ECS 122A B01-B03 FQ 2021 Homework 02

Geoffrey Mohn

TOTAL POINTS

100 / 100

QUESTION 1

1 Q1 **20 / 20**

✓ **+ 20 pts** *Correct*

QUESTION 2

2 Q2 **20 / 20**

✓ **+ 20 pts** *Correct*

QUESTION 3

3 Q3 **20 / 20**

✓ **+ 20 pts** *Correct*

QUESTION 4

4 Q4 **20 / 20**

✓ **+ 20 pts** *All Correct*

QUESTION 5

5 Q5 **20 / 20**

✓ **+ 20 pts** *Complete*

Hw 2

1.) $T(n) = T(n-1) + n$ is $O(n^2)$

Prove $T(n) \leq cn^2$ for some constant c

by I.H. assume $T(n) \leq cn^2$ for all positive numbers less than n $\therefore$ $T(n-1) \leq c((n-1)^2)$

$(n-1)^2 = n^2 - n - n + 1$

$$T(n) \leq c((n-1)^2) + n$$

$$T(n) \leq c(n^2 - 2n + 1) + n$$

$$T(n) \leq cn^2 - cn + c \quad (\text{for } c \geq 1)$$

for $n \geq n_0$ set $n_0 = 1$

$$T(1) = T(0) + 1 \quad \leq (1)^2$$

2.) $T(n) = T(n/2) + 1$ is $O(\log n)$

Base case: $T(1) = T(1/2) + 1$ is $O(\log n)$

Weak Induction

$$T(k) \leq c \log k$$

To prove $T(k+1) \leq c \log(k+1)$

use strong induction

$$\forall\ 1 \leq n \leq k,\ T(n) \leq c \log n$$

To prove $T(k+1) \leq c \log(k+1)$

$$T(k+1) \leq T\left(\frac{k+1}{2}\right) + 1 \quad \text{substitute } T\left(\frac{k+1}{2}\right)$$

$$\leq c \log\left(\frac{k+1}{2}\right) + 1 \quad \log \frac{a}{b} = \log a - \log b$$

$$c \log(k+1) - c + 1$$

1 Q1 **20 / 20**

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Hw 2

1.) $T(n) = T(n-1) + n$ is $O(n^2)$

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$\forall\ 1 \leq n \leq k,\ T(n) \leq c \log n$

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$$\leq c \log\left(\frac{k+1}{2}\right) + 1 \qquad \log \frac{a}{b} = \log a - \log b$$

$$c \log(k+1) - c + 1$$

2 Q2 **20 / 20**

✓ **+ 20 pts** *Correct*

3. $T(n) = T(n/2) + n^2$ is $O(n \log n)$

Assume $T(K-1) \leq c(K-1)$

Prove $T(K) \leq c(K)$

IH Let $K \geq 1$ $\forall\ 1 \leq n \leq K-1$, $T(n) \leq c\, n \log n$

$$T(K) = T\left(\frac{K}{2}\right) + K^2$$

$$\leq c\left(\frac{K}{2} \log K/2\right) + K^2$$

$$c\left[\frac{K}{2} \log K - \frac{K}{2} \underset{(1)}{\log 2}\right] + K^2$$

$$c\left(\frac{K}{2} \log K\right) + K^2 - \left(\frac{cK}{2}\right)$$

To make $c\left(\frac{K}{2} \log K\right) - \frac{cK}{2} + K^2 \leq cK \log K$

$K=2$

$T(n)$ is not $O(n \log n)$ $\quad c(1) - 1 + 4$

$c(4) \leq c(2)$!=

4) $T(n) = 3T(n/2) + n$ is $O(n^{\log 3})$

add lower order term

Assume $1 \leq n \leq K-1$, $T(n) \leq c\, n^{\log 3} + dn$

$d \in \mathbb{R}$

To Prove $T(K) \leq cK^{\log 3} + dK$

$$T(K) = 3T\left(\frac{K}{2}\right) + K$$

IH $$\leq 3\left(c\left(\frac{K}{2}\right)^{\log 3} + d\left(\frac{K}{2}\right)\right) + K$$

