Solving HACT models with bankruptcy choice*

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Abstract

We introduce bankruptcy choice to the heterogeneous agent in continuous time (HACT) framework developed in Achdou et al. (2022). We demonstrate that real-options-like problems such as the decision to declare bankruptcy can be efficiently solved using the "value-matching" condition only (unlike alternative methods that require both value matching and "smooth pasting"). Moreover, we show that under certain conditions, smooth-pasting may not hold. Given this, we recommend (and demonstrate the use of) linear complementarity problem (LCP) solvers for real-option like problems, especially in settings where control variables depend on the slope of the value function. We show that this approach is more flexible and computationally efficient than other popular solution methods. In particular, it is less prone to errors in settings that have corner solutions.

Keywords: Incomplete markets, Bankruptcy, Real options, Continuous time, Heterogeneous agent models

JEL Classifications: D14, E21, K35, C61, C63

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1 Introduction

Continuous-time methods for efficiently solving heterogeneous agent models have transformed the macroeconomic research agenda in recent years Achdou et al. (2022), Ahn et al. (2018) and Fernández-Villaverde et al. (2023). In this note we introduce bankruptcy (default) choice to the standard consumption-savings problem, in the heterogeneous agent in continous time (HACT) setting developped by Achdou et al. (2022). The contributions of this note are twofold. First, we demonstrate that real-options-like problems such as the decision to file for bankruptcy can be efficiently solved using the value-matching condition only (unlike alternative methods that require both value matching and smooth pasting). Moreover, we show that under certain conditions, smooth-pasting may not hold. This allows us to present a theoretical yet intuitive understanding of boundary conditions in models with incomplete markets in continuous time.

Second, given that smooth pasting may not hold at the threshold between default and no-default, we recommend computing real-option like problems such the decision to default with linear complementary problem (LCP) solvers. This framework is both more flexible and also reduces the computational cost relative to other methods implemented in HACT models that feature bankruptcy choice. For example, Nuno and Thomas (2015) use a method that starts by imposing smooth pasting and repeatedly varies the grid endpoints until value matching is satisfied.[1] Our proposed LCP approach is more flexible as the grid can remain fixed, can be generalized to a larger state vector and can solve for bankruptcy choice without needing to verify smooth pasting. The LCP approach also has benefits over the splitting method (SM), used for example by Bornstein (2020). SM keeps grid points fixed, is straightforward and intuitive to implement but is computationally demanding as it requires iterating the Hamilton-Jacobi-Bellman (HJB) equation with a small updating parameter.[2] The LCP approach is more common in the finance literature - see Huang and Pang (1998) - but has been used less extensively in economic settings (with the exception of Moll (2016), Kaplan et al. (2017), Shaker-Akhtekhane (2017) and Mellior (2023)). Since alternative methods are currently in use, we provide novel theoretical results on boundary conditions and numerical comparisons to show that the LCP method, which is not reliant on verifying smooth pasting and is more flexible, has benefits in terms of computational efficiency.

In Section 2 we introduce bankruptcy choice to an otherwise standard consumption-savings problem. Section 3 shows cases where bankruptcy is optimal but where smooth pasting cannot be satisfied and discusses the intuition behind boundary conditions in HACT models. Section 4 numerically compares the computational efficiency of alternative solution methods used in the HACT literature.

2 The model

The model builds on Achdou et al. (2022) where agents maximize utility from consumption, c_t, discounted at rate ρ, subject to a flow budget constraint. Agents experience idiosyncratic income shocks. Income z follows a two point jump process, where $z_H > z_L$ and λ_L and λ_H are the Poisson rates of jumps from low to high and high to low income, respectively. Agents can save by accumulating wealth a (a negative value of a means an agent is in debt). There is an exogenous debt limit, $\underline{a}$, such that $a_t \geq \underline{a}$, where $-\frac{z_L}{r} < \underline{a} < 0$ and r is the interest rate. In contrast to Achdou et al. (2022), we assume agents can choose to file for bankruptcy. Filing

[1] This approach is not scalable when the state space has a dimension bigger than one. For instance, it cannot handle settings where agents may default at different values of wealth as the income state changes.

