Assignment 2 Computational Models - Spring 2022

Exercise 1

Define the language L_n over alphabet $\sum_n = \{1, 2, ..., n\}$ to be the set of words which do not contain all the letters from $\sum_n$, now lets describe DFA $M = (Q, \sum, \delta, q_0, F)$ s.t M accept L_n.

$$Q = \{q_k | k \subseteq \{\Sigma_n\}\}, \Sigma = \Sigma_n, q_0 = q_\phi, F = \{Q/\{1, 2, 3 \dots n\}\}$$

$$\delta(q_k, \sigma) = \begin{cases} q_{k\cup\sigma} & \text{if } \sigma \notin \{k\} \\ q_k & \text{if } \sigma \in \{k\} \end{cases}$$

its sufficient to see that the following claim hold to proof M accept L_n

claim 0.1. *the DFA M **not** accept w iff $\{\sigma : \sigma \in w\} = \{k_n\}$ when k_n is the set of all the letters from Σ_n*

Proof. First lets notice that Q contain all the subset from Σ_n including the empty set witch correspond the empty world i.e $\{\sigma\}$ so in total we looking at $O(2^n)$ state for finite alphabet size n.
$\Rightarrow w \notin L_m$ if $w \in \Sigma_n^*$ since M have only one state lets call it q_n which is not accepting state, hence $\hat{\delta}(q_0, w) = q_n$ we must go through at least n state to achieve q_n.
now using reduction on the number of district steps we done for $\hat{w} = \phi$ $\delta(q_0, \phi) = q_0$ and for some $\hat{w}, \hat{\sigma}$ s.t

$$\hat{\sigma} \notin |\{\sigma : \sigma \in \hat{w}\}| = |k|, \hat{\delta}(q_0, \hat{w}) = q_k \Rightarrow \delta(q_k, \hat{\sigma}) = q_{k\cup\hat{\sigma}} = \hat{\delta}(q_0, (\hat{w}||\hat{\sigma})), \{\sigma : \sigma \in \hat{w}||\hat{\sigma}\}| = |k+1|$$

hence we done n district steps so $|\{\sigma : \sigma \in w\}| = n$ since M is DAG (except the self loops) w contain all the letters from the alphabet
$\Leftarrow w := \forall\sigma \in \Sigma_n | \sigma \in \{w\}$ now lets look at the first $\sigma \in w$, so $\delta(q_0, \sigma) = q_\sigma \in Q$ if the next letter in w hold $\hat{\sigma} \neq \sigma, \delta(q_\sigma, \hat{\sigma}) = q_\sigma$ but we know that w have n district letters so in total we get

$$\hat{\delta}(q_0, w) = \hat{\delta}(q_{\sigma_1}, \underbrace{\hat{w}}_{\{\hat{w}\} = \{w/\sigma_1\}}) = \hat{\delta}(q_{\{\sigma_1, \sigma_2 \dots \sigma_k\}}, \underbrace{\hat{w}}_{\{\hat{w}\} = \{w/\sigma_1, \sigma_2 \dots \sigma_k\}}) = \delta(q_{\{w/\sigma_n\}}, \sigma_n) = q_n \notin F$$

hence $w \notin L_m$.

□

Exercise 2

When A,B regular languages over same alphabet. $A \diamond B = \{xy : x \in A \wedge y \in B \wedge |x| = |y|\}$ lets define

$$\Sigma = \{0, 1\}, A = L(0^*), B = L(1^*)$$

by assuming that $A \diamond B$ is regular language, using the property of regular language operation we get.

$$(A \diamond B) \cap \{0^*, 1^*\} = \{0^i, 1^i | i \geq 0\}$$

which lead to contradiction since $\{0^i, 1^i | i \geq 0\}$ is not regular language

