Advanced Theoretical Physics - Exam

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October 23, 2024

1 Black-body radiation in non-interacting Yang-Mills theory

Consider the SU(N_c) Yang-Mills theory in the continuum, described by the Euclidean gauge action:

$$S_G = \int_0^\beta \mathrm{d}x_0 \int \mathrm{d}^3x \; \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \,, \quad a = 1, \ldots, N_c^2 - 1 \,, \tag{1}$$

where the field strength is a field living in the algebra of the gauge group, namely

$$F_{\mu\nu} = F^a_{\mu\nu} T^a \,, \quad T^a \in \mathfrak{su}(N) \,, \tag{2}$$

and it is defined as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \,, \tag{3}$$

with g being the continuum bare coupling and f_{abc} the structure constants of the group. Since we will deal with a gas of free gluons, we can send $g \to 0$ and the last term in (3) drops out. The gluonic pressure of the gas can be computed by formulating the problem with a path integral representation. Given the partition function Z of the system[1]

$$Z = \int [DA] \, e^{-S_G} \,, \tag{4}$$

we can define its free energy density as

$$f = \frac{\mathrm{d}}{\mathrm{d}V} \left(T \ln Z \right) \,, \tag{5}$$

thus the pressure follows from the thermodynamic relation

$$p = -f \,. \tag{6}$$

We start by expanding the gluon modes in $\mathbb{R}^4$:

$$A_\mu(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{ip\cdot x} \tilde{A}_\mu(p) \,, \quad \tilde{A}_\mu(p) \equiv \tilde{A}^a_\mu(p) T^a \,, \tag{7}$$

where $p = (p_0, \boldsymbol{p})$ is the four-momentum, $p \cdot x$ denotes the Euclidean scalar product between four-vectors and $\tilde{A}_\mu(p)$ are the gluon field Fourier modes. We work at a finite temperature

[1] The integration measure over gauge field configurations assumes formal sense in a lattice regularization.

by compactifying the temporal extension L_0 and by imposing periodic boundary conditions on fields along the time direction:

$$A_\mu(x_0, \boldsymbol{x}) = A_\mu(x_0 + L_0, \boldsymbol{x})\,; \tag{8}$$

the temperature of the system is given by $T = 1/L_0$. This implies a quantization of the momenta along the time coordinate, namely p_0 can now only assume discrete values:

$$\omega_n = \frac{2\pi}{L_0} n = 2\pi T n\,, \tag{9}$$

known as Matsubara frequencies (for a bosonic field). We could carry out the computation of the pressure in the continuum, but in order to be completely formal when treating the integration measure of the path integral representation, we will discretize the spatial coordinates as well. It follows that the dimensionless gluon modes are:

$$A_\mu(x_0, \boldsymbol{x}) = \frac{1}{\sqrt{VT}} \sum_n \sum_{\boldsymbol{p}} e^{i(\omega_n x_0 + \boldsymbol{p}\boldsymbol{x})} \tilde{A}_\mu(p)\,, \tag{10}$$

where V is the spatial volume, the momentum $p = (\omega_n, \boldsymbol{p})$ is such that $p^2 = \omega_n^2 + \boldsymbol{p}^2$ and $\tilde{A}_\mu^{a*}(p) = \tilde{A}_\mu^a(-p)$ since the gluon field is real. By using the definitions of Eqs. (2) and (10) in relation (1), the action can be recast as ($g = 0$):

$$\begin{aligned} S &= \int_0^\beta \mathrm{d}x_0 \int_V \mathrm{d}^3\boldsymbol{x}\, \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}\,, \\ &= \frac{1}{4}\frac{1}{VT} \int_0^\beta \mathrm{d}x_0 \int_V \mathrm{d}^3\boldsymbol{x} \left(\sum_{n,\boldsymbol{p}} i(p_\mu \tilde{A}^a_\nu - p_\nu \tilde{A}^a_\mu) e^{i(\omega_n x_0 + \boldsymbol{p}\boldsymbol{x})} \right) \times \\ &\qquad\qquad \times \left(\sum_{m,\boldsymbol{q}} i(q_\mu \tilde{A}^a_\nu - q_\nu \tilde{A}^a_\mu) e^{i(\omega_m x_0 + \boldsymbol{q}\boldsymbol{x})} \right)\,. \end{aligned}$$