[2] Hurtado et al. (2023) present another method that keeps grids fixed but assumes that opportunities to default arrive randomly.

for bankruptcy is immediately followed by discharge of all debts, the agent obtains the value of default, V^D, and loses the value of not being in default, V^N. Formally, the agent's problem is given by

$$V_i^N(a_t) = \max_{c,T} E_t \left[\int_t^T e^{-\rho(s-t)} u(c_s) \mathrm{d}s + e^{-\rho(T-t)} V^D(a_T) \right], \tag{1}$$

$$\text{s.t.} \quad \frac{\mathrm{d}a_t}{\mathrm{d}t} = z_{it} + r_t a_t - c_t, \tag{2}$$

where $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ and $\sigma > 0$. Without loss of generality, we parameterize the model such that only the low income type ever files for bankruptcy. Furthermore, we assume a debt elastic interest rate function given by.[3]

$$r(a) = \overline{r} + \gamma_0 e^{-\gamma_1(a-\gamma_2)}. \tag{3}$$

The stationary Hamilton-Jacobi-Bellman (HJB) equation of the no-default value function is

$$\rho V_i^N = \max_c \; u(c) + \frac{\partial V_i^N}{\partial a} S_i + \lambda_i [V_j^N - V_i^N], \qquad i = L, H \quad i \neq j. \tag{4}$$

For notational convenience we drop time subscripts and denote the drift as S instead of $\frac{\mathrm{d}a}{\mathrm{d}t}$. In the no default region we have the first order condition in consumption (FOC) given by

$$u'(c_i) = \frac{\partial V_i^N}{\partial a}. \tag{5}$$

The value function of no-default must satisfy the following constraint

$$V_i^N(a) \geq V^D(a). \tag{6}$$

Denote a^* the default threshold. When $a^* > \underline{a}$, there is an interior solution and V^N satisfies the following optimality conditions:

$$V_L^N(a^*) = V^D(a^*), \tag{7}$$

$$\frac{\partial V_L^N(a^*)}{\partial a} = \frac{\partial V^D(a^*)}{\partial a}. \tag{8}$$

These conditions are known as value matching (7) and smooth pasting (8).[4] If the default threshold is at $\underline{a}$, smooth pasting may not be satisfied. We re-express the agent's problem as a variational inequality as follows

$$\min \left\{ \rho V_i^N - \; u(c) - \frac{\partial V_i^N}{\partial a} S_i - \lambda_i [V_j^N - V_i^N], V_i^N - V^D \right\} = 0. \tag{9}$$

Equation (9) can be conveniently solved as a LCP. Note that this formulation does not impose the smooth pasting condition; it is a byproduct (in cases were default takes place in the interior of the state space) Moll (2016).

[3]The debt elastic interest rate is not strictly necessary. However, as it can generate more curvature in the value function it simplifies obtaining corner or interior solutions by changing only one parameter.

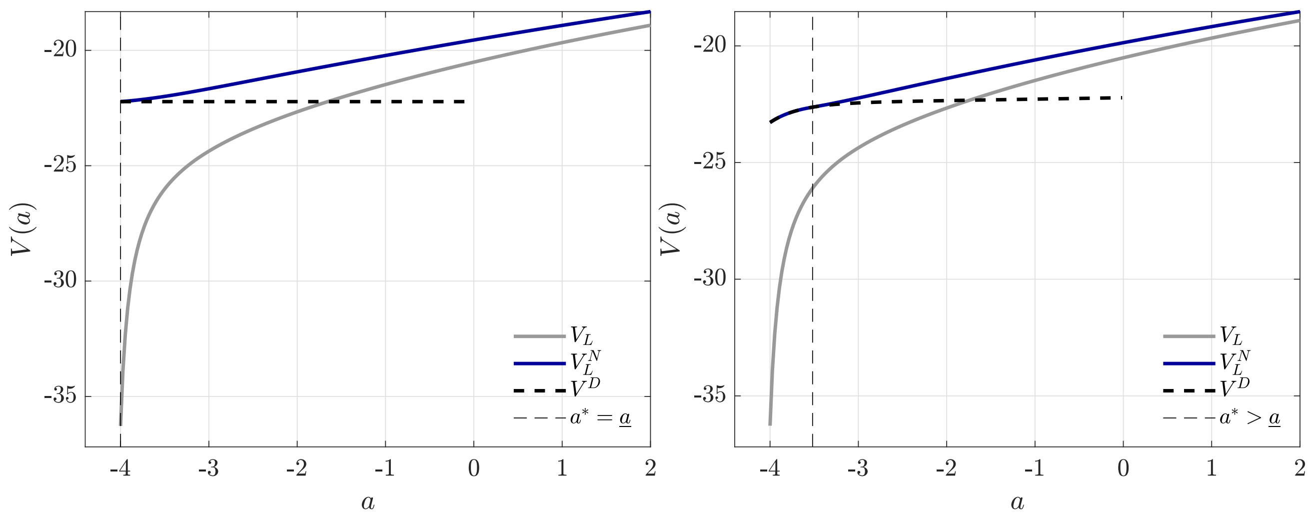
[4]See Dixit and Pindyck (1994).