Exercise 3

(a) $L_1 = \{w : \#a(w) \geq \#b(w)\}$ over $\Sigma = \{a, b, c\}$
L_1 is not regular while assuming it is. for fix ℓ lets $w = b^\ell a^\ell$,$w \in L_1$ based on Pumping Lemma we can notice $|w| > \ell$, and for any patrion of $w = xyz$ such that $|y| > 0, |xy| \leq \ell$ hence y is in the form of b^k now lets look at

$$b^{\ell+k}a^\ell \Rightarrow \#_b(b^{\ell+k}a^\ell) > \#_a(b^{\ell+k}a^\ell) \Rightarrow b^{\ell+k}a^\ell \notin L_1$$

we got an contradiction hence L_1 is not regular

(b) $L_2 = \{w : |w| \in \mathbb{N}_{even} \wedge w = w^R)\}$ over $\Sigma = \{0, 1\}$
L_2 is not regular while assuming it is. for fix ℓ lets $w = 1^\ell 001^\ell$,$w \in L_2$ based on Pumping Lemma we can notice $|w| > \ell$, and for any patrion of $w = xyz$ such that $|y| > 0, |xy| \leq \ell$ hence y is in the form of 1^k now lets look at

$$(1^{k+\ell}001^\ell) \neq (1^{k+\ell}001^\ell)^R \Rightarrow 1^{k+\ell}001^\ell \notin L_2$$

we got an contradiction hence L_2 is not regular

(c) $L_3 = \{w : |w| \in \mathbb{N}_{even} \wedge w = w^R)\}$ over $\Sigma = \{0\}$
L_3 is regular language since we can be written as regular expression

$$L_3 = L((00)^*) = \{00\}^* = \{\epsilon, 00, 0000, \dots\}$$

(d) $L_4 = \{w : |w| \in \mathbb{N} \text{ s.t } |w| = n^3\}$ over $\Sigma = \{0, 1\}$
L_4 is not regular while assuming it is. for fix ℓ lets $w = 0^{\ell^3}$,$w \in L_4$ based on Pumping Lemma we can notice $|xyz| > \ell$, and for any patrion of $w = xyz$ such that $|y| > 0, |xy| \leq \ell$. hence for some k,m such that $k + m \leq \ell$ this means that :

$$x = 0^k, y = 0^m, z = 0^{\ell^3-k-m} \Rightarrow, xyz = 0^{\ell^3}$$

by our assumption for any $n \in \mathbb{N}$, $xy^nz = 0^{\ell^3+m(n-1)} \in L_4$, lets choose n=2

$$\text{since } 1 \geq m \geq \ell \text{ we will get for some } t \Rightarrow \underline{t^3 = \ell^3 + m}$$

$$\text{but } t^3 \geq (\ell+1)^3 = \ell^3 + 3\ell^2 + 3\ell + 1 > \ell^3 + \ell \geq \ell^3 + m \Rightarrow \underline{t^3 > \ell^3 + m}$$

we got an contradiction hence L_4 is not regular

Exercise 4

claim 0.2. *For language L s.t L satisfies pumping lemma with pumping constant ℓ , $L||L$ satisfies pumping lemma with pumping constant 2ℓ*

Proof. Lets $w_1, w_2 \in L'$ i.e $w_1 \in L \wedge w_2 \in L$ and lets assume $|w_1w_2| \geq 2\ell$ to imply the pumping lemma, now we can notice that there is 2 possible scenario

- $|w_1| \geq \ell$ hence we can write w_1 as $w_1 = xyz \Rightarrow$ and we can write $w_1 w_2$ as $w_1 w_2 = xyz||w_2$ which stand with the lemma property

- $|w_1| < \ell$ so $|w_2| > \ell$ can write w_2 as $w_2 = xyz$ and $|xy| \leq \ell \Rightarrow |w_1 xy| \leq 2\ell, |y| > 0$ we can write $w_1 w_2$ as $\underbrace{w_1 x}_{x'} yz$ which stand with the claim property

$\square$

Exercise 5

(a)

Lets L be a regular language over alphabet Σ. and we will proof the language L' define $L' = \{xyz : xy^R z \in L\}$ is regular. since L is regular $\Rightarrow$ exists some DFA that accept L $M = (Q, \Sigma, \delta, q_0, F)$ now lets define few new DFA's s.t:

