ECS 20 Homework 3

ECS 20 — Fall 2016
Section A01

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Problem 1

Show that this implication is a tautology, by using a table of truth: $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$.

Let LHS be $[(\text{p} \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)]$

p	q	r	$p \vee q$	$p \rightarrow r$	$q \rightarrow r$	LHS=$[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)]$	$LHS \rightarrow r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

Problem 2

Show that $[(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$ is a tautology

Let LHS be (p $\vee$ q) $\wedge(\neg p \vee r)$

p	q	r	$p \vee q$	$\neg p \vee r$	LHS=$(p \vee q) \wedge (\neg p \vee r)$	$q \vee r$	$LHS \rightarrow (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T
T	F	T	T	T	T	T	T
T	F	F	T	F	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	F	T	T
F	F	F	F	T	F	F	T

Problem 3

a) Let x be a real number. Show that " if x^2 is irrational, it follows that x is irrational."

Let $p : x^2$ is irrational
let $q : x$ is irrational.
prove $p \to q$. Use indirect proof ($\neg q \to \neg p$.)

$\neg q = x$ is rational. There exists an integer a and a non-zero integer b such that $x = \frac{a}{b}$. Then $x^2 = \frac{a^2}{b^2}$. Since a^2 and b^2 are integers, x^2 is a rational number. $\neg p$ is true. Then $\neg q \to \neg p$ is true, thus $p \to q$ is true.

b) Based on question a), can you say that " if x is irrational, it follows that x^2 is irrational."

It is not a valid proposition. The statement in a) can be simplified as $p \to q$, while the second statement is $q \to p$ so they are not equivalent.

Problem 4

Prove that a square of an integer ends with a 0, 1, 4, 5 6 or 9. (Hint: let $n = 10k + l$, where $l = 0, 1, \ldots ,9$)

Let n be an integer; there exists two integers k and l such that $n = 10k + l$ where $0 \leq l \leq 9$. We get:

$$\begin{aligned} n^2 &= (10k + l)^2 \\ &= 100k + 20kl + l^2 \\ &= k \times 100 + 2kl \times 10 + l^2 \end{aligned}$$

$k \times 100$ and $2kl \times 10$ are multiples of 10. Therefore, n^2 ends as l^2. In the following table, we show that l^2 always end with a 0, 1, 4, 5, 6, or 9.

l	l^2	end
0	0	0
1	1	1
2	4	4
3	9	9
4	16	6
5	25	5
6	36	6
7	49	9
8	64	4
9	81	1

Problem 5

Prove that if n is a positive integer, then n is even if and only if $5n+6$ is even.

Let p be the proposition "n is even"
q be the proposition "$5n+6$ is even".
show that $p \leftrightarrow q$, which is logically equivalent to show that $p \rightarrow q$ and $q \rightarrow p$.

a) show $p \rightarrow q$:

Assue p is true, i.e. n is even. when n is even, there exists an integer k such that $n = 2k$.

$$\begin{aligned} 5n+6 &= 5(2k)+6 \\ &= 10k+6 \\ &= 2 \times (5k+3) \end{aligned}$$

Since $5k+3$ is an integer, and $5n+6$ is a multiple of 2, it is always even.

b) show $q \rightarrow p$:

Assume q is true, i.e. $5n+6$ is even. As $5n+6$ is even, there exists an integer k such that $5n+6 = 2k$.

$$\begin{aligned} 5n &= 2k-6 \\ n &= 2k-6-4n \\ n &= 2 \times (k-3-2n) \end{aligned}$$

Since $k-3-2n$ is an integer, n is a multiple of 2: it is even.

n is even $\leftrightarrow$ $5n+6$ is even.

Problem 6

Prove that either $3 \times 100^{450} + 15$ or $3 \times 100^{450} + 16$ is not a perfect square.

Let $n = 3 \times 100^{450} + 15$. The two numbers are n and $n+1$.

