

Magnitude Homology - Report

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1 Goals

1. Develop a library to efficiently compute magnitude homology. To do this we construct a matrix representing the MH boundary operator.
2. Use MH for network analysis. Our hope is that MH can give us information about the structure of a network and how a particular structure influences information flow.

2 Magnitude Homology framework

Definition 1. Let $G = (V, E)$ be a simple graph. We define the (k, ℓ) -magnitude chain, $MC_{k, \ell}(G)$, as the free abelian group generated by the $(k + 1)$ -tuples of vertices of G such that the path (x_0, \dots, x_k) has length ℓ . That is,

$$MC_{k, \ell} = \langle (x_0, \dots, x_k) : (x_0, \dots, x_k) \in V^{k+1}, x_i \neq x_{i+1}, l(x_0, \dots, x_k) = \ell \rangle.$$

The ℓ -magnitude complex $MC_{*, \ell}(G)$ is the direct sum over k of all (k, ℓ) -magnitude chains,

$$MC_{*, \ell}(G) = \bigoplus_{k \geq 0} MC_{k, \ell}(G).$$

Definition 2. The boundary operator $\partial_{k, \ell} : MC_{k, \ell}(G) \rightarrow MC_{k-1, \ell}(G)$ is defined as follows:

$$\partial_{k, \ell}(x_0, \dots, x_k) = \sum_{i=0}^k a_i(x_0, \dots, \hat{x}_i, \dots, x_k), \text{ where}$$

$$a_i = \begin{cases} (-1)^i, & \text{if } l(x_0, \dots, \hat{x}_i, \dots, x_k) = \ell \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3. The (k, ℓ) -magnitude homology group, $MH_{k, \ell}(G)$, is defined as

$$MH_{k, \ell}(G) = \frac{\ker \partial_{k, \ell}}{\text{im } \partial_{k+1, \ell}}$$

2.1 Operative definition

Notice that the information provided by $MC_{k,l}$ is extremely noisy. This is because the definition only asks that each vertex x_i of the k -tuple is different from x_{i+1} , and this does not prevent going back and forth between the same two vertices. That is, a generator of $MC_{4,4}$ is for example $(x_0, x_1, x_0, x_1, x_0)$.

We attempt to overcome this issue with a slight modification of the definition, in which we erase from $MC_{k,l}$ all paths revisiting a vertex.

Definition 4. We define the (k, ℓ) -reduced magnitude chain, $MC_{k,\ell}^{red}(G)$, as the free abelian group generated by the $(k+1)$ -tuples of different vertices of G such that the path (x_0, \dots, x_k) has length ℓ . That is,

$$MC_{k,\ell}^{red} = \langle (x_0, \dots, x_k) : (x_0, \dots, x_k) \in V^{k+1}, x_i \neq x_j, l(x_0, \dots, x_k) = \ell \rangle.$$

The ℓ -reduced magnitude complex $MC_{*,\ell}^{red}(G)$ is the direct sum over k of all (k, ℓ) -reduced magnitude chains,

$$MC_{*,\ell}^{red}(G) = \bigoplus_{k \geq 0} MC_{k,\ell}^{red}(G).$$

Since from now on we will only rely on $MC_{k,\ell}^{red}(G)$, we will indicate (with an abuse of notation) the (k, ℓ) -reduced magnitude chain as $MC_{k,\ell}(G)$.

3 Matrix representing $\partial_{k,\ell}$

MODIFY HERE AND DESCRIBE SPARSE MATRIX

The matrix $\Delta_{k,\ell}$ representing $\partial_{k,\ell}$ is constructed using the following algorithm:

1. find the tuples generating $MC_{k,\ell}(G)$ and $MC_{k-1,\ell}(G)$
2. initialize an all-zeros matrix of dimension $MC_{k-1,\ell}(G) \times MC_{k,\ell}(G)$
3. for $t \in MC_{k,\ell}(G)$, if $\partial_{k,\ell}(t) = t' \in MC_{k-1,\ell}(G)$ change the entry (t', t) to -1 .

3.1 Complexity

3.2 Code

The code can be found in this repository.

3.3 Experiments

This matrix was proven to be effective in the computation of magnitude Betti numbers $\beta_{k,\ell}$ using examples taken from the paper “Categorifying the magnitude of a graph”, Hepworth and Willerton (arXiv: 1505.04125v2).

4 Interpretation of the rank of $MH_{k,l}$.

4.1 Diagonal $MH_{k,k}$

As for the original definition given by Hepworth and Willerton, the ranks of the $(0,0)$ and $(1,1)$ MH groups of a graph $G = (V, E)$ represent the cardinality of V and the cardinality of V , respectively.

$MH_{k,k}$ provides us with the precise number of 3-cycles and 4-cycles contained in a graph (modulo automorphisms). Indeed, consider the following facts:

- the dimension of the kernel of our matrix $\Delta_{k,k}$ is equal to the number of all-zero columns plus the number of columns that are "copies" of a previously written column.
- the number of all-zero columns is (modulo automorphisms) equal to the number of triangles contained in the graph. This is because if a (k,k) -tuple (x_0, x_1, \dots, x_k) is sent to zero after removing a vertex x_i , it means that the shortest path between x_{i-1} and x_{i+1} has length smaller than 2. So there exists an edge (x_{i-1}, x_{i+1}) and equivalently a triangle (x_{i-1}, x_i, x_{i+1}) .
- the number of "repeated" columns indicates how many (k,k) -tuples $(x_0, \dots, x_i, \dots, x_k)$ are sent to the same $(k-1, k)$ -tuple $(x_0, \dots, \hat{x}_i, \dots, x_k)$, and this provides us with upper and lower bounds for the number (modulo automorphisms) of 4-cycles (EXPAND).

