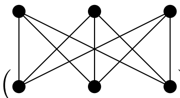
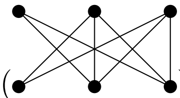
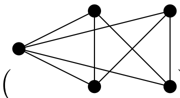
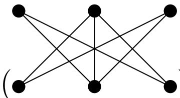
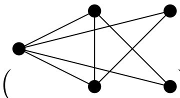
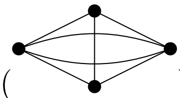
图论作业(5.5)

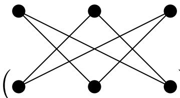
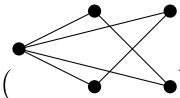
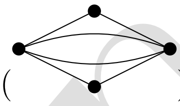
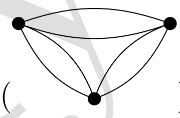
中国人民大学 信息学院 崔冠宇 2018202147

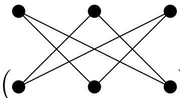
P48, T10 求 $K_{3,3}$ 生成树的个数.

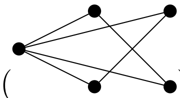
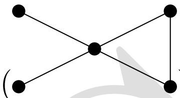
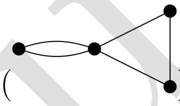
解: 由 Cayley 公式:

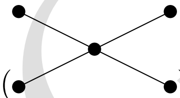
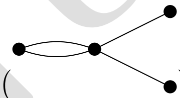
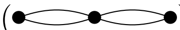
$\tau($  $) = \tau($  $) + \tau($  $)$

$= \tau($  $) + 2\tau($  $) + \tau($  $)$

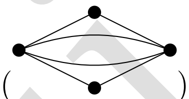
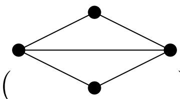
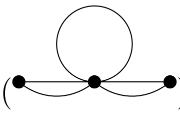
$= \tau($  $) + 3\tau($  $) + 3\tau($  $) + \tau($  $)$

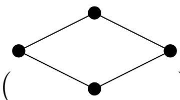
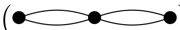
易知: $\tau($  $) = 6$(因为这是个圈),

$\tau($  $) = \tau($  $) + \tau($  $)$

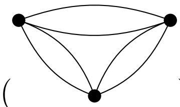
$= \tau($  $) + 2\tau($  $) + \tau($  $)$

$= 1 + 2 \times 2$(因为左边重边二选一) $+ 2^2$(左右各二选一) $= 9$,

$\tau($  $) = \tau($  $) + \tau($  $)$

$= \tau($  $) + 2\tau($  $)$

$= 4$(因为这是个圈) $+ 2 \times 2^2$(左右各二选一) $= 12$,

$\tau($  $) = C_3^2 \times 2^2$(三组重边选两组, 再分别二选一) $= 12$.

所以 $\tau(K_{3,3}) = 6 + 3 \times 9 + 3 \times 12 + 12 = 81.$[1]

[1]事实上, 由矩阵树定理(Matrix-Tree Theorem)可以证得: $\tau(K_{m,n}) = m^{n-1}n^{m-1}$.

P49, T29 证明: 若 T 是顶数不小于 3 的树, 则 T 的直径是 2 的充分必要条件是 T 是星.

证明:

- T 是星 $\Rightarrow T$ 的直径为 2: 这是显然的.

- T 是星 $\Leftarrow T$ 的直径为 2: 任取两片叶 $l_1, l_2 \in V(T)$, u_1, u_2 分别与二者相邻, 我们证明 $u_1 = u_2$. 若不然, 因为 T 是树, u_1, u_2 之间必存在长度大于 0 的路 $P(u_1, u_2)$(因为 l_1, l_2 均为叶子, 不可能在路上), 取 $(l_1, u_1) \cup P(u_1, u_2) \cup (u_2, l_2)$, 长度大于 2, 与 $d(T) = 2$ 矛盾, 故 $u_1 = u_2$. 由于叶子是任取的, 这说明 T 的所有叶子有共同的相邻点, 即 T 是星.

□

P49, T30 若 G 是加权连通图, 且有一个长为 m 的圈 C, C 上的边的权相等, 是 $E(G)$ 中边权最小值, 则 G 中至少有 m 棵不同的最优树, 试加证明.

证明: (构造性证明) 由题设, 不妨设圈上的边为 $e_1, e_2, \ldots, e_m$, 其它边为 $e_{m+1}, \ldots e_s$. 考虑他们从小至大排序, 圈上每个 $e_i (i = 1, 2, \ldots, m)$ 都可能处在第 m 位. 用 Kruskal 算法构造最优树时, 圈上 $m - 1$ 条边都会被纳入最小生成树中, 另一条边不会被纳入最小生成树, 这样的情况有 m 种, 对应的不同的最小生成树就至少有 m 棵. □