HW-4
CELESTIAL MECHANICS

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Exercise: *five ground stations (GS) have performed instantaneous range measurements to an Erth satellite. The observations are contained in the file range.dat, where the first column indicates the observation time in seconds from the epoch $t_0 = 0$, the second column the ID of the observing GS, while the third column contains the measured range in meters. It is known that the satellite is orbiting on the aquatorial plane of a point-mass Earth (with $R_\oplus =$ 6378.137 km and $\omega_\oplus = 7.29115854579367 \cdot 10^{-5}$). The GS's are also located on the equator. At each epoch $t_0 = 0$ the raference meridian of the Earth-fixed reference system is aligned with inertial x-axis. Our aim is to estimenate the initial state of the satellite and correction to the approximate values of GM ($= 398600.00 Km^3/s^2$) and the coordinates of GS's 2 to 5.*

Solution:

The east longitudes of the five GS in the Eart-fixed reference frame are:

- $\lambda_1 = 8.0^o$
- $\lambda_2 = 79.8^o$
- $\lambda_3 = 151.9^o$
- $\lambda_4 = 224.1^o$
- $\lambda_5 = 296.2^o$

and the state vector, which rappresents the positions and the velocity, is:

- $x_0 = 8285.00$ Km
- $y_0 = 4783.00$ Km
- $\dot{x_0} = -4.60$ Km/s
- $\dot{y_0} = 7.90$ Km/s.

In general, the dynamics and the measurmentes involve significant nonlinear relationships. If we consider the dynamical state $\bar{X}(t) = (\bar{r}(t), \bar{v}(t))$ it is governed by nonlinear dynamical equations:

$$\dot{\bar{X}}(t) = \bar{F}(\bar{X}, \bar{P}, t) \qquad \bar{X}(t_0) = \bar{X}_0 \tag{1}$$

where $\bar{P}$ is a vector of costant parameters decribing the force model (value of GM) and $\bar{X}_0$ is the state vector (position and velocity) at the epoch t_0. Instead the measurements can be modeled by the function $\bar{G}(\bar{X}, \bar{Q}, t)$:

$$\bar{Y}(t) = \bar{G}(\bar{X}, \bar{Q}, t) + \epsilon(t) \tag{2}$$

where $\bar{Y}$ is the vector of observations, $\bar{Q}$ is an other vector of costants which contains the geometric parameters (or non-dynamical) and ϵ represents the measurement errors.

Our goal is to improve initial value for the initial state vector and the paramenter $\bar{P}$, $\bar{Q}$ using a set of measurements and assuming that the reference solution (or nominal orbit) $\bar{X}^*(t, \bar{P^*})$ and the measurement model $\bar{G}(t, \bar{X}^*, \bar{Q}^*)$ are known. In our program these results are given by the functions $EoM.m$, which represents the equation of motion, and $G.m$, which evaluates the measurement model.

Now we introduce the variations of the state vector, the observations and the parameters $\bar{x}, \bar{y}, \bar{p}$ and $\bar{q}$. Expanding the force and the measurment model to the first order and after a little manipulation we obtain the state deviation and the differential measurement:

$$\dot{\bar{x}}(t) = \bar{A}(t)\bar{x}(t) + \bar{B}(t)\bar{p} \tag{3}$$

$$\bar{y}_i = \tilde{\bar{H}}_i\bar{x}_i + \bar{K}_i\bar{q} + \epsilon_i \tag{4}$$

where:

$$\bar{A}(t) = \left[\frac{\partial \bar{F}(t)}{\partial \bar{X}(t)}\right]^* \qquad \bar{B}(t) = \left[\frac{\partial \bar{F}(t)}{\partial \bar{P}}\right]^* \tag{5}$$

$$\tilde{\bar{H}}_i = \left[\frac{\partial \bar{G}}{\partial \bar{X}}\right]^*_i \qquad \bar{K}_i = \left[\frac{\partial \bar{G}}{\partial \bar{Q}}\right]^*_i \tag{6}$$

For each matrix we wrote a matlab fuction which calculates all partial derivatives: $AMAT.m$, $BMAT.m$, $KMAT.m$ and $HtMAT.m$.