Therefore, to obtain the number of 3-cycles contained in our graph we need to divide the number of all-zero columns by 6, i.e. by the cardinality of the automorphisms group of the triangle D_3 .

Remark 5. *We can link this value to the global and local clustering coefficients, and to the cycle ratio defined in [PUT REFERENCE HERE]. In case of a directed graph (i.e. network) this value accounts for transitivity.*

To obtain the number of 4-cycles we first divide the number of "repeated" columns by two, in order to disregard the orientation, and then we again divide by two, so that two $(k-1, k)$ -paths are glued into the same 4-cycle.

Example 6. *PUT HERE EXAMPLE OF ICOSAHEDRAL GRAPH.*

Although the (k,k) -magnitude homology groups $MH_{k,k}$ all provide the same information, we point out that the rank of $MK_{k,k}$ might be different from the rank of $MK_{k',k'}$ when $k \neq k'$, and in particular $\text{rank}(MH_{k,k}) < \text{rank}(MH_{k',k'})$ for $k < k'$. Consider for example the case of the icosahedral graph presented above. Here we have $\text{rank}(MH_{2,2}) = 180$ and $\text{rank}(MH_{3,3}) = 252$, and this is because some cycles are counted more than once in $MH_{3,3}$. For example, the tuples $(11, 7, 9, 10)$, $(4, 10, 9, 7) \in MC_{3,3}$ are both sent to 0 because of the triangle $(7, 9, 10)$.

Given this and considering the (obvious) fact that $MH_{2,2}$ is much faster to compute, in the future analysis we will just make use of $MH_{2,2}$.

4.2 Second diagonal $MH_{k-1,k}$

We are not yet able to provide a general interpretation for all magnitude homology groups $MH_{k-1,k}$ on the second diagonal, but we believe that $MH_{2,3}$ contains information regarding the number of 4-cliques, 5-cliques and 6-cliques in the graph.

Consider the following chain

$$\begin{array}{ccccccc} \dots & \rightarrow & MC_{3,3} & \rightarrow & MC_{2,3} & \rightarrow & MC_{1,3} & \rightarrow & 0 \\ & & (x_0, x_1, x_2, x_3) & \rightarrow & (x_0, \hat{x}_1, x_2, x_3) & \rightarrow & (x_0, \hat{x}_1, \hat{x}_2, x_3) & & \end{array}$$

and assume $x_3 \neq \hat{x}_1$.

If $(x_0, \hat{x}_1, x_2, x_3) \in \ker(\partial_{2,3})$ then one of the following is true:

- Two different tuples in $MC_{2,3}$ are sent to the same tuple in $MC_{1,3}$, which means there is either a 4-cycle or a 6-cycle.
- The considered tuple in $MC_{2,3}$ is sent to zero, which means there exists either a 4-cycle or a 5-cycle.

Now, when we quotient by the image of $\partial_{3,3}$ we are in fact disregarding the elements $(x_0, \hat{x}_1, x_2, x_3) \in MC_{2,3}$ such that $\ell(x_0, \hat{x}_1, x_2, x_3) = \ell(x_0, x_1, x_2, x_3)$. That is, we are disregarding the tuples that do now contain the triangle (x_0, x_1, x_2) , and that therefore cannot be part of a clique.

Summarizing, $MH_{2,3}$ contains information about 4, 5, 6-cycles that contain all triangles, and this means counting 4-cliques and candidates 5, 6-cliques.

Remark 7. We point out that the hypothesis " $x_3 \neq \hat{x}_1$ " is crucial to obtain this interpretation. Indeed, without this assumption it could happen that $x_3 = \hat{x}_1$, which would mean "revisiting an edge" and adding a lot of noise to $MH_{2,3}$.

We recall the definition of *diagonal graph* introduced by Hepworth and Willerton in [ADD REF].

Definition 8. A graph G is called *diagonal* if $MH_{k,l}(G) = 0$ whenever $k \neq l$.

What we noticed until now suggest the following fact

Proposition 9. If a graph G is diagonal, then it is clique-free.

Proof. Suppose G is diagonal, then $MH_{k,l}(G) = 0$ whenever $k \neq l$. In particular $MH_{2,3} = 0$, meaning the graph contains no 4-clique, and therefore no bigger clique. \square

5 Problems to solve

1. The information in the magnitude homology groups $MH_{2,2}$ and $MH_{2,3}$ is "not divided", meaning we are just given the dimension of the kernel without the distinction between all-zero columns and repeated columns.
2. Add in the software the hypothesis " $x_3 \neq \hat{x}_1$ ". This is not trivial because the software doesn't really "see" \hat{x}_1 , it just computes the length of the path supported by the tuple $(x_0, \hat{x}_1, x_2, x_3)$.

6 Ideas for the future

1. Network analysis:
 - Construct a time series using active nodes of a network
 - Detect small cycles and cliques using MH
 - Detect persistent structures using PH
2. Prove that (see if.. but I think so) MH is a stable tool:
 - Define an "interaction index" which should be a measure the tendency of a general vertex v to interact with other vertices in the graph (maybe taking inspiration from connectivity index [ADD RED] and cycle index [ADD RED]). Maybe the global clustering coefficient can be used?
 - Define a distance between two graphs G and G' using the interaction index
 - See if a small variation in the interaction implies a small variation in MH
 - Maybe define a "MH diagram" following the idea of persistence diagrams and do the above point for this MH diagram