- $M_{q_0,q_k} = (Q, \Sigma, \delta, q_0, q_k)$ for any $q_k \in Q$ this DFA will cover any path that accessible from q_0

- now lets look at $M_{q_k,q_j} = (Q, \sum, \delta, q_k, q_j)$lets the language of M_{q_k,q_j} is define by $L(M_{q_k,q_j}) = \{w : \exists q_k, q_j \in Q|$ exists path s.t $q_k \rightsquigarrow q_j\}$,since $L(M_{q_k,q_j})$ is regular $\Rightarrow$ $rev(L(M_{q_k,q_j}))$ is regular (Recitation 3 ex.1). for any $q_k, q_j \in Q$ lets define the following language $L(M_{q_k,q_j})^R$

- $M_{q_k,F} = (Q, \Sigma, \delta, q_k, F)$ for any $q_k \in Q$ this DFA will cover any path that end in F

claim 0.3. $L' = \cup_{q_i,q_j} L(M_{q_0,q_i})||L(M_{q_i,q_j})^R||L(M_{q_j,F})$

Proof.

$$w = xyz \in L' \Leftrightarrow xy^R z \in L \Leftrightarrow \exists \hat{\delta}(\hat{\delta}(\hat{\delta}(q_0, x), y^R), z) \in F_m \Leftrightarrow$$

$$\exists M_{q_0,q_i} : \hat{\delta}(q_0, x) \in F_{M_{q_0,q_i}} \wedge \exists L(M_{q_i,q_j})^R : y \in L(M_{q_i,q_j})^R \wedge \exists M_{q_j,F} : \hat{\delta}(q_j, z) \in F_{M_{q_j,F}} = F_m$$

$$\Leftrightarrow xyz \in \cup_{q_i,q_j} L(M_{q_0,q_i})||L(M_{q_i,q_j})^R||L(M_{q_j,F})$$

$\square$

(b)

Lets L be a regular language over alphabet Σ. and we will proof the language L' define $L' = \{xy \in \Sigma^* : (x \in L) \text{ XOR } (y \in L)\}$ is regular. using the closure property of regular language

- $L = \{x \in \Sigma^* : x \in L\}$ and $\overline{L} = \{x \in \Sigma^* : x \notin L\}$ are regular.

- $L||\overline{L} = \{xy \in \Sigma^* : x \in L \wedge y \notin L\}$ and $\overline{L}||L = \{xy \in \Sigma^* : x \notin L \wedge y \in L\}$ are regular.

- $(L||\overline{L}) \cup (\overline{L}||L) = \{xy \in \Sigma^* : (x \in L \wedge y \notin L) \vee (x \notin L \wedge y \in L) = (x \in L) \text{ XOR } (y \in L)\}$ is regular

Exercise 6

(a) $\{w \in \Sigma^* : \#_0(w) \leq 3\}$
lets express the following as regular expression

$$\underbrace{1^*}_{I} \cup \underbrace{1^*01^*}_{II} \cup \underbrace{1^*01^*01^*}_{III} \cup \underbrace{1^*01^*01^*01^*}_{IV}$$

(I) is the regular expression of the language that contain non zeros at all.
(II) is the regular expression of the language that contain exactly 1 zero.
(III) is the regular expression of the language that contain exactly 2 zero.
(IV) is the regular expression of the language that contain exactly 3 zero.
hence combine them all together will hold regular expression of the language that contain at most 3 zeros.

(b) $\{w \in \Sigma^* : |w| \bmod 4 = 2\}$
lets express the following as regular expression

$$\underbrace{(0 \cup 1)(0 \cup 1)}_{(I)} \underbrace{((0 \cup 1)(0 \cup 1)(0 \cup 1)(0 \cup 1))^*}_{II}$$

(I) is the regular expression of the language that size is exactly 2.
(II) is the regular expression of the language from size 0,4,8..
hence the concatenate of (I) and (II) will hold the $|w| \bmod 4 = 2$ property