We need to relate the measurement deviation at time t_i to the state deviation $\bar{x}_0$ and parameter deviation, therefore we compute the dynamical state transition matrix $\Phi_{xx}(t, t_0)$ and the parameters state transition matrix $\Phi_{xp}(t, t_0)$ solving the following systems:

$$\dot{\bar{\Phi}}_{xx}(t, t_0) = \bar{A}(t)\bar{\Phi}_{xx}(t, t_0) \qquad \bar{\Phi}_{xx}(t_0, t_0) = I \tag{7}$$

$$\dot{\bar{\Phi}}_{xp}(t, t_0) = \bar{A}(t)\bar{\Phi}_{xp}(t, t_0) + \bar{B}(t) \qquad \bar{\Phi}_{xp}(t_0, t_0) = 0. \tag{8}$$

We can now write the solution of VE (variational equations) as:

$$\bar{x}(t) = \bar{\Phi}_{xx}(t, t_0)\bar{x}_0 + \bar{\Phi}(t, t_0)\bar{p}. \tag{9}$$

Furthermore, the differential measurement equation can be written in this form:

$$\bar{y}_i = \tilde{\bar{H}}_i\bar{\Phi}_{xx}(t, t_0)\bar{x}_0 + \tilde{\bar{H}}_i\bar{\Phi}_{xp}(t, t_0)\bar{p} + \bar{K}_i\bar{q} + \epsilon_i \tag{10}$$

and if we define:

$$\bar{H}_i = (\bar{H}_i^x \qquad \bar{H}_i^p \qquad \bar{K}_i) \tag{11}$$

where $\bar{H}_i^x = \tilde{\bar{H}}_i\bar{\Phi}_{xx}(t, t_0)$ and $\bar{H}_i^p = \tilde{\bar{H}}_i\bar{\Phi}_{xp}(t, t_0)$.
We can write:

$$\bar{y}_i = \bar{H}_i\bar{x}_0 + \epsilon_i \tag{12}$$

where $\bar{x}_0$ is the system deviation vector.

In our problem we have n parameter and m total observations with $m > n$. That means we can apply the least square method to solve our system of equations. This method allows us to select $\hat{x}$ of x as that value that minimizes the sum of the squares of the calulated observation residuals. We consider the performance index:

$$J(\bar{x}) = \frac{1}{2}\epsilon^T\bar{W}\epsilon \tag{13}$$

where W is measurement weight matrix but in our problem we have W = I and we can write the performace index in the following form:

$$J(\bar{x}) = \frac{1}{2}(\bar{y} - \bar{H}\bar{x})^T(\bar{y} - \bar{H}\bar{x}). \tag{14}$$

It is easy to show that, if the normal matrix $\bar{H}^T\bar{H}$ is positve define it will have an inverse and $\hat{x}$ is exactly the solution that minimizes the sum of square:

$$\hat{x} = (\bar{H}^T\bar{H})^{-1}\bar{H}^T\bar{y} \qquad (1) \tag{15}$$

In the main of our Matlab progam ($MAIN.m$) we have implemented an algorithm called batch processor. It takes all the observations, maps them back and forward to the initial time using the state transition matrix (how we have previus explained) and then solves Equation 1 once with all the data.
The basic steps for implementing the algorithm follow below:

1. Integrate from t_0 to t_{max} and from t_{min} to t_0:

$$\dot{\bar{X}}(t) = \bar{F}(\bar{X}, \bar{P}, t) \tag{16}$$

$$\dot{\bar{\Phi}}_{xx}(t, t_0) = \bar{A}(t)\bar{\Phi}_{xx}(t, t_0) \qquad \bar{\Phi}_{xx}(t_0, t_0) = I \tag{17}$$

$$\dot{\bar{\Phi}}_{xp}(t, t_0) = \bar{A}(t)\bar{\Phi}_{xp}(t, t_0) + \bar{B}(t) \qquad \bar{\Phi}_{xp}(t_0, t_0) = 0. \tag{18}$$

2. Write a cycle to accumulate all observations:

$$\bar{y}_i = \bar{Y}_i - \bar{G}(\bar{X}_i, t_i) \qquad \bar{H}_i = (\bar{H}_i^x \qquad \bar{H}_i^p \qquad \bar{K}_i) \tag{19}$$

$$\bar{L} = \sum_{i=1}^{m} \bar{H}_i^T \bar{H}_i \qquad \bar{N} = \sum_{i=1}^{m} \bar{H}_i^T \bar{y}_i \tag{20}$$

3. Solve the least square:

$$\hat{x}_k = (\bar{L}^{-1}\bar{N}, \tag{21}$$

   where k is the number of the iteration. In our work the method converge so rapidly that we stop the processor aleady after the third iteration.

