Mathematics Study Notes

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1 Logarithm Rules

Logarithms answer the question: "To what power must a base be raised to produce a given number?"

1.1 Definition

If $b^x = a$, then $\log_b(a) = x$, where:

- $b > 0$, $b \neq 1$ (the base),
- $a > 0$ (the argument).

1.2 Logarithm Properties

1. **Product Rule:** $\log_b(xy) = \log_b(x) + \log_b(y)$
2. **Quotient Rule:** $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
3. **Power Rule:** $\log_b(x^n) = n\log_b(x)$
4. **Change of Base Rule:** $\log_b(x) = \frac{\log_k(x)}{\log_k(b)}$, for any base $k > 0$.

1.3 Example

Calculate $\log_2(32)$:

$$\log_2(32) = \log_2(2^5) = 5\log_2(2) = 5$$

2 Derivatives

A derivative measures the rate of change of a function with respect to its variable.

2.1 Definition

For a continuous function $f(x)$, the derivative is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2.2 Basic Rules

1. **Constant Rule:** If $f(x) = c$, then $f'(x) = 0$.
2. **Power Rule:** If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
3. **Sum Rule:** If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
4. **Product Rule:** If $f(x) = g(x)h(x)$, then $f'(x) = g'(x)h(x) + g(x)h'(x)$.
5. **Quotient Rule:** If $f(x) = \frac{g(x)}{h(x)}$, then:

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2}$$

6. **Chain Rule:** If $f(x) = g(h(x))$, then $f'(x) = g'(h(x))h'(x)$.

2.3 Example

Find the derivative of $f(x) = x^3 + 2x$:

$$f'(x) = 3x^2 + 2$$

3 Matrices

A matrix is a rectangular array of numbers arranged in rows and columns. Matrices are fundamental tools in linear algebra, commonly used to solve systems of equations, perform transformations, and model real-world problems.

3.1 Notation and Dimensions

A matrix is generally represented as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where:

- a_{ij} represents the element in the ith row and jth column.
- m is the number of rows, and n is the number of columns.

The dimensions of the matrix are written as $m \times n$ (rows $\times$ columns).

3.2 Types of Matrices

- **Square Matrix:** A matrix with the same number of rows and columns ($m = n$).
- **Row Matrix:** A matrix with only one row ($1 \times n$).
- **Column Matrix:** A matrix with only one column ($m \times 1$).
- **Diagonal Matrix:** A square matrix where all off-diagonal elements are zero.
- **Identity Matrix (I):** A square matrix with 1's on the diagonal and 0's elsewhere.
- **Zero Matrix:** A matrix where all elements are zero.

3.3 Matrix Operations

1. **Addition:** Matrices of the same dimensions can be added element by element:

$$A + B = \begin{bmatrix} a_{11} + b_{11} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

2. **Scalar Multiplication:** Multiply each element of a matrix by a scalar k:

$$kA = \begin{bmatrix} k \cdot a_{11} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

3. **Matrix Multiplication:** Multiply two matrices A $(m \times n)$ and B $(n \times p)$:

$$C = AB, \quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

   A and B can only be multiplied if the number of columns in A equals the number of rows in B.

4. **Transpose:** Flip a matrix over its diagonal:

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots \\ a_{12} & a_{22} & \cdots \end{bmatrix}$$

3.4 Determinant of a Square Matrix

The determinant is a scalar value that provides important information about a matrix. For a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(A) = ad - bc$$

For larger matrices, determinants are computed using cofactor expansion.

3.5 Properties of Determinants

- $\det(I) = 1$, where I is the identity matrix.
- If a matrix has a row or column of all zeros, $\det(A) = 0$.
- If two rows or columns are identical, $\det(A) = 0$.
- Swapping two rows or columns changes the sign of the determinant.

4 Matrix Inverse

The inverse of a square matrix A is a matrix A^{-1} such that:

$$AA^{-1} = A^{-1}A = I$$

where I is the identity matrix.

4.1 Steps to Calculate the Inverse

1. **Determinant:** Compute $\det(A)$.
2. **Cofactor Matrix:** Find the cofactor of each entry of A.
3. **Adjugate Matrix:** Transpose the cofactor matrix to get $\operatorname{adj}(A)$.
4. **Inverse Formula:** If $\det(A) \neq 0$, then:

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$$

4.2 Example

Given $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$:

- Compute the determinant: $\det(A) = (2)(4) - (3)(1) = 8 - 3 = 5$.
- Find the cofactors: $\operatorname{Cof}(A) = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$.
- Transpose the cofactor matrix: $\operatorname{adj}(A) = \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$.
- Apply the formula:

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix}.$$

4.3 Inverse of a 3×3 Matrix

For a square matrix A of dimensions $n \times n$, the inverse can be calculated using the formula:

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$$

where:

- $\det(A)$ is the determinant of A.
- $\operatorname{adj}(A)$ is the adjugate matrix, which is the transpose of the cofactor matrix.

4.3.1 Steps for a 3×3 Matrix

Given a matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$:

1. Calculate the determinant:

$$\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg)$$

2. Compute the cofactor matrix: Each element C_{ij} is given by:

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

   where M_{ij} is the minor matrix obtained by removing the i-th row and j-th column from A.

