2.7 Fourier Transform

Consider the Fourier series, where we replace X_n by its expression over the period $-T/2 < t < T/2$.

$$x(t) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{\frac{-j2\pi nt}{T}} dt \right) e^{\frac{j2\pi nt}{T}} \tag{2.65}$$

Now let us consider an arbitrary signal $x(t)$ which is aperiodic. An aperiodic signal may be considered to be a periodic one whose period is infinite. Therefore we can rewrite (2.65) as:

$$x(t) = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \left(\frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{\frac{-j2\pi nt}{T}} dt \right) e^{\frac{j2\pi nt}{T}} \tag{2.66}$$

In the limit as T goes to infinity, $1/T$ becomes df which is an infinitesimally thin slice of frequency and n/T is the nth frequency slice while $(n+1)/T$ is the $(n+1)$th frequency slice (which does not overlap with its neighbours). Therefore if n goes from $-\infty$ to ∞, then n/T represents all of the infinitesimally thin but non-overlapping frequency slices from $-\infty$ to ∞ and can therefore be represented by the variable f whose units are in Hz. Then the summation over n becomes an integral over f. Therefore (2.66) becomes

$$x(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \, e^{j2\pi ft} df \tag{2.67}$$

We rewrite (2.67) as

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} dt \tag{2.68}$$

where

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \tag{2.69}$$

Equation (2.69) is called the Fourier Transform of x(t) and (2.68) is called the inverse Fourier transform of $X(f)$.

Example 2.9

Find the Fourier Transforms of $x(t) = \delta(t)$ and $y(t) = \Pi(t)$ using (2.69).

Solution

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt \\ &= \int_{-\infty}^{\infty} \delta(t) dt \qquad * \\ &= 1 \end{aligned}$$

where the second line denoted by * is a direct result of (2.2).

$$
\begin{aligned}
Y(f) &= \int_{-\infty}^{\infty} \Pi(t) e^{-j2\pi f t} dt \\
&= \int_{-1/2}^{1/2} e^{-j2\pi f t} dt \\
&= -\frac{1}{j2\pi f} e^{-j2\pi f t}\Big|_{-1/2}^{1/2} \\
&= \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} \\
&= \frac{\sin(\pi f)}{\pi f} = \text{sinc}(f)
\end{aligned}
$$

Exercise 2.15

Find the Fourier Transform of $x(t) = e^{-at}u(t)$, $a > 0$.

2.7.1 Fourier Transform Properties

Table 2.1 lists the properties of the Fourier Transform that students were introduced to in their Signals and Systems class.

Table 2.1: Fourier Transform Properties

Property	$\boldsymbol{x(t)}$	$\boldsymbol{X(f)}$
1 Linearity	$ax_1(t)+bx_2(t)$	$aX_1(f)+bX_2(f)$
2 Time Shift	$x(t-\tau)$	$X(f)e^{-j2\pi f\tau}$
3 Time Scale	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)$
4 Frequency shift	$x(t)e^{j2\pi f_o t}$	$X(f-f_o)$
5 Time convolution	$x(t)*y(t)$	$X(f)Y(f)$
6 Time multiplication	$x(t)y(t)$	$X(f)*Y(f)$
7 Differentiation	$dx(t)/dt$	$j2\pi fX(f)$
8 Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j2\pi f}X(f)+\frac{1}{2}X(0)\delta(f)$
9 Duality	$X(t)$	$x(-f)$
10 Complex conjugate	$x^*(t)$	$X^*(-f)$

Example 2.10

Prove Properties 1, 2, 3 and 5

Solution

1) $x_3(t) = a x_1(t) + b x_2(t)$

$$X_3(f) = \int_{-\infty}^{\infty} (a x_1(t) + b x_2(t)) e^{-j2\pi f t} dt$$

$$= \int_{-\infty}^{\infty} \left(a x_1(t) e^{-j2\pi f t} + b x_2(t) e^{-j2\pi f t} \right) dt$$

$$= \int_{-\infty}^{\infty} a x_1(t) e^{-j2\pi f t} dt + \int_{-\infty}^{\infty} b x_2(t) e^{-j2\pi f t} dt$$

$$= a \int_{-\infty}^{\infty} x_1(t) e^{-j2\pi f t} dt + b \int_{-\infty}^{\infty} x_2(t) e^{-j2\pi f t} d$$