Recalling that

$$\int_0^\beta \mathrm{d}x_0\; e^{i(\omega_n + \omega_m)x_0} = \frac{1}{T}\delta_{n,-m}\,, \qquad \int_V \mathrm{d}^3\boldsymbol{x}\; e^{i(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{x}} = V\delta_{\boldsymbol{p},-\boldsymbol{q}}\,, \tag{11}$$

some algebraic steps yield to

$$S \equiv \frac{1}{2T^2} \sum_{n,\boldsymbol{p}} \tilde{A}^{a*}_\mu(p) M_{\mu\nu} \tilde{A}^a_\nu(p)\,, \quad \text{where} \quad M_{\mu\nu} \equiv p^2 \delta_{\mu\nu} - p_\mu p_\nu\,. \tag{12}$$

In order to define a propagator for the gluon fields, one would need to invert the kernel $M_{\mu\nu}$ by imposing

$$I_{\mu\nu} M_{\nu\rho} = \delta_{\mu\rho}\,; \tag{13}$$

since $I_{\mu\nu}$ is a SO(4) rank-2 tensor, it must be of the form

$$I_{\mu\nu} = c_1(p)\delta_{\mu\nu} + c_2(p)\frac{p_\mu p_\nu}{p^2}\,, \tag{14}$$

where $c_1(p)$ and $c_2(p)$ are momentum-dependent coefficients. It is easy to show that relation (13) cannot be satisfied by such an object, hence we find that $M_{\mu\nu}$ is not invertible. Consequently, the Gaussian integral for the definition of the partition function Z is ill-defined due to the non-invertibility of the kinetic term. Moreover, even if the latter were well-defined, the integration over gauge configurations in Eq. (4) suffers from the well known Gribov ambiguity, i.e. the configurational integral would lead to incorrect result due to the integration over physically equivalent gauge configurations.

In order to overcome the aforementiond problems, Yang-Mills theories require a gauge fixing procedure, which can be implemented through the so-called Faddeev-Popov prescription. First, we define a function $\Lambda^a(x)$ of the path integral variables, e.g.

$$\Lambda^a(x) = \partial_\mu A^a_\mu(x) \, , \tag{15}$$

such that it has a unique solution for $A^a_\mu(x)$ - otherwise the Gribov ambiguity still holds. The Faddeev-Popov prescription consists in the insertion of the following unitary factor in the partition function:

$$\prod_{\substack{x,y\\a,b}} \delta(\partial_\mu A^a_\mu(x) - \Lambda^a(x)) \det\left[\frac{\delta \Lambda^a(x)}{\delta \theta^b(y)}\right] \, , \tag{16}$$

where the delta reinforces the gauge-fixing condition, the second term takes into account the variation of the latter with respect to a generic infinitesimal gauge transformation $G(x) \simeq \mathbb{1} + i\theta^a(x) T^a(x)$, so that their combination takes care of the redundancy in the integration. This factor does not change the expectation value of gauge invariant quantities, as it introduces at most an overall constant in the partition function, and it is independent on the particular choice of $\Lambda^a(x)$. Manipulating the second term in (16), one can show it can be expressed as

$$\det\left[\partial_\mu D_\mu\right] \, , \tag{17}$$

where $D_\mu \cdot = \partial_\mu \cdot + ig\left[A_\mu(x), \cdot\right]$ is the usual covariant derivative acting on a field in the adjoint representation of $\mathrm{SU}(N_c)$. The latter can be expressed as the result of an integral over Grassmann variables $c(x) = c^a(x) T^a$, $\bar{c}(x) = \bar{c}^a(x) T^a$ known as ghost fields[2], namely in the continuum:

$$\begin{aligned}
\det\left[\partial_\mu D_\mu\right] &= \int [D\bar{c}][Dc] \, \exp\left\{\frac{2}{g^2} \int \mathrm{d}^4x \, \mathrm{Tr}\left[\bar{c}(x)[\partial_\mu D_\mu](x) c(x)\right]\right\} \, , \\
&= \int [D\bar{c}][Dc] \, \exp\left\{-\frac{2}{g^2} \int \mathrm{d}^4x \, \mathrm{Tr}\left[\partial_\mu \bar{c}(x) D_\mu c(x)\right]\right\} \, , \\
&\equiv \int [D\bar{c}][Dc] \, \exp\{-S_{\mathrm{FP}}\} \, .
\end{aligned}$$