3 Default as an interior vs corner solution

This section shows three distinct cases where default takes place, with particular consideration of when smooth pasting is or is not satisfied. If bankruptcy is optimal, the default threshold is either $a^* > \underline{a}$ or $a^* = \underline{a}$. In both cases we have to compute how agents behave at $\underline{a}$. Consider the following parameterization of the default value function

$$V^D = \frac{u(z^d + \psi r a)}{\rho}, \tag{10}$$

where z^d and ψ capture income when in default and how much the debt position at the moment of default affects V^D, respectively. The latter can be thought of as the marginal penalty of entering default with more debt. We use the following calibration throughout this note $\rho = 0.05$, $\sigma = 2$, $z_L = 0.75$, $z_H = 1.25$, $z^d = 0.9$ $\gamma_0 = 0.0075$, $\gamma_1 = 2.7$, $\gamma_2 = -3$, $\bar{r} = 0.035$, $\lambda_L = \lambda_H = 0.25$ and $\underline{a} = -4$. We vary ψ, which can generate a corner solution when it is relatively small (Figures 1a and 2) and interior solutions at larger values, as in Figure 1b.



(a) Corner solution & smooth pasting not satisfied

(b) Interior solution & smooth pasting satisfied

Fig. 1: Default as an interior vs corner solution

In Figures 1a and 1b, V_L depicts the value function of a low-income-type agent in an environment where bankruptcy is not possible. V_L^N and V^D depict the value function of the low income type when bankruptcy is allowed and that of default, respectively. Figure 1a illustrates how a constant value of default gives rise to a corner solution. A constant value function, generated by setting $\psi = 0$, implies that the marginal penalty of more debt in the default regime is zero. In such cases agents push the default threshold, a^*, to the boundary $\underline{a}$. The smooth pasting condition would erroneously imply infinite consumption at $\underline{a}$, violating the agent's consumption smoothing motive. Hence, if $\frac{\partial V^D(a)}{\partial a}$ is relatively flat, optimizing agents would rather increase consumption for all time periods from t_0 until the moment of bankruptcy filing, instead of engaging in an infinite consumption spike at $\underline{a}$. As a result, the level of $V_L^N(a)$ shifts up for the entire state space. Economic factors such as bankruptcy laws and risk premia will affect the curvature of $V^D(a)$ and $V_L^N(a)$ and determine whether and where default takes place. For instance, as we increase ψ, it becomes less attractive to default with larger amounts of debt. Figure 1b shows that increasing ψ to 0.007 leads to an interior solution where both value matching and smooth pasting are satisfied.

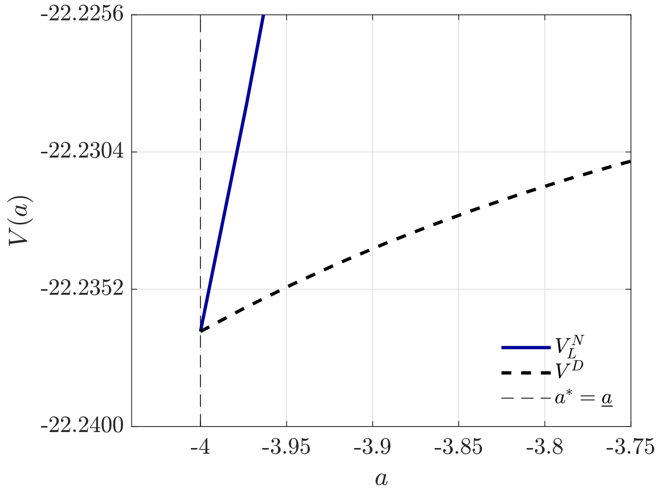


Fig. 2: Corner solution - Smooth pasting is not satisfied even though the slope of the value of default is positive.