$$= a X_1(f) + b X_2(f)$$

2) $x_1(t) = x(t - \tau)$

$$X_1(f) = \int_{-\infty}^{\infty} x(t-\tau) e^{-j2\pi f t} dt$$

$u = t - \tau$

$du = dt$

$$X_1(f) = \int_{-\infty}^{\infty} x(u) e^{-j2\pi f (u+\tau)} du$$

$$= \int_{-\infty}^{\infty} x(u) e^{-j2\pi f u} e^{-j2\pi f \tau} du$$

$$= e^{-j2\pi f \tau} \int_{-\infty}^{\infty} x(u) e^{-j2\pi f u} du$$

$$= e^{-j2\pi f \tau} X(f)$$

3) $a > 0 \rightarrow a = |a|$

$x_1(t) = x(|a| t)$

$$X_1(f) = \int_{-\infty}^{\infty} x(|a| t) e^{-j2\pi f t} dt$$

$u = |a| t$

$du = |a| dt$

$$X_1(f) = \int_{-\infty}^{\infty} x(u) e^{-j2\pi f \frac{t}{|a|}} \frac{du}{|a|}$$

$$= \frac{1}{|a|} \int_{-\infty}^{\infty} x(u) e^{-j2\pi \frac{f}{|a|} u} du$$

$$= \frac{1}{|a|} X\left(\frac{f}{|a|}\right), \quad a > 0$$

$a<0 \rightarrow a=-|a|$

$$x_1(t)=x(-|a|t)$$

$$X_1(f)=\int_{-\infty}^{\infty} x(-|a|t)e^{-j2\pi ft}\,dt$$

$u=-|a|t \rightarrow t=\infty \rightarrow u=-\infty$

$du=-|a|\,dt \qquad t=-\infty \rightarrow u=\infty$

$$X_1(f)=\int_{\infty}^{-\infty} x(u)e^{-j2\pi f\frac{u}{-|a|}}\frac{du}{-|a|}$$

$$=-\int_{-\infty}^{\infty} x(u)e^{-j2\pi\frac{f}{-|a|}u}\frac{du}{-|a|}$$

$$=\frac{1}{|a|}\int_{-\infty}^{\infty} x(u)e^{-j2\pi\left(\frac{f}{-|a|}\right)u}\,du$$

$$=\frac{1}{|a|}X\left(\frac{f}{-|a|}\right) \qquad a<0$$

$$X_1(f)=\begin{cases}\frac{1}{|a|}X\left(\frac{f}{|a|}\right) & a>0\\ \frac{1}{|a|}X\left(\frac{f}{-|a|}\right) & a<0\end{cases}$$

$$X_1(f)=\frac{1}{|a|}X\left(\frac{f}{a}\right)$$

5) $Z(f)=X(f)Y(f)$

$$z(t)=\int_{-\infty}^{\infty} X(f)Y(f)e^{j2\pi ft}\,df$$

$$Y(f)=\int_{-\infty}^{\infty} y(\lambda)e^{-j2\pi f\lambda}\,d\lambda$$

$$z(t)=\int_{-\infty}^{\infty} X(f)\int_{-\infty}^{\infty} y(\lambda)e^{-j2\pi f\lambda}\,d\lambda\, e^{j2\pi ft}\,df$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f)y(\lambda)e^{-j2\pi f\lambda}e^{j2\pi ft}\,df\,d\lambda$$

$$=\int_{-\infty}^{\infty} y(\lambda)\int_{-\infty}^{\infty} X(f)e^{+j2\pi f(t-\lambda)}\,df\,d\lambda$$

$$=\int_{-\infty}^{\infty} y(\lambda)x(t-\lambda)\,d\lambda$$

$$=x(t)*y(t)$$

Exercise 2.16

Prove properties 4 and 6.

2.7.2 Using the Fourier Transform Properties

Many Fourier Transforms can be determined using the Fourier Transform properties of section 2.7.1. Based on (2.69) and the Fourier transform properties, a list of important Fourier transform pairs is given in Table 2.2.