Prove through contradiction: Assume that both n and $n+1$ are perfect squares:

$$\exists k \in \mathbb{Z}, k^2 = n$$
$$\exists l \in \mathbb{Z}, l^2 = n+1$$

Then

$$\begin{aligned} l^2 &= k^2+1 \\ (l-k)(l+k) &= 1 \end{aligned}$$

Since l and k are integers, there are only two cases:

- $l-k = 1$ and $l+k = 1$, i.e. $l = 1$ and $k = 0$. Then we would have $k^2 = 0$, $n = 0$: contradiction
- $l-k = -1$ and $l+k = -1$, i.e. $l = -1$ and $k = 0$. contradiction.

The proposition is true.

Problem 7

Prove or disprove that if a and b are rational numbers, then a^b is also rational.

It is not true. Let $a = 2$ and $b = 1/2$, both a and b are rational numbers. $a^b = \sqrt{2}$ which is irrational.

Problem 8

Prove that at least one of the real numbers $a_1, a_2, \ldots a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

proof through contradiction.

Assume none of the real numbers a_1, a_2, ..., a_n is greater than or equal to the average of these numbers, shown by E

By definition

$$E = \frac{a_1 + a_2 + ... + a_n}{n}$$

Assume

$$\begin{aligned} a_1 &< E \\ a_2 &< E \end{aligned}$$

a$_n < E$

take sum

$$a_1 + a_2 + ... + a_n < n * E$$

put E the equivalence of E

$$a_1 + a_2 + ... + a_n < a_1 + a_2 + ... + a_n$$

a number cannot be smaller than itself so it is not possible. If the contradiction assumption is wrong then the original proposition is correct.

Problem 9

The proof below has been scrambled. Please put it back in the correct order statements order: 3, 5, 4, 2, 1.

Problem 10

Prove that these four statements are equivalent: (i) n^2 is odd, (ii) $1-n$ is even, (iii) n^3 is odd, (iv) n^2+1 is
given four propositions are:

- p : n^2 is odd
- q : $1-n$ is even
- r : n^3 is odd
- s : n^2+1 is even

then show that:

- $q \leftrightarrow p$
- $q \leftrightarrow r$
- $q \leftrightarrow s$

If these three are logical equivalent then the propositions are true.

1) **Proof 1**: $1-n$ is even $\leftrightarrow$ n^2 is odd.

   two implications: (1) $1-n$ is even implies n^2 is odd
   n^2 is odd implies that $1-n$ is even.

   a) **Implication 1**: $q \rightarrow p$
      direct proof.
      Assume q is true, i.e. $1-n$ is even.
      There exists an integer k such that $1-n = 2k$. $n = 1-2k$. Squares on each side:

$$n^2 = (1-2k)^2 = 4k^2 - 2k + 1 = 2(2k^2 - k) + 1$$

      n^2 is odd, so $q \rightarrow p$.

   b) **Implication 2**: $p \rightarrow q$.
      indirect proof: $\neg q \rightarrow \neg p$.

      $\neg q$: $1-n$ is odd
      $\neg p$: n^2 is even.

      assume $1-n$ is odd. There exists an integer k such that $1-n = 2k+1$; $n = -2k$. square: $n^2 = 4k^2$, so n^2 is even.
      $\neg q \rightarrow \neg p$is true so $p \rightarrow q$ is true.
      $q \rightarrow p$ and $p \rightarrow q$, then $p \Leftrightarrow q$.

2) **Proof 2**: $1-n$ is even $\leftrightarrow n^3$ is odd. $q \leftrightarrow$r
two implications: (1) $1-n$ is even implies n^2 is odd and (2), n^2 is odd implies that $1-n$ is even.

   a) **Implication 1**: $q \to r$
   direct proof.
   Assume q is true, i.e. $1-n$ is even. There exists an integer k such that $1-n=2k$. $n=1-2k$. Take cubes

   $$n^3 = (1-2k)^3 = -8k^3 + 12k^2 - 6k + 1 = 2(-4k^3 + 6k^2 - 3k) + 1$$

   n^3 is odd. $q \to r$.

   b) **Implication 2**: $r \to q$.
   indirect proof, $\neg q \to \neg r$.
   · $\neg q$: $1-n$ is odd
   · $\neg p$: n^3 is even.
   Assume $1-n$ is odd. There exists an integer k such that $1-n=2k+1$; so $n=-2k$. Tke cube: $n^3 = 8k^3 = 2(4k^3)$, thus n^3 is even.
   $\neg q \to \neg r$; then $r \to q$ is true.