3. Form the adjugate matrix: The adjugate is the transpose of the cofactor matrix:

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}^T$$

4. Apply the formula for the inverse:

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

4.3.2 Example: Inverse of a 3×3 Matrix

Let:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 3 & 4 & -1 \end{bmatrix}$$

1. **Calculate the determinant:**

$$\det(A) = 2((-1)(-1) - (4)(2)) - 1((1)(-1) - (3)(2)) + 3((1)(4) - (3)(-1))$$

   Simplify:

$$\det(A) = 2(1 - 8) - 1(-1 - 6) + 3(4 + 3) = 2(-7) - 1(-7) + 3(7)$$

$$\det(A) = -14 + 7 + 21 = 14$$

2. **Compute the cofactor matrix:**

$$C = \begin{bmatrix} (-1)^2 \det \begin{bmatrix} -1 & 2 \\ 4 & -1 \end{bmatrix} & (-1)^3 \det \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} & (-1)^4 \det \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix} \\ (-1)^3 \det \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} & (-1)^4 \det \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix} & (-1)^5 \det \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \\ (-1)^4 \det \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} & (-1)^5 \det \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} & (-1)^6 \det \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix}$$

   Calculate each determinant and form the cofactor matrix.

3. **Transpose the cofactor matrix to form the adjugate:**
$$\text{adj}(A) = C^T$$

4. **Compute the inverse:**
$$A^{-1} = \frac{1}{14}\text{adj}(A)$$

4.3.3 Calculating the Adjugate Matrix Step by Step

The adjugate matrix $\text{adj}(A)$ is formed by:

1. Computing the cofactor matrix: Find the cofactor C_{ij} for each element a_{ij} of the matrix.

2. Transposing the cofactor matrix: Flip the rows and columns of the cofactor matrix.

For a 3×3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$:

- Each cofactor C_{ij} is calculated as:
$$C_{ij} = (-1)^{i+j} \det(M_{ij}),$$
where M_{ij} is the minor matrix formed by removing the i-th row and j-th column from A.

4.3.4 Example: Calculating the Adjugate Matrix

Let:
$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 3 & 4 & -1 \end{bmatrix}$$

1. **Find the cofactor for each element:**
   - $C_{11} = (-1)^{1+1} \det \begin{bmatrix} -1 & 2 \\ 4 & -1 \end{bmatrix} = \det \begin{bmatrix} -1 & 2 \\ 4 & -1 \end{bmatrix} = (-1)(-1) - (2)(4) = 1 - 8 = -7$
   - $C_{12} = (-1)^{1+2} \det \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = -\det \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = -(1(-1) - (2)(3)) = -(-1 - 6) = 7$
   - $C_{13} = (-1)^{1+3} \det \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix} = \det \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix} = (1)(4) - (-1)(3) = 4 + 3 = 7$
   - Repeat this process for all elements:
$$\text{Cofactor Matrix} = \begin{bmatrix} -7 & 7 & 7 \\ -14 & 11 & -5 \\ -11 & -8 & -3 \end{bmatrix}$$

2. **Transpose the cofactor matrix:** Swap rows and columns:
$$\text{adj}(A) = \begin{bmatrix} -7 & -14 & -11 \\ 7 & 11 & -8 \\ 7 & -5 & -3 \end{bmatrix}$$

4.3.5 Final Formula for the Inverse

Once the adjugate matrix is calculated, use the formula:

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$$

Substitute $\det(A) = 14$ and $\text{adj}(A)$:

$$A^{-1} = \frac{1}{14}\begin{bmatrix} -7 & -14 & -11 \\ 7 & 11 & -8 \\ 7 & -5 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -1 & -\frac{11}{14} \\ \frac{1}{2} & \frac{11}{14} & -\frac{4}{7} \\ \frac{1}{2} & -\frac{5}{14} & -\frac{3}{14} \end{bmatrix}$$

4.3.6 Signs of Cofactors

The sign of each cofactor in a matrix is determined by the **alternating sign rule**:

$$C_{ij} = (-1)^{i+j}\det(M_{ij}),$$

where:

- C_{ij} is the cofactor of the element a_{ij}.
- M_{ij} is the minor matrix obtained by removing the i-th row and j-th column.
- The sign is positive if $i + j$ is even, and negative if $i + j$ is odd.

4.3.7 Checkerboard Pattern for Cofactor Signs

For any matrix, the signs of the cofactors form a **checkerboard pattern**. For a 3×3 matrix:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

4.3.8 Example: Signs of Cofactors in a 3×3 Matrix

Let:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The cofactors of each element a_{ij} will have the following signs:

- **First row:**

$$C_{11} = +\det\begin{bmatrix} e & f \\ h & i \end{bmatrix}, \quad (i + j = 2, \text{even, positive})$$
$$C_{12} = -\det\begin{bmatrix} d & f \\ g & i \end{bmatrix}, \quad (i + j = 3, \text{odd, negative})$$
$$C_{13} = +\det\begin{bmatrix} d & e \\ g & h \end{bmatrix}, \quad (i + j = 4, \text{even, positive})$$

- **Second row:**

$$C_{21} = -\det \begin{bmatrix} b & c \\ h & i \end{bmatrix}, \quad (i+j=3, \text{odd, negative})$$
$$C_{22} = +\det \begin{bmatrix} a & c \\ g & i \end{bmatrix}, \quad (i+j=4, \text{even, positive})$$
$$C_{23} = -\det \begin{bmatrix} a & b \\ g & h \end{bmatrix}, \quad (i+j=5, \text{odd, negative})$$

- **Third row:**

$$C_{31} = +\det \begin{bmatrix} b & c \\ e & f \end{bmatrix}, \quad (i+j=4, \text{even, positive})$$
$$C_{32} = -\det \begin{bmatrix} a & c \\ d & f \end{bmatrix}, \quad (i+j=5, \text{odd, negative})$$
$$C_{33} = +\det \begin{bmatrix} a & b \\ d & e \end{bmatrix}, \quad (i+j=6, \text{even, positive})$$

4.3.9 General Rule

For any square matrix:

- Positive $(+)$ if $i + j = \text{even}$.
- Negative $(-)$ if $i + j = \text{odd}$.