In the limit of free gluons $g \to 0$, we have that $\det\left[\partial_\mu D_\mu\right] \to \det\left[\partial_\mu \partial_\mu\right]$, which does not depend on gauge fields anymore, hence ghost and gluons are decoupled. Ghost fields do not propagate nor create physical states, but they are necessary to remove the contributions from unphysical gluon polarizations. Therefore, inserting the Faddeev-Popov factor (16) into the partition function (4), we now have:

$$\begin{aligned}
Z &= \int [DA] \exp\left\{-\int_0^\beta \mathrm{d}x_0 \int \mathrm{d}^3x \, \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}\right\} \, , \\
&= \int [D\bar{c}][Dc] \, \exp\{-S_{\mathrm{FP}}\} \delta(\partial_\mu A_\mu - \Lambda) \int [DA] \exp\{-S_G\} \, .
\end{aligned}$$

[2]These fields do not create physical states, as they violate the spin-statistics theorem: they are Grassmann variables described by the Klein-Gordon action for massless bosonic degrees of freedom.

By giving a Gaussian weight to all the possible fields $\Lambda(x) = \Lambda^a(x)T^a$ we end up with

$$
\begin{aligned}
Z &= \int [D\bar{c}][Dc]\,[DA]\,\exp\{-S_{\rm FP} - S_G\}\exp\Big\{-\frac{1}{2\alpha}\int \mathrm{d}^4x\,(\partial_\mu A_\mu)^2\Big\}\,,\\
&= \int [D\bar{c}][Dc]\,[DA]\,\exp\{-S_{\rm FP} - S_G - S_{\rm GF}\}\,,
\end{aligned}
$$

where α is a gauge-fixing parameter (we set $\alpha = 1$ to work in the Feynman gauge).

Finally, to compute the pressure we need to evaluate the partition function. Going to momentum space in a discrete space-time one can show that:

$$
S_{\rm GF} + S_G = \frac{1}{2T^2}\prod_{n,\boldsymbol{p}} \tilde{A}^a_\mu(-p)(p^2)\tilde{A}^a_\nu(p)\,, \tag{18}
$$

which is the real scalar field action for $4(N_c^2-1)$ fields. The gluonic contribution to the partition function is then:

$$
\begin{aligned}
Z_{\rm gluons} &= \int \prod_{\mu,a,\boldsymbol{p}} \delta\tilde{A}^a_\mu(p)\prod_{n,\boldsymbol{p}}\exp\Big\{-\frac{1}{2}\tilde{A}^a_\mu(-p)\left(\frac{p^2}{T^2}\right)\tilde{A}^a_\nu(p)\Big\}\,,\\
&= \mathcal{N}\left[\prod_{n,\boldsymbol{p}}\frac{\omega_n^2+\boldsymbol{p}^2}{T^2}\right]^{-\frac{1}{2}\left[4(N_c^2-1)\right]}\,,
\end{aligned}
$$

where $\mathcal{N}$ is an irrelevant factor and the exponent $4(N_c^2-1)$ comes from the number of bosonic degrees of freedom which contribute to the Gaussian integral. It immediately follows that:

$$
\ln(Z_{\rm gluons}) = -2(N_c^2-1)\sum_{n=-\infty}^{\infty}\sum_{\boldsymbol{p}}\ln\left(\frac{\omega_n^2+\boldsymbol{p}^2}{T^2}\right) + \text{const}\,. \tag{19}
$$

Similarly, the ghost fields contribution reads:

$$
\begin{aligned}
Z_{\rm ghost} &= \int \prod_{a,\boldsymbol{p}} \delta\bar{c}^a(p)\delta c^a(p)\prod_{n,\boldsymbol{p}}\exp\Big\{-\bar{c}^a(p)\left(\frac{p^2}{T^2}\right)c^a(p)\Big\}\,,\\
&= \mathcal{N}'\left[\prod_{n,\boldsymbol{p}}\frac{\omega_n^2+\boldsymbol{p}^2}{T^2}\right]^{(N_c^2-1)}\,;
\end{aligned}
$$

notice that the exponent shows no $-1/2$ factor in front of the number of degrees of freedom, as ghosts are Grassmann variables. Hence:

$$
\ln(Z_{\rm ghost}) = (N_c^2-1)\sum_{n=-\infty}^{\infty}\sum_{\boldsymbol{p}}\ln\left(\frac{\omega_n^2+\boldsymbol{p}^2}{T^2}\right) + \text{const}'\,. \tag{20}
$$

