drift at $\underline{a}$.

Step 2: The modified $\boldsymbol{A}$ matrix

The $\boldsymbol{A}$ matrix describes the flows from one point to another in the state space[12]. We can compactly write the HJB equation as

$$\rho \boldsymbol{v} = u(\boldsymbol{c}) + \boldsymbol{A}\boldsymbol{v}$$

where we solve implicitly for the value function as shown next.

[10] Of course, there may be cases where the slopes of $V_L^N(a)$ and $V^D(a)$ coincide at $\underline{a}$.

[11] As mentioned in Moll (2016), one can show that the HJBVI implies smooth pasting and value matching. One can also verify that this holds in our numerical results.

[12] In order to decrease notational burden the $\boldsymbol{A}$ matrix shown here ignores income transitions. The superscript n on $\boldsymbol{A}$ represents the nth step of the iteration.

$$\frac{V_i^{n+1}-V_i^n}{\Delta t}+\rho V_i^{n+1}=u(c^n)+\frac{\partial V_i^{n+1}}{\partial a}[z+ra_i-c_i^n] \tag{11}$$

$$V_i^{n+1}=\left[\boldsymbol{I}\left(\rho+\frac{1}{\Delta t}\right)-\boldsymbol{A^n}\right]^{-1}\left[u(\boldsymbol{c^n})+\frac{V_i^n}{\Delta t}\right] \tag{12}$$

As in Achdou (2017) et al. and Nuño and Thomas (2015) let the the outflows and inflows be represented as follows[13].

$$x_i=-\min\left\{\frac{z+ra_i-c_i^B}{\Delta a},0\right\} \qquad \text{inflow from } a_i \text{ to } a_{i-1}$$

$$y_i=\min\left\{\frac{z+ra_i-c_i^B}{\Delta a},0\right\}-\max\left\{\frac{z+ra_i-c_i^F}{\Delta a},0\right\} \qquad \text{outflow from } a_i$$

$$z_i=\max\left\{\frac{z+ra_i-c_i^F}{\Delta a},0\right\} \qquad \text{inflow from } a_i \text{ to } a_{i+1}$$

where these flows are captured by the matrix $\boldsymbol{A}$.

$$\boldsymbol{A}=\begin{bmatrix} y_1 & z_1 & 0 & \cdots & \cdots & \cdots & 0 \\ x_2 & y_2 & z_2 & 0 & \cdots & \cdots & 0 \\ 0 & x_3 & y_3 & z_3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & x_{i_{max}} & y_{i_{max}} \end{bmatrix}$$

The last modification concerns the very first entry, y_1. In the benchmark case we set the backward based drift as zero, which effectively imposes the debt limit and forbids accumulating more debts at $\underline{a}$. In our setting the agent will be allowed to optimize at this very last instant prior to exiting. Augmenting the $\boldsymbol{A}$ matrix to capture such a feature would leave us with an x_1 term representing the backward drift as shown next. Let I and $\boldsymbol{I}$ denote the number of points in the asset grid and the identity matrix of size $I \times I$.

$$\underbrace{A_{\text{augmented}}}_{(I+1)\times(I+1)}=\begin{bmatrix} y_0 & z_0 & 0 & \cdots & \cdots & \cdots & 0 \\ \boldsymbol{x_1} & y_1 & z_1 & 0 & \cdots & \cdots & 0 \\ 0 & x_2 & y_2 & z_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & x_{i_{max}} & y_{i_{max}} \end{bmatrix} \quad \underbrace{A_{\text{modified}}}_{I\times I}=\begin{bmatrix} y_1+\boldsymbol{x_1} & z_1 & 0 & \cdots & \cdots & \cdots & 0 \\ x_2 & y_2 & z_2 & 0 & \cdots & \cdots & 0 \\ 0 & x_3 & y_3 & z_3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & x_{i_{max}} & y_{i_{max}} \end{bmatrix}$$

We can trim the augmented matrix and work with a A_{modified}, by adding x_1 to the first entry of $\boldsymbol{A}$ and subtracting $u'(c^*)x_1\Delta a$ to the right hand side of expression (11). Notice that when we show this in (11) it is clear that we are just adding the utility flow stemming from borrowing one last instant before becoming bankrupt. We know what x_1 is since we know $c(\underline{a})$ from the value matching equation (10). Adding x_1 to y_1 makes the first entry of A_{modified} equal to zero. Now, let A_{modified} be represented as $\boldsymbol{A}_m$ This leaves us with the following system of equations.

$$V_i^{n+1}=\left[\boldsymbol{I}\left(\rho+\frac{1}{\Delta t}\right)-\boldsymbol{A_m^n}\right]^{-1}\left[u(\boldsymbol{c^n})+\frac{V_i^n}{\Delta t}-u'(c^*)\boldsymbol{x_1\Delta a}\mathbb{1}_{\boldsymbol{i=1}}\right]$$

$$V_i^{n+1}=\left[\boldsymbol{I}\left(\rho+\frac{1}{\Delta t}\right)-\boldsymbol{A_m^n}\right]^{-1}\left[u(\boldsymbol{c^n})+\frac{V_i^n}{\Delta t}+u'(c^*)[z+ra-c^*]\mathbb{1}_{\boldsymbol{i=1}}\right] \tag{14}$$

[13] This follows from the upwind scheme. See the numerical appendix in Achdou (2017) et al.

Recall that $u'(c^*) = \frac{V_1^{n+1} - V_0^{n+1}}{\Delta a} = u'(c(\underline{a}))$. Hence, the RHS has an extra term in the first grid point capturing the utility flow from moving into bankruptcy. Remember that c^* is the value such $F(\underline{c}) = 0$ (default) or, if there is no solution, the minimum of $F(\underline{c})$ (Huggett). In the latter case, the zero drift eliminates the extra term on the RHS and we are back in the standard case of no default. Remark that the $\boldsymbol{A}_m$ matrix is ill suited for obtaining the wealth distribution if default does take place. For further reference see Ben Moll's treatment of the KFE in the liquid-illiquid note on the HACT project website and Nuño and Thomas (2015).

The algorithm

The main file is `mainfile.m`. It first runs a standard Huggett model, which will be used as a starting guess for the HJBVI. The second part, which compares the values of $V_i^N(a)$ and $V^D(a)$, calls on `cTsolver.m` (for getting consumption at $\underline{a}$) and the `LCP.m` function. In order to illustrate how to solve for the bankruptcy choice we have set $V^D(a)$ and interest payments on debt as given. Note that the shape of these will have a major impact on whether it is optimal to file for bankruptcy at $\underline{a}$ or at the interior of the state space. Hence, the algorithm lets you use different, arbitrarily defined, risk premium and $V^D(a)$ functions. You may affect these choices by setting `risk-premium` and `Vstarflat` to 1 or 0.

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