Figure 2 shows that we can also have corner solutions in cases where V^D has a positive slope ($\psi = 0.001$). In cases with corner solutions we will not be able to rely on smooth pasting. However, we can still use value matching, equation (7). To do so, define $F(c(\underline{a}))$ as the discrepancy in value matching at $\underline{a}$ as follows

$$F(\underline{c}) = V_L^N(\underline{a}) - V^D, \tag{11}$$

where $\underline{c} = c(\underline{a})$. Plug in (4), the HJB equation for V_L^N and the FOC (5) into $F(c)$ to obtain the next expression.

$$F(\underline{c}) = \frac{u(\underline{c}) + u'(\underline{c})S_L + \lambda_L V_H^N(\underline{a})}{\rho + \lambda_L} - V^D \tag{12}$$

Let c^* represent the level of consumption at $\underline{a}$ that minimizes the residual of $F(\underline{c})$. If default is optimal then $F(c^*) = 0$ and value matching is satisfied. Otherwise, $F(\underline{c}) > 0$ and we recover the boundary condition of the standard consumption-savings problem without default. To stress the link between both cases, consider the slope of (12) with respect to $\underline{c}$ in Figure (3), given by

$$\frac{\partial F(\underline{c})}{\partial \underline{c}} = \frac{u''(\underline{c})\,(z_L + r\underline{a} - \underline{c})}{\rho + \lambda_L}. \tag{13}$$

Since $u'' < 0$, then $F(\underline{c})$ is U-shaped and reaches a minimum at $\underline{c} = z_L + r\underline{a}$ (i.e., at $S_L = 0$). Figure 3 shows that when the agent chooses bankruptcy the drift will be negative and $c(\underline{a}) > z_L + r\underline{a}$. Thus in models where agents are allowed to choose bankruptcy, generating a negative drift at the lower boundary, or even in models where the debt limit can never be violated (non-negative drift at $\underline{a}$), we can solve for consumption at $\underline{a}$ by exploiting (11) and restricting solutions to those where (13) is non-negative.[5]

[5] When the consumption policy implies a negative (positive) drift on wealth, the upwind scheme approximates the derivative of V with a backward (forward) finite difference (see Achdou et al. 2022). In general we use boundary conditions to determine consumption at the boundary. But with bankruptcy choice we do not know in advance which boundary condition should be implemented at $\underline{a}$ nor can we compute the backward looking derivative of V. The approach we propose lets the value matching condition select the correct boundary condition whether bankruptcy is or is not allowed.

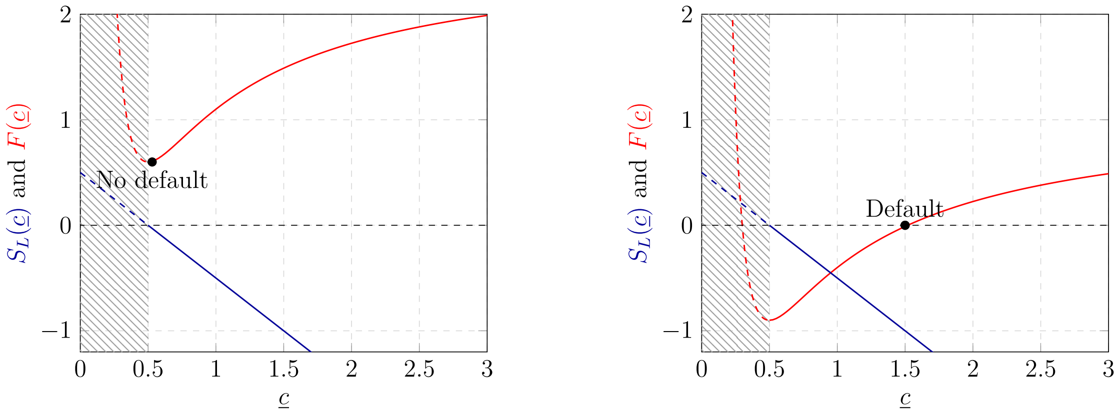


Fig. 3: The value matching equation determines consumption at the boundary $\underline{a}$, yielding the standard boundary condition in Achdou et al. (2022) when bankruptcy is not optimal (left panel) and when bankruptcy is optimal (right panel), implying a negative drift at the boundary.