Combining both contributions we get the result for the full Yang-Mills theory

$$
\ln Z = \ln(Z_{\rm gluons}) + \ln(Z_{\rm ghost}) = -(N_c^2-1)\sum_{n=-\infty}^{\infty}\sum_{\boldsymbol{p}}\ln\left(\frac{\omega_n^2+\boldsymbol{p}^2}{T^2}\right) + \text{const}''\,, \tag{21}
$$

which has to be compared with the result found in Lecture 3 for a single massless scalar field

$$\ln Z = 2(N_c^2 - 1)\ln(Z_{\text{scalar}})\,. \tag{22}$$

The gauge-fixing procedure produced the correct counting for the physical degrees of freedom: each of the (N_c^2-1) gluon fields has 4 polarizations, but only two of them are physical. The unphysical polarizations are cancelled by the ghost fields contribution introduced with the Faddeev-Popov prescription. Performing the Matsubara sums in Eqs. (19) and (20) [1], going to the thermodynamic limit and performing the momentum integral yields to the well known result for the pressure of a free gluons gas:

$$p = 2(N_c^2 - 1)\frac{\pi^2 T^4}{90}\,. \tag{23}$$

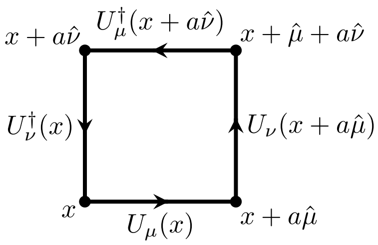
In perturbation theory, it is the standard approach (although there are some exceptions [2]) to reinforce a gauge-fixing procedure: this introduces new gluon-ghost vertices into the game, hence presumibly a perturbative computation of the gluonic pressure would lead to the correct result we just showed.

2 Lattice partition function of Yang-Mills theory

Consider the discretization of the SU(N_c) Yang-Mills theory on a $N_s^3 \times N_T$ lattice with lattice spacing a. The most simple action is given by the Wilson action

$$S_g[U] = \sum_x \sum_{\mu,\nu} \beta \left(1 - \frac{1}{3}\,\mathrm{Re}\,\mathrm{Tr}\, U_p(x)\right)\,, \tag{24}$$

where $U_p(x)$ is the standard lattice plaquette



$$U_p(x) = U_\mu(x)U_\nu(x + a\hat{\mu})U_\mu^\dagger(x + a\hat{\nu})U_\nu^\dagger(x)\,, \tag{25}$$

and $\beta = \frac{2N_c}{g^2}$ is the lattice bare gauge coupling. The corresponding partition function is then given by

$$Z = \int [DU] e^{-S_g[U]}\,, \qquad [DU] \equiv \prod_{\tau,\boldsymbol{x}} \left[\delta U_0(\tau,\boldsymbol{x}) \prod_{i=1}^{3} \delta U_i(\tau,\boldsymbol{x})\right]\,. \tag{26}$$

Periodic Boundary Conditions (PBC) are imposed along all directions:

$$\begin{cases} U_\mu(\tau,\boldsymbol{x}) = U_\mu(\tau + N_T,\boldsymbol{x}) \\ U_\mu(\tau,\boldsymbol{x}) = U_\mu(\tau,\boldsymbol{x} + N_s) \end{cases}, \qquad \forall\,(\boldsymbol{x},\tau)\,, \tag{27}$$

where τ corresponds to a single time-slice and $\boldsymbol{x}$ is the 3D space coordinate.

The transfer matrix formalism allows us to link the path-integral representation of the theory to an Hamiltonian formalism. To this end, it is useful to rewrite the Wilson action in Eq. (24) as a sum over time-slices, where the contributions coming from time-like and space-like plaquettes are separated [3]

$$S_g[U] = \sum_\tau L[U_i(\tau+1), U_0(\tau), U_i(\tau)] \,, \tag{28}$$

where $U_i(\tau)$ stands for the spatial links completely contained in the time-slice τ and $U_0(\tau)$ corresponds to the temporal link connecting time-slices τ and $\tau+1$. The functional L is defined as

$$\begin{aligned} L[U_i(\tau+1), U_0(\tau), U_i(\tau)] = \frac{1}{2}L_1[U_i(\tau+1)] + \frac{1}{2}L_1[U_i(\tau)] + \\ + L_2[U_i(\tau+1), U_0(\tau), U_i(\tau)] \,, \end{aligned} \tag{29}$$