Table 2.2: Important Fourier Transform Pairs

***x*(*t*)**	***X*(*f*)**
$A\delta(t)$	A
A	$A\delta(f)$
$A\Pi(t)$	$A\text{sinc}(f)$
$A\text{sinc}(t)$	$A\Pi(f)$
$A\Lambda(t)$	$A\text{sinc}^2(f)$
$A\text{sinc}^2(t)$	$A\Lambda(f)$
$e^{-at}u(t),\ a > 0$	$\frac{1}{a+j2\pi f}$
$e^{-a\|t\|},\ a > 0$	$\frac{2a}{a^2+(2\pi f)^2}$
$Ae^{j2\pi f_o t}$	$A\delta(f-f_o)$
$A\cos(2\pi f_o t)$	$\frac{A}{2}\delta(f-f_o)+\frac{A}{2}\delta(f+f_o)$
$A\sin(2\pi f_o t)$	$\frac{A}{2j}\delta(f-f_o)-\frac{A}{2j}\delta(f+f_o)$
$A\cos(2\pi f_o t+\phi)$	$\frac{Ae^{j\phi}}{2}\delta(f-f_o)+\frac{Ae^{-j\phi}}{2}\delta(f+f_o)$
$\text{sgn}(t)$	$\frac{1}{j\pi f}$
$u(t)$	$\frac{1}{j2\pi f}+\frac{1}{2}\delta(f)$

Example 2.11

Use the Fourier Transform properties and the Fourier Transforms for A and $e^{-at}u(t)$ to find the Fourier Transforms of $A\cos(2\pi f_o t)$ and $e^{-a|t|}$ ($a > 0$).

Solution

$$\begin{aligned}\mathcal{F}\{A\cos(2\pi f_o t)\} &= \mathcal{F}\left\{\frac{A}{2}e^{j2\pi f_o t}+\frac{A}{2}e^{-j2\pi f_o t}\right\}\\ &= \mathcal{F}\left\{\frac{A}{2}e^{j2\pi f_o t}\right\}+\mathcal{F}\left\{\frac{A}{2}e^{-j2\pi f_o t}\right\}\\ &= \frac{A}{2}\delta(f-f_o)+\frac{A}{2}\delta(f+f_o)\end{aligned}$$

(Uses linearity and frequency shift properties).

For $e^{-a|t|}$, we can rewrite it as $e^{-at}u(t) + e^{at}u(-t)$. Therefore $\mathcal{F}\{e^{-a|t|}\} = \mathcal{F}\{e^{-at}u(t)\} + \mathcal{F}\{e^{at}u(-t)\}$. We can see that $e^{at}u(-t)$ is $e^{-at}u(t)$ with t replaced by $-t$. Therefore we can find its Fourier transform using the time scaling property.

$$\mathcal{F}\{e^{at}u(-t)\} = \frac{1}{a - j2\pi f}$$

Therefore $\mathcal{F}\{e^{-a|t|}\}$ is given by

$$\mathcal{F}\{e^{-a|t|}\} = \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} = \frac{2a}{a^2 + (2\pi f)^2}$$

Example 2.12

If $\mathcal{F}\{x(t)\} = X(f)$, what is $\mathcal{F}\{x(at - b)\}$? What about $\mathcal{F}\{x[a(t - b)]\}$?

Solution

To solve this one, we need to use both the time scaling and time shifting properties, but the order in which we apply them is important and often leads to errors.

$$\mathcal{F}\{x(at - b)\} = \int_{-\infty}^{\infty} x(at - b) e^{-j2\pi f t} dt$$

Let $u = at-b$. Then $t = (u + b)/a = (u/a) + (b/a)$ and $dt = du/a$. Assuming a is positive then

$$\mathcal{F}\{x(at - b)\} = \int_{-\infty}^{\infty} x(u) e^{-j2\pi f\left(\left(\frac{u}{a}\right) + \left(\frac{b}{a}\right)\right)} du/a = e^{-j2\pi f b/a} \left[\frac{1}{a}\int_{-\infty}^{\infty} x(u) e^{-j2\pi\left(\frac{f}{a}\right)u} du\right]$$

where the term in the square bracket is the Fourier transform of $x(at)$ for $a > 0$. Therefore $\mathcal{F}\{x(at - b)\} = \frac{e^{-j2\pi f b/a}}{|a|} X\left(\frac{f}{a}\right)$. It can then be shown that $\mathcal{F}\{x[a(t - b)]\} = \frac{e^{-j2\pi f b}}{|a|} X\left(\frac{f}{a}\right)$.