Figure 3 shows that when bankruptcy is optimal, $F(c^*)$ will have two roots. The smaller root can be ignored since it yields a positive drift at the debt limit. A positive drift at $\underline{a}$ implies we are moving away from the default boundary and runs into a contradiction with exercising the option to default.[6] Instead, we keep the root that yields a negative drift and is consistent with exercising the option to default.[7]

Our method captures the correct boundary condition depending on whether $F(\underline{c})$ crosses the zero line, and if it does, selecting the root where (13) is positive. The insights from this section are summarized in Table 1.[8]

Table 1
Boundary conditions.

Default	Value matching	Smooth pasting	Drift at $\underline{a}$
At $a^* > \underline{a}$ (interior solution)	yes	yes	<0
At $a^* = \underline{a}$ (corner solution)	yes	no ($V_L^{N\prime}(\underline{a}) > V^{D\prime}(\underline{a})$)	<0
No	no ($V_L^N(a) > V^D(a)$)	no	≥ 0

4 Comparing solution methods

Table 2 compares the computational cost of solving V^N with the splitting method (SM), the method devised by Hurtado et al. (2023) (HNT) and the LCP approach described above. We compute the HJB equation absolute and relative residuals for all methods. The convergence criteria is set to a maximum absolute change in the value function, per iteration, of 1e-6.[9] Both the SM and HNT methods are sensitive to the update step, Δ. The smaller the step is, the closer we get to the true solution, but at the cost of additional iterations. This trade-off worsens in cases where there is a corner solution, especially for the SM method.

[6]In Figure 3 we make explicit the dependence of the drift on $\underline{c}$ to highlight the connection between its sign and bankruptcy choice.

[7]Wälde (2010) shows the HJB equation in a standard consumption-savings problem without default has two roots in consumption, but one of them implies a non-concave value function. In our model we pick the root that is consistent with bankruptcy choice.

[8]Appendix A shows how default at the boundary requires modifying the entries of the so called A matrix.

[9]Computations are done with a Toshiba Portege X30-E with Intel Core i7-8550U processor (1.8 GHz) and 16 GB of RAM. We define a grid in wealth with 300 points. Replication codes are available at `https://github.com/GMellior/Bankruptcy_in_HACT`.

Table 2
Computational cost of V^N.

	Update parameter Δ	Time (seconds)	Sup norm†	Rel error‡	Iterations	Boundary
		Case A: Interior solution with $a^* = -3.52$ and $\psi = 0.007$				
LCP	∞	$1.39e-1$	$1.59e-9$	$7.55e-11$	13	-3.52
SM*	$1.00e-3$	3.12e2	$5.06e-3$	$2.18e-4$	68839	-3.52
SM	$5.00e-2$	5.95	$2.47e-1$	$1.06e-2$	1613	-3.49
SM	$1.00e-1$	2.01	$4.77e-1$	$2.04e-2$	859	-3.44
HNT ($\gamma = 46.25$)	$4.00e-2$	3.74	$2.50e-5$	$1.48e-6$	2671	-3.52
HNT ($\gamma = 23.12$)	$8.00e-2$	2.02	$1.25e-5$	$7.39e-7$	1366	-3.49
		Case B: Corner solution with $a^* = \underline{a} = -4$ and $\psi = 0.001$				
LCP	∞	$3.03e-1$	$6.90e-10$	$3.33e-11$	15	-4.00
SM	$5.00e-5$	1.53e3	$5.70e-2$	$2.56e-3$	446750	-4.00
SM	$4.00e-4$	3.51e2	$1.56e-2$	$7.01e-4$	106449	-3.97
HNT ($\gamma = 185.00$)	$5.00e-3$	1.55e1	$2.00e-4$	$1.19e-5$	11844	-3.97
HNT ($\gamma = 92.50$)	$2.00e-2$	8.07	$4.99e-5$	$2.98e-6$	4591	-3.95
		Case C: Corner solution with $a^* = \underline{a} = -4$ and $\psi = 0$				
LCP	∞	$8.12e-1$	$3.10e-9$	$1.51e-10$	18	-4.00
SM	$1.00e-5$	1.20e3	$1.00e-1$	$4.46e-3$	599281	-3.95
SM	$2.00e-4$	1.19e3	$7.66e-3$	$3.45e-4$	183700	-3.57
HNT ($\gamma = 185.00$)	$5.00e-3$	3.63e1	$2.00e-4$	$1.19e-5$	11818	-3.97
HNT ($\gamma = 92.50$)	$2.00e-2$	4.36	$4.99e-5$	$2.98e-6$	4578	-3.95