where

$$L_1[U_i(\tau+1)] = -\frac{\beta}{N_c}\sum_{p(\tau)} \mathrm{Re}\,\mathrm{Tr}\, U_p \,, \tag{30}$$

$$L_2[U_i(\tau+1), U_0(\tau), U_i(\tau)] = -\frac{\beta}{N_c}\sum_{p(\tau,\tau+1)} \mathrm{Re}\,\mathrm{Tr}\, U_p \,, \tag{31}$$

with $p(\tau)$ being the space-like plaquettes contained in the time slice τ and $p(\tau,\tau+1)$ the time-like plaquettes joining time-slices τ and $\tau+1$.

Before we introduce the transfer matrix, following [4] we can build the Hilbert space of the theory as the space of all the L^2-complex-valued wave functions $\psi[U]$ of the space-like links $U_i(\boldsymbol{x}) \in \mathrm{SU}(N_c)$. This space is embedded with the following scalar product

$$\langle\phi|\psi\rangle = \int [DU_i]\phi^*[U_i]\psi[U_i] \,, \qquad [DU_i] \equiv \prod_{\boldsymbol{x}}\prod_{i=1}^{3} \delta U_i(\boldsymbol{x}) \,. \tag{32}$$

In order to define the functions $\psi[U]$ on which the transfer matrix $\hat{T}$ will act, a set of vectors $|U\rangle$ that diagonalize the gauge field operator $\hat{U}_i(\boldsymbol{x})\ \forall\, \boldsymbol{x}$ is introduced:

$$\hat{U}_i(\boldsymbol{x})\,|U\rangle = U_i(\boldsymbol{x})\,|U\rangle \ ; \tag{33}$$

with the following normalization

$$\langle U|\psi\rangle = \psi[U] \,. \tag{34}$$

Physical states in a gauge theory must satisfy a gauge invariance condition, for instance here we require

$$\psi[U^G] = \psi[U] \qquad \forall\, G(\boldsymbol{x}) \in \mathrm{SU}(N_c), \tag{35}$$

where $U^G(\boldsymbol{x}) = G(\boldsymbol{x})U(\boldsymbol{x})G^\dagger(\boldsymbol{x})$ is a gauge transformed field. It is also useful to define a projector onto the subspace of gauge-invariant states, namely

$$\langle U|\hat{\mathbb{P}}_G|\psi\rangle = \int [DG]\psi[U^G] \,, \qquad [DG] \equiv \prod_{\boldsymbol{x}} \delta G(\boldsymbol{x}) \,, \tag{36}$$

and of course $\hat{\mathbb{P}}_G^2 = \hat{\mathbb{P}}_G$. The transfer matrix acts on wave-functions $\psi[U]$ by translating the corresponding states by a single time-slice:

$$\hat{T}\,|\psi[U_i(\tau,\boldsymbol{x})]\rangle = |\psi[U_i(\tau+1,\boldsymbol{x})]\rangle \ ; \tag{37}$$

and the matrix elements of $\hat{T}$ between two neighbouring time-slices τ and $\tau+1$ are defined as

$$T[U_i(\tau+1), U_i(\tau)] = \int DU_0(\tau) \exp\{-L[U_i(\tau+1), U_0(\tau), U_i(\tau)]\} \ . \tag{38}$$

Note that the transfer matrix, by definition, evolves a state by a time step a, hence we can write $\hat{T} = e^{-a\hat{H}}$. With this setup, we can now show how the partition function is related to the transfer matrix. Recall the expression for Z given in (26) and write the integration measure explicitly as $[DU] = \prod_{\tau=0}^{N_T-1}[DU_0(\tau)][DU_i(\tau)]$, it follows that

$$Z = \int \prod_{\tau=0}^{N_T-1} [DU_0(\tau)][DU_i(\tau)] e^{-S_g[U]} \,, \tag{39}$$

$$= \int \prod_{\tau=0}^{N_T-1} [DU_0(\tau)][DU_i(\tau)] \exp\left\{-\sum_{\tau} L[U_i(\tau+1), U_0(\tau), U_i(\tau)]\right\} \,, \tag{40}$$

$$= \int \prod_{\tau=0}^{N_T-1} [DU_i(\tau)] T[U_i(\tau+1), U_i(\tau)] \,, \tag{41}$$

