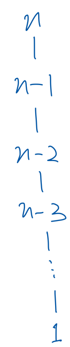
ECS 122A – Algorithm & Analysis
Homework 03 Solution

Question 1 (10 points each)

For the following recurrences, use recursion tree to find the tightest possible Big-O. (Hint: Check the Big-O guesses for the recurrences in Homeowrk 2.)

1. $T(n) = T(n-1) + n$

   Answer:



$$T(n) = n + (n-1) + (n-2) + ... + 1$$
$$= \frac{n(n+1)}{2}$$

$T(n)$ is $O(n^2)$

2. $T(n) = T(n/2) + 1$

   Answer:

$$\begin{array}{c} 1 \\ \downarrow \\ 1 \\ \downarrow \\ 1 \\ \downarrow \\ \vdots \\ \downarrow \\ 1 \end{array}$$

$$T(n) = 1 \times logn = logn$$

   $T(n)$ is $O(logn)$.

3. $T(n) = T(n/2) + n^2$

   Answer:

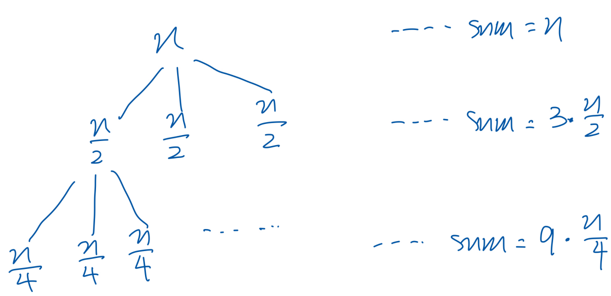
$$\begin{array}{c} n^2 \\ | \\ (\frac{n}{2})^2 \\ | \\ (\frac{n}{4})^2 \\ | \\ (\frac{n}{8})^2 \\ | \\ \vdots \\ | \\ 1 \end{array}$$

$$
\begin{aligned}
T(n) &= n^2 + (\frac{n}{2})^2 + (\frac{n}{4})^2 + (\frac{n}{8})^2 + ... + 1 \\
&= n^2(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + ... + 1) \\
&= n^2 \sum_{i=0}^{logn} \frac{1}{4^i} \\
&= n^2 \frac{1 - (\frac{1}{4})^{logn+1}}{1 - \frac{1}{4}} \\
&= \frac{4}{3} n^2 (1 - \frac{1}{4n^2})
\end{aligned}
$$

$T(n)$ is $O(n^2)$.

4. $T(n) = 3T(\frac{n}{2}) + n$

   Answer:

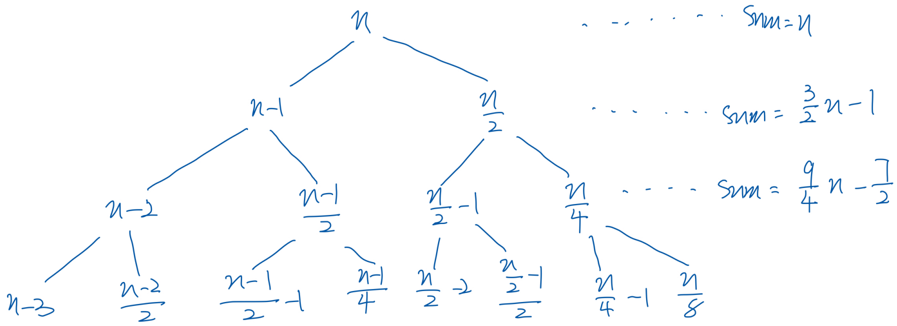


$$
\begin{aligned}
T(n) &= n + \frac{3}{2}n + \frac{9}{4}n + ... + 1 \\
&= n \sum_{i=1}^{logn} (\frac{3}{2})^i \\
&= n \frac{1 - (\frac{3}{2})^{logn+1}}{1 - \frac{3}{2}} \\
&= 2n((\frac{3}{2})^{logn+1} - 1) \\
&= 3^{logn+1} - 2n \\
&= 3 \cdot 3^{logn} - 2n \\
&= 3 \cdot n^{log3} - 2n
\end{aligned}
$$

$T(n)$ is $O(n^{log3})$.

5. $T(n) = T(n-1) + T(\frac{n}{2}) + n$

   Answer:



$$
\begin{aligned}
T(n) &= (n + \frac{3}{2}n + \frac{9}{4}n + ... + 1) - (\text{some number, can be ignored}) \\
&= n \sum_{i=1}^{n} (\frac{3}{2})^i \text{ (Note: the length of the longest branch is) } n \\
&= n \frac{1 - (\frac{3}{2})^{n+1}}{1 - \frac{3}{2}} \\
&= 3n(\frac{3}{2})^n - 2n
\end{aligned}
$$

$T(n)$ is $O(n(\frac{3}{2})^n)$.

Question 2 (10 points each)

For the following recurrences, use the master method to find the Big-Θ if possible. If not, explain why. Assume that $T(n)$ is constant for sufficiently small n.

1. $T(n) = 2T(n/4) + 1$

   Answer:

$$f(n) = 1$$
$$n^{log_b a} = n^{log_4 2} = n^{\frac{1}{2}}$$

   $f(n)$ is smaller. Try Case 1.
   Let $\epsilon = \frac{1}{4}$. Then 1 is $O(n^{\frac{1}{2}-\frac{1}{4}}) = O(n^{\frac{1}{4}})$. (Any $0 < \epsilon \leq \frac{1}{2}$ works.)
   So Case 1 applies. $T(n)$ is $\Theta(n^{\frac{1}{2}})$.

2. $T(n) = 2T(n/4) + \sqrt{n}$

   Answer:

$$f(n) = n^{\frac{1}{2}}$$
$$n^{log_b a} = n^{\frac{1}{2}}$$

   Case 2 applies. $T(n)$ is $\Theta(n^{\frac{1}{2}} log n)$.

3. $T(n) = 2T(n/4) + n$

   Answer:

$$f(n) = n$$

$$n^{log_b a} = n^{\frac{1}{2}}$$

$f(n)$ is larger. Try Case 3.

- Let $\epsilon = \frac{1}{4}$. Then n is $\Omega(n^{\frac{1}{2}+\frac{1}{4}}) = \Omega(n^{\frac{3}{4}})$. (Any $0 < \epsilon \leq \frac{1}{2}$ works.)
- Regularity condition:

  $$af(\frac{n}{b}) = 2f(\frac{n}{4}) = \frac{n}{2}$$

  Let $c = \frac{1}{2}$:

  $$cf(n) = \frac{n}{2}$$

  (Any $\frac{1}{2} \leq c < 1$ works.) The regularity condition holds.

So Case 3 applies. $T(n)$ is $\Theta(n)$.

4. $T(n) = 2T(n/4) + n^2$

   Answer:

   $$f(n) = n^2$$
   $$n^{log_b a} = n^{\frac{1}{2}}$$

   $f(n)$ is larger. Try Case 3.

   - Let $\epsilon = \frac{1}{4}$. Then n^2 is $\Omega(n^{\frac{1}{2}+\frac{1}{4}}) = \Omega(n^{\frac{3}{4}})$. (Any $0 < \epsilon \leq \frac{3}{2}$ works.)
   - Regularity condition:

     $$af(\frac{n}{b}) = 2f(\frac{n}{4}) = 2 \times (\frac{n}{4})^2 = \frac{n^2}{8}$$

     Let $c = \frac{1}{8}$:

     $$cf(n) = \frac{n^2}{8}$$

     (Any $\frac{1}{8} \leq c < 1$ works.)

   So Case 3 applies. $T(n)$ is $\Theta(n^2)$.