† and ‡ are the max of the absolute value of the HJB equation residuals in absolute and relative terms, respectively. The arrival rate of bankruptcy opportunities is denoted by γ. SM* has a stopping criteria of 1e-7.

The HNT method requires adjusting both the time step and rate of arrival of bankruptcy opportunities γ. We want this rate to be as large as possible to approximate the bankruptcy choice model, but doing so requires reducing Δ, thus increasing the number of iterations. The LCP method is not sensitive to Δ, is more accurate, requires less iterations and is thus faster than the other two alternatives.

5 Conclusion

If the value of default is high enough to satisfy value matching and has a positive slope, then the default threshold a^* may be in the interior of the state space or at the left boundary $\underline{a}$. Smooth pasting will be satisfied in the former but generally not in the latter. Regardless, we still have to compute consumption at $\underline{a}$. The approach outlined above takes care of finding consumption at this point and is general enough to find the solution to models with debt limits, whether bankruptcy is or is not allowed. Checking value matching and smooth pasting at a^* is not necessary when default occurs in the interior; as it is a by-product of the variational inequality, which can be handled by the LCP solver. We thus propose this approach, as it is more general and computationally efficient relative to two other popular methods that have been used in recent HACT models that feature bankruptcy choice.

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Appendix A

Allowing for default at $\underline{a}$ requires a modification to the finite difference scheme used in Achdou et al. (2022). Without loss of generality, suppose that we have only one income level and that we have I grid points in a. The HJB equation of the no default value function is given by

$$\rho V = \max_c \; u(c) + \frac{\partial V}{\partial a}[z + ra - c]. \tag{14}$$

We can compactly write the HJB equation as

$$\rho \boldsymbol{V} = u(\boldsymbol{c}) + \boldsymbol{A}(\boldsymbol{V})\boldsymbol{V},$$

where $\boldsymbol{V} = [V_1, V_2, ..., V_I]'$ and $\boldsymbol{c} = [c_1, c_2, ..., c_I]'$ are the discretized value function and consumption policy at gridpoints $i = 1, 2, .., I$ and $\boldsymbol{A}(\boldsymbol{V})$ is the discretized operator that computes the drift times the partial derivative of $\boldsymbol{V}$. For ease of notation we refer to $\boldsymbol{A}(\boldsymbol{V})$ as $\boldsymbol{A}$.

Following Achdou et al. (2022) we approximate the derivative of the value function with respect to a with an upwind scheme. This means that when the consumption policy implies a negative (positive) drift on wealth, the upwind scheme approximates the derivative of V with a backward (forward) finite difference. Remember that we compute consumption, with the FOC, as a function of the partial derivative of V with respect to a

$$c = (u')^{-1}\left(\frac{\partial V}{\partial a}\right) \tag{15}$$

Let c^B and c^F represent consumption based on the backward and forward finite difference of V, respectively. As in Achdou et al. (2022) and Hurtado et al. (2023) let the the outflows and inflows at gridpoint i be depicted as follows.[10]

$$x_i = -\min\left\{\frac{z + ra_i - c_i^B}{\Delta a}, 0\right\} \qquad \text{inflow from } a_i \text{ to } a_{i-1}$$

$$y_i = \min\left\{\frac{z + ra_i - c_i^B}{\Delta a}, 0\right\} - \max\left\{\frac{z + ra_i - c_i^F}{\Delta a}, 0\right\} \qquad \text{outflow from } a_i$$

$$z_i = \max\left\{\frac{z + ra_i - c_i^F}{\Delta a}, 0\right\} \qquad \text{inflow from } a_i \text{ to } a_{i+1}$$