$$= \mathrm{Tr}\left\{\hat{T}^{N_T}\right\} \,, \tag{42}$$

where N_T corresponds to the number of time-slices in the lattice and Eqs. (28) and (38) have been inserted in intermediate steps. We can also recover this relation the other way around. Expand the trace of $\hat{T}^{N_T}$ over a complete set of states, for instance

$$\mathrm{Tr}\left\{\hat{T}^{N_T}\right\} = \int [DU_i(N_T)] \, \langle U_i(N_T, \boldsymbol{x}) | \hat{T}^{N_T} | U_i(N_T, \boldsymbol{x}) \rangle \ ; \tag{43}$$

inserting $(N_T - 1)$ completeness relations of the form $\mathbb{1} = \int [DU_i(\tau')] \, |U_i(\tau', \boldsymbol{x})\rangle\langle U_i(\tau', \boldsymbol{x})|$ in between each $\hat{T}$ ($\tau' = N_T - 1, \ldots, 1$) and using PBC on the last state ($|U_i(N_T, \boldsymbol{x})\rangle = |U_i(0, \boldsymbol{x})\rangle$) leads to

$$\mathrm{Tr}\left\{\hat{T}^{N_T}\right\} = \int [DU_i(N_T)] \cdots [DU_i(1)] \, \langle U_i(N_T, \boldsymbol{x}) | \hat{T} | U_i(N_T - 1, \boldsymbol{x}) \rangle \times \tag{44}$$

$$\cdots \langle U_i(1, \boldsymbol{x}) | \hat{T} | U_i(0, \boldsymbol{x}) \rangle \ , \tag{45}$$

$$= \int \prod_{\tau'=1}^{N_T} [DU_i(\tau')] T[U_i(\tau'), U_i(\tau' - 1)] = Z \,. \tag{46}$$

Notice that PBC have been crucial for recostructing the trace.

With regard to numerical computations, in general the gauge fixing procedure is not required for a Monte Carlo (MC) estimation of an integral over gauge configurations of the form $\int [DU] f[U]$. Indeed, thanks to the lattice regularization, the theory is already finite and well-defined without choosing any particular gauge. Furthermore, gauge fixing introduces long-range interactions which are not well simulated by a local MC algorithm: even if the final results are gauge independent and must not change when fixing the gauge, the computational performance can be significantly slower (see e.g. [5, 6]). Eventually, one can show [4] that the transfer matrix elements are gauge invariant, i.e.

$$T[U(\tau+1), U(\tau)] = T[U^{G'}(\tau+1), U^{G}(\tau)] \,, \quad \Rightarrow \quad Z = \mathrm{Tr}\left\{\left[\hat{T}\hat{\mathbb{P}}_G\right]^{N_T}\right\} \,, \tag{47}$$

meaning that there is no need to choose a particular gauge for the simulation.

References

[1] Owe Philipsen. “Lattice QCD at non-zero temperature and baryon density”. In: *Les Houches Summer School: Session 93: Modern perspectives in lattice QCD: Quantum field theory and high performance computing.* Sept. 2010, pp. 273–330. arXiv: 1009.4089 [hep-lat].

[2] G. Parisi and Yong-shi Wu. “Perturbation Theory Without Gauge Fixing”. In: *Sci. Sin.* 24 (1981), p. 483.

[3] I. Montvay and G. Munster. *Quantum fields on a lattice.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, Mar. 1997. ISBN: 978-0-521-59917-7, 978-0-511-87919-7. DOI: 10.1017/CBO9780511470783.

[4] Michele Della Morte and Leonardo Giusti. “Symmetries and exponential error reduction in YangMills theories on the lattice”. In: *Computer Physics Communications* 180.6 (June 2009), pp. 819–826. ISSN: 0010-4655. DOI: 10.1016/j.cpc.2009.03.009. URL: http://dx.doi.org/10.1016/j.cpc.2009.03.009.

[5] M. Creutz. “Monte Carlo Study of Quantized SU(2) Gauge Theory”. In: *Phys. Rev. D* 21 (1980), pp. 2308–2315. DOI: 10.1103/PhysRevD.21.2308.

[6] A. Vladikas. “MONTE CARLO SIMULATIONS AND GAUGE FIXING”. In: *Phys. Lett. B* 169 (1986), pp. 93–96. DOI: 10.1016/0370-2693(86)90692-1.