$$= \frac{3}{2}cK^{\log 3} + \frac{3}{2}dK + K$$

To Show $\frac{3}{2}cK^{\log 3} + \frac{3}{2}dK + K \leq cK^{\log 3} + dK$

$$\frac{1}{2}cK^{\log 3} + \frac{1}{2}dK + K \leq 0$$

$K \geq 0$

$\frac{1}{2}dK \leq 0 \quad d \leq -2$

3 Q3 **20 / 20**

✓ **+ 20 pts** *Correct*

3. $T(n) = T(n/2) + n^2$ is $O(n \log n)$

Assume $T(k-1) \le c(k-1)$

Prove $T(k) \le c(k)$

IH Let $k \ge 1$ $\forall\ 1 \le n \le k-1$, $T(n) \le c\, n \log n$

$$T(k) = T\left(\frac{k}{2}\right) + k^2$$
$$\le c\left(\frac{k}{2} \log k/2\right) + k^2$$
$$c\left[\frac{k}{2} \log k - \frac{k}{2} \underset{(1)}{\log 2}\right] + k^2$$
$$c\left(\frac{k}{2} \log k\right) + k^2 - \left(\frac{ck}{2}\right)$$

To make $c\left(\frac{k}{2} \log k\right) - \frac{ck}{2} + k^2 \le ck \log k$

$k = 2$

$T(n)$ is not $O(n \log n)$ $\qquad c(1) - 1 + 4$

$c(4) \le c(2)\ !=$

4) $T(n) = 3T(n/2) + n$ is $O(n^{\log 3})$

add lower order term

Assume $1 \le n \le k-1$, $T(n) \le c\, n^{\log 3} + dn$ $\qquad d \in \mathbb{R}$

To Prove $T(k) \le ck^{\log 3} + dk$

$$T(k) = 3T\left(\frac{k}{2}\right) + k$$

IH $\qquad \le 3\left(c\left(\frac{k}{2}\right)^{\log 3} + d\left(\frac{k}{2}\right)\right) + k$

$$= \frac{3}{2} ck^{\log 3} + \frac{3}{2} dk + k$$

To Show $\frac{3}{2} ck^{\log 3} + \frac{3}{2} dk + k \le ck^{\log 3} + dk$

$$\frac{1}{2} ck^{\log 3} + \frac{1}{2} dk + k \le 0$$

$k \ge 0$

$\frac{1}{2} dk \le 0 \qquad d \le -2$

4 Q4 **20 / 20**

✓ **+ 20 pts** *All Correct*

5) $T(n) = T(n-1) + T(n/2) + n$ is $O(n2^n)$

add lower order term

Assume $1 \le n \le k-1$, $T(n) \le cn2^n + dn$

$d \in \mathbb{R}$

To Prove $T(k) \le ck2^k + dk$

$$T(k) = T(k-1) + T(\tfrac{k}{2}) + k$$

$$T(k) \le c(k-1)2^{k-1} + d(k-1) + c(\tfrac{k}{2})2^{(k/2)} + d(k/2) + k$$

$$(2ck-2c)^{k-1} + dk - d + (ck)^{k/2} + \frac{dk}{2} + k$$

To Show $2c(k-1)^{k-1} + d(k-1) + (ck)^{k/2} + \frac{dk}{2} + k \le$

$$ck2^k + dk$$

let $k=2$
$d=1$

$$2c(1)^1 + 1 + 2c + 1 + 2 \le c8 + 2$$

$$4c + 4 \le 8c + 2$$

-2 $-8c$ -2

$$-4c + 2 \le 0$$

$\forall c \ge 1$

5 Q5 **20 / 20**

✓ **+ 20 pts** *Complete*