The flows above are represented as

$$\boldsymbol{A} = \begin{bmatrix} y_1 & z_1 & 0 & \cdots & \cdots & \cdots & 0 \\ x_2 & y_2 & z_2 & 0 & \cdots & \cdots & 0 \\ 0 & x_3 & y_3 & z_3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & x_I & y_I \end{bmatrix}.$$

In models without default the drift at $\underline{a}$ is non-negative and this is imposed with a boundary condition that enforces a debt limit. The backward based derivative of V at the boundary $\underline{a}$ is set such that c^B implies a drift equal to zero (you consume your income net of interest payments).

$$u'(z + r\underline{a}) = \frac{\partial V^B(\underline{a})}{\partial a} \tag{16}$$

[10] This follows from the upwind scheme. See the numerical appendix in Achdou et al. (2022).

When we allow for bankruptcy we obtain consumption at $\underline{a}$ via the value matching equation, using $F(c)$. If bankruptcy is chosen, the implied drift will not be equal to zero.[11] Remember that the rows in $\boldsymbol{A}$ must sum to zero (Achdou et al. (2022)). Default at $\underline{a}$ makes the sum of the first row of the $\boldsymbol{A}$ matrix different from zero, so the inflows and outflows at this grid point are no longer balanced. We can fix this by introducing a so called 'ghost node'.[12] As we can now have a negative drift at $\underline{a}$ the first grid point of the discretized HJB equation would have the following terms on the RHS

$$u(c^*) + \left(\frac{V_1 - V_0}{\Delta a}\right) S^B(\underline{a}), \tag{17}$$

where $S^B(\underline{a})$ is the backward drift at $\underline{a}$ and V_0 is the ghost node consistent with c^*. We do not need to compute V_0, as we already have c^* and the expression above can thus be rewritten as

$$u(c^*) + u'(c^*) S^B(\underline{a}). \tag{18}$$

If we keep the terms above outside of the $\boldsymbol{A}$ matrix, we restore balance in its first row and in addition we take into account the utility flow stemming from borrowing one last instant before becoming bankrupt at $\underline{a}$. When bankruptcy is not allowed the drift $S^B(\underline{a})$ is equal to zero and thus these terms vanish. Hence, when we allow default at $\underline{a}$ we have to modify the first entry of the $\boldsymbol{A}$ matrix and add a new term to (14). We update the matrix, now represented as $\boldsymbol{A_m}$, by adding an x_1 term to its first entry.

$$\boldsymbol{A}_{\mathrm{m}} = \begin{bmatrix} y_1 + x_1 & z_1 & 0 & \cdots & \cdots & \cdots & 0 \\ x_2 & y_2 & z_2 & 0 & \cdots & \cdots & 0 \\ 0 & x_3 & y_3 & z_3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & x_{I_{max}} & y_{I_{max}} \end{bmatrix}$$

This modification cancels out the negative drift at $\underline{a}$ in the $\boldsymbol{A}$ matrix. By doing so we restore balance on the first row and we take into account the utility flow from the last instant before becoming bankrupt. Recall that $\frac{V_1 - V_0}{\Delta a} = u'(c^*)$.

$$\rho \boldsymbol{V} = u(\boldsymbol{c}) + \boldsymbol{A_m V} + u'(c^*)[z + r\underline{a} - c^*]\mathbb{1}_{\boldsymbol{i=1}} \tag{20}$$

Remember that c^* is the value of consumption at $\underline{a}$ such that $|F(c)| = 0$ (default) or, if there is no solution, the minimum of $|F(c)|$ (no default). In the latter case, the zero drift eliminates the extra term on the RHS and we are back in the standard case of no default. Finally, note that the transpose of the $\boldsymbol{A}_m$ matrix is ill suited for obtaining the wealth distribution if default does take place. Depending on where agents go to in the state space after declaring default, we have to move the x_1 term in the first row of $\boldsymbol{A_m}$ to the corresponding column that points to where new bankrupt agents go. This will guide the flow of bankrupt agents at $\underline{a}$ to such points.[13]

[11] See Figure 3 and the discussion in Section 3.

[12] See Nuno and Thomas (2015) for an implementation of ghost nodes in HACT models.

[13] If default takes place in the interior of the state space we redirect the flows into bankruptcy in a similar way, but in such cases, using the row of the $\boldsymbol{A}$ matrix that corresponds to the default threshold a^*.