$q \to r$ and $r \to q$, then r $\Leftrightarrow q$.

1) **Proof 3**: $1-n$ is even $\leftrightarrow n^2+1$ is even.
two implications: (1) $1-n$ is even implies n^2+1 is even and (2), n^2+1 is even implies that $1-n$ is even.

   a) **Implication 1**: $q \to s$
   direct proof.
   Assume q is true, i.e. $1-n$ is even. There exists an integer k such that $1-n=2k$. Therefore $n=1-2k$. Squares on each side:

   $$n^2 = (1-2k)^2 = 4k^2 - 2k + 1 = 2(2k^2 - k) + 1$$

   Therefore:

   $$n^2 + 1 = 2(2k^2 - k) + 1 + 1 = 2 * (2k^2 - k + 1)$$

   n^2+1 is even. so $q \to s$.

   b) **Implication 2**: $s \to q$.
   indirect proof: $\neg q \to \neg s$.
   · $\neg q$: $1-n$ is odd
   · $\neg s$: n^2+1 is odd.
   Assume $1-n$ is odd. There exists an integer k such that $1-n=2k+1$; therefore $n=-2k$. Take square so that $n^2=4k^2$, then $n^2+1=4k^2+1$. n^2+1 is odd.
   $\neg q \to \neg s$; is true so $s \to q$ is true .

$q \to s$ and $p \to s$, so $s \Leftrightarrow q$.

Extra Credit

Use Exercise 8 to show that if the first 10 strictly positive integers are placed around a circle, in any order, then there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 17.

Let a_1, a_2, ..., a_{10} be an order of 10 positive integers from 1 to 10 being placed around a circle the ten numbers a relate first 10 positive integers to get:

$$a_1 + a_2 + ... + a_{10} = 1 + 2 + ... + 10 = 55 \tag{1}$$

a_1, a_2, ..., a_{10} are not order 1, 2, ..., 10 but the sum is 55

There are 10 sums:

$$\begin{aligned} S_1 &= a_1 + a_2 + a_3 \\ S_2 &= a_2 + a_3 + a_4 \\ S_3 &= a_3 + a_4 + a_5 \\ S_4 &= a_4 + a_5 + a_6 \\ S_5 &= a_5 + a_6 + a_7 \\ S_6 &= a_6 + a_7 + a_8 \\ S_7 &= a_7 + a_8 + a_9 \\ S_8 &= a_8 + a_9 + a_{10} \\ S_9 &= a_9 + a_{10} + a_1 \\ S_{10} &= a_{10} + a_1 + a_2 \end{aligned}$$

compute the sum of these numbers:

$$\begin{aligned} S_1 + S_2 + ... + S_{10} &= (a_1 + a_2 + a_3) + (a_2 + a_3 + a_4) + ... + (a_{10} + a_1 + a_2) \\ &= 3 * (a_1 + a_2 + ... + a_{10}) \\ &= 3 * 55 \\ &= 165 \end{aligned}$$

The average of S_1, S_2, ..., S_{10} is:

$$\begin{aligned} \overline{S} &= \frac{S_1 + S_2 + ... + S_{10}}{10} \\ &= \frac{165}{10} \\ &= 16.5 \end{aligned}$$

at least one of S_1, S_2, ..., S_{10} is greater than or equal to $\overline{S}$, S_1. It cannot be equal to 16.5. So at least one of S_1, S_2, ..., S_{10} is greater to or equal to 17.