4. Update nominal trajectory and go to step 1:

$$\bar{X}_0^* = \bar{X}_0^* + \hat{x}_0 \qquad \bar{x}_0 = \bar{x}_0 - \hat{x}_0 \tag{22}$$

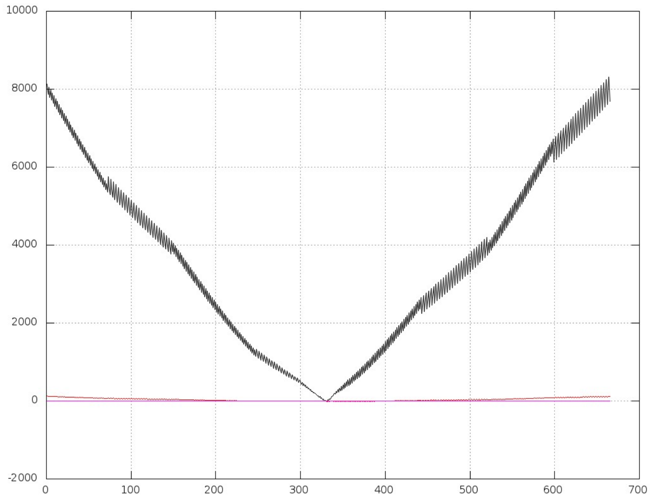

Figure 1: This figure shows the three iteration of the batch processor: the first one in black, the second one in red and the third one in magenta.

The results we have obtained from our Matlab program are shown in the following tables and at the end we have plotted the orbit obtain from the given state vactor and the orbit that we have determineted using the statistical method.

	1^{st} Iteration	2^{nd} Iteration	3^{rd} Iteration
x_0 (Km)	8283.238938	8285.458844	8285.443002
y_0 (Km)	4794.621047	4783.606118	4783.602737
$\dot{x}_0$ (Km/s)	-4.619632	-4.609806	-4.609807
$\dot{y}_0$ (Km/s)	7.985906	7.984426	7.984419
GM (Km)	399491.511738	398600.577923	398600.441335
GS_2 (rad)	1.397214	1.396263	1.396263
GS_3 (rad)	2.652654	2.652902	2.652900
GS_4 (rad)	3.910588	3.909539	3.909538
GS_5 (rad)	5.166602	5.166175	5.166175
$R_\oplus$ (Km)	6378.137	6378.137	6378.137
$\omega_\oplus$ rad/s	7.2911585457936695e-05	7.2911585457936695e-05	7.2911585457936695e-05
θ (rad)	0.000000	0.000000	0.000000
GS_1 (rad)	1.396263e-01	1.396263e-01	1.396263e-01

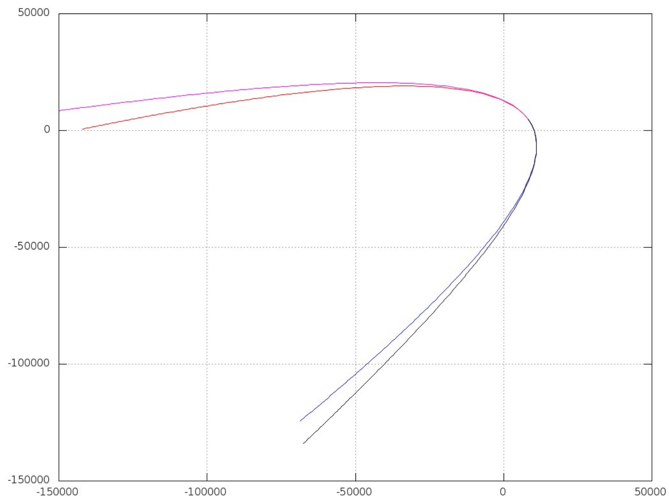

Figure 2: The red-blue orbit represents the given orbit and the magenta-black one is the orbit obtained using the given observations. They are both drawn in the Earth-fixed frame.