2.7.3 Fourier Transform of a Periodic Signal

Consider a periodic signal $x(t)$ that has period T. In other words, $x(t) = x(t+T)$ for all t. The fundamental frequency of $x(t)$ is $f_o = 1/T$. The signal $x(t)$ can be represented by its Fourier series. Therefore

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_o t} \tag{2.70}$$

The Fourier transform of $x(t)$ is $X(f)$ which is given by:

$$
\begin{aligned}
X(f) &= \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_o t}\right\} \\
&= \sum_{n=-\infty}^{\infty} X_n \mathcal{F}\left\{e^{j2\pi n f_o t}\right\} \\
&= \sum_{n=-\infty}^{\infty} X_n \delta(f - n f_o)
\end{aligned} \tag{2.71}
$$

where the second step of (2.71) is due to the linearity property and the third is from the 8th row of Table 2.2.

Exercise 2.17

Find the Fourier Transform of the periodic signals of example 2.8 and Exercise 2.9.

2.7.4 Frequency Response of a Linear Time-Invariant System

We know from (2.23) that the input-output relationship of linear time-invariant systems is $y(t) = x(t)*h(t)$. If we take the Fourier Transform of both sides of (2.23) we obtain:

$$\boxed{Y(f) = X(f)H(f)} \tag{2.72}$$

where $Y(f) = \mathcal{F}\{y(t)\}$, $X(f) = \mathcal{F}\{x(t)\}$, and $H(f) = \mathcal{F}\{h(t)\}$. Equation (2.72) is a result of the convolution property of the Fourier Transform. The frequency response of a linear time-invariant system is defined as $Y(f)/X(f)$, therefore **the frequency response of a linear time-invariant system is $H(f) = \mathcal{F}\{h(t)\}$.**

Consider the case where $x(t) = A\cos(2\pi f_o t + \phi)$ which is input to the circuit shown in Figure 2.15. The output $y(t)$ can be found from $Y(f)$ in (2.72).

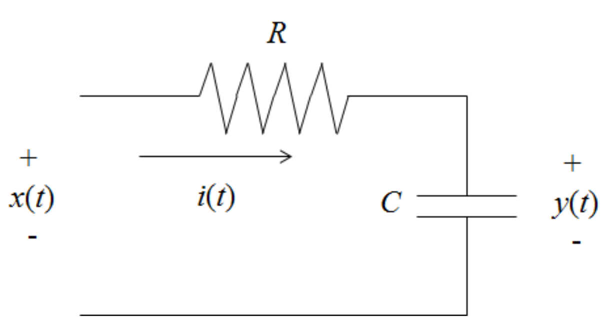

Figure: 2.15: Circuit whose frequency response is to be determined.

We know from circuit theory that

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \tag{2.73}$$

and

$$x(t) = i(t)R + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \tag{2.74}$$

Taking the Fourier Transforms of both sides and assuming that *i*(*t*) has no DC component (which is expected since *x*(*t*) has no DC component either), we get

$$Y(f) = \frac{I(f)}{j2\pi fC} \tag{2.75}$$

and

$$X(f) = I(f)R + \frac{I(f)}{j2\pi fC} = I(f)\left(R + \frac{1}{j2\pi fC}\right) \tag{2.76}$$

Dividing (2.75) by (2.76) we get

$$H(f) = \frac{1}{1+j2\pi fRC} \tag{2.77}$$

and therefore the impulse response of the circuit of Fig. 2.14 is:

$$h(t) = \frac{1}{RC}e^{-\left(\frac{t}{RC}\right)}u(t) \tag{2.78}$$

From Table 2.1, $X(f) = \frac{A}{2}\delta(f - f_o)e^{j\phi} + \frac{A}{2}\delta(f + f_o)e^{-j\phi}$, therefore $Y(f)$ is given by:

$$Y(f) = \frac{A}{2(1+j2\pi f_o RC)}\delta(f - f_o)e^{j\phi} + \frac{A}{2(1+j2\pi f_o RC)}\delta(f + f_o)e^{-j\phi} \tag{2.79}$$

Complex functions $A(f)+jB(f) = \sqrt{A^2(f) + B^2(f)}e^{j\tan^{-1}(B(f)/A(f))}$ and $1/(A(f)+jB(f))$ is therefore given by:

$$\frac{1}{A(f)+B(f)} = \frac{1}{\sqrt{A^2(f)+B^2(f)}}e^{-j\tan^{-1}(B(f)/A(f))} \tag{2.80}$$

Therefore we can rewrite (2.79) as:

$$\begin{aligned} Y(f) = {} & \frac{A}{2\sqrt{1+(2\pi f_o RC)^2}}\delta(f - f_o)e^{j\left(\phi - \tan^{-1}(2\pi f_o RC)\right)} \\ & + \frac{A}{2\sqrt{1+(2\pi f_o RC)^2}}\delta(f + f_o)e^{-j(\phi - \tan^{-1}(2\pi f_o RC))} \end{aligned} \tag{2.81}$$

Taking the inverse Fourier transform of (2.81) we get

$$y(t) = \frac{A}{\sqrt{1+(2\pi f_o RC)^2}}\cos(2\pi f_o t + \phi - tan^{-1}(2\pi f_o RC)) \tag{2.82}$$

We can see from (2.82) that when a sinusoid is input to a linear time-invariant system, the output is also sinusoid with the same frequency but with a different amplitude and a different phase. We state that the amplitude response of the circuit of Fig. 2.14, $|H(f)|$, is given by:

$$|H(f_o)| = \frac{1}{\sqrt{1+(2\pi f_o RC)^2}} \tag{2.83}$$

and the phase response of the circuit is given by $\angle H(f)$ below:

$$\angle H(f_0) = -\tan^{-1}(2\pi f_o RC) \tag{2.84}$$

For an input sinusoid with frequency f_o, we multiply its amplitude by (2.83) and we add (2.84) to its phase to find the output sinusoid. For any frequency f, we can find the amplitude and phase responses for the circuit in Fig. 2.14 from (2.78). They are simply (2.83) and (2.84) with f_o replaced by f.

For any linear time-invariant system with frequency response $H(f)$, the amplitude response and phase responses are given by

$$|H(f)| = \sqrt{Re\{H(f)\}^2 + Im\{H(f)\}^2} \tag{2.85}$$

$$\angle H(f) = \tan^{-1}(Im\{H(f)\}/Re\{H(f)\}) \tag{2.86}$$

We can also write $H(f)$ as:

$$H(f) = |H(f)|e^{j\angle H(f)} \tag{2.87}$$

Example 2.13

A linear time-invariant system has impulse response $h(t)$ = 12sinc(40t-5). What are $H(f)$, $|H(f)|$ and $\angle H(f)$? What is $y(t)$ if $x(t)$ = 3cos(2π12t)? If $x(t)$ = 4cos(2π18t)? If $x(t)$ = 3cos(2π12t) + 4cos(2π18t)?

Solution

$$h(t) = 12\,\mathrm{sinc}(40t-5)$$

from example 2.12

de l'exemple 2.12

$$H(f) = \frac{12}{40}\,\Pi\left(\frac{f}{40}\right)e^{-j\frac{\pi}{4}f}$$

$$= 0.3\,\Pi\left(\frac{f}{40}\right)e^{-j\frac{\pi}{4}f}$$

$$H(12) = 0.3\,\Pi\left(\frac{12}{40}\right)e^{-j3\pi}$$

$$= 0.3\,e^{-j3\pi}$$ since / car $\Pi(0.3) = 1$

$$= |H(12)|\,e^{j\angle H(12)}$$

therefore / donc

$|H(12)| = 0.3$

$\angle H(12) = -3\pi$

$x(t) = 3\cos(2\pi 12t)$

$y(t) = (3 \times 0.3)\cos(2\pi 12t - 3\pi)$

$= 3 \times 0.3 \cos(2\pi 12t - 2\pi - \pi)$

$= 3 \times 0.3 \cos(2\pi 12t - \pi)$

$= -0.9\cos(2\pi 12t)$

$H(18) = 0.3\Pi\left(\frac{18}{40}\right)e^{-j4.5\pi} = 0.3e^{-j4.5\pi}$

$x(t) = 4\cos 2\pi 18t$

$y(t) = 1.2\cos(2\pi 18t - 4.5\pi)$

$= 1.2\cos(2\pi 18t - 4\pi - 0.5\pi)$

$= 1.2\cos(2\pi 18t - 0.5\pi)$

$= 1.2\sin(2\pi 18t)$

$x(t) = 3\cos(2\pi 12t) + 4\cos(2\pi 18t)$

Since system is LTI

Parce que le système est LIT

$y(t) = -0.9\cos(2\pi 12t) + 1.2\sin(2\pi 18t)$

Exercise 2.18

The impulse response of an LTI system is $h(t) = \Lambda(6t\text{-}2)$. What is the output if the input is $x(t) = 8\cos(2\pi 4t) + 6\cos(